ON THE SPARSITY OF POSITIVE-DEFINITE AUTOMORPHIC FORMS WITHIN A FAMILY

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Abstract. In [BM90], Baker and Montgomery prove that almost all Fekete polynomials with respect to a certain ordering have at least one zero on the interval $(0,1)$. A Fekete polynomial has no zeros on the interval $(0,1)$ if and only if the corresponding automorphic form is positive-definite. Generalizing [BM90], we formulate an axiomatic result about sets of automorphic forms π satisfying certain averages when suitably ordered that ensures that almost all π's are not positive-definite within such sets. We then apply the result to various families, including the family of holomorphic cusp forms, the family of the Hilbert class characters of imaginary quadratic fields, and the family of elliptic curves. In the appendix, we apply the result to general families of automorphic forms defined in [SST13].

1. Introduction

1.1. **Definition of positive-definiteness.** Let π be a self-dual automorphic form on $GL_m/\mathbb{Q}$ and let $\Lambda(s,\pi)$ be its completed standard L-function. Denote by $\epsilon_\pi = \pm 1$ the root number of π and by N_π the conductor of π. Then $\Lambda(s,\pi)$ satisfies the functional equation:

$$N_\pi^{\frac{1-s}{2}}\Lambda(1-s,\pi) = \epsilon_\pi N_\pi^{\frac{s}{2}}\Lambda(s,\pi). \tag{1.1}$$

Let $\Lambda_0(s,\pi) = \Lambda(s,\pi)\left(s(1-s)/2\right)^k$, where k is the order of the pole of $\Lambda(s,\pi)$ at $s=1$. We say π (or equivalently $\Lambda(s,\pi)$) is *positive-definite* if $\Lambda_0\left(\frac{1}{2}+it,\pi\right)$ is a positive-definite function of t in the additive group $\mathbb{R}$. In other words, π (or $\Lambda(s,\pi)$) is positive-definite if for all $l \in \mathbb{N}$ and for every set of real numbers $t_1, \cdots, t_l \in \mathbb{R}$, the following matrix is a positive-definite hermitian matrix:

$$\left(\Lambda_0\left(\frac{1}{2}+i(t_j-t_k),\pi\right)\right)_{\substack{j=1,\cdots,l\\k=1,\cdots,l}}.$$

We first remark that if π is a positive-definite automorphic form, then $\Lambda_0(s,\pi)$ has no real zeros. For instance, when π is a cuspidal automorphic form, we have $\Lambda_0(s,\pi) = \Lambda(s,\pi)$, and we can represent $\Lambda(s,\pi)$ as the Mellin transform of a

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bounded rapidly decreasing function $\phi_\pi(y) \in C^\infty(0,\infty)$ (see §3 for details):

$$\Lambda(s,\pi) = \int_0^\infty \phi_\pi(y)\, y^{s-1}\, dy.$$

Observe that by Bochner's theorem, the condition for π being positive-definite is equivalent to $\phi_\pi(y)$ being a positive function. Now the positivity of $\phi_\pi(y)$ implies that

$$\Lambda(\sigma,\pi) = \int_0^\infty \phi_\pi(y)\, y^{\sigma-1}\, dy > 0$$

for any $\sigma > 0$. Therefore from (1.1), we see that $\Lambda(s,\pi)$ has no real zero if π is positive-definite. In particular, if $\pi = \chi$ is a Dirichlet character that is positive-definite, then $L(s,\chi)$ has no Siegel zero.

More remarks from [Sar11]*:*

(1) If a cuspidal representation π of $GL_2/\mathbb{Q}$ corresponds to a classical holomorphic Hecke cuspform $f_\pi(z)$ on $\mathbb{H}$, then $\Lambda(s,\pi)$ is the Mellin transform of $f_\pi(iy)$. Hence π is positive-definite if and only if f_π has no zeros on the imaginary axis.

(2) Likewise, when π corresponds to a classical Maass-Hecke cuspform ϕ_π, then π is positive-definite if and only if the nodal line of ϕ_π does not meet the imaginary axis. This is especially interesting when ϕ_π is a Maass-Hecke cusp form with eigenvalue 1/4. (Such forms are expected to correspond exactly to even 2-dimensional Galois representations $\rho : Gal\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to GL(2,\mathbb{C})$, whose image is finite.) Assuming Selberg's eigenvalue conjecture[Sel65], these ϕ_π's have the smallest nonzero eigenvalue on the corresponding modular surface, hence by Courant's nodal domain theorem, these will have very few nodal domains.

(3) The fundamental example of a positive-definite π is $\pi = 1$, the trivial representation of GL_1, for which $L(s,\pi) = \zeta(s)$, the Riemann zeta function (See [Bom03, Sar11] for the explicit expression of $\phi_\pi(y)$ in this case.)

(4) Such positive-definite L-functions have nice queueing theoretic properties (Chapter 5, [Bac11]).

(5) Since

$$\Lambda_0(s,\pi_1 \boxplus \pi_2) = \Lambda_0(s,\pi_1)\,\Lambda_0(s,\pi_2),$$

if π_1 and π_2 are positive-definite, then so is the isobaric sum $\pi_1 \boxplus \pi_2$ by Bochner's theorem.

(6) Most π's of small conductor are positive-definite. For instance, when $\pi = \chi_d$ is a quadratic Dirichlet character of conductor d, among first 168 π's with prime conductors, 145 of them are positive-definite [CGPS00]. Similarly, let π correspond to an elliptic curve E_π defined over $\mathbb{Q}$. Among such curves of conductor at most 1000, 319 have rank 0, (a necessary condition in view of the Birch and Swinnerton-Dyer Conjecture) and of those 283 are positive-definite [Sar11].

(7) The question of the existence of positive-definite χ_d's was first asked by Fekete: he proposed that all χ_d's are positive-definite. Pólya [Pól19] showed that this is false for infinitely many χ_d's. Later Chowla [Cho35] repeated this false conjecture and Heilbronn [Hei37] disproved it similarly. Baker and Montgomery [BM90] show that almost all χ_d's are not positive-definite in the sense of natural density.

1.2. **Family of automorphic forms.** The central question about positive definiteness is:

Question 1. *Fix $m \geq 1$. Is the set of cuspidal automorphic forms on $GL_m/\mathbb{Q}$ which are positive-definite, finite or infinite?*

We study this question in the context of families of automorphic forms. We refer the reader to [SST13] for the definition of a family of automorphic forms. Our aim is to show that almost all members are not positive-definite within such family. To this end, we formulate an axiomatic result about a set of automorphic forms $\mathscr{F}$ with an ordering $\mathscr{N}$ satisfying certain average conditions. We first assume that we are given a pair $(\mathscr{F}, \mathscr{N})$ which satisfies the following conditions.

- $\mathscr{F}$ is a countable set of cuspidal automorphic forms on $GL_m/\mathbb{Q}$ for some fixed $m \geq 1$.
- Each $\pi \in \mathscr{F}$ is self-dual.
- $\mathscr{N} : \mathscr{F} \to \mathbb{N}$ is an ordering of $\mathscr{F}$ such that
$$S(X) = \{\pi \in \mathscr{F} \mid \mathscr{N}(\pi) < X\}$$
is a finite set for any $X > 0$.
- $\gamma(s) = L(s, \pi_\infty)$ is the same for every $\pi \in \mathscr{F}$. We further assume that π_∞ is tempered so that $\gamma(s)$ has no pole when $\mathrm{Re}(s) > 0$.

For any map $F : \mathscr{F} \to \mathbb{C}$, we define the "average" of $F(\pi)$ to be
$$\mathbb{E}(F(\pi)) = \mathbb{E}_{(\mathscr{F},\mathscr{N})}(F(\pi)) = \lim_{X\to\infty} \frac{1}{|S(X)|} \sum_{\pi \in S(X)} F(\pi),$$
whenever the limit exists. For example, if we denote by $\lambda_\pi(n)$ the n-th Dirichlet coefficient of $L(s, \pi)$, $\mathbb{E}(\lambda_\pi(n))$ is the average of $\lambda_\pi(n)$ over the family, which often can be computed via trace formulae. We define the density of a subset $A \subset \mathscr{F}$ by $\mathbb{E}(\mathbf{1}_A(\pi))$ where $\mathbf{1}_A(\pi) = 1$ if $\pi \in A$ and 0 otherwise.

We say $(\mathscr{F}, \mathscr{N})$ is statistically balanced and fluctuating if the assumptions A and B concerning the averages of the coefficients of the π's when ordered by $\mathscr{N}$ in Definition 2.1 are satisfied. Essentially, the first assumption guarantees the existence of the limiting distribution X_p of $\lambda_\pi(p)$ for each fixed p and the second assumption controls the mean and variance of X_p uniformly in p.

1.3. **Statement of the results.** In Section 4, we prove that positive-definiteness is sparse in such set of automorphic forms $\mathscr{F}$.

Lemma 1.1. *If $(\mathscr{F}, \mathscr{N})$ is statistically balanced and fluctuating then the set of positive-definite automorphic forms $\mathscr{P} \subset \mathscr{F}$ is a density 0 subset of $\mathscr{F}$.*

Note that if $m = 1$ and $\mathscr{F}$ is the (universal) family of real Dirichlet characters ordered by conductor, then Lemma 1.1 is a result of Baker and Montgomery [BM90]. They examine the fluctuation of logarithmic derivative of $L(s, \chi)$ on the interval $(1/2, 1)$ near $1/2$. One of the main inputs in their work is the strong density theorem for the location of the zeros of these functions. For the more general families that we study, such density theorems are not known. We therefore give a more direct treatment of Lemma 1.1 that avoids these density theorems but uses a number of probabilistic ideas from [BM90].

Our first results are proven by separately verifying that various families $\mathscr{F}$ of π's are statistically balanced and fluctuating. We use spectral techniques (trace formulae) and arithmetic geometric techniques (monodromy). We state our results as follows:

Theorem 1.2. *The set of positive-definite members in the family $\mathscr{F}$ has density* 0 *in the following cases:*

(1) $m = 2$; $\mathscr{F} = \bigcup_{q:squarefree} \mathscr{F}(k,q)$, *where $\mathscr{F}(k,q)$ is the set of primitive holomorphic cusp forms of weight k and level q. The ordering is given by $|q|$.*
(2) $m = 2$; $\mathscr{F} = \bigcup_{D} \mathscr{F}(D)$, *where $\mathscr{F}(D)$ is the set of holomorphic modular forms of weight 1 which are associated to a Hilbert class character of an imaginary quadratic field of odd discriminant $-D$. The ordering is given by $|D|$.*
(3) $m = 3,4,5$; $\mathscr{F}_{m-1} = \bigcup_{q:squarefree} \mathscr{F}_{m-1}(k,q)$, *where $\mathscr{F}_{m-1}(k,q)$ is the set of automorphic forms corresponding to the L-functions of the form $L\left(s,\pi,sym^{m-1}\right)$ (these are automorphic forms on $GL_m\backslash\mathbb{Q}$* [KS00] *), where π is a primitive holomorphic cusp form of weight k and level q. The ordering is given by $|q|$.*

Remark 1.3. All the automorphic forms in the theorem are self-dual and have either trivial or quadratic central character.

Now we consider families of elliptic curves.

Theorem 1.4. *Let $\mathscr{F}$ be the set of automorphic L-functions attached to elliptic curves, endowed with the order by naive height. Then the set of positive-definite L-functions in $\mathscr{F}$ has density* 0.

Theorem 1.5. *Let $\{E(t)\}_{t\in\mathbb{Q}}$ be the one-parameter family of elliptic curves given by some fixed polynomials a and b such that*

$$E(t):\ y^2 = x^3 - a(t)x + b(t),$$

for which the j-invariant is non-constant and $4a(t)^3 - 27b(t)^2 \neq 0$ for $t \in \mathbb{Q}$. If we order $\{E(t)\}_{t\in\mathbb{Q}}$ by the height of t,

$$\mathscr{N}(t) = \max\left\{|n|,|m| : (n,m)=1, t=\frac{n}{m}\right\},$$

then the set of positive-definite L-functions attached to $E(t)$ has density 0 *within the set of L-functions attached to $E(t)$.*

Lastly, in Appendix A (joint work with Shin), we apply Lemma 1.1 to the general families of automorphic forms defined in [SST13] via Langlands functoriality. Assuming that a version of the functoriality conjecture holds in the case at hand (Hypothesis A.1), we prove that such families are statistically balanced and fluctuating (Theorem A.8).

2. Preliminaries

2.1. **Asymptotic analysis and distribution of π_p.** Let the local L-function attached to π on $GL_m/\mathbb{Q}$ at a finite prime p be given by

$$L(s,\pi_p) = \prod_{j=1}^{m}\left(1 - \frac{\alpha_j(\pi_p)}{p^s}\right)^{-1}$$

for some complex numbers $\alpha_1(\pi_p), \cdots, \alpha_m(\pi_p)$ and let the Dirichlet series for $L(s,\pi)$ be given by

$$L(s,\pi) = \prod_{p:\ \text{finite prime}} L(s,\pi_p) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}.$$

Let $\Psi(\pi_p)$ be the m-tuple:

$$\Psi(\pi_p) = (\alpha_1(\pi_p), \cdots, \alpha_m(\pi_p)).$$

Let $\overline{\mathbb{D}} \subset \mathbb{C}$ be the set of complex numbers of modulus less than or equal to 1. Now we define some conditions on the family.

Definition 2.1. For any given $(\mathscr{F}, \mathscr{N})$, we define (A) and (B) as:

- (A): There exists a permutation invariant probability measure μ_p on $\mathbb{C}^m$ supported on $\overline{\mathbb{D}}^m$ for each prime p such that, for any finite set of primes S, the $|S|$-tuple $(\Psi(\pi_p))_{p\in S}$ is equidistributed with respect to $\prod_{p\in S}\mu_p$.
- (B): The following estimates concerning the asymptotic distribution of $\lambda_\pi(p)$ are satisfied:

$$\mathbb{E}_{(\mathscr{F},\mathscr{N})}\left(\lambda_\pi(p)^2\right) \gg 1,$$
$$\mathbb{E}_{(\mathscr{F},\mathscr{N})}\left(\lambda_\pi(p)\right) \ll p^{-\frac{1}{2}}.$$

Here and elsewhere, we write $A \ll_\tau B$ to mean $|A| \le C(\tau) B$ for some constant $C(\tau)$ depending only on τ.

Remark 2.2. It is necessary to specify the ordering to deal with the limiting distributions of countable sets. For instance,

$$\{1,0,0,1,0,0,1,0,0,\cdots\}$$

and

$$\{1,0,1,0,1,0,1,0,\cdots\}$$

have different limiting distribution although they are equal as sets.

Remark 2.3. (A) implies that for any continuous function $F:\mathbb{C}^m \to \mathbb{C}$,

$$\mathbb{E}\left(F(\alpha_1(\pi_p),\cdots,\alpha_m(\pi_p))\right) = \int_{\mathbb{C}^m} F(z_1,\cdots,z_m)\, d\mu_p.$$

Also note that for any finite set of primes $S = \{p_1, \cdots, p_k\}$ and any continuous functions $F_p : \mathbb{C}^m \to \mathbb{C}$, it follows from (A) that

$$\begin{aligned}\mathbb{E}\left(\prod_{p\in S} F_p(\alpha_1(\pi_p),\cdots,\alpha_m(\pi_p))\right) &= \int_{(\mathbb{C}^m)^k} \prod_{p\in S} F_p(z_1,\cdots,z_m)\, d\mu_{p_1}\cdots d\mu_{p_k} \\ &= \prod_{p\in S}\int_{\mathbb{C}^m} F_p(z_1,\cdots,z_m)\, d\mu_p \\ &= \prod_{p\in S} \mathbb{E}\left(F_p(\alpha_1(\pi_p),\cdots,\alpha_m(\pi_p))\right).\end{aligned}$$

Let $d_k(n)$ be the n-th Dirichlet coefficient of the k-th power of Riemann zeta function $\zeta(s)^k$ in the region $\mathrm{Re}(s) > 1$:

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

Let $R(n)$ be a multiplicative function defined by $R(n) = \prod_{p \| n} p$.

Proposition 2.4. *Assume that $(\mathscr{F}, \mathscr{N})$ satisfies (A) and (B). Then there exists a sufficiently large integer b such that*

$$|\mathbb{E}(\lambda_\pi(n_1)\lambda_\pi(n_2))| \le \frac{d_b(n_1 n_2)}{\sqrt{R(n_1 n_2)}}. \tag{2.1}$$

Proof. Firstly observe that

$$\lambda_\pi(p^k) = \sum_{\substack{\kappa_j \ge 0 \\ \kappa_1 + \cdots + \kappa_m = k}} \alpha_1^{\kappa_1}(\pi_p) \cdots \alpha_m^{\kappa_m}(\pi_p),$$

and that from the assumption (A),

$$|\mathbb{E}(\alpha_1^{\kappa_1}(\pi_p) \cdots \alpha_m^{\kappa_m}(\pi_p))| \le 1$$

for any choice of $\kappa_1, \cdots, \kappa_m \ge 0$. Therefore

$$\begin{aligned}
\left|\mathbb{E}\left(\lambda_\pi\left(p^k\right)\lambda_\pi\left(p^l\right)\right)\right| &\le \sum_{\substack{\kappa_j \ge 0 \\ \kappa_1 + \cdots + \kappa_m = k}} \sum_{\substack{\kappa'_j \ge 0 \\ \kappa'_1 + \cdots + \kappa'_m = l}} \left|\mathbb{E}\left(\alpha_1^{\kappa_1 + \kappa'_1}(\pi_p) \cdots \alpha_m^{\kappa_m + \kappa'_m}(\pi_p)\right)\right| \\
&\le \sum_{\substack{\kappa_j \ge 0 \\ \kappa_1 + \cdots + \kappa_m = k}} \sum_{\substack{\kappa'_j \ge 0 \\ \kappa'_1 + \cdots + \kappa'_m = l}} 1 \\
&= \binom{k+m-1}{m-1}\binom{l+m-1}{m-1} = d_m\left(p^k\right) d_m\left(p^l\right).
\end{aligned}$$

For $k + l = 1$,

$$\left|\mathbb{E}\left(\lambda_\pi\left(p^k\right)\lambda_\pi\left(p^l\right)\right)\right| \le C p^{-\frac{1}{2}}$$

for some constant $C > 0$ uniformly in p from the assumption (B). Now let b be sufficiently large such that

$$d_m\left(p^k\right) d_m\left(p^l\right) \le d_b\left(p^{k+l}\right)$$

and

$$C < d_b(p).$$

Then for any $k, l \ge 0$ we have an upper bound of the form:

$$\left|\mathbb{E}\left(\lambda_\pi\left(p^k\right)\lambda_\pi\left(p^l\right)\right)\right| \le \frac{d_b\left(p^{k+l}\right)}{\sqrt{R\left(p^{k+l}\right)}}.$$

Now (2.1) follows from the following equation:

$$\mathbb{E}(\lambda_\pi(n)\lambda_\pi(m)) = \prod_{p^k \| n,\ p^l \| m} \mathbb{E}\left(\lambda_\pi\left(p^k\right)\lambda_\pi\left(p^l\right)\right),$$

which is a consequence of the assumption (A). $\square$

2.2. **Lemmata from probability theory.** In this section we list lemmas without proofs that will be used in subsequent chapters. We refer the reader to [BM90] for the proofs.

Lemma 2.5 (Lemma 5, [BM90])**.** *Suppose that for $r = 1, 2, 3, \ldots,$ the random variables Z_{rn} are independent, where $1 \leq n \leq N_r$, and put*

$$Z_r = \sum_{n=1}^{N_r} Z_{rn}.$$

Suppose that $\mathbb{E}(Z_{rn}) = 0$ for all n and r, and that $\mathbb{P}(|Z_{rn}| \leq c_n) = 1$ for all n and r, where $c_n \geq 0$ are constants such that

$$\sum_{n=1}^{\infty} c_n^3 < \infty.$$

Let

$$\sigma(r) = \left(\sum_{n=1}^{N_r} \mathrm{Var}(Z_{rn}) \right)^{1/2}$$

denote the standard deviation of Z_r, and suppose that $\sigma(r) \to \infty$ as $r \to \infty$. Then the distribution of the random variable $Z_r/\sigma(r)$ tends to the normal distribution with $\mu = 0$ and $\sigma = 1$ as $r \to \infty$.

Definition 2.6. For a sequence of real numbers $a_1, \cdots, a_n$,

$$S^{-}(a_1, \cdots, a_n)$$

is the number of sign changes in the sequence with zero terms deleted, and

$$S^{+}(a_1, \cdots, a_n)$$

is the maximum number of sign changes with zero terms replaced by numbers of arbitrary sign. If f is a real-valued function defined on an interval (a, b), then $S^{\pm}(f; a, b)$ denotes the supremum of

$$S^{\pm}(f(a_1), \cdots, f(a_n))$$

over all finite sequences for which $a < a_1 < \cdots < a_n < b$.

Lemma 2.7 (Lemma 6, [BM90])**.** *Let $\delta > 0$, and suppose that $Z_1, \cdots, Z_R$ are independent random variables such that $\mathbb{P}(Z_r > 0) \geq \delta$ and $\mathbb{P}(Z_r < 0) \geq \delta$ for all r. Then*

$$\mathbb{P}\left(S^{-}(Z_1, \cdots, Z_R) \leq \frac{1}{5}\delta R \right) \ll e^{-\delta R/3}$$

uniformly in δ and R.

Lemma 2.8 ([Kar68])**.** *Let f be a real-valued function defined on $\mathbb{R}$ that is Riemann integrable on finite intervals, and suppose that the Laplace transform*

$$\mathscr{L}(s) = \int_{-\infty}^{\infty} f(x) e^{-sx}\, dx$$

converges for all $s > 0$. Then

$$S^{-}(f; -\infty, +\infty) \geq S^{+}(\mathscr{L}; 0, \infty).$$

3. Estimates

Recall from §1.2 that $(\mathscr{F}, \mathscr{N})$ satisfies the following conditions.

- $\mathscr{F}$ is a countable set of cuspidal automorphic forms on $GL_m/\mathbb{Q}$ for some fixed $m \geq 1$.
- Each $\pi \in \mathscr{F}$ is self-dual.
- $\mathscr{N} : \mathscr{F} \to \mathbb{N}$ is an ordering of $\mathscr{F}$ such that
$$S(X) = \{\pi \in \mathscr{F} | \mathscr{N}(\pi) < X\}$$
is a finite set for any $X > 0$.
- $\gamma(s) = L(s, \pi_\infty)$ is the same for every $\pi \in \mathscr{F}$. We further assume that π_∞ is tempered so that $\gamma(s)$ has no pole when $\mathrm{Re}(s) > 0$.

The Archimedean factor $L(s, \pi_\infty)$ of an automorphic form on $GL_m/\mathbb{Q}$ is given by

$$L(s, \pi_\infty) = \pi^{-ms/2} \prod_{j=1}^{m} \Gamma\left(\frac{s + \kappa_j}{2}\right) \tag{3.1}$$

for some complex numbers $\kappa_1, \cdots, \kappa_m$. In terms of these parameters, the temperedness condition is equivalent to saying $\mathrm{Re}(\kappa_j) \geq 0$ for all $j = 1, \cdots, m$.

Note that the completed L-function $\Lambda(s, \pi)$ is given by

$$\Lambda(s, \pi) = L(s, \pi_\infty) L(s, \pi).$$

For Sections 3 and 4, we further assume that $(\mathscr{F}, \mathscr{N})$ satisfies A and B.

3.1. **Mellin transform of an automorphic L-function.**

Let

$$\phi_\pi(y) = \frac{1}{2\pi i} \int_{(2)} \Lambda(s, \pi) y^{-s} ds$$

and

$$W(y) = \frac{1}{2\pi i} \int_{(2)} \gamma(s) y^{-s} ds,$$

where (σ) denotes the contour given by $\mathrm{Re}(s) = \sigma$ oriented upward. Because $L(s, \pi)$ is of finite order and $\gamma(s)$ is a product of gamma functions with no poles on $\mathrm{Re}(s) > 0$, both integrals converge absolutely for all $y > 0$. By shifting the contour to the right, we see that $W(y)$ is rapidly decreasing in y, and by shifting the contour to $(-\epsilon)$ for sufficiently small $\epsilon > 0$, we see that $W(y)$ is bounded.

From the bound $|\alpha_j(\pi_p)| < \sqrt{p}$ [JS81], we have $\lambda_\pi(n) = O_\epsilon\left(n^{1/2+\epsilon}\right)$. Therefore

$$\phi_\pi(y) = \sum_{n=1}^{\infty} \lambda_\pi(n) W(ny)$$

converges absolutely for all $y > 0$, and $\phi_\pi(y)$ is rapidly decreasing in y.

Now applying the Mellin inversion formula to the functional equation (1.1) of $\Lambda(s, \pi)$, we obtain

$$\phi_\pi(y) = \frac{\epsilon_\pi}{N_\pi^{1/2} y} \phi\left(\frac{1}{N_\pi y}\right),$$

from which it follows that $\phi_\pi(y) \ll_k y^k$ for all $k > 0$ as $y \to 0$. This implies that for all $s \in \mathbb{C}$,

$$\Lambda(s, \pi) = \int_0^\infty \phi_\pi(y) y^{s-1} dy,$$

and that the integral converges absolutely.

We denote by $\phi_\pi^N(y)$ the summation of $\lambda_\pi(n)W(ny)$ over N-smooth numbers:

$$\phi_\pi^N(y)=\sum\nolimits^N\lambda_\pi(n)W(ny)=\sum_{n:p|n\Rightarrow p<N}\lambda_\pi(n)W(ny).$$

Let

$$L^N(s,\pi)=\prod_{p<N}\prod_{j=1}^m\left(1-\alpha_j(\pi_p)p^{-s}\right)^{-1}=\sum\nolimits^N\frac{\lambda_\pi(n)}{n^s}$$

and let

$$\Lambda^N(s,\pi)=L^N(s,\pi)L(s,\pi_\infty).$$

Then it follows that

$$\phi_\pi^N(y)=\frac{1}{2\pi i}\int_{(2)}\Lambda^N(s,\pi)y^{-s}ds.$$

Lemma 3.1. *There exists a constant $A>0$ such that*

$$W(y)\ll e^{-m\pi y^{2/m}}\left(y^A+1\right)$$

uniformly in $y>0$.

Proof. Since $W(y)$ is bounded, we assume that $y>1$. Recall from (3.1) that the Archimedean factor is given by

$$\gamma(s)=\pi^{-ms/2}\prod_{j=1}^m\Gamma\left(\frac{s+\kappa_j}{2}\right)$$

with $\mathrm{Re}(\kappa_j)\geq 0$ for $j=1,\cdots,m$. Together with the assumption that π is self-dual, we see that

$$\frac{1}{m}\sum_{j=1}^m(\kappa_j-1)=\kappa$$

is a real number that is at least -1.

Assuming that $\mathrm{Re}(s)>1$, we give an upper bound for the product of gamma functions using Stirling's formula [Leb65]:

$$\begin{aligned}\prod_{j=1}^m\Gamma\left(\frac{s+\kappa_j}{2}\right)&\ll\left|\exp\left(\sum_{j=1}^m\frac{s+\kappa_j-1}{2}\log\frac{s}{2}-\frac{ms}{2}\right)\right|\\&=\left|\exp\left(\frac{m(s+\kappa)}{2}\log\frac{s}{2}-\frac{ms}{2}\right)\right|.\end{aligned}$$

Let $\sigma=\mathrm{Re}(s)$, $t=\mathrm{Im}(s)$, and $\tau=\frac{t}{\sigma}$. Let $\psi(x)=\log\sqrt{1+x^2}-x\arctan x$. Then we may rewrite the above expression as

$$\begin{aligned}&\exp\left(\frac{m(\sigma+\kappa)}{2}\log\frac{\sigma\sqrt{1+\tau^2}}{2}-\frac{m\sigma\tau}{2}\arctan\tau-\frac{m\sigma}{2}\right)\\=&\exp\left(\frac{m(\sigma+\kappa)}{2}\log\frac{\sigma}{2}-\frac{m\sigma}{2}+\frac{m\kappa}{2}\log\sqrt{1+\tau^2}+\frac{m\sigma}{2}\psi(\tau)\right).\end{aligned}$$

We use this estimate to give an upper bound for $W(y)$ as follows:

$$\begin{aligned} W(y) =& \frac{1}{2\pi i}\int_{(2)} \gamma(s)\, y^{-s} ds \\ &\ll \left(y\pi^{m/2}\right)^{-\sigma} \exp\left(\frac{m(\sigma+\kappa)}{2}\log\frac{\sigma}{2} - \frac{m\sigma}{2}\right) \\ &\times \int_{-\infty}^{\infty} \sigma\left(1+\tau^2\right)^{m\kappa/4} \exp\left(\frac{m\sigma}{2}\psi(\tau)\right) d\tau. \end{aligned}$$

Because $W(y)$ is independent of the choice of σ, we choose $\sigma = 2\pi y^{2/m}$. Observe that $\psi(\tau) \le 0$ and that $\sigma > 2\pi$ from the assumption that $y > 1$. Therefore we conclude

$$\begin{aligned} W(y) \ll& e^{-m\pi y^{2/m}} y^{\kappa+2/m} \int_{-\infty}^{\infty} \left(1+\tau^2\right)^{m\kappa/4} \exp\left(m\pi\psi(\tau)\right) d\tau \\ \ll& e^{-m\pi y^{2/m}} y^{\kappa+2/m}, \end{aligned}$$

where we used the fact that $\psi(\tau) < -|\tau|$ is satisfied for all large $|\tau|$ in the last inequality. □

Lemma 3.2. *For any positive $\epsilon > 0$, we have the following estimate:*

$$\phi_\pi^N(y) \ll_{N,\epsilon} y^{-\left(1/2-1/\left(m^2+1\right)\right)-\epsilon}.$$

Proof. Recall that

$$\phi_\pi^N(y) = \frac{1}{2\pi i}\int_{(2)} L(s,\pi_\infty) \prod_{p<N} L(s,\pi_p)\, y^{-s} ds. \tag{3.2}$$

From the following estimate [LRS99]

$$|\alpha_j(\pi_p)| \le p^{\frac{1}{2}-\frac{1}{m^2+1}},$$

we have the following upper bound for each p and for $\mathrm{Re}(s) = 1/2 - 1/\left(m^2+1\right) + \epsilon$ with $\epsilon > 0$:

$$|L(s,\pi_p)| \le \left(1-p^{-\epsilon}\right)^{-m}.$$

Since $L(s,\pi_\infty)$ is exponentially decaying in $\mathrm{Im}(s)$ for any fixed $\mathrm{Re}(s) > 0$ (Stirling's formula [Leb65]), we can shift the contour of the integral in (3.2) to $\left(1/2 - 1/\left(m^2+1\right) + \epsilon\right)$ for any $\epsilon > 0$, so that

$$\begin{aligned} \phi_\pi^N(y) &\ll y^{-\left(1/2-1/\left(m^2+1\right)\right)-\epsilon} \prod_{p<N} \left(1-p^{-\epsilon}\right)^{-m} \\ &\ll_{\epsilon,N} y^{-\left(1/2-1/\left(m^2+1\right)\right)-\epsilon}. \end{aligned}$$

□

3.2. **Approximation.** In this section, we assume that $N > 0$ is large and fixed, and study how $\phi_\pi(y)$ is well approximated by $\phi_\pi^N(y)$. Firstly, note that the first N terms of $\phi_\pi^N(y)$ and $\phi_\pi(y)$ agree. We therefore expect that $\phi_\pi(y) - \phi_\pi^N(y)$ is negligible in the range $y > N^{-1/2}$. We quantify this as follows.

Lemma 3.3. *For s in the region $1/2 \le \mathrm{Re}(s) \le 3/2$ we have:*

$$\int_{N^{-1/2}}^{\infty} \left(\phi_\pi(y) - \phi_\pi^N(y)\right) y^{s-1} \log y\, dy = O\left(e^{-mN^{1/m}}\right).$$

Proof. Using the bound $\lambda_\pi(n) = O(n)$ and the assumption that $1/2 \le \sigma = \mathrm{Re}(s) \le 3/2$, we have

$$\begin{aligned}\left|\int_{N^{-1/2}}^{\infty}\left(\phi_\pi(y)-\phi_\pi^N(y)\right)y^{s-1}\log y\,dy\right| &\le \int_{1/\sqrt{N}}^{\infty}\sum_{n\ge N}\left|\lambda_\pi(n)W(ny)y^{\sigma-1}\log y\right|\,dy\\ &\ll \int_{N^{-1/2}}^{\infty}\sum_{n\ge N}|W(ny)|\,ny^{\sigma-1}\left(y^{-1/2}+y^{1/2}\right)\,dy\\ &\ll \int_{N^{-1/2}}^{\infty}\sum_{n\ge N}|W(ny)|\,n\left(y^{-1}+y\right)\,dy.\end{aligned}$$

Note that in the range $n \ge N$ and $y > N^{-1/2}$, we have $ny > \sqrt{N}$ and $(ny)^3 > ny^{-1}$. We may therefore bound the last expression using Lemma 3.1 by

$$\begin{aligned}&\int_{N^{-1/2}}^{\infty}\sum_{n>N}(ny)^{A+3}\exp\left(-m\pi(ny)^{2/m}\right)\,dy\\ \ll&\int_{N^{-1/2}}^{\infty}\int_{N}^{\infty}(xy)^{A+3}\exp\left(-m\pi(xy)^{2/m}\right)\,dxdy\\ =&\frac{m^2}{4}\int_{N^{-1/m}}^{\infty}\int_{N^{2/m}}^{\infty}(xy)^{\frac{m}{2}(A+4)-1}\exp\left(-m\pi xy\right)\,dxdy.\end{aligned}$$

Since x and y satisfy $xy > N^{1/m} > 1$, we bound the polynomial part in the last integral by

$$\exp\left((\pi-1)mxy\right) \gg (xy)^{\frac{m}{2}(A+4)-1}.$$

Also, because $y > N^{-1/m}$, we have $N^{2/m}y > 1$. So we may conclude that

$$\begin{aligned}&\int_{N^{-1/2}}^{\infty}\left(\phi_\pi(y)-\phi_\pi^N(y)\right)y^{s-1}\log y\,dy\\ \ll&m^2\int_{N^{-1/m}}^{\infty}\int_{N^{2/m}}^{\infty}\exp\left(-mxy\right)\,dxdy\\ \ll&m^2\int_{N^{-1/m}}^{\infty}\int_{N^{2/m}}^{\infty}N^{2/m}y\exp\left(-mxy\right)\,dxdy\\ =&e^{-mN^{1/m}}.\end{aligned}$$

□

Since we are fixing $N > 0$, by varying π over $\mathscr{F}$ we may treat $\phi_\pi^N(y)$ as a random series constructed from finitely many random variables.

Lemma 3.4. *Fix s such that $\mathrm{Re}(s) = \sigma \ge \frac{1}{2} + \epsilon_0$ for some $\epsilon_0 > 0$. There exists $D > 0$, depending only on $(\mathscr{F}, \mathscr{N})$ such that*

$$\mathbb{E}\left(\left|\int_0^{1/\sqrt{N}}\phi_\pi^N(y)\,y^{s-1}\log y\,dy\right|\right) \ll \frac{1}{N^{\epsilon_0/4}\epsilon_0^D} \tag{3.3}$$

holds, where the implicit constant does not depend on s.

Proof. Firstly, recall from Lemma 3.1 that

$$W(y) \ll e^{-2y^{2/m}}.$$

It follows that

$$\left|\phi_\pi^N(y)\right|^2 = \left(\sum\nolimits^N \lambda_\pi(n) W(ny)\right)^2 \ll \sum_{n_1,n_2} n_1 n_2 e^{-2(n_1 y)^{2/m} - 2(n_2 y)^{2/m}}$$

is absolutely convergent. By the dominated convergence theorem, we may exchange the sum and $\mathbb{E}_{(\mathscr{F},\mathscr{N})}$ in $\mathbb{E}\left(|\phi_\pi^N(y)|^2\right)$. Therefore from Proposition 2.4, we have

$$\begin{aligned}\mathbb{E}\left(\left|\phi_\pi^N(y)\right|^2\right) &\le \sum_{n_1,n_2\ge 1} \frac{d_b(n_1 n_2)}{\sqrt{R(n_1 n_2)}} \left|W(n_1 y) W(n_2 y)\right| \\ &= \sum_{n=1}^{\infty} \frac{d_b(n)}{\sqrt{R(n)}} \sum_{n_1 n_2 = n} \left|W(n_1 y) W(n_2 y)\right|.\end{aligned}$$

Again by Lemma 3.1,

$$W(n_1 y) W(n_2 y) \ll e^{-2(n_1 y)^{2/m} - 2(n_2 y)^{2/m}} \le e^{-4(\sqrt{n_1 n_2} y)^{2/m}}$$

hence

$$\sum_{n_1 n_2 = n} \left|W(n_1 y) W(n_2 y)\right| \ll d_2(n) e^{-4\left(ny^2\right)^{1/m}},$$

and

$$\mathbb{E}\left(\left|\phi_\pi^N(y)\right|^2\right) \ll \sum_{n=1}^{\infty} \frac{d_b(n) d_2(n)}{\sqrt{R(n)}} e^{-4\left(ny^2\right)^{1/m}}. \tag{3.4}$$

Consider a Dirichlet series with coefficients $\frac{d_b(n) d_2(n)}{\sqrt{R(n)}}$:

$$\begin{aligned}L_b(s) &= \sum_{n=1}^{\infty} \frac{d_b(n) d_2(n)}{\sqrt{R(n)}} \frac{1}{n^s} \\ &= \prod_p \left(1 + \frac{2 d_b(p)}{p^{\frac{1}{2}+s}} + \frac{3 d_b(p^2)}{p^{2s}} + \frac{4 d_b(p^3)}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{2b}{p^{\frac{1}{2}+s}} + \frac{3b(b+1)/2}{p^{2s}} + O_b\left(p^{-3\sigma}\right) + \cdots\right).\end{aligned}$$

Observe that

$$L_b(s) \zeta(s+1/2)^{-2b} \zeta(2s)^{-3b(b+1)/2} = \prod_p \left(1 + O_b\left(p^{-3\sigma}\right)\right)$$

is absolutely convergent in the region $\mathrm{Re}(s) > 1/3$, and hence $L_b(s)$ has meromorphic continuation to $\mathrm{Re}(s) > 1/3$ with a possible pole of finite order at $s = 1/2$. Therefore by a standard Tauberian theorem (Theorem 5.11, [MV07]), there exists a constant $B > 0$ such that for any $Y > 2$,

$$\sum_{n<Y} \frac{d_b(n) d_2(n)}{\sqrt{R(n)}} \ll Y^{1/2} \log^B Y.$$

Now by applying summation by parts to (3.4) we obtain

$$\mathbb{E}\left(\left|\phi_\pi^N(y)\right|^2\right) \ll \frac{1}{y} \log^B \frac{1}{y} \tag{3.5}$$

for $y < 1/2$.

From Lemma 3.2 and the dominated convergence theorem, we may interchange $\mathbb{E}(\cdot)$ and the integral in (3.3), so that

$$\mathbb{E}\left(\left|\int_0^{1/\sqrt{N}} \phi_\pi^N(y)\, y^{s-1}\log y\, dy\right|\right) \leq \int_0^{1/\sqrt{N}} \mathbb{E}\left(\left|\phi_\pi^N(y)\right|\right) y^{\sigma-1}\log\frac{1}{y}\, dy.$$

We apply Jensen's inequality and (3.5) to obtain an upper bound:

$$\begin{aligned}\int_0^{1/\sqrt{N}} \mathbb{E}\left(\left|\phi_\pi^N(y)\right|\right) y^{\sigma-1}\log\frac{1}{y}\, dy &\leq \int_0^{1/\sqrt{N}} \mathbb{E}\left(\left|\phi_\pi^N(y)\right|^2\right)^{1/2} y^{\sigma-1}\log\frac{1}{y}\, dy \\ &\ll \int_0^{1/\sqrt{N}} y^{\sigma-3/2}\log^{\frac{B}{2}+1}\frac{1}{y}\, dy.\end{aligned}$$

Let $\frac{B}{2}+2=D$. Then from the assumption $\sigma \geq \frac{1}{2}+\epsilon_0$, and from the inequality

$$\log x \leq \frac{2(D-1)}{\epsilon_0} x^{\frac{\epsilon_0}{2(D-1)}}$$

for $x>1$, we conclude that

$$\begin{aligned}\int_0^{1/\sqrt{N}} y^{\sigma-3/2}\log^{\frac{B}{2}+1}\frac{1}{y}\, dy &= \int_0^{1/\sqrt{N}} y^{\sigma-3/2}\log^{D-1}\frac{1}{y}\, dy \\ &\ll \frac{1}{\epsilon_0^{D-1}}\int_0^{1/\sqrt{N}} y^{\sigma-3/2-\epsilon_0/2}\, dy \\ &\ll \frac{1}{\epsilon_0^{D-1}}\int_0^{1/\sqrt{N}} y^{-1+\epsilon_0/2}\, dy \\ &\ll \frac{1}{N^{\epsilon_0/4}\epsilon_0^{D}}.\end{aligned}$$

□

4. The sparsity of the positive-definite forms

4.1. **Sketch of proof.** In this section, we prove Lemma 1.1. In order to prove that almost all automorphic forms in given family are not positive-definite, we investigate sign changes of $\phi_\pi(y)$ on $(0,\infty)$. In [BM90], the authors prove that for almost all Dirichlet characters $\pi=\chi$,

$$\int_0^\infty \phi_\pi(y)\, y^{s-1}\log y\, dy = \frac{\Lambda'}{\Lambda}(s,\pi)$$

has a sufficient number of sign changes as $s \to 1/2+0^+$, and hence so does ϕ_π by Lemma 2.8. However, for such an argument one must ensure that $L(s,\pi)$ does not have zeros when s is near $1/2$ in order to have a convergent integral. In [BM90], the zero density estimate for Dirichlet L-function is used so that the above expression is convergent near the critical line for almost all Dirichlet characters. Since this is not known in general, we consider

$$\int_{N^{-1/2}}^\infty \phi_\pi(y)\, y^{s-1}\log y\, dy$$

instead, where N is a large fixed parameter.

From Lemma 3.3 and 3.4, this integral can be approximated by

$$\int_0^\infty \phi_\pi^N(y)\, y^{s-1}\log y\, dy,$$

with a small number of exceptional forms. Express the integral as follows:

$$\begin{aligned}\int_0^\infty \phi_\pi^N(y)\, y^{s-1}\log y\, dy &= \frac{d}{ds}\int_0^\infty \phi_\pi^N(y)\, y^{s-1}\, dy\\ &= \frac{d}{ds}\Lambda^N(s,\pi)\\ &= \gamma(s)\, L^N(s,\pi)\left(\frac{\gamma'}{\gamma}(s) + \frac{L^{N\prime}}{L^N}(s,\pi)\right).\end{aligned}$$

The advantage of considering the truncated L-function $\Lambda^N(s,\pi)$ instead of $\Lambda(s,\pi)$ is that $\Lambda^N(s,\pi)$ does not have zeros on $[1/2, 1]$, hence the integral is always absolutely convergent.

Now we study the oscillation of $\frac{L^{N\prime}}{L^N}(s,\pi)$ for s near $1/2$, using the statistical arguments from [BM90]. We prove that it has large sign changes; after claiming that $L^N(s,\pi)$ is not too small for most π's, we deduce that

$$\int_{N^{-1/2}}^\infty \phi_\pi(y)\, y^{s-1}\log y\, dy$$

has sufficient number of sign changes.

4.2. **Estimates with specified parameters.** We first specify the choice of parameters that are used in subsequent sections. For $M > 0$, put $M_1 = [\frac{M}{5}]$, and for integers $M_1 < n \le M$, let $\sigma_n = \frac{1}{2} + \exp(-4^n)$. Define

$$\begin{aligned}u(\sigma) &= \exp\left(\left(\sigma - \frac{1}{2}\right)^{-1/2}\right),\\ v(\sigma) &= \exp\left(\left(\sigma - \frac{1}{2}\right)^{-2}\right),\end{aligned}$$

and let $N = v(\sigma_M)$ and $\epsilon_0 = \exp\left(-4^M\right) = 1/\sqrt{\log N}$. In order to detect sign changes of $\phi_\pi(y)$, we relate an integral transform of $\phi_\pi(y)$ and the Mellin transform of $\phi_\pi^N(y)$.

Lemma 4.1. *For all but $O(1/\log N)$ forms $\pi \in \mathscr{F}$, the following inequality simultaneously holds for all $M_1 < n \le M$:*

$$\left|\int_0^\infty \phi_\pi^N(y)\, y^{\sigma_n-1}\log y\, dy - \int_{1/\sqrt{N}}^\infty \phi_\pi(y)\, y^{\sigma_n-1}\log y\, dy\right| < \exp\left(-\sqrt{\log N}/8\right).$$

Proof. Note that for $M_1 < n \le M$, we have $\sigma_n \in [1/2+\epsilon_0, 1]$. Hence for each fixed $M_1 < n \le M$, we have from Lemma 3.3 and 3.4 that

$$\begin{aligned}\mathbb{E}\left(\left|\int_0^\infty \phi_\pi^N(y)\, y^{\sigma_n-1}\log y\, dy - \int_{N^{-1/2}}^\infty \phi_\pi(y)\, y^{\sigma_n-1}\log y\, dy\right|\right) &\ll \frac{1}{N^{\epsilon_0/4}\epsilon_0^D}\\ &= \log^{D/2} N \exp\left(-\sqrt{\log N}/4\right).\end{aligned}$$

So by Chebyshev's inequality, all but $O\left(\log^{D/2} N \exp\left(-\sqrt{\log N}/8\right)\right)$ forms $\pi \in \mathscr{F}$ satisfy

$$\left|\int_0^\infty \phi_\pi^N(y)\, y^{\sigma_n-1}\log y\, dy - \int_{1/\sqrt{N}}^\infty \phi_\pi(y)\, y^{\sigma_n-1}\log y\, dy\right| < \exp\left(-\sqrt{\log N}/8\right).$$

Since $M = O(\log\log\log N)$, this implies that for all but

$$\ll \exp\left(-\sqrt{\log N}/8\right)\log^{D/2} N \log\log\log N \ll 1/\log N$$

forms $\pi \in \mathscr{F}$, the inequality holds simultaneously in $M_1 < n \leq M$. $\square$

Lemma 4.2. *There exists a constant $c_0 > 0$ such that the inequality*

$$L^N(\sigma_n, \pi) > \frac{1}{\log^{c_0} N}$$

is satisfied for all $M_1 < n \leq M$ but $O\left(\frac{1}{\log N}\right)$ forms $\pi \in \mathscr{F}$.

Proof. Let $1/2 \leq \sigma \leq 1$ be fixed. Recall that

$$L^N(\sigma, \pi) = \prod_{p<N} L(\sigma, \pi_p) = \prod_{p<N}\prod_{j=1}^{m}\left(1 - \frac{\alpha_j(\pi_p)}{p^\sigma}\right)^{-1}.$$

Since $|\alpha_j(\pi_p)| < \sqrt{p}$, $L^N(\sigma, \pi)$ is positive for $\sigma \in (1/2, 1]$. From the assumption B, we have

$$\mathbb{E}(\lambda_\pi(p)) = O\left(p^{-1/2}\right)$$

and

$$\mathbb{E}\left(\left|\sum_{\substack{i_1+\cdots+i_m=k\\ 0\leq i_j\leq 1}} \alpha_1(\pi_p)^{i_1}\cdots\alpha_m(\pi_p)^{i_m}\right|\right) \ll 1$$

for $0 \leq k \leq m$. Therefore there exists a constant $c > 0$ such that

$$\begin{aligned}
\mathbb{E}\left(L^N(\sigma,\pi)^{-1}\right) &= \prod_{p<N}\mathbb{E}\left(L(\sigma,\pi_p)^{-1}\right)\\
&= \prod_{p<N}\left(1 - \frac{\mathbb{E}(\lambda_\pi(p))}{p^\sigma} + O\left(1/p^{2\sigma}\right)\right)\\
&\ll \prod_{p<N}\left(1 + \frac{c}{p}\right)\\
&\ll \log^c N,
\end{aligned}$$

where we used Mertens' second theorem in the last estimate. By Chebyshev's inequality, the inequality

$$L^N(\sigma,\pi)^{-1} < \log^{c+2} N$$

is satisfied for all but $O\left(\frac{1}{\log^2 N}\right)$ forms $\pi \in \mathscr{F}$. Since the implicit constant does not depend on σ, we conclude that

$$L^N(\sigma_n,\pi)^{-1} < \log^{c+2} N$$

is satisfied for all $M_1 < n \leq M$ but $O\left(\frac{\log\log\log N}{\log^2 N}\right)$ forms. The result follows by taking $c_0 = c + 2$. $\square$

4.3. **Oscillation of $-\frac{L^{N\prime}}{L^N}(\sigma,\pi)$.** Note that

$$-\frac{L^{N\prime}}{L^N}(\sigma,\pi)=\sum_{p<N}\sum_{k=1}^{\infty}\frac{\Lambda_\pi\left(p^k\right)}{p^{k\sigma}},$$

where

$$\Lambda_\pi\left(p^k\right)=\log p\sum_{j=1}^{m}\alpha_j\left(\pi_p\right)^k.$$

Observing that $\Lambda_\pi(p)=\lambda_\pi(p)\log p$, we rearrange the sum as follows:

$$\begin{aligned}-\frac{L^{N\prime}}{L^N}(\sigma,\pi)&=\sum_{p<N}\sum_{k=1}^{\infty}\frac{\Lambda_\pi\left(p^k\right)}{p^{k\sigma}}\\&=\sum_{p<N}\frac{\left(\lambda_\pi(p)-\mathbb{E}\left(\lambda_\pi(p)\right)\right)\log p}{p^\sigma}\\&\quad+\sum_{p<N}\frac{\mathbb{E}\left(\lambda_\pi(p)\right)\log p}{p^\sigma}+\sum_{p<N}\sum_{k=2}^{\infty}\frac{\Lambda_\pi\left(p^k\right)}{p^{k\sigma}}.\end{aligned}$$

From A, the condition that $|\alpha_j(\pi_p)|<3/2$ for all $j=1,\cdots,m$ and for all $p<N$ holds for almost all $\pi\in\mathscr{F}$. Also recall that

$$\mathbb{E}\left(\lambda_\pi(p)\right)=O\left(p^{-1/2}\right).$$

It follows that for almost all $\pi\in\mathscr{F}$,

$$\begin{aligned}&\sum_{p<N}\frac{\mathbb{E}\left(\lambda_\pi(p)\right)\log p}{p^\sigma}+\sum_{p<N}\sum_{k=2}^{\infty}\frac{\Lambda_\pi\left(p^k\right)}{p^{k\sigma}}\\&\ll\sum_{p<N}\frac{\log p}{p^{1/2+\sigma}}+\sum_{p<N}\frac{\log p}{p^{2\sigma}}\\&\ll\frac{1}{2\sigma-1}\end{aligned}$$

holds uniformly in $1/2<\sigma<1$, where we used Merten's theorem in the last estimate.

Let $C>0$ be a sufficiently large constant for which

$$\left|\sum_{p<N}\frac{\mathbb{E}\left(\lambda_\pi(p)\right)\log p}{p^\sigma}+\sum_{p<N}\sum_{k=2}^{\infty}\frac{\Lambda_\pi\left(p^k\right)}{p^{k\sigma}}\right|<\frac{C}{2\sigma-1}\tag{4.1}$$

holds for all $1/2<\sigma<1$ and for almost all $\pi\in\mathscr{F}$.

We now study the first sum using the probabilistic arguments from [BM90].

Lemma 4.3. *There exist $M_1<n_1(\pi)<n_2(\pi)<n_3(\pi)<M$ such that*

$$\frac{2C}{2\sigma_{n_i(\pi)}-1}<(-1)^i\sum_{p<N}\frac{\left(\lambda_\pi(p)-\mathbb{E}\left(\lambda_\pi(p)\right)\right)\log p}{p^{\sigma_{n_i(\pi)}}}$$

for all but $O(1/M)$ forms $\pi\in\mathscr{F}$.

Proof. For any $\frac{1}{2} < \sigma < 1$ such that $v(\sigma) < N$, we split

$$\sum_{p<N} \frac{(\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))) \log p}{p^\sigma}$$

into three parts:

$$\begin{aligned} I_1(\sigma,\pi) &= \sum_{p\le u(\sigma)} \frac{(\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))) \log p}{p^\sigma}, \\ I_2(\sigma,\pi) &= \sum_{u(\sigma)\le p<v(\sigma)} \frac{(\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))) \log p}{p^\sigma}, \\ I_3(\sigma,\pi) &= \sum_{v(\sigma)<p<N} \frac{(\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))) \log p}{p^\sigma}. \end{aligned}$$

Assuming that σ is fixed, and using Mertens' theorem and summation by parts to estimate summation over primes, we have

$$\mathbb{E}\left(I_1(\sigma,\pi)^2\right) \ll \sum_{p\le u(\sigma)} \frac{(\log p)^2}{p} \ll (\log u(\sigma))^2 \ll \frac{1}{2\sigma-1}$$

and

$$\begin{aligned} \mathbb{E}\left(I_3(\sigma,\pi)^2\right) &\ll \sum_{p>v(\sigma)} \frac{(\log p)^2}{p^{2\sigma}} \\ &\ll (2\sigma-1)^{-1} v(\sigma)^{1-2\sigma} (\log v(\sigma))^2 \\ &\ll (2\sigma-1)^{-3} \exp\left(-2(\sigma-1/2)^{-1}\right) \\ &\ll \exp\left(-(\sigma-1/2)^{-1}\right) \ll 2\sigma-1. \end{aligned}$$

So by Chebyshev's inequality,

$$|I_1(\sigma_n,\pi)| + |I_3(\sigma_n,\pi)| < \frac{1}{2\sigma_n-1}$$

holds for all $M_1 < n \le M$ but

$$\ll M(2\sigma_{M_1}-1) = 2M\exp\left(-4^{M_1}\right) \ll 1/M$$

forms $\pi \in \mathscr{F}$.

Now we consider I_2. Recall from A that there exists μ_p supported in $\overline{\mathbb{D}}^m$ for which $\Psi(\pi_p)$ is equidistributed. Hence we can find the limiting distribution $\tilde{X}_p$ of $\lambda_\pi(p)$, which is supported in $[-m,m]$. Let X_p be the limiting distribution of $\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))$, so that $\mathbb{E}(X_p) = 0$. Observe from B that

$$\mathbb{E}\left(X_p^2\right) = \mathbb{E}\left(\lambda_\pi(p)^2\right) - \mathbb{E}(\lambda_\pi(p))^2 = \mathbb{E}\left(\lambda_\pi(p)^2\right) + O\left(p^{-1}\right)$$

and therefore there exists a constant $c > 0$ such that

$$\mathbb{E}\left(X_p^2\right) > c^2 \tag{4.2}$$

for all sufficiently large p. Note that for any fixed σ, the asymptotic distribution of $I_2(\sigma, \pi)$ is the distribution of

$$X(\sigma) = \sum_{u(\sigma) \leq p < v(\sigma)} \frac{\log p}{p^{\sigma}} X_p.$$

Let $\rho(\sigma)$ be the standard deviation of $X(\sigma)$. Then by Lemma 2.5,

$$X(\sigma)/\rho(\sigma)$$

converges to the normal distribution $N(0,1)$. By (4.2) and by applying Mertens' theorem with summation by parts, the inequality

$$\rho(\sigma) > c \left(\sum_{u(\sigma) \leq p < v(\sigma)} \frac{(\log p)^2}{p^{2\sigma}} \right)^{\frac{1}{2}} \sim \frac{c}{2\sigma - 1},$$

is satisfied as σ tends to $1/2$ from the right. Let $\Phi(x)$ be the cumulative normal distribution function. Pick δ so that $0 < \delta < \Phi\left(-\frac{2C+1}{c}\right)$. Then assuming that M is sufficiently large, we have

$$\mathbb{P}\left(X(\sigma_n) > \frac{2C+1}{2\sigma_n - 1}\right) \geq \delta,\ \mathbb{P}\left(X(\sigma_n) < -\frac{2C+1}{2\sigma_n - 1}\right) \geq \delta,$$

for each $M_1 < n \leq M$. Define B_n as follows:

$$B_n = \begin{cases} 1 & \text{if } X(\sigma_n) > \frac{2C+1}{2\sigma_n - 1}, \\ -1 & \text{if } X(\sigma_n) < -\frac{2C+1}{2\sigma_n - 1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since the intervals $(u(\sigma_n), v(\sigma_n)]$ are disjoint, the variables $X(\sigma_n)$ are independent. Hence by Lemma 2.7,

$$\mathbb{P}\left(S^-(B_{M_1+1}, B_{M_1+2}, \cdots, B_M) \leq \delta(M - M_1)/5\right) \ll \exp(-\delta(M - M_1)/5).$$

Therefore we can find $M_1 < n_1(\pi) < n_2(\pi) < n_3(\pi) < M$ such that

$$\frac{2C+1}{2\sigma_{n_i(\pi)} - 1} < (-1)^i I_2\left(\sigma_{n_i(\pi)}, \pi\right)$$

for all but $O(1/M)$ forms $\pi \in \mathscr{F}$.

Because

$$I_1(\sigma, \pi) + I_2(\sigma, \pi) + I_3(\sigma, \pi) = \sum_{p<N} \frac{(\lambda_\pi(p) - \mathbb{E}(\lambda_\pi(p))) \log p}{p^{\sigma}}$$

and

$$|I_1(\sigma_n, \pi)| + |I_3(\sigma_n, \pi)| < \frac{1}{2\sigma_n - 1}$$

holds for all $M_1 < n \leq M$ but $O(1/M)$ forms $\pi \in \mathscr{F}$, we obtain the assertion. $\square$

4.4. **Proof of Lemma 1.1.** We are now ready to prove Lemma 1.1. From (4.1) and Lemma 4.3, for all but $O(1/M)$ forms $\pi \in \mathscr{F}$ there exist $M_1 < n_1(\pi) < n_2(\pi) < n_3(\pi) \leq M$ such that

$$\frac{C}{2\sigma_{n_i(\pi)} - 1} < (-1)^i \frac{L^{N\prime}}{L^N}\left(\pi, \sigma_{n_i(\pi)}\right).$$

Recall that

$$\int_0^\infty \phi_\pi^N(y)\, y^{\sigma-1} \log y\, dy = \gamma(\sigma) L^N(\sigma,\pi)\left(\frac{\gamma'}{\gamma}(\sigma) + \frac{L^{N\prime}}{L^N}(\sigma,\pi)\right).$$

From the assumption that π_∞ is tempered, $\gamma(s)$ is bounded away from 0, and $\gamma'(s)/\gamma(s) = O(1)$ for $s \in (1/2, 1)$. Therefore by Lemma 4.2, for such $n_1(\pi) < n_2(\pi) < n_3(\pi)$, the inequality

$$\frac{c_1}{(\log N)^{c_0}} < (-1)^i \int_0^\infty \phi_\pi^N(y)\, y^{\sigma_{n_i(\pi)}-1} \log y\, dy$$

is satisfied with some constant $c_1 > 0$, for all but $\ll \frac{1}{\log N} + \frac{1}{M} \ll \frac{1}{M}$ forms $\pi \in \mathscr{F}$.

Observing that

$$\exp\left(-\sqrt{\log N}/8\right) \ll \frac{c_1}{(\log N)^{c_0}}$$

for all sufficiently large N, we apply Lemma 4.1 to deduce that

$$\int_{1/\sqrt{N}}^\infty \phi_\pi(y)\, y^{\sigma-1} \log y\, dy$$

has at least two sign changes in $\sigma \in (1/2, 1)$ for all but $O(1/M)$ forms. So by Lemma 2.8, except for $O(1/M)$ forms $\pi \in \mathscr{F}$, $\phi_\pi(y)$ has at least one sign change on $(0,\infty)$. Since M can be chosen arbitrarily large, we conclude that almost all π are not positive-definite.

Remark 4.4. We can relax A so that μ_p exists for all but finitely many primes p, since a finite collection of primes has only a finite contribution to $I_1(\pi, \sigma_n)$ for all sufficiently large n.

Remark 4.5. Because what we proved above is $\Lambda(2+it,\pi)$ being a positive-definite function in $t \in \mathbb{R}$ for almost all π's, we can replace the assumption of every π being cuspidal by the assumption that $\Lambda(s,\pi)$ has no poles (or π is cuspidal) for almost all π's, and obtain the same conclusion. For such π's, $\Lambda(\pi, 2+it)$ being a positive-definite function is equivalent to π being positive-definite.

5. Result I

In this section, we prove Theorem 1.2 by verifying A and B for each given family.

5.1. **Holomorphic modular form.** Let

$$\mathscr{F} = \bigcup_{q:\text{square-free}} \mathscr{F}(k,q),$$

where $\mathscr{F}(k,q)$ is the set of primitive holomorphic cusp forms of weight k and level q, and let $\mathscr{N}(\pi)$ be the level of π.

In this case, equidistribution of $\lambda_\pi(p)$ and equidistribution of $(\alpha_1(\pi_p), \alpha_2(\pi_p))$ are equivalent due to the fact that

$$\lambda_\pi(p) = \alpha_1(\pi_p) + \alpha_2(\pi_p)$$
$$1 = \alpha_1(\pi_p)\alpha_2(\pi_p)$$

when $p \nmid \mathscr{N}(\pi)$ and

$$\lambda_\pi(p) = \alpha_1(\pi_p) + \alpha_2(\pi_p)$$

with $\{|\alpha_1(\pi_p)|, |\alpha_2(\pi_p)|\} = \left\{\frac{1}{\sqrt{p}}, 0\right\}$ when $p \mid \mathscr{N}(\pi)$; consequently, $\lambda_\pi(p)$ determines $(\alpha_1(\pi_p), \alpha_2(\pi_p))$ uniquely.

When q runs over the square-free integers that are coprime to p, $\lambda_\pi(p)$ is equidistributed with respect to the probability measure [Ser97]

$$\frac{p+1}{\pi}\frac{\sqrt{1-x^2/4}}{\left(p^{1/2}+p^{-1/2}\right)^2 - x^2}\,dx.$$

When q runs over square-free integers that are multiples of p, from [Ham98], average of $\lambda_\pi(p)$ is 0, which implies in this case that $\lambda_\pi(p)$ is equidistributed with respect to the probability measure

$$\frac{1}{2}\delta_{-1/\sqrt{p}} + \frac{1}{2}\delta_{1/\sqrt{p}},$$

where δ_a is the Dirac delta measure concentrated at a.

So if the following limit exists

$$\alpha_p = \lim_{X\to\infty}\frac{\sum_{\substack{q<X, p\nmid q\\ q:\text{square-free}}} |\mathscr{F}(k,q)|}{\sum_{\substack{q<X\\ q:\text{square-free}}} |\mathscr{F}(k,q)|},$$

we see that $\lambda_\pi(p)$ is equidistributed with respect to the measure

$$X_p = \alpha_p\frac{p+1}{\pi}\frac{\sqrt{1-x^2/4}}{\left(p^{1/2}+p^{-1/2}\right)^2 - x^2}\,dx + (1-\alpha_p)\left(\frac{1}{2}\delta_{-1/\sqrt{p}} + \frac{1}{2}\delta_{1/\sqrt{p}}\right).$$

We compute this explicitly using the dimension formula for $\mathscr{F}(k,q)$ in [Mar05], and obtain $\alpha_p = 1/\left(1+1/p-1/p^2\right)$. From this, $\mathbb{E}\left(\lambda_\pi(p)^2\right) = 1 + O(1/p)$ and $\mathbb{E}(\lambda_\pi(p)) = 0$, and so B is satisfied.

In order to verify A, it is sufficient to check if

$$\mathbb{E}\left(\lambda_\pi\left(\prod_{r=1}^k p_r^{\alpha_r}\right)\right) = \prod_{r=1}^k \mathbb{E}\left(\lambda_\pi\left(p_r^{\alpha_r}\right)\right)$$

holds. From [Ser97, Ham98], when $\prod_{r=1}^k p_r^{\alpha_r} = n$ is non-square, both quantities are zero, and when n is square, both quantities are $1/\sqrt{n}$.

Therefore $(\mathscr{F}, \mathscr{N})$ satisfies A and B.

5.2. **Dihedral forms.** For a squarefree integer $D > 3$ with $D \equiv 3 \mod 4$ let ψ be a character of the ideal class group of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-D}\right)$ that is not a genus character. For such ψ, one can associate a primitive holomorphic cusp form in $S_1(\Gamma_0(D), \chi_{-D})$ where χ_{-D} is the unique primitive quadratic character modulo D. Let $\mathscr{F}(D)$ be the set of all such forms. Note that the size of $\mathscr{F}(D)$ is given by

$$|\mathscr{F}(D)| = h_{-D} - 2^{\omega(D)}$$

where $\omega(n)$ is the number of distinct prime divisors of n. By Siegel's theorem, $h_{-D} \gg_\epsilon D^{1/2-\epsilon}$ for all $\epsilon > 0$, and so we may neglect effect of genus characters assuming that D is sufficiently large.

Now put $\mathscr{F} = \bigcup \mathscr{F}(D)$, where $-D$ runs over all negative odd fundamental discriminant less than -3, and let $\mathscr{N}(\pi) = D$ if and only if $\pi \in \mathscr{F}(D)$. We confine ourselves to odd discriminants in order to simplify the computations. One may include any fundamental discriminants D via following the exact same argument with some extra care.

5.2.1. *Summation of the class numbers in arithmetic progression.*

Lemma 5.1. *Assume that* $(a,b) = 1$ *with* $4|b$ *and* $a \equiv 3 \mod 4$. *Then there exists a constant* $A_{b,a}$, *which can be explicitly computed, such that*

$$\sum_{\substack{D:squarefree,\ 0<D<X \\ D\equiv a \mod b}} L(1,\chi_{-D}) \sim A_{b,a} X$$

holds as $X \to \infty$.

Proof. We study the asymptotic behavior of

$$\sum_{\substack{D:\text{squarefree},\ 0<D<X \\ D\equiv a \mod b}} L(1,\chi_{-D})$$

as $X \to \infty$ assuming $(a,b) = 1$ with $4|b$ and $a \equiv 3 \mod 4$. First, note that

$$\begin{aligned} L(1,\chi_{-D}) &= \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{1}{n} e^{-n/X} + \frac{1}{2\pi i} \int_{(-1/2)} L(s+1,\chi_{-D}) \Gamma(s) X^s \, ds \\ &= \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{1}{n} e^{-n/X} + R(D,X) \end{aligned}$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

One can prove the sum of the error term $R(D,X)$ over D is negligible compared to X from the arguments given in [Sar85], i.e.

$$\sum_{\substack{D:\text{squarefree},\ 0<D<X \\ D\equiv a \mod b}} |R(D,X)| = o(X).$$

We first re-express the leading term as follows:

$$\begin{aligned} &\sum_{\substack{D:\text{squarefree},\ 0<D<X \\ D\equiv a \mod b}} \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{1}{n} e^{-n/X} \\ &= \sum_{\substack{0<D<X \\ D\equiv a \mod b}} \sum_{m^2|D} \mu(m) \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{1}{n} e^{-n/X} \\ &= \sum_{n=1}^{\infty} \frac{e^{-n/X}}{n} \sum_{\substack{m<\sqrt{X} \\ (m,bn)=1}} \mu(m) \sum_{\substack{s<X/m^2 \\ s\equiv a\tilde{m}^2 \mod b}} \left(\frac{-s}{n}\right) \\ &= (EQ). \end{aligned}$$

Here $\tilde{m}$ is a multiplicative inverse of m modulo b. For $(b,r)=1$, put

$$\alpha_{b,r}(n)=\frac{1}{8}\sum_{j=1}^{8n}\left(\frac{bj-r}{n}\right).$$

It is easy to check that $\alpha_{b,r}(n)$ is a multiplicative function in n defined for odd p by

$$\alpha_{b,r}\left(p^{2k+1}\right)=\begin{cases}p^{2k+1}\left(\frac{-r}{p}\right) & \text{if } p|b\\ 0 & \text{if } p\nmid b\end{cases}$$

$$\alpha_{b,r}\left(p^{2k}\right)=\begin{cases}p^{2k} & \text{if } p|b\\ p^{2k}-p^{2k-1} & \text{if } p\nmid b\end{cases}$$

and by

$$\alpha_{b,r}\left(2^{2k+1}\right)=\begin{cases}2^{2k+1}\left(\frac{-r}{2}\right) & \text{if } 8|b\\ 0 & \text{if } 4||b\end{cases}$$

$$\alpha_{b,r}\left(2^{2k}\right)=2^{2k}$$

Observe that $\alpha_{b,r}(n)=\alpha_{b,rs^2}(n)$ if $(s,b)=1$. Using Lemma 2.3 in [Sar85],

$$\sum_{\substack{s<X/m^2\\ s\equiv a\tilde{m}^2 \mod b}}\left(\frac{-s}{n}\right)=\alpha_{b,a}(n)\frac{X}{bm^2n}+O\left(\frac{n\log n}{\sqrt{k(n)}}\right),$$

where $k(n)$ is implicitly defined by the following relation:

$$\sum_{n=1}^{\infty}\frac{k(n)^{-1/2}}{n^s}=\frac{\zeta(2s)\zeta(s+1/2)}{\zeta(2s+1)}.$$

Let $1/2>\delta>0$ be a constant to be determined. We have

$$\begin{aligned}&\sum_{\substack{m<\sqrt{X}\\ (m,bn)=1}}\mu(m)\sum_{\substack{s<X/m^2\\ s\equiv a\tilde{m}^2 \mod b}}\left(\frac{-s}{n}\right)\\ =&\sum_{\substack{m<X^{1/2-\delta}\\ (m,bn)=1}}\mu(m)\left(\alpha_{b,a}(n)\frac{X}{bm^2n}+O\left(\frac{n\log n}{\sqrt{k(n)}}\right)\right)+\sum_{\substack{X^{1/2-\delta}<m<X^{1/2}\\ (m,bn)=1}}O\left(\frac{X}{m^2}\right)\\ =&\frac{X\alpha_{b,a}(n)}{bn}\sum_{\substack{m<X^{1/2-\delta}\\ (m,bn)=1}}\frac{\mu(m)}{m^2}+O\left(X^{1/2-\delta}\frac{n\log n}{\sqrt{k(n)}}\right)+O\left(X^{1/2+\delta}\right).\end{aligned}$$

Therefore

$$\begin{aligned}(EQ)=\frac{X}{b}\sum_{n=1}^{\infty}\frac{e^{-n/X}\alpha_{b,a}(n)}{n^2}&\sum_{\substack{m<X^{1/2-\delta}\\ (m,bn)=1}}\frac{\mu(m)}{m^2}\\ &+O_\epsilon\left(X^{1-\delta+\epsilon}\right)+O\left(X^{1/2+\delta}\log X\right).\end{aligned}$$

Putting $\delta = 1/4$, we get the following asymptotic when $X \to \infty$:

$$\sum_{\substack{D:\text{squarefree, } 0<D<X \\ D\equiv a \mod b}} L\left(1, \chi_{-D}\right) \sim \frac{6X}{\pi^2 b} \sum_{n=1}^{\infty} \frac{\alpha_{b,a}\left(n\right)}{n^2} \prod_{p|bn} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{5.1}$$

Let $\beta_b(n)$ be a multiplicative function in n defined by

$$\beta_b\left(n\right) = \prod_{p|n, p\nmid b} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

Then (5.1) can be further simplified to

$$\begin{aligned} &= \frac{6X}{\pi^2 b} \prod_{p|b} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{p} \sum_{k=0}^{\infty} \frac{\alpha_{b,a}\left(p^k\right)\beta_b\left(p^k\right)}{p^{2k}} \\ &= A_{b,a} X. \end{aligned}$$

□

We compute $A_{b,a}$ for some specified values:

$$A_{4,3} = \frac{8}{3\pi^2} \prod_{q:odd} \frac{q^3 + q^2 - 1}{\left(q^2 - 1\right)\left(q + 1\right)},$$

$$A_{4p,a} = \begin{cases} \frac{p^2(p+1)}{(p^3+p^2-1)(p-1)} A_{4,3} & \text{if } \left(\frac{-a}{p}\right) = 1, \\ \frac{p^2}{p^3+p^2-1} A_{4,3} & \text{if } \left(\frac{-a}{p}\right) = -1, \end{cases}$$

where p is an odd prime. Using the summation by parts and the identity

$$L\left(1, \chi_{-D}\right) = \frac{\pi h_{-D}}{\sqrt{D}},$$

we conclude from Lemma 5.1 the following lemma.

Lemma 5.2. *Let*

$$\tilde{\mathscr{F}}\left(p, \pm 1\right) = \bigcup_{\left(\frac{-D}{p}\right) = \pm 1} \mathscr{F}\left(D\right),$$

$$\tilde{\mathscr{F}}\left(p\right) = \bigcup_{p|D} \mathscr{F}\left(D\right).$$

Then $\tilde{\mathscr{F}}\left(p, 1\right)$, $\tilde{\mathscr{F}}\left(p, -1\right)$, and $\tilde{\mathscr{F}}\left(p\right)$ have asymptotic density

$$\frac{\left(p + 1\right)p^2}{2\left(p^3 + p^2 - 1\right)}, \ \frac{\left(p - 1\right)p^2}{2\left(p^3 + p^2 - 1\right)}, \ \text{and } \frac{p^2 - 1}{p^3 + p^2 - 1}$$

respectively.

5.2.2. *Distribution of the coefficients.* Let $T\left(D\right)$ be the set of characters of the ideal class group of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-D}\right)$. Since $T\left(D\right)$ is a group, we have

$$\sum_{\psi \in T(D)} \psi\left(\mathfrak{J}\right) = h_{-D}\omega\left(\mathfrak{J}\right),$$

where $\omega\left(\mathfrak{J}\right) = 1$ if $\mathfrak{J}$ is a principal ideal and 0 otherwise. For $\pi \in \mathscr{F}\left(D\right)$ corresponding to $\psi \in T\left(D\right)$, we have

$$\lambda_\pi\left(m\right) = \sum_{N(\mathfrak{J}) = m} \psi\left(\mathfrak{J}\right).$$

Therefore

$$\sum_{\pi\in\mathscr{F}(D)}\lambda_\pi(m)=h_{-D}\sum_{N(\mathfrak{J})=m}\omega(\mathfrak{J})+O\left(d_2(m)\,d_2(D)\right).$$

If m is an integer that is not a square and $D>m$, then $x^2+Dy^2=m$ has no integral solution. If m is a square and $D>m$, then there exists exactly one solution (up to unit) for $x^2+Dy^2=m$. Hence:

Lemma 5.3. *Fix an integer $m>0$. Then for all $\epsilon>0$,*

$$\frac{1}{|\mathscr{F}(D)|}\sum_{\pi\in\mathscr{F}(D)}\lambda_\pi(m)=\chi(m)+O_\epsilon\left(d_2(D)/D^{1/2-\epsilon}\right),$$

where $\chi(m)=1$ if m is a square and 0 otherwise.

Now recall that the local L-function of π is given by

$$L(s,\pi_p)=\begin{cases}\left(1-\lambda_\pi(p)\,p^{-s}+p^{-2s}\right)^{-1} & \text{if } \left(\frac{-D}{p}\right)=1,\\ \left(1-p^{-2s}\right)^{-1} & \text{if } \left(\frac{-D}{p}\right)=-1,\\ \left(1-\lambda_\pi(p)\,p^{-s}\right)^{-1} & \text{otherwise.}\end{cases}$$

Together with Lemma 5.2 and Lemma 5.3, this yields the limiting distribution of $\lambda_\pi(p)$ for any odd prime p:

$$X_p=\frac{p^3+p^2}{2\left(p^3+p^2-1\right)}\frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\,dx+\frac{p^2-1}{2\left(p^3+p^2-1\right)}\left(\delta_{-1}+\delta_1\right).$$

We get the asymptotic independence of $\lambda_\pi(p)$ again from Lemma 5.3. Summing up, $(\mathscr{F},\mathscr{N})$ satisfies A and B.

5.3. **Symmetric powers.** Let π be a primitive holomorphic cusp form with square-free level q and fixed weight k. Let $\mathscr{F}_m=\bigcup_{q:\text{squarefree}}\mathscr{F}_m(k,q)$, where $\mathscr{F}_m(k,q)$ is the set of automorphic forms corresponding to the L-functions of the form $L(s,\pi,\mathrm{sym}^m)$. (For $m=2,3,4$, these are automorphic forms on GL_{m+1} over $\mathbb{Q}$ [KS00].)

The local factors of symmetric m-th power L-function at finite places are as follows:

$$L_p(s,\pi,\mathrm{sym}^m)=\begin{cases}\prod_{i=0}^m\left(1-\alpha_1(\pi_p)^{m-i}\,\alpha_2(\pi_p)^i\,p^{-s}\right)^{-1} & \text{if } (p,q)=1,\\ \left(1-\lambda_\pi(p)^m\,p^{-s}\right)^{-1} & \text{if } p|q.\end{cases}$$

The Dirichlet series of $L(s,\pi,\mathrm{sym}^m)$ is

$$L(s,\pi,\mathrm{sym}^m)=\sum_{n=1}^\infty\frac{\lambda_{\mathrm{sym}^m\pi}(n)}{n^s}.$$

From Section 5.1, we know that $\mathscr{F}_m$ satisfies A. Also, from the following relation

$$\begin{aligned}\lambda_{\mathrm{sym}^m\pi}(p)^2&=\lambda_\pi(p^m)^2\\&=\sum_{j=0}^m\lambda_\pi\left(p^{2j}\right)\end{aligned}$$

for π unramified at p, we have

$$\mathbb{E}\left(\lambda_{\mathrm{sym}^m\pi}(p)^2\right) = 1 + O\left(\frac{1}{p}\right)$$
$$\mathbb{E}\left(\lambda_{\mathrm{sym}^m\pi}(p)\right) = O\left(p^{-m/2}\right),$$

and therefore (B) is also satisfied.

6. Result II

In this section, we prove Theorem 1.4 and 1.5.

6.1. **Two-parameter family of elliptic curves.** For each elliptic curve E defined over $\mathbb{Q}$, there exists a unique pair of integers a and b ($4a^3 \neq 27b^2$) such that E is isomorphic to the curve $E_{a,b}$ defined by

$$y^2 = x^3 - ax + b$$

and such that $p^{12} \nmid \left(a^3, b^2\right)$ for each prime p. We call such a pair of integers (a, b) *minimal*. We define the naive height of the elliptic curve $E_{a,b}$ by

$$H\left(E_{a,b}\right) = \max\left\{4|a|^3, 27b^2\right\}.$$[1]

Let $\mathscr{F}$ be the set of $E_{a,b}$ for which (a, b) is minimal. We define the ordering for $\mathscr{F}$ by the height, so that $\mathscr{N} = H$.

It is known that for every elliptic curve E, the (normalized) L-function $L(E, s)$ attached to E is automorphic [BCDT01]. In other words, for each E, there exists $\pi \in S_2(\Gamma_0(N))$ where N is the conductor of the curve E, such that $L(E, s) = L(s, \pi)$. Therefore we may treat this two-parameter family of elliptic curves $(\mathscr{F}, \mathscr{N})$ as a family of automorphic forms, and we prove Theorem 1.4 via verifying (A) and (B) for $(\mathscr{F}, \mathscr{N})$.

Note that if two curves are isogenous, then the corresponding π is the same. Hence the set of L-functions corresponding to each elements of $\mathscr{F}$ is a multi-set. It might be possible that even if the positive-definite L-functions consist of density 0 set in this multi-set, when we count them without multiplicity, the density of the positive-definite L-functions becomes positive. However, one can check that this is not the case, using the fact that the size of the isogeny class of elliptic curves is bounded by 8 [Ken82].

Remark 6.1. There are several ways to order elliptic curves; for instance, one may order curves by height, discriminant, or conductor. It is expected that these orderings are comparable in the sense that the average of the quantities related to curves (average number of the points over $\mathbb{F}_p$, for instance) should be the same regardless which ordering we choose. However, among these orderings, especially when dealing with the automorphic forms or the L-functions associated to curves, we might want to choose the conductor. Nevertheless, we do not deal with this case in this article due to technical difficulties, although we expect Theorem 1.4 to hold even when curves are ordered by the conductor.

---

[1] We choose such scaling since the discriminant of the curve $E_{a,b}$ is given by $\Delta = 16\left(4a^3 - 27b^2\right)$. See Remark 6.1.

6.1.1. *Preparation.* For any integers a and b with $4a^3 \neq 27b^2$, we define an elliptic curve $E_{a,b}$ by the equation

$$y^2 = x^3 - ax + b.$$

We write $E \sim E'$ to indicate that the elliptic curves E and E' are isomorphic. Then

$$E_{a,b} \sim E_{c,d}$$

if and only if there exists a rational number t such that

$$(a,b) = \left(ct^4, dt^6\right).$$

For any pair of integers a and b we put,

$$h(a,b) = \max\left\{4|a|^3, 27b^2\right\}.$$

For simplicity, for integers a and b such that $4a^3 \neq 27b^2$, we write

$$\lambda_{a,b}(n) = \lambda_{E_{a,b}}(n),$$

where $\lambda_{E_{a,b}}(n)$ is the n-th normalized Dirichlet coefficient of the standard L-function attached to $E_{a,b}$. When $4a^3 = 27b^2$, we simply put $\lambda_{a,b}(n) = 0$.

Now for each prime $p > 3$, let

$$\alpha(p,k) = \frac{1}{p^2} \sum_{a,b} \lambda_{a,b}(p)^k,$$

where (a,b) runs over each equivalence class of (a,b) modulo p exactly once with $4a^3 \neq 27b^2$ and $p^{12} \nmid \left(a^3, b^2\right)$. Note that when $(a,b) \equiv (0,0) \mod p$, we have $\lambda_{a,b}(p) = 0$. Likewise we define $\alpha(p_1p_2\cdots p_r, k)$ to be the average of $\lambda_{a,b}(p_1\cdots p_r)^k$ over the equivalence classes of (a,b) modulo $p_1\cdots p_k$.

Lemma 6.2. *For any fixed set of primes $p_1, p_2, \cdots, p_r > 3$, we have*

$$\frac{1}{X^{5/6}2^{4/3}3^{-3/2}} \sum_{\substack{h(a,b)<X \\ 4a^3\neq 27b^2}} \lambda_{a,b}(p_1\cdots p_r)^k = \prod_{j=1}^{r}\left(1 - \frac{1}{p_j^{12}}\right)^{-1} \alpha(p_j,k) + O\left(X^{-1/3}\right) \tag{6.1}$$

where the implied constant depends only on $p_1, \cdots, p_r$ and k.

Proof. Let $\hat{\lambda}_{a,b}(p_1\cdots p_r)$ be defined by

$$\hat{\lambda}_{a,b}(p_1\cdots p_r) = \begin{cases} 0 & \text{if } p_j|a \text{ and } p_j|b \text{ for some } 1 \leq j \leq r, \\ \lambda_{a,b}(p_1\cdots p_r) & \text{otherwise}. \end{cases}$$

From the definition of $\alpha(p_1p_2\cdots p_r,k)$, the sum of $\hat{\lambda}_{a,b}(p_1\cdots p_r)^k$ over any box of size $p_1\cdots p_r \times p_1\cdots p_r$ is equal to $(p_1\cdots p_r)^2\,\alpha(p_1p_2\cdots p_r,k)$.

Therefore

$$\begin{aligned}\sum_{h(a,b)<X} \hat{\lambda}_{a,b}(p_1\cdots p_r)^k =& X^{5/6}2^{4/3}3^{-3/2}\alpha(p_1p_2\cdots p_r,k) \\ &+ O\left(p_1\cdots p_r\left(X^{1/3} + X^{1/2}\right)\right).\end{aligned}$$

By the Chinese remainder theorem, we have

$$\alpha(p_1p_2\cdots p_r,k) = \alpha(p_1,k)\cdots\alpha(p_r,k).$$

Observe that $\hat{\lambda}_{a,b}(p_1 \cdots p_r)^k$ and $\lambda_{a,b}(p_1 \cdots p_r)^k$ are different only if $(a,b) = (p_j^4 a', p_j^6 b')$ for some $1 \le j \le r$ and $a', b'\mathbb{Z}$. We apply the same argument to such pairs inductively to conclude (6.1). $\square$

6.1.2. *Distribution of $\lambda_{a,b}(p)$.* Let S be the set of minimal pairs in $\mathbb{Z}^2$.

Lemma 6.3. *Let $F : \mathbb{Z}^2 \to \mathbb{C}$ be a function that satisfies*

$$F(a,b) = F\left(n^4 a, n^6 b\right)$$

for any $a, b \in \mathbb{Z}$ such that $4a^3 \ne 27b^2$ and for any positive integer n. Then

$$\sum_{\substack{(a,b)\in S\\ h(a,b)<X}} F(a,b) = \sum_{n=1}^{\infty} \mu(n) \sum_{\substack{h(a,b)<X/n^{12}\\ 4a^3\ne 27b^2}} F(a,b)$$

Proof. Let n act on $(a,b) \in \mathbb{Z}^2$ by

$$n * (a,b) = \left(n^4 a, n^6 b\right)$$

Then by the minimality of S,

$$\mathbb{Z}^2 = \bigcup_{n=1}^{\infty} n * S \bigcup \left\{(a,b) \,|4a^3 = b^2\right\},$$

where each set is disjoint. We therefore have

$$\sum_{\substack{h(a,b)<X\\ 4a^3\ne 27b^2}} F(a,b) = \sum_{n=1}^{\infty} \sum_{\substack{(a,b)\in S\\ h(a,b)<X/n^{12}}} F(a,b).$$

Applying Möbius inversion, we get the assertion. $\square$

Now we combine Lemma 6.3 and 6.2 to compute the moments of Dirichlet coefficients.

Lemma 6.4. *For any fixed set of primes $p_1, p_2, \cdots, p_r > 3$,*

$$\begin{aligned}\mathbb{E}_{(\mathscr{F},\mathscr{N})}\left(\lambda_{a,b}(p_1 \cdots p_r)^k\right) &= \prod_{j=1}^{r} \left(1 - \frac{1}{p_j^{12}}\right)^{-1} \alpha(p_j, k)\\ &= \prod_{j=1}^{r} \mathbb{E}_{(\mathscr{F},\mathscr{N})}\left(\lambda_{a,b}(p_j)^k\right).\end{aligned}$$

We therefore conclude that the family satisfies A and B, using the fact that [Bir68]:

$$\begin{aligned}\alpha(p,1) &= 0\\ \alpha(p,2) &= 1 + O\left(\frac{1}{p}\right).\end{aligned}$$

6.2. **One-parameter family of elliptic curves.** Let $\{E(t)\}_{t\in\mathbb{Q}}$ be the one parameter family of elliptic curves given by some fixed polynomials a and b such that

$$E(t): \ y^2 = x^3 - a(t)\,x + b(t)\,,$$

where the j-invariant is non-constant and $4a(t)^3 - 27b(t)^2 \neq 0$ for all $t \in \mathbb{Q}$. We order $\{E(t)\}_{t\in\mathbb{Q}}$ by the height of t:

$$\mathscr{N}(t) = \max\left\{|n|, |m| : (n,m) = 1, t = \frac{n}{m}\right\}.$$

Using similar ideas from the previous section, one can verify A for the family using the periodicity of $a(t)$ and $b(t)$ modulo $m \in \mathbb{N}$ and the Chinese remainder theorem.

For B, note that for all but $O(1/p)$ of t's, $E(t)$ is minimal modulo p. (In fact, except $O(1/p)$ of t's, $a(t) \not\equiv 0 \mod p$ and $b(t) \not\equiv 0 \mod p$.) Therefore if we let $\alpha(p,k)$ to be the average over $\lambda_{E(t)}\left(p^k\right)$ as t runs over $\mathbb{F}_{p^k}$,

$$\mathbb{E}\left(\lambda_{E(t)}\left(p^k\right)\right) - \alpha(p,k) \ll_k \frac{1}{p}.$$

Now we recall from [Kat90] that

$$\alpha(p,k) \ll p^{-k/2}.$$

Therefore

$$\mathbb{E}\left(\lambda_{E(t)}(p)\right) \ll p^{-1/2}$$

and

$$\begin{aligned}\mathbb{E}\left(\lambda_{E(t)}(p)^2\right) &= 1 + \mathbb{E}\left(\lambda_{E(t)}\left(p^2\right)\right)\\ &= 1 + \alpha(p,2) + O(1/p)\\ &= 1 + O(1/p)\,.\end{aligned}$$

Hence B holds, and therefore Theorem 1.5 follows.

Remark 6.5. Unlike other results regarding family of L-functions, the conductor of the L-function is irrelevant in our theorem because the positive-definiteness of an L-function can be determined by the values of the L-function off the critical strip.

Appendix A. More examples

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In this appendix, we give more examples of families $(\mathscr{F},\mathcal{N})$ to which the main results of the paper apply. As in [ST], the examples are constructed via Langlands functoriality (Hypothesis A.1).

A.1. **Families to be considered.** Let G be a split reductive group over $\mathbb{Q}$ such that $G(\mathbb{R})$ admits discrete series. We assume that the center $Z(G)$ is anisotropic over $\mathbb{Q}$ for simplicity.[2] For $n \in \mathbb{N}$, define an open compact subgroup of $G(\mathbb{A}^\infty)$ by

$$U(n) := \ker\left(G\left(\widehat{\mathbb{Z}}\right) \to G(\mathbb{Z}/n\mathbb{Z})\right).$$

Let $r : \widehat{G} \to GL_m(\mathbb{C})$ be a faithful irreducible representation of the dual group of G such that

$$r \simeq r^\vee.$$

Fix an irreducible algebraic representation ξ of $G \otimes_{\mathbb{Q}} \mathbb{C}$. Assume that the highest weight of ξ is regular. Let $\Pi_\infty(\xi)$ be the set of discrete series of $G(\mathbb{R})$ whose infinitesimal character and central character are the same as $\xi^\vee$. Then $\Pi_\infty(\xi)$ is an L-packet. For an automorphic representation π of $G(\mathbb{A})$, define $m(\pi)$ to be the multiplicity in the discrete L^2-spectrum of $G(\mathbb{Q})\backslash G(\mathbb{A})$, and $N(\pi)$ to be the least $N \in \mathbb{N}$ such that $\pi^{U(N)} \neq 0$. Let $\{n_k\}$ be an increasing sequence of positive integers. Assume

- each prime p divides only finitely many n_k.

Define $\mathcal{F}_k$ to be the multi-set of all discrete automorphic representations π of $G(\mathbb{A})$ such that $\pi_\infty \in \Pi_\infty(\xi)$ in which π appears with multiplicity

$$a_{\mathcal{F}_k}(\pi) := m(\pi)\dim(\pi^\infty)^{U(n_k)}.$$

Hypothesis A.1. For each $k \geq 1$ and $\pi \in \mathcal{F}_k$, there is an isobaric automorphic representation Π of $GL_m(\mathbb{A})$ (the functorial lift of π under r) such that

1. at all finite places v where G, r and π are unramified, the Satake parameter for π_v transfers to that of Π_v via r,
2. the L-parameter for π_∞ transfers to that for Π_∞ via r (where the L-parameters are given by the local Langlands correspondence for real reductive groups).

This is the same as Hypothesis 10.1 of [ST], to which we refer the reader for more details on conditions (1) and (2). The functorial lift Π as above is denoted $r_*\pi$. Put $\mathscr{F}_k := r_*\mathcal{F}_k$ (as a multi-set) and $\mathscr{F} := \{\mathscr{F}_k\}_{k\geq 1}$.

Example A.2. Let G be a split symplectic group, or a split orthogonal group in n variables where n is either odd or divisible by 4. Take r to be the standard representation of the dual group $\widehat{G}$. Then $G(\mathbb{R})$ contains a compact maximal torus, and so admits discrete series. Moreover, Hypothesis A.1 is known in this case by the work of Arthur [Art13] conditionally on the stabilization of the twisted trace

[2] In fact, [ST] (in the level aspect) works with a reductive group over a totally real field that admits discrete series at all infinite places without assuming that $Z(G)$ is anisotropic or that G is split. Though our results should extend to that setting without difficulty (and in particular should include the case of quasi-split unitary groups by using [Mok]), we chose to restrict ourselves to split groups in favor of simplicity and clarity.

formula, the weighted fundamental lemma (being written up by Chaudouard and Laumon), and a technical result in harmonic analysis. (See [BMM11, 1.18] for a detailed discussion of these conditions.)

A.2. **Satake transforms.** Let p be a prime and G be a Chevalley reductive group over $\mathbb{Z}_p$. Let T be a split maximal torus of G over $\mathbb{Q}_p$ in a good relative position to $G(\mathbb{Z}_p)$ and $B \supset T$ a Borel subgroup. Let $X_*(T)$ and $X_*(T)^+$ be the cocharacter group of T and its subset of B-dominant members, respectively. Denote by Ω the associated Weyl group, which is equipped with sign character $\mathrm{sgn} : \Omega \to \{\pm 1\}$. Write ρ for the half sum of all B-positive roots of T in G. Put $K_p := G(\mathbb{Z}_p)$. Write $\mathcal{H}_p^{\mathrm{ur}}(G)$ for the unramified Hecke algebra of bi-K_p-invariant functions on $G(\mathbb{Q}_p)$ with values in $\mathbb{C}$. Similarly $\mathcal{H}_p^{\mathrm{ur}}(T)$ is the algebra of functions on $T(\mathbb{Q}_p)$ bi-invariant under $T(\mathbb{Q}_p) \cap K_p$. There is an obvious action of Ω on each of $\mathcal{H}_p^{\mathrm{ur}}(T)$ and $X_*(T)$. For $\mu \in X_*(T)^+$, define $\tau_\mu^G \in \mathcal{H}_p^{\mathrm{ur}}(G)$ to be the characteristic function on $K_p\mu(p)K_p$, and define $\chi_\mu \in \mathbb{C}[X_*(T)]^\Omega$ by the formula

$$\chi_\mu \sum_{\omega\in\Omega} \mathrm{sgn}(\omega)\,\omega\mu = \sum_{\omega\in\Omega} \mathrm{sgn}(\omega)\,\omega(\rho+\mu)$$

in the group algebra $\mathbb{C}[X_*(T)]$. It is known (cf. [Kat82, p.465]) that $\{\chi_\mu\}_{X_*(T)^+}$ forms a $\mathbb{C}$-basis of $\mathbb{C}[X_*(T)]^\Omega$. The Satake isomorphism is a canonical $\mathbb{C}$-algebra isomorphism

$$\mathcal{S} : \mathcal{H}_p^{\mathrm{ur}}(G) \xrightarrow{\sim} \mathbb{C}[X_*(T)]^\Omega.$$

We refer the reader to [Bor79] or [Car79] for details. If we write $\widehat{G}_{\mathrm{ss}}$ for the set of semisimple elements of $\widehat{G}$, there is a canonical isomorphism between $\mathbb{C}[X_*(T)]^\Omega$ and the sub $\mathbb{C}$-algebra in the space of functions on $\widehat{G}_{\mathrm{ss}}$ generated by the finite dimensional irreducible characters of $\widehat{G}$ ([Bor79, §6]). For $r : \widehat{G} \to GL_n(\mathbb{C})$ an irreducible representation of complex Lie groups, write $\mathrm{tr}\ r$ for its character viewed as an element of $\mathbb{C}[X_*(T)]^\Omega$.

Lemma A.3. *Assume that r is as above. Let $\mu \in X_*(T)^+$.*

(1) *Suppose that r has highest weight μ. Then $\mathcal{S}^{-1}(\mathrm{tr}\ r) = \chi_\mu$.*
(2) *Suppose that $\mu \neq 0$ (equivalently, r is not the trivial representation). Then there exists a constant $C(\mu) > 0$ depending only on μ and the root datum of G such that*
$$|\chi_\mu(1)| \leq C(\mu)\,p^{-1}.$$
(3) *If $\mu = 0$, then $\chi_\mu(1) = 1$.*

Remark A.4. The point of (2) is that $C(\mu)$ is independent of p when one starts with a $\mathbb{Q}$-split group and considers it over $\mathbb{Q}_p$ as p varies.

Proof. Parts (1) and (2) are the lemmas 2.1 and 2.9 in [ST]. The last assertion is obvious since χ_0 is the identity in the group algebra $\mathbb{C}[X_*(T)]$, which corresponds to the characteristic function on K_p. □

A.3. **Sparsity of positive definite members.** Going back to the setup of §A.1, we aim to show that almost all members of $\mathscr{F}_k$ are not positive definite as $k \to \infty$. For any function F on the set of automorphic representations of $GL_n(\mathbb{A})$, let us define

$$\mathbb{E}_{\mathscr{F}}(F(\pi)) \stackrel{\mathrm{def}}{=} \lim_{k\to\infty} \frac{1}{|\mathscr{F}_k|} \sum_{\pi\in\mathscr{F}_k} F(\pi),$$

which will play the role of $\mathbb{E}_{(\mathscr{F},\mathscr{N})}(F(\pi))$ of §2.1.[3] Note that properties Ⓐ and Ⓑ of Definition 2.1 still make sense. Also, for the family given in §A.1, we have $\iota_1 = 1$, hence almost all members in $\mathscr{F}_k$ are cuspidal as k goes to ∞ [SST13]. From Remark 4.5, we therefore have an analogue of Lemma 1.1 in our setting using exactly the same argument presented in §2, 3, and 4. Recall that ξ is fixed so that all members of $\mathscr{F}$ have the same infinite component by part (2) of Hypothesis A.1.

Lemma A.5. *Let $\mathscr{F}$ be a family as in §A.1 satisfying Ⓐ and Ⓑ. Then almost all members in $\mathscr{F}$ are not positive definite (as $k \to \infty$) in the following sense: Let $B_k \subset \mathscr{F}_k$ be the sub multi-set of positive-definite members. Then $\lim_{k\to\infty} |B_k|/|\mathscr{F}_k| = 0$.*

Our final task is to verify properties Ⓐ and Ⓑ for the family $\{\mathscr{F}_k\}$. Actually, we prove stronger assertions, as can be easily seen from the proofs.

Lemma A.6. *The family $\mathscr{F}$ satisfies Ⓐ.*

Proof. Take μ_p in Ⓐ to be the Plancherel measure on the unramified unitary dual of $G(\mathbb{Q}_p)$. For each prime p, note that p doesn't divide the level for $k \gg 0$ by assumption. So the lemma is exactly the level aspect in Corollary 9.22 of [ST]. □

Lemma A.7. *The family $\mathscr{F}$ satisfies Ⓑ.*

Proof. Corollary 9.22 of [ST] (+ Lemma A.3.(i)) gives us

$$\mathbb{E}_{\mathscr{F}}(\lambda(p)) = \widehat{\mu}_p^{\mathrm{pl}}(\chi_r), \tag{A.1}$$

$$\mathbb{E}_{\mathscr{F}}\left(\lambda(p)^2\right) = \widehat{\mu}_p^{\mathrm{pl}}(\chi_{r\otimes r}). \tag{A.2}$$

Here $\chi_{r\otimes r} := \sum_{r'} a_{r'}\chi_{r'}$, where $\oplus_{r'} a_{r'} r'$ is the decomposition of $r\otimes r$ into irreducible representations with multiplicity $a_{r'} \in \mathbb{Z}_{\geq 0}$. The Plancherel formula satisfied by the Plancherel measure tells us that $\widehat{\mu}_p^{\mathrm{pl}}(\chi_r) = \chi_r(1)$ and $\widehat{\mu}_p^{\mathrm{pl}}(\chi_{r\otimes r}) = \sum_{r'} a_{r'}\chi_{r'}(1)$. From (A.1) and Lemma A.3.(2),

$$\mathbb{E}_{\mathscr{F}}(\lambda(p)) = O\left(p^{-1}\right).$$

Since r is self-dual, $a_{r'} = 1$ when r' is the trivial representation. (To see this, observe that $\mathrm{Hom}_{\widehat{\mathrm{G}}}(\mathrm{r}, \mathrm{r}^{\vee})$ is one-dimensional if nonzero, provided that r is irreducible.) From (A.2) and Lemma A.3.(2)(3),

$$\mathbb{E}_{\mathscr{F}}\left(\lambda(p)^2\right) = 1 + O\left(p^{-1}\right)$$

, where the implicit constant is dependent only on the decomposition $r \otimes r = \oplus_{r'} a_{r'} r'$ and the constants $C(\mu)$ as μ ranges over the highest weights corresponding to r' with $a_{r'} > 0$. The latter two are clearly independent of p. □

Theorem A.8. *Let $\{\mathscr{F}_k\}$ be a family of §A.1. Under Hypothesis A.1, almost all members of $\mathscr{F}$ are not positive definite.*

Proof. Apply Lemma A.5 along with Lemmas A.6 and A.7. □

In particular, the conclusion of the theorem is true for Example A.2, conditional on the expected results as described in that example. This provides a large number of examples in addition to Theorem 1.2, 1.4, and 1.5.

[3] The only difference in the setting is that we have $S(X) \subset S(Y)$ whenever $X < Y$, where there is no such relation among $\mathscr{F}_k$. Note that we do not make use of this fact in the proof of Lemma 1.1.

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