

$RR = \#BS$ VIA LOCALIZATION OF INDEX

TAKAHIKO YOSHIDA

For a pre-quantized closed symplectic manifold a *Riemann-Roch number* is defined to be the index of a Spin^c Dirac operator with coefficients in the pre-quantization line bundle.

Suppose the pre-quantized symplectic manifold is equipped with a structure of a Lagrangian fiber bundle. A fiber of the Lagrangian fiber bundle is said to be *Bohr-Sommerfeld* if the restriction of the pre-quantization line bundle to the fiber is trivially flat. Bohr-Sommerfeld fibers appear discretely. Then Andersen showed in [1] that the Riemann-Roch number is equal to the number of Bohr-Sommerfeld fibers.

Similar phenomena have been observed for several examples of Lagrangian fiber bundles with singular fibers, such as,

- moment maps of toric varieties [2],
- the Gelfand-Cetlin system for complex flag manifolds [7],
- Goldman's Hamiltonian system on the moduli space of flat $\text{SU}(2)$ -bundle on a Riemann surface [5, 9],

and for the following generalizations

- pre-symplectic toric manifolds [10],
- Spin^c manifolds [6],
- torus manifolds [11, 8],

and so on.

In this series of lectures I explain an approach to investigate these phenomena by using a localization of index. The plan of this lectures is as follows

- § 1 Hamiltonian mechanics on symplectic manifolds
- § 2 Lagrange fiber bundles
- § 3 Quantization
- § 4 An approach to $RR = \#BS$ by localization of index.

In § 1 I briefly explain the Hamiltonian mechanics on symplectic manifolds. § 1 is an introduction to symplectic manifolds and also necessary for § 3. In § 2 I introduce a notion of a Lagrange fiber bundle and prove its classification theorem. In § 3 I explain some background of the above phenomena. In § 4 I explain a localization technique of the index of a Dirac-type operator and its application to $RR = \#BS$. § 4 is based on a joint work [3, 4] with Mikiyo Furuta and Hajime Fujita. I show that if a prequantized symplectic manifold admits a structure of a singular Lagrangian fiber bundle, then, the Riemann-Roch number is localized on nonsingular Bohr-Sommerfeld fibers and singular fibers. The idea used here is the Witten deformation that is used to prove the Morse inequalities in [12].

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, MEIJI UNIVERSITY, 1-1 HIGASHIMITA 1-CHOME, TAMA-KU, KAWASAKI, 214-8571, JAPAN
E-mail address: takahiko@math.meiji.ac.jp