Manifolds covered by lines, defective manifolds and a restricted Hartshorne Conjecture

(joint work with F. Russo)

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Context: $k = \mathbb{C}$.
$X \subset \mathbb{P}^N$ smooth irreducible nondegenerate of dimension n and codimension c.

Prime Fano manifolds of high index

Definition

$X \subset \mathbb{P}^N$ is a *prime Fano manifold of index* $i(X)$ if its Picard group is generated by the hyperplane section class H and $-K_X = i(X)H$ for some positive integer $i(X)$.

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We say that X has "high index" if $i(X) \geq \dfrac{n+3}{2}$. Dual defective and some special secant defective manifolds (to be defined later) provide interesting examples.

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- In the special case of prime Fanos, $\dim \mathcal{L}_x = i(X) - 2$. So, $i(X) \geq \dfrac{n+3}{2}$ is equivalent to $\dim \mathcal{L}_x \geq \dfrac{n-1}{2}$.

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Remark

$a \geq 0$; $a = \dim \mathcal{F}_x$, where $x \in X$ is a general point.

Theorem (A)

Assume $a \geq \dfrac{n-1}{2}$. *Then*

1. (Beltrametti–Sommese–Wiśniewski) *There is a Mori contraction* $\mathrm{cont}_{\mathcal{F}} : X \to Z$ *of lines from* $\mathcal{F}$; F *general fiber of* $\mathrm{cont}_{\mathcal{F}}$, $\dim F = f$, *with* $a + 1 \leq f \leq n$;

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Conjecture

(HC) *If* $n \geq 2c + 1$, *then* $X \subset \mathbb{P}^N$ *is a complete intersection.*

Definition

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1. *If $a \geq \dfrac{n-1}{2}$, then all lines in $\mathcal{F}$ are contact lines;*
2. *if $\mathrm{Pic(X)}$ is cyclic, then $a \leq \dfrac{3(n-2)}{4}$. Accepting the truth of the Hartshorne Conjecture, the better bound $a \leq \dfrac{2(n-1)}{3}$ holds.*

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X is called *conic-connected* if two general points $x, x' \in X$ belong to a conic contained in X.

Theorem (2)

For $X \subset \mathbb{P}^N$ the following results hold:

1. *If $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is nonempty, it is set theoretically defined by (at most) d equations; in particular, we have $a \geq n - 1 - d$.*

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   - *$X \subset \mathbb{P}^N$ is a prime Fano manifold and $i(X) = a+2$;*
   - *the following conditions are equivalent:*
     - (i) *$X \subset \mathbb{P}^N$ is a complete intersection;*
     - (ii) *$\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection of codimension d;*
     - (iii) *$a = n-1-d$.*

Theorem (3)

Assume that $a \geq \dfrac{n-1}{2}$ *and* $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ *is a nondegenerate complete intersection. Then* X *is conic-connected,* $a \leq n - c - 1$ *and* $n \geq 2c + 1$.

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- If $n \geq degree(X) + 1$ then X is a complete intersection, unless it is projectively equivalent to $\mathbb{G}(1,4) \subset \mathbb{P}^9$;
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- If X is covered by lines, $a \geq \dfrac{n-1}{2}$ and $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a nondegenerate complete intersection, then X is a complete intersection too.

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The following results show that all expectations are fulfilled in the quadratic case.

Theorem (4)

Assume that X is quadratic. Then:

1. *If $n \geq c+1$ then X is covered by lines. Moreover, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is scheme theoretically defined by c independent quadratic equations.*

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   (ii) $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ *is a complete intersection;*
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3. (HC) *If $n \geq 2c+1$ then X is a complete intersection.*
4. *If X is a prime Fano manifold of index $i(X) \geq \frac{2n+5}{3}$, then X is a complete intersection.*

Theorem (5)

Assume that X is quadratic. If $n = 2c$ and X is not a complete intersection, then it is projectively equivalent to one of the following:

1. $\mathbb{G}(1,4) \subset \mathbb{P}^9$, *or*
2. $S^{10} \subset \mathbb{P}^{15}$.

Defective manifolds

Definition

Secant variety of X: SX = closure of the locus of secants to $X \subset \mathbb{P}^N$, $\dim SX = 2n + 1 - \delta$, $\delta \geq 0$ *secant defect*.
If $\delta > 0$, X is *secant defective*.

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Classification results when n is small were classically obtained (Severi, Scorza). In general understanding secant defective manifolds is difficult, so we study a special case; it leads to rationally connected manifolds, where tools from Mori theory may be applied.

- For a general point $p \in SX$ consider the cone $C_p(X)$ of secants through p. Its trace on X is the entry locus with respect to p, denoted $\Sigma_p(X)$. $\Sigma_p(X)$ is of pure dimension δ and connects two general points of X. Not much is known on the structure of $\Sigma_p(X)$.

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- We may consider the case when the cone $C_p(X)$ is linear as being the simplest. Then $\Sigma_p(X)$ is a quadric (assume X is not a hypersurface).

This leads to the following definition:

Definition

X is a *local quadratic entry locus variety* (LQEL) if for any $x, x' \in X$ general points there is a quadric $Q^{\delta}_{xx'}$, $x, x' \in Q^{\delta}_{xx'} \subseteq X$.

- For a general point $p \in SX$ consider the cone $C_p(X)$ of secants through p. Its trace on X is the entry locus with respect to p, denoted $\Sigma_p(X)$. $\Sigma_p(X)$ is of pure dimension δ and connects two general points of X. Not much is known on the structure of $\Sigma_p(X)$.
- We may consider the case when the cone $C_p(X)$ is linear as being the simplest. Then $\Sigma_p(X)$ is a quadric (assume X is not a hypersurface).

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Remark

δ is the maximal possible value.

The next theorem may be found in the following papers:

F. Russo, Varieties with quadratic entry locus. I, *Math. Ann.* 2009.
——, F. Russo, Varieties with quadratic entry locus. II, *Compositio Math.* 2008.
——, F. Russo, Conic-connected manifolds, *J. Reine Angew. Math.* 2010.
B. Fu, Inductive characterizations of hyperquadrics, *Math. Ann.* 2008.

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4. *if $\delta \geq 3$ then $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is LQEL, $S\mathcal{L}_x = \mathbb{P}^{n-1}$, $\dim \mathcal{L}_x = \dfrac{n+\delta}{2} - 2$, $\delta(\mathcal{L}_x) = \delta - 2$. If $\delta \geq \dfrac{n}{2}$, complete classification; it implies Zak's classification of Severi varieties;*

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5. *X complete intersection iff $X \cong \mathbb{Q}^n (\delta = n)$;*
6. *$X \not\cong \mathbb{Q}^n$ then $\delta \leq \dfrac{n+8}{3}$; equality cases are classified.*

Conjecture

Assume X is a LQEL manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}\langle H\rangle$. Then X is obtained by linear sections/isomorphic projections from a rational homogeneous manifold.

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"Corollary"

Complete classification of LQEL manifolds; $\delta \leq 8$ if $X \not\cong \mathbb{Q}^n$.

Definition

The *dual variety* of $X \subset \mathbb{P}^N$, denoted $X^\vee$, is $q(Z) \subset (\mathbb{P}^N)^\vee$ where

$$Z = \{(x, H) \mid T_{X,x} \subset H\} \subset X \times (\mathbb{P}^N)^\vee$$

and $q : Z \to (\mathbb{P}^N)^\vee$ is the natural projection.
$\dim Z = N - 1$.
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When $k > 0$, X is *dual defective*.

Classical fact: If $k > 0$, X is covered by linear $\mathbb{P}^k$'s.

Definition

$X \subset \mathbb{P}^N$ is a *scroll* if

$$X = \mathbb{P}(E) \to Y \tag{1}$$

where E is a vector bundle of rank at least two over the projective manifold Y and the fibers of (1) are linearly embedded.

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- Later on Landman and Zak proved that $n \equiv k \pmod 2$ if $k > 0$.
- Zak's theorem on tangencies yields $k \leq c - 1$.
- In two famous papers (1985–86) L. Ein proved the following theorems.

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If X *is not a scroll then* $k \leq \dfrac{n+2}{3}$*; moreover, equality holds iff* $X \cong S^{10}$.

Remark

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Theorem (Beltrametti–Fania–Sommese)

Let X be a dual defective manifold and consider the contraction $\mathrm{cont}_{[\ell]} : X \to Z$ (which exists by a previous result). If F is a general fiber of the contraction, we have

$$k(F) = k(X) + \dim Z.$$

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Let X be a dual defective manifold and consider the contraction $\mathrm{cont}_{[\ell]} : X \to Z$ (which exists by a previous result). If F is a general fiber of the contraction, we have

$$k(F) = k(X) + \dim Z.$$

This remarkable result reduces the classification of dual defective manifolds to the case when $\mathrm{Pic}(X) = \mathbb{Z}\langle H \rangle$.

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"Corollary"

Complete classification of dual defective manifolds; $k \leq 4$ if X is not a scroll.

- The bounds $\delta \leq \dfrac{n+8}{3}$ and $k \leq \dfrac{n+2}{3}$ coincide when $\delta = k + 2$.

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- The bound expresses precisely the "Hartshorne condition" $\dim \mathcal{L}_x \leq 2\,\mathrm{codim}\,(\mathcal{L}_x, \mathbb{P}^{n-1})$!
- This is compatible with the fact that a LQEL may be a complete intersection only if it is a quadric, while a dual defective manifold may be a complete intersection only if it is linear (a case leading to scrolls).