NUMBER OF NODAL DOMAINS OF EIGENFUNCTIONS ON NON-POSITIVELY CURVED SURFACES WITH CONCAVE BOUNDARY

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ABSTRACT. It is an open problem in general to prove that there exists a sequence of Δ_g-eigenfunctions φ_{j_k} on a Riemannian manifold (M, g) for which the number $N(\varphi_{j_k})$ of nodal domains tends to infinity with the eigenvalue. Our main result is that $N(\varphi_{j_k}) \to \infty$ along a subsequence of eigenvalues of density 1 if the (M, g) is a non-positively curved surface with concave boundary, i.e. a generalized Sinai or Lorentz billiard. Unlike the recent closely related work of Ghosh-Reznikov-Sarnak and of the authors on the nodal domain counting problem, the surfaces need not have any symmetries.

1. Introduction

Let (M, g) be a surface with non-empty smooth boundary $\partial M \neq \emptyset$. We consider the eigenvalue problem,

$$\begin{cases} -\Delta\varphi_\lambda = \lambda^2\varphi_\lambda, \\ B\varphi_\lambda = 0 \ \text{ on } \ \partial M \end{cases},$$

where B is the boundary operator, e.g. $B\varphi = \varphi|_{\partial M}$ in the Dirichlet case or $B\varphi = \partial_\nu\varphi|_{\partial M}$ in the Neumann case. We denote by $\{\varphi_j\}$ an orthonormal basis of eigenfunctions, $\langle\varphi_j, \varphi_k\rangle = \delta_{jk}$, with $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ counted with multiplicity. The inner product is defined by $\langle f, g\rangle = \int_M f\bar{g}dA$ where dA is the area form of g. We denote the nodal line of φ_λ by

$$Z_{\varphi_\lambda} = \{x : \varphi_\lambda(x) = 0\}.$$

This paper makes use of recent joint work of the second author with several collaborators. Theorem 1.7 is recent joint work with X. Han, A. Hassell and H. Hezari [XH]. It also uses calculations in recent work [HZ12] with H. Hezari. Theorem 1.2 is joint work in progress with C. D. Sogge [SZb]. The boundary quantum ergodicity theorem and boundary local Weyl law is joint work with A. Hassell [HZ04] and is closely related to recent joint work with H. Christianson and J. Toth [CTZ13]. We would also like to thank N. Simanyi for helpful correspondence regarding [Kra89] and L. Stoyanov for correspondence on billiard problems. Research partially supported by NSF grant DMS-1206527. The first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP)(No. 2013042157). The first author was also partially supported by TJ Park Post-doc Fellowship funded by POSCO TJ Park Foundation.

The nodal domains of φ_λ are the connected components of $M\backslash(Z_{\varphi_\lambda} \cup \partial M)$; We denote the number of nodal domains by $N(\varphi_\lambda)$. We further denote by

$$\Sigma_{\varphi_\lambda} = \{x \in Z_{\varphi_\lambda} : d\varphi_\lambda(x) = 0\}$$

the singular set of φ_λ. The purpose of this article, developing the method of [JZ13], is to prove that $N(\varphi_j) \to \infty$ along a subsequence of density one of the eigenvalues for certain surfaces (M, g) with boundary $\partial M \neq \emptyset$ and with ergodic billiard flow. A significant gain in the billiard case is that we do not require the surface (or eigenfunctions) to have a symmetry.

The first result states sufficient conditions under which the number of nodal domains tends to infinity along a subsequence of 'almost all' eigenfunctions of any orthonormal basis.

Theorem 1.1. *Let (M, g) be a surface with non-empty smooth boundary ∂M. Let $\{\varphi_j\}$ be an orthonormal eigenbasis of Dirichlet (resp. Neumann) eigenfunctions. Assume that (M, g) satisfies the following conditions:*

(i) *The billiard flow G^t is ergodic on S^*M with respect to Liouville measure;*

(ii) *There does not exist a self-focal point $q \in \partial M$ for the billiard flow, i.e. a point q such that the set $\mathcal{L}_q$ of loop directions $\eta \in B_q^*\partial M$ has positive measure in $B_q^*\partial M$.*

(iii) *The Cauchy data*

$$(\varphi_j|_{\partial M}, \lambda_j^{-1}\partial_\nu \varphi_j|_{\partial M})$$

of the eigenfunctions is strictly sub-maximal in growth, i.e. both components are $o(\lambda_j^{\frac{1}{2}})$ as $\lambda_j \to \infty$.

Then there exists a subsequence $A \subset \mathbb{N}$ of density one so that

$$\lim_{\substack{j\to\infty \\ j\in A}} N(\varphi_j) = \infty.$$

To draw specific conclusions we need to show that the conditions are satisfied in natural examples. In this article we consider "Sinai-Lorentz" billiards, i.e. non-positively curved surfaces with concave boundary. They are known to satisfy (i) and (ii) is easily extracted from a number of articles. Condition (iii) is the analogue for Cauchy data of the sup norm results on manifolds without boundary proved in [SZ02, SZa]. In the forthcoming article [SZb] the following is proved:

Theorem 1.2. *Let (M, g) be a Riemannian manifold of dimension n with geodesically concave boundary. Suppose that there exist no self-focal points $q \in \partial M$. Then the sup-norms of Cauchy data of Dirichlet, resp. Neumann, eigenfunctions are $o(\lambda_j^{\frac{n-1}{2}})$.*

It is possible that concavity of the boundary implies non-existence of self-focal points on the boundary. The conclusion of Theorem 1.2 should be true for general Riemannian manifolds manifolds with boundary with no boundary self-focal points, but we only state the result needed for the present article.

For Sinai-Lorentz billiards we conjecture that the sup norms are of order $O(\frac{\lambda_j^{\frac{n-1}{2}}}{\log \lambda_j})$. This would be useful for getting logarithmic lower bounds on numbers of nodal domains.

1.1. **Surfaces of non-positive curvature with concave bounary.** The hypothesis of Theorem 1.1 is satisfied by surfaces of non-positive curvature with concave boundary. By this we mean a non-positively curved surface,

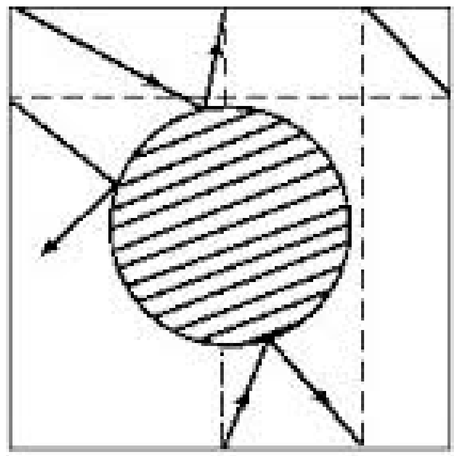
$$M = X \backslash \bigcup_{j=1}^{r} \mathcal{O}_j, \tag{1.1}$$

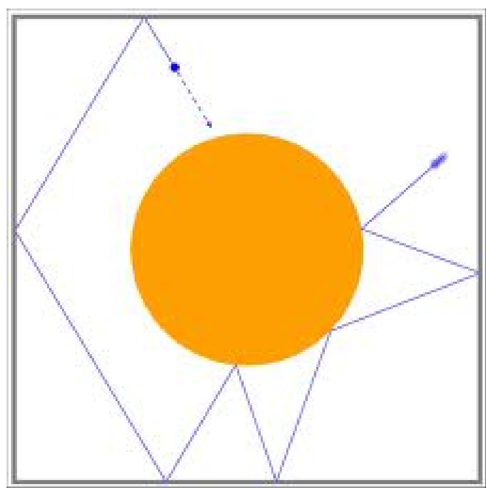
obtained by removing a finite union $\mathcal{O} := \bigcup_{j=1}^{r} \mathcal{O}_j$ of embedded nonintersecting geodesically convex domains (or 'obstacles') $\mathcal{O}_j$ from a closed non-positively curved surface (X, g). We denote the scalar curvature of (X, g) by K and assume $K \leq 0$. In the case where X is a flat torus or a square, such a billiard is called a Sinai billiard. It is proved in [Sin, CS87, BSC90] that the billiard flow of a Sinai billiard is ergodic. When the ambient surface (X, g) is non-positively curved, ergodicity of the billiard flow is proved in [Kra89]. The result will be discussed in more detail in §2.

Billiards on (M, g) of the form (1.1) also satisfy condition (ii) on non-existence of self-focal points. Self-focal points q are necessarily self-conjugate, i.e. there exists a broken Jacobi field along a geodesic billiard loop at q vanishing at both endpoints. But as we review in §2, non-positively curved dispersive billiards as above do not have conjugate points.

Thus (assuming Theorem 1.2) we have,

Corollary 1.3. *The conclusion of Theorem 1.1 holds for a non-positively curved surface* (1.1) *with concave boundary.*





We expect Theorem 1.1 to generalize to all hyperbolic billiards, such as planar billiard tables with concave walls. We also expect it to generalize to the Bunimivoch stadium. The only new complication is to the proof of Theorem 1.2, since the corners complicate sup norm estimates of eigenfunctions. In fact, we only need the sup norm estimates away from the corners. The rest of the proof of Theorem 1.1 extends with no further work to domains with Lipschitz boundaries. Since sup norm estimates are of independent interest, we postpone the proof of Theorem 1.2 and further results on sup norms of Cauchy data eigenfunctions to a subsequent work.

1.2. **Outline of the proof of Theorem 1.1.** The overall argument follows the strategy of [JZ13], and is based on counting the zeros of the Cauchy data φ_j^b of an ergodic sequence of eigenfunctions on ∂M. The presence of the boundary creates both new complications and new simplifications. Perhaps surprisingly, the main difficulty in the proof of Theorem 1.1 that does not occur in [JZ13] is to prove that the standard sup norm bound $O(\lambda_j^{\frac{1}{2}})$

on φ_j can be improved to $o(\lambda_j^{\frac{1}{2}})$ under the assumptions of the theorem. By generalizing the arguments of [SZ02, STZ11, SZa] to Cauchy data of Dirichlet or Neumann eigenfunctions on surfaces with boundary, we prove that the Cauchy data have sup norms which are $o(\lambda_j^{\frac{1}{2}})$ as long as no self-focal points occur.

To begin with, we recall the definition and properties of Cauchy data of ergodic sequences of eigenfunctions along the boundary. We use the following notation:

Definition 1.4. *The Cauchy data of Neumann (resp. Dirichlet) eigenfunctions of (M, g) is defined by*

$$\begin{cases} \varphi_j^b = \varphi_j|_{\partial M}, & \textit{Neumann boundary conditions}, \\ \\ \varphi_j^b = \lambda_j^{-1}\partial_\nu \varphi_j|_{\partial M}, & \textit{Dirichlet boundary conditions}, \end{cases}$$

In [HZ04] (see also [Bur05]) it is proved that if the geodesic billiard flow of a Riemannian manifold (M, g) is ergodic, then there exists a subsequence $A \subset \mathbb{N}$ of density one so that, for any $f \in C(M)$,

$$\lim_{\substack{j\to\infty \\ j\in A}} \int_{\partial M} f|\varphi_j^b|^2 ds = \omega_B(f), \tag{1.2}$$

where $\omega_B(f)$ is a 'limit state' (i.e. a positive measure viewed as a linear functional on $C(B^*\partial M)$) which depends on the boundary condition B. We only apply the result to 'multiplication operators' by $f \in C(\partial M)$ in this article. The result in more detail is in §3. It has recently been extended to general hypersurfaces of M (with or without boundary) in [CTZ13]. We refer to the density one sequence (1.2) as a "boundary ergodic sequence" of eigenfunctions.

The main input in Theorem 1.1 is the following result about boundary nodal points:

Theorem 1.5. *Let (M, g) be a Riemannian surface which satisfies three assumptions given in Theorem 1.1. Then for any given orthonormal eigenbasis of Neumann eigenfunctions $\{\varphi_j\}$, there exists a subsequence $A \subset \mathbb{N}$ of density one such that*

$$\lim_{\substack{j\to\infty \\ j\in A}} \#Z_{\varphi_j} \cap \partial M = \infty.$$

Furthermore, there are an infinite number of zeros where $\varphi_j|_{\partial M}$ changes sign. For Dirichlet eigenbasis $\{\varphi_j\}$, there exists a subsequence $A \subset \mathbb{N}$ of density one such that

$$\lim_{\substack{j\to\infty \\ j\in A}} \#\Sigma_{\varphi_j} \cap \partial M = \infty.$$

Given Theorem 1.5, the remainder of the proof of Theorem 1.1 is topological. As in [GRS13, JZ13], we use an Euler characteristic argument, Theorem 7.2, to obtain a lower bound on the number of nodal domains from the lower bound on the number of boundary zeros. The fact that the zeros occur on the boundary allows us to simplify the proof in [JZ13] and to omit the symmetry assumption. We recall that in [JZ13], we proved that $N(\varphi_\lambda)$ tends to infinity along a density one sequence of even or odd eigenfunctions on a Riemannian surfaces (M, g, σ) of negative curvature with an isometric orientation reversing involution σ with separating fixed point set. We used the involution to relate signed zeros of restrictions of eigenfunctions to the fixed point set with nodal domains. In the boundary case, we relate number of signed zeros along the boundary to nodal domains without any symmetry assumptions.

For instance, assume that Z_φ is non-singular. Then the number of connected components of Z_φ essentially gives a lower bound for the number of nodal domains. In [JZ13], the assumption of the existence of σ is necessary in order to ensure that there are at most two intersections between $Fix(\sigma)$ and a closed nodal curve, which allows one to relate the number of nodal domains and number of intersections. Such symmetry assumption is not necessary in the boundary case, because a nodal segment which has an end point at ∂M must terminate in ∂M.

The intersection points above are the points where the nodal set touches the boundary in the sense of [TZ09]. That article shows that when the boundary is piecewise real analytic, then the number of intersection points is $\leq C_M \lambda$ for some constant depending only on the domain. In theorem 1.5, we obtain a qualitative lower bound on the number of intersection points.

1.3. **Outline of the proof of Theorem 1.5.** The key step in proving Theorem 1.5 is the following

Proposition 1.6. *Let (M, g) be a Riemannian surface with smooth boundary and ergodic billiards. Then exists a subsequence of density one of the Neumann eigenfunctions such that, for any fixed interval $\beta \subset \partial M$,*

$$\int_\beta |\varphi_j| ds > \left| \int_\beta \varphi_j ds \right|.$$

Moreover, there exists a density one subsequence of Dirichlet eigenfunctions such that for any fixed interval $\beta \subset \partial M$,

$$\int_\beta |\partial_\nu \varphi_j| ds > \left| \int_\beta \partial_\nu \varphi_j ds \right|.$$

That is,

$$\int_\beta |\varphi_j^b| ds > \left| \int_\beta \varphi_j^b ds \right|. \tag{1.3}$$

The Proposition clearly implies that the Neumann eigenfunctions φ_λ must have a sign-changing zero on β for a density one subsequence of eigenfunctions. Similarly for normal derivatives of Dirichlet eigenfunctions. Theorem 1.5 is a direct consequence of Proposition 1.6.

To prove Proposition 1.6 we combine three results, two of which are already proved elsewhere and one which we prove here. The first is the quantum ergodic restriction theorem (1.2). The second ingredient is the following Kuznecov sum formula, extending the result of [Zel92] to manifolds with boundary. It is an immediate consequence of Theorem 1 of [XH]:

Theorem 1.7 ([XH])**.** *Let (M, g) be a Riemannian surface with smooth boundary ∂M. Let $\{\varphi_j\}$ be an orthonormal basis of Neumann eigenfunctions. Let $f \in C_0^\infty(\partial M)$. Then there exists a constant $c > 0$ such that,*

$$\sum_{\lambda_j < \lambda} \left| \int_{\partial M} f \varphi_j ds \right|^2 = \left(\frac{2}{\pi} \int_{\partial M} f^2 ds \right) \lambda + o(\lambda).$$

Let $\{\varphi_j\}$ be an orthonormal basis of Dirichlet eigenfunctions. Then there exists a constant $c > 0$ such that,

$$\sum_{\lambda_j < \lambda} \left| \lambda_j^{-1} \int_{\partial M} f \partial_\nu \varphi_j ds \right|^2 = \left(\frac{2}{\pi} \int_{\partial M} f^2 ds \right) \lambda + o(\lambda).$$

We recall the proof in §5. We view Theorem 1.7 as an asymptotic mean formula for a sequence of probability measures, stating that on average, $\left| \int_{\partial M} f \varphi_j^b ds \right|^2$ is of order λ_j^{-1}. An application of Chebychev's inequality then gives

Corollary 1.8. *There exists a constant $c = c_f > 0$ depending only on f such that, for each $M > 0$ there exists a subsequence of proportion $\geq 1 - \frac{c}{M}$ of the $\{\varphi_j\}$ for which*

$$\left| \int_{\partial M} f \varphi_j^b ds \right| \leq M \lambda_j^{-\frac{1}{2}}$$

We only use Corollary 1.8 in the proof of Theorem 1.1.

We now assemble Theorems 1.7, 1.2 and the Quantum Ergodic Restriction (QER) theorem of [HZ04] to given an outline of the proof Proposition 1.6. For simplicity we assume that the boundary conditions are Neumann. From the third assumption in Theorem 1.1, we have

$$||\varphi_j|_{\partial M}||_{L^\infty} \leq \lambda_j^{1/2} o(1).$$

We then prove that for any $M > 0$ there exists a subsequence of density $\geq 1 - \frac{1}{M}$ so that for any arc $\beta \subset H$,

$$| \int_\beta \varphi_j ds| \leq C \lambda_j^{-1/2} M$$

and there exists a subsequence of density one for which

$$\int_\beta |\varphi_j| ds \geq ||\varphi_j||_{C^0(\beta)}^{-1} ||\varphi_j||^2_{L^2(\beta)} \geq C\lambda_j^{-1/2} \frac{1}{o(1)},$$

This is a contradiction if φ_j has no sign change on β.

It follows that for any $M > 0$ there exists a subsequence of density $\geq 1-\frac{1}{M}$ for which

$$\int_\beta |\varphi_j| ds > |\int_\beta \varphi_j ds|.$$

This implies the existence of a subsequence of density one with this property.

2. Billiards on negatively curved surfaces exterior to convex obstacles

Before discussing billiards on these surfaces, we introduce some general notation and background.

2.1. **Billiard map.** We denote by

$$\Phi^t : S^*M \to S^*M \tag{2.1}$$

the billiard (or broken geodesic) flow on S^*M. The trajectory $\Phi^t(x, \xi)$ consists of geodesic motion between impacts with the boundary, with the usual reflection law at the boundary.

The billiard map β is defined on the unit ball bundle $B^*\partial M$ of ∂M as follows: given $(y, \eta) \in B^*\partial M$, i.e. with $|\eta| < 1$, we let $(y, \zeta) \in S^*M$ be the unique inward-pointing unit covector at y which projects to (y, η) under the map $T^*_{\partial M}\overline{M} \to T^*\partial M$. Then we follow the geodesic

$$(y, \eta) \in B^*(\partial M) \to \Phi^t(q(y), \zeta(y, \eta)) \tag{2.2}$$

until the projected billiard orbit first intersects the boundary again; let $y' \in \partial M$ denote this first intersection. We denote the inward unit normal vector at y' by $\nu_{y'}$, and let $\zeta' = \zeta + 2(\zeta \cdot \nu_{y'})\nu_{y'}$ be the direction of the geodesic after elastic reflection at y'. Also, let η' be the projection of η' to $T^*_{y'}\partial M$. Then

$$\beta(y, \eta) := (y', \eta'). \tag{2.3}$$

The directions tangent to ∂M cause singularities (discontinuities) in β, and the billiard map is not apriori well-defined on initial directions which are tangent to ∂M, i.e. $(y, \eta) \in S^*\partial M$ with $|\eta| = 1$. To obtain a smooth dynamical system, one often removes the tangential directions $S^*\partial M$. Billiard trajectories starting on ∂M in non-tangential directions may become tangential at some future intersection and so we puncture out all trajectories which in the past or future become tangential. We define $B^0_q = B^*_q\partial M \backslash S^*_q\partial M$ to be the projections of non-tangential directions and define $\mathcal{R}^1 \subset B^0_q$ to be $\beta^{-1}(B^0_q)$ and, more generally, $\mathcal{R}^{k+1} = \beta^{-1}(\mathcal{R}^k)$ for natural numbers k. Thus $\mathcal{R}^k$ consists of the points where β^k is well defined and maps to B^0_q. Similarly we define $\mathcal{R}^{-1} = \beta_-^{-1} B^0_q$ and $\mathcal{R}^{-k-1} = \beta_-^{-1}(\mathcal{R}^{-k})$. Clearly $\mathcal{R}^1 \supset \mathcal{R}^2 \supset \dots,$

$\mathcal{R}^{-1} \supset \mathcal{R}^{-2} \supset \dots$ and it is shown in [CFS82] that each $\mathcal{R}^k$ has full measure. Consequently, if we define

$$\mathcal{R}^\infty = \bigcap_k \mathcal{R}^k,$$

then β is a symplectic diffeomorphism on $\mathcal{R}^\infty$. As we now explain, we may define β at tangent vectors to ∂M so that the billiard trajectory is simply a geodesic of (X, g) which hits ∂M tangentially. The billiard map is then defined on all of $B^*\partial M$ but is discontinuous along the set of tangential directions.

2.2. **Ergodicity of the billiard map for non-positively curved surfaces with concave boundary.** We need the following result of Kramli-Simonyi-Szasz [Kra89].

Proposition 2.1 ([Kra89])**.** *Billiards on a non-positively curved surface with concave boundary are ergodic.*

For the sake of completeness, we briefly review the proof. The main result of the [Kra89] is a proof of the "Fundamental Theorem for Dispersing Billiards" (Theorem 5.1). The term is taken from work of Bunimovich-Sinai and roughly asserts the existence almost everywhere of stable/unstable foliations for the billiard map and some of its quantitative properties. In section 6 of [Kra89] the authors show how ergodicity of the billiards follows from Theorem 5.1 by the Hopf-Sinai argument.

The surfaces in [Kra89] are assumed to be the exterior (1.1) of a finite union of convex obstacles (i.e. the boundary curves have strictly positive geodesic curvature from the inside). However, they are more general than the non-positively curved surfaces assumed here. The billiards are only assumed to satisfy 'Vetier's conditions', which are conditions implying that 'no focal points arise'. The conditions are stated precisely in Condition 1.2-1.4 in [Kra89]. Condition (1.2) is that the distance between obstacles is bounded below by some $\tau_{\min} > 0$, which is obvious for a compact surface when the obstacles do not intersect. Condition (1.3) is that there exists $\tau_{\max}$ so that any geodesic must intersect ∂M in time $\leq \tau_{\max}$. This condition can be removed if the curvature is strictly negative. Condition (1.4) is a curvature condition which is satisfied as long as $K \leq 0$. In this case, Condition (1.3) becomes irrelevant. In fact, $K \leq 0$ along implies the fundamental theorem and ergodicity of the billiard flow.

For $(x, \xi) \in S^*_x M$, the stable (resp. unstable) fiber $H^{(s)}(x, \xi)$ (resp. $H^{(u)}(x, \xi)$) through x is the set of $(y, \eta) \in S^*M$ so that

$$\lim_{t \to +\infty} d(\Phi^t(y, \eta), \Phi^t(x, \xi)) = 0$$

(resp. $t \to -\infty$). The stable leaf through x is $\bigcup_{t \in \mathbb{R}} \Phi^t(H^{(s)}(x))$. Similarly for the unstable leaf. Vetier proved that under the conditions above, there exist stable and unstable fibers through almost $(x, \xi) \in S^*M$ which are C^1

curves. It follows that through almost every $(q,\eta) \in B^*\partial M$ there exist stable/unstable leaves for the billiard map, which are C^1 curves. In particular, this is the case for non-positively curved surfaces with concave boundary.

2.3. Absence of self-focal points non-positively curved surfaces with concave boundary.

We also need

Lemma 2.2. *There do not exist partial self-focal points on a non-positively curved surfaces with concave boundary.*

We use broken Jacobi fields for the billiard flow and begin with some background from [Woj94, ZL, Bia13]. The Lemma is also proved by L. Stoyanov in his articles [Sto99, Sto89]. A (normal) Jacobi field is an orthogonal vector field $J(t)$ along a billiard trajectory γ with transversal reflections at ∂M, which satisfies the Jacobi equation $\frac{D^2}{dt^2} J + K(\gamma(t))J = 0$ away from the elastic impacts, and which is reflected by the law

$$\begin{pmatrix} J \\ J' \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ \frac{2K(s)}{\sin\varphi(s)} & -1 \end{pmatrix} \begin{pmatrix} J \\ J' \end{pmatrix} = \begin{pmatrix} -J \\ \frac{2K(s)}{\sin\varphi(s)} J - J' \end{pmatrix}$$

at the reflection point. Here φ is the angle that $\gamma'(t)$ makes with the boundary at the impact time. Recalling that a Jacobi field is the variation vector field $J(t) = \frac{\partial}{\partial\varepsilon}\gamma_\varepsilon(t)$ of a 1-parameter family of billiard trajectories, we see that the reflection law is the derivative in ε of the reflection law for the curves γ_ε.

Proof. Assume first that Y is a simply connected non-positively curved surface, and that $\mathcal{O}_1, \cdots, \mathcal{O}_m \subset Y$ are disjoint obstacles. Fix a point $q \in \partial\mathcal{O}_1$ and consider billiard trajectories of (q,η) on $Y \setminus \bigcup_{j=1}^m \mathcal{O}_j$. For a given billiard trajectory of (q,η), we correspond a sequence $\{a_j(\eta)\}_{j\geq 0}$ with $1 \leq a_j(\eta) \leq m$ such that j-th impact occurs on the boundary of $\mathcal{O}_{a_j}$ (we assume that $a_0(\eta) = 1$.)

For a given sequence $B = \{b_j\}_{0\leq j\leq M}$ with $1 \leq b_j \leq m$, let

$$S_B := \{\eta \mid a_j(\eta) = b_j,\ j = 0, \cdots, M\}.$$

Let $q_j(\eta) \in \partial\mathcal{O}_{a_j(\eta)}$ be the j-th impact point. Since Y is simply connected, the length of billiard trajectory from q to $q_M(\eta)$ is bounded from above by some constant for $\eta \in S_B$. Therefore, if there are infinitely many $\eta \in S_B$ such that $q_M(\eta) = q'$ with same q', then q and q' are conjugate. However, this is impossible, since the norm of the Jacobi field is monotonically increasing between impacts and only grows at an impact. Hence along any billiard trajectory from any $q \in \partial M$, it cannot vanish for $t > 0$.

Now let $\tilde{X}$ be the universal covering of X and let $\tilde{\mathcal{O}}_j$ for $j = 1, 2, \cdots$ be the lift of obstacles on X. Fix $q \in \partial M$. For each $(q,\eta) \in S^*M$ whose billiard trajectory is a loop, we correspond a finite length sequence $\{a_j(\eta)\}_{j\leq T}$ as above, where $T > 0$ is the first index such that $q_T(\eta)$ is a preimage of q on $\tilde{X}$. Then from above, we infer that there are finitely many η such that

$\{a_j(\eta)\}_{j\leq T} = B$ for any given B. Since the set of B is countable, this proves that there are at most countably many η, for which billiard trajectory of (q,η) is a loop.

□

3. Quantum ergodic restriction theorems

We now review the quantum ergodic restriction theorem to the boundary of [HZ04]. Roughly speaking, the results says that if the billiard ball map β on $B^*\partial M$ is ergodic, then the boundary values u_j^b of eigenfunctions are quantum ergodic.

Define

$$\gamma(q) = \sqrt{1-|\eta|^2}, \quad q = (y,\eta). \tag{3.1}$$

Boundary Values			
B	$B\varphi_\lambda$	φ_λ^b	$d\mu_B$
Dirichlet	$u\|_Y$	$\lambda^{-1}\partial_\nu\varphi_\lambda\|_Y$	$\gamma(q)d\sigma$
Neumann	$\partial_\nu\varphi_\lambda\|_Y$	$\varphi_\lambda\|_Y$	$\gamma(q)^{-1}d\sigma$

Theorem 3.1. *Let $M \subset \mathbb{R}^n$ be a compact manifold with boundary and with ergodic billiard map. Let $\{\varphi_j^b\}$ be the boundary values of the eigenfunctions $\{\varphi_j\}$ of Δ_B on $L^2(M)$ in the sense of the table above. Let A_h be a semiclassical operator of order zero on ∂M. Then there is a subset S of the positive integers, of density one, such that*

$$\lim_{j\to\infty, j\in S}\langle A_{h_j}\varphi_j^b, \varphi_j^b\rangle = \omega_B(A), \tag{3.2}$$

where $h_j = \lambda_j^{-1}$ and ω_B is the classical state on the table above.

4. Restriction of the wave group of a Sinai billiard to the boundary

The billiard flow arises in spectral problems because the singularities of the fundamental solution of the wave equation propagate along billiard trajectories. The billiard flow relevant to our problem is thus determined by propagation of singularities for the wave equation on a non-positively curved surface (1.1) in the exterior of a finite union of disjoint convex obstacles. In this case, the singularities which intersect the boundary are known as grazing rays. It was proved independently by R. B. Melrose and M. E. Taylor [Mel75] that a singularity propagating along a grazing ray simply continues along the same geodesic of the ambient space when it touches the boundary. Consequently, the dynamical billiard flow of the previous section coincides with the propagation of singularities.

For the purposes of this article, and in particular for the estimate of sup norms, we only need two aspects of the wave kernel:

- The wave front set of the wave kernel and its restriction to the boundary.
- A small time parametrix and symbol calculation for the restriction of the wave kernel to the diagonal of the boundary sufficient for the calculation of the principal symbol at $t = 0$.

We do not need the parametrix construction of [Mel75] which is valid near grazing rays, i.e. billiard trajectories intersecting ∂M tangentially. In fact, the wave front set calculation we need is classical, while the symbol calculation is contained in [HZ12, XH]. We will only add to the known results when we adapt the sup norm estimate of [SZ02] to the Cauchy data of eigenfunctions.

We now briefly review the results we need for the sake of completeness.

4.1. **Wave front set of the wave group.** We denote by

$$E_B(t) = \cos\left(t\sqrt{-\Delta_B}\,\right), \quad \text{resp.} \;\; S_B(t) = \frac{\sin\left(t\sqrt{-\Delta_B}\right)}{\sqrt{-\Delta_B}} \tag{4.1}$$

the even (resp. odd) wave operators (M, g) with boundary conditions B. The wave group $E_B(t)$ is the solution operator of the mixed problem

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) E_B(t, x, y) = 0, \\ \\ \quad E_B(0, x, y) = \delta_x(y), \quad \frac{\partial}{\partial t} E_B(0, x, y) = 0, \;\; x, y \in M; \\ \\ B E_B(t, x, y) = 0, \;\; x \in \partial M \end{cases}$$

The wave front sets of $E_B(t, x, y)$ and $S_B(t, x, y)$ are determined by the propagation of singularities theorem of [MS78] for the mixed Cauchy Dirichlet (or Cauchy Neumann) problem for the wave equation. We from [Hör90] (Vol. III, Theorem 23.1.4 and Vol. IV, Proposition 29.3.2) that

$$WF(E_B(t, x, y)) \subset \bigcup_{\pm} \Lambda_{\pm}, \tag{4.2}$$

where $\Lambda_{\pm} = \{(t, \tau, x, \xi, y, \eta) : \; (x, \xi) = \Phi^t(y, \eta), \; \tau = \pm|\eta|_y\} \subset T^*(\mathbb{R} \times \Omega \times \Omega)$ is the graph of the generalized (broken) geodesic flow, i.e. the billiard flow Φ^t. The same is true for $WF(S_B)$. As mentioned above, the broken geodesics in the setting of (1.1) are simply the geodesics of the ambient negatively curved surface, with the equal angle reflections at the boundary; tangential rays simply continue without change at the impact.

4.2. **Restriction of wave kernels to the boundary.** The key tool in the proof of Theorem 1.2 is the analysis of the restriction of the Schwartz kernel $E_B(t, x, y)$ of $\cos t\sqrt{\Delta_B}$ to $\mathbb{R} \times \partial M \times \partial M$ and further to $\mathbb{R} \times \Delta_{\partial M \times \partial M}$, where $\Delta_{\partial M \times \partial M}$ is the diagonal of $\partial M \times \partial M$. We denote by dq the surface measure on the boundary ∂M, and by $ru = u|_{\partial M}$ the trace operator. We denote by $E_B^b(t,q',q) \in \mathcal{D}'(\mathbb{R}\times\partial M\times\partial M)$ the following boundary traces of the Schwartz kernel $E_B(t,x,y)$ defined in (4.1):

$$E_B^b(t,q',q) = \begin{cases} r_{q'} r_q \partial_{\nu_{q'}} \partial_{\nu_q} E_D(t,q',q), & \text{Dirichlet} \\ \\ r_{q'} r_q \;\; E_N(t,q',q), & \text{Neumann} \end{cases} \tag{4.3}$$

The subscripts q', q refer to the variable involved in the differentiating or restricting. Henceforth we use the notation γ_q^B for the boundary trace. Thus, $\gamma_q^B = r_q$ in the Neumann case and $\gamma_q^B = r_q \partial_{\nu_q}$ in the Dirichlet case.

The sup norm bounds of Theorem 1.2 are derived from an analysis of the singularities of the boundary trace of the wave kernel along the diagonal,

$$E_B^b(t,q,q) = \sum_{j=1}^{\infty} \cos(t\lambda_j)|\varphi_j^b(q)|^2. \tag{4.4}$$

Here as above, $\varphi_j^b(q) = \gamma_q^B \varphi_j$. We need sufficient information on (4.4) to be able to adapt the proof of [SZ02] to the setting of Cauchy data of eigenfunctions. The Kuznecov asymptotics of Theorem 1.7 is proved by studying

$$\int_{\partial M}\int_{\partial M} E_B^b(t,q,q') ds(q) ds(q') = \sum_{j=1}^{\infty} \cos(t\lambda_j) \left| \int_{\partial M} \varphi_j^b(q) ds(q) \right|^2. \tag{4.5}$$

4.3. **Wave front set of the restricted wave kernel.** The first and simplest piece of information is the wave front set of (4.4). It follows from (4.2) and from standard results on pullbacks of wave front sets under maps, the wave front set of $E_B^b(t,q,q')$ consists of co-directions of broken trajectories which begin and end on ∂M. That is,

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t,q,q')) \quad \subset \quad & \{(t,\tau,q,\eta,q',\eta') \in B^*\partial M \times B^*\partial M : \\ & [\Phi^t(q,\xi(q,\eta))]^T = (q',\eta'),\ \tau = -|\xi|\}. \end{aligned} \tag{4.6}$$

Here, the superscript T denotes the tangential projection to $B^*\partial M$. We refer to Section 2 of [HZ12] for an extensive discussion. It follows from (4.6) that

$$\begin{aligned} WF(\gamma_q^B \gamma_{q'}^B E(t,q,q)) \quad \subset \quad & \{(t,\tau,q,\eta,q,\eta') \in B_q^*\partial M \times B_q^*\partial M : \\ & [G^t(q,\xi(q,\eta))]^T = (q,\eta'),\ \ \tau = -|\xi(q,\eta)|\}. \end{aligned} \tag{4.7}$$

Thus, for $t \neq 0$, the singularities of the boundary trace $\gamma_q^B \gamma_{q'}^B E(t,q,q)$ at $q \in \partial M$ to broken bicharacteristic loops based at q in $\overline{M}$. When $t = 0$ all inward pointing co-directions belong to the wave front set.

Remark 4.1. *One of the principal features of the boundary trace $E_B^b(t,q,q)$ along the diagonal is that the singularity at $t = 0$ becomes uniformly isolated* *from other singularities, while the interior kernel $E_B(t,x,x)$ has singularities at $t = 2d(x)$ arbitrarily close to $t = 0$.*

4.4. **Wave group of a non-positively curved surface exterior to convex obstacles.** Althought it is not really necessary for the purpose of this article, it is illuminating to observe that the wave kernel in our setting of a non-positively curved surface exterior to convex obstacles can be obtained by summing the much simpler wave kernel on the universal cover in the exterior of one (or a finite number) of convex obstacles. It is possible that this Poincaré series type formula will produce logarithmic improvements on sup norm estimates, which are at present unknown for any boundary problem. We plan to investigate this in future work.

We denote by $(\tilde{X}, \tilde{g})$ the universal Riemannian cover of (X, g). For instance, $\tilde{X} = \mathbb{R}^2$ in the case of a flat torus or $\tilde{X} = \mathbf{H}^2$ in the case of a hyperbolic surface of genus $g \geq 2$. If $\pi : \tilde{X} \to X$ then

$$\pi^{-1}(\mathcal{O}) = \bigcup_{\gamma \in \Gamma} \gamma \tilde{\mathcal{O}}$$

where the union is over the elements of the deck transformation group $\Gamma \simeq \pi_1(X)$, and where

$$\tilde{\mathcal{O}} = \bigcup \tilde{\mathcal{O}}_j,$$

with $\tilde{\mathcal{O}}_j$ a choice of one component of $\pi^{-1}(\mathcal{O}_j)$. The union is a disjoint one by the assumption that each $\mathcal{O}_j$ is embedded in X. We may (and will) pick a connected fundamental domain $\mathcal{D} \subset \tilde{X}$ for Γ in such a way that $\mathcal{O}_j \subset \mathcal{D}^o$ (the interior of $\mathcal{D}$ for all j.

We consider the Laplacian

$$\tilde{\Delta}_B \text{ on } \tilde{X} \backslash \tilde{\mathcal{O}}$$

with Dirichlet, resp. Neumann, boundary conditions B. Let

$$\tilde{E}_B(t,x,y) \text{ be the Schwartz kernel of } \cos t \sqrt{\tilde{\Delta}_B} \text{ on } \tilde{X} \backslash \tilde{\mathcal{O}}. \tag{4.8}$$

Proposition 4.2. *Let $E_B(t,x,y)$ be the cosine wave kernel, i.e. the Schwartz kernel of $\cos t\sqrt{\Delta_B}$, on (M,g) with Dirichlet, resp. Neumann, boundary conditions on $\partial M = \partial \mathcal{O}$ and let $\tilde{E}_B(t,x,y)$ (4.8) be the cosine wave kernel on $\tilde{X} \backslash \tilde{\mathcal{O}}$. Then,*

$$E_B(t,x,y) = \sum_{\gamma \in \Gamma} \tilde{E}_B(t, \tilde{x}, \gamma \tilde{y}),$$

where $\tilde{x}, resp. \tilde{y} \in \mathcal{D} \in \pi^{-1}(x), resp. \pi^{-1}(y)$.

Proof. Both sides solve the homogeneous wave equation and so we only need to prove that they have the same initial and boundary conditions. When $t = 0$, we need to verify that $\delta_y(x) = \sum_{\gamma \in \Gamma} \delta_{\tilde{y}}(\gamma x)$ for all $y, x, \in M$. It suffices to check this in $\mathcal{D}$ where it is obvious. We then need to check the boundary condition, but again it suffices to check within $\mathcal{D}$ and again they match.

$\square$

We now consider boundary restrictions. On $\tilde{X}$ we consider the restriction of $\tilde{E}^b(t,q,q')$ to $\partial\tilde{\mathcal{O}}$ (with the exterior normal derivative in the Dirichlet case).

Corollary 4.3. *With the same notation,*

$$E^b_B(t,q,q') = \sum_{\gamma\in\Gamma} \tilde{E}^b_B(t,\tilde{q},\gamma\tilde{q}').$$

4.5. **Local Weyl law.** The boundary local Weyl law gives an asymptotic formula for the spectral averages of the expected value of an observable A_h relative to boundary traces of eigenfunctions. The relevant algebra of observables in our setting as in [HZ04] is the algebra $\Psi^0_h(\partial M)$ of zeroth order semiclassical pseudodifferential operators on ∂M, depending on the parameter $h \in [0,h_0]$. We denote the symbol of $A = A_h \in \Psi^0_h(\partial M)$ by $a = a(y,\eta,h)$. Thus $a(y,\eta) = a(y,\eta,0)$ is a smooth function on $T^*\partial M$; we may without loss of generality assume it is compactly supported. We further define states on the algebra $\Psi^0_h(\partial M)$ by

$$\omega_B(A) = \frac{4}{\operatorname{vol}(S^{n-1})\operatorname{vol}(M)} \int_{B^*\partial M} a(y,\eta)d\mu_B. \tag{4.9}$$

Here, as in (3.1),

$$d\mu_B = \gamma(q)d\sigma \text{ (Dirichlet)}, \quad d\mu_B = \gamma(q)^{-1}d\sigma \text{ (Neumann)}$$

where

$$\gamma(q) = \sqrt{1-|\eta|^2}, \quad q = (y,\eta).$$

For either Dirichlet or Neumman boundary conditions, the local Weyl law is proved in Lemma 1.2 of [HZ04]:

Proposition 4.4. *Let A_h be a zeroth order semiclassical operator on ∂M. Then,*

$$\lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j\le\lambda} \langle A_{h_j}\varphi^b_j, \varphi^b_j\rangle \to \omega_B(A) \tag{4.10}$$

Note that in [HZ04] the kernel of A_h was assumed to be disjoint from the singular set of ∂M, but in this article the singular set is empty.

5. Kuznecov sum formula for the boundary integral: Proof of Theorem 1.7

The general Kuznecov formula in [Zel92] for C^∞ Riemannian manifolds (M,g) without boundary is a singularity expansion for the distribution

$$S_H(t) = \int_H \int_H E(t,q,q')ds(q')ds(q), \tag{5.1}$$

where where $H \subset M$ is a smooth submanifold and where

$$E(t) = \cos t\sqrt{\Delta}$$

is the even wave kernel. The singularities of $S_H(t)$ in the boundaryless case were shown to correspond to trajectories of the geodesic flow which intersect H orthogonally at two distinct times, and to be singular at the difference T of these times. We refer to such trajectories as H-*orthogonal* geodesics.

Theorem 1.7 is a generalization of the Kuznecov formula of [Zel92] to boundary traces on surfaces with concave boundary. We do not consider the full singularity expansion as in [Zel92] but only the singularity at $t = 0$ of

$$\begin{aligned} S_f(t) : &= \int_{\partial M} \int_{\partial M} E_B^b(t, q, q') f(q) f(q') ds(q) ds(q') \\ &= \sum_j \cos t\lambda_j \left| \int_{\partial M} f(q) \varphi_j^b(q) ds(q) \right|^2 . \end{aligned} \tag{5.2}$$

where $E_B(t) = \cos t\sqrt{\Delta_B}$ is the even wave kernel with either Dirichlet or Neumann boundary conditions.

In fact, Theorem 1.7 is a corollary of Theorems 1, Proposition 2 and Theorem 3 of [XH], which are proved for general manifolds with boundary.

Theorem 5.1. *Let $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\hat{\rho}$ is identically 1 near 0, and has sufficiently small support. Let $f \in C^\infty(\partial M)$. Then for either the Dirichlet or Neumann boundary conditions,*

$$f(x) = \lim_{\lambda \to \infty} \frac{\pi}{2} \sum_j \rho(\lambda - \lambda_j) \langle \varphi_j^b, f \rangle \varphi_j^b(x), \tag{5.3}$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\partial M}$ denotes the inner product in $L^2(\partial M)$.

Evidently, Theorem 1.7 follows by taking the inner product with f on both sides of the equation. For the sake of completeness we sketch a proof which (slightly) exploits the concavity of the boundary.

5.1. Sketch of the proof.

Lemma 5.2. *There exists $\varepsilon_0 > 0$ so that the*

$$sing\ supp S_f(t) \cap (-\varepsilon_0, \varepsilon_0) = \{0\}.$$

Proof. By (4.7) and standard pullback and pushforward calculations for wave front sets as in [Zel92], the singular support of $S_f(t)$ consists of $t = 0$ together with the 'sojourn times' equal to lengths of billiard trajectories which hit the boundary orthogonally at both endpoints. Such a billiard trajectory either (i) intersects two distinct components of ∂M, in which case its length is bounded below by the minimum distance $d_{\mathcal{O}}$ between the components, or (ii) intersects the same component orthogonally. However if it starts off orthogonally to the boundary, it cannot intersect the boundary again until it departs from a Fermi normal coordinate chart along the boundary, i.e. the radius of the maximal tube around each component which is embedded in M. The minimum $\varepsilon(M, g)$ over components of the maximal embedding radius gives a geometric lower bound for its length in this case.

Thus, we may let $\varepsilon_0 = \min\{\varepsilon(M, g), d_{\mathcal{O}}\}$.

□

Therefore it suffices to determine the singularity at $t = 0$ of $S_f(t)$. Equivalently, we prove a smoothed version and then use a cosine Tauberian theorem. As mentioned in the introduction, we only need a sufficiently accurate asymptotic expansion and remainder to prove Corollary 1.8.

To study the singularity at $t = 0$, we introduce a smooth cutoff $\rho \in \mathcal{S}(\mathbb{R})$ with supp$\hat{\rho} \subset (-\varepsilon, \varepsilon)$, where $\hat{\rho}$ is the Fourier transform of ρ and $\varepsilon < \varepsilon_0$. With no loss of generality we assume that $\hat{\rho} \in C_0^\infty(\mathbb{R})$ is a positive even function such that $\hat{\rho}$ is identically 1 near 0, has support in $[-1, 1]$ and is decreasing on $\mathbb{R}_+$. We then study

$$S_f(\lambda, \rho) = \int_{\mathbb{R}} \hat{\rho}(t)\ S_f(t) e^{it\lambda} dt. \tag{5.4}$$

Our purpose is to obtain an asymptotic expansion of $S_f(\lambda, \rho)$ as $\lambda \to \infty$.

Proposition 5.3. *$S_f(\lambda, \rho)$ is a semi-classical Lagrangian distribution whose asymptotic expansion in both the Dirichlet and Neumann cases is given by*

$$S_f(\lambda, \rho) = \frac{\pi}{2} \sum_j (\rho(\lambda - \lambda_j) + \rho(\lambda + \lambda_j)) |\langle \varphi_j^b, f \rangle|^2 = ||f||^2_{L^2(\partial M)} + o(1), \tag{5.5}$$

Proof. For $\varepsilon < \varepsilon_0$ in Lemma 5.2, we only need to determine the contribution of the main singularity of $S_f(t)$ (5.2) at $t = 0$. As in [SZ02], the $\rho(\lambda + \lambda_j)$ term contributes $\mathcal{O}(\lambda^{-M})$ for all $M > 0$ and therefore may be neglected.

To show that $S_f((t)$ is a Lagrangian distribution and to determine its singularity at $t = 0$, it suffices to construct a sufficiently precise parametrix for $\chi(q, D_t, D_q) E_B^b(t, q, q')$ for small t in some neighborhood of the diagonal in $\partial M \times \partial M$. By Proposition 4.2, it suffices to construct a small time parametrix in the universal Riemannian cover of (X, g) for the mixed Cauchy-Dirichlet (or -Neumann) problem in the exterior of a convex obstacle. Such a parametrix has been constructed by Melrose [Mel75] and Taylor [Tay] using 'Fourier-Airy' integral operators to deal with the grazing rays, i.e. geodesics which intersect $\mathcal{O}$ tangentially.

However, we are only concerned with the boundary trace of the wave kernel, $E_B^b(t, q, q')$ for small times and it is possible to construct a simpler parametrix. Indeed, in the case of a concave boundary, for small times $E_B^b(t, q, q')$ is singular only when $t = 0$ and $q = q' \in \partial M$. This follows from (4.6) and the fact that there do not exist any broken geodesic billiard trajectories from q to q' for small t except when $t = 0, q = q'$. Thus it suffices to determine the singularity at $t = 0$.

We introduce a pseudo-differential cutoff $\chi(q, D_t, D_q)$ on $\mathbb{R} \times \partial M$ whose symbol vanishes in an arbitrarily small δ-neighborhood of the tangential directions to ∂M. More precisely, as in [XH], we let $\chi(y, D_t, D_y)$ be a pseudodifferential operator on $\mathbb{R} \times \partial M$ with symbol of the form

$$\chi(y, \tau, \eta) = \zeta(|\eta|^2_{\tilde{g}}/\tau^2)(1 - \varphi(\eta, \tau)), \tag{5.6}$$

where $\zeta(s)$ is supported where $s \leq 1-\delta$ for some positive δ, and $\varphi \in C_c^\infty(\mathbb{R}^n)$ is equal to 1 near the origin.

We then decompose $E_B^b(t,q,q')$ into an almost tangential part and a part with empty wave front set in tangential directions,

$$E_B^b(t,q,q') = (I - \chi(q, D_t, D_q))E_B^b(t,q,q') + \chi(q, D_t, D_q))E_B^b(t,q,q').$$

We then have corresponding terms in $S_f(t)$,

$$S_f^\varepsilon(t) = \int_{\partial M}\int_{\partial M} (I - \chi(q, D_t, D_q))E_B^b(t,q,q')f(q)f(q')dS(q)dS(q')$$

and

$$S_f^{>\varepsilon}(t) = \int_{\partial M}\int_{\partial M} \chi(q, D_t, D_q)E_B^b(t,q,q')f(q)f(q')dS(q)dS(q').$$

As in [Zel92] (1.6) we express $S_f(t)$ and $S_f(\lambda,\rho)$ in terms of pushforward under the submersion

$$\pi : \mathbb{R} \times \partial M \times \partial M \to \mathbb{R}, \quad \pi(t,q,q') = t.$$

From (4.6) we find that for $t \in (-\varepsilon, \varepsilon)$,

$$WF(S_f^\varepsilon(t)) = \\ \{(0,\tau) : \pi^*(0,\tau) = (0,\tau,0,0) \in WF(I - \chi(q, D_t, D_q))E_B^b(t,q,q')\}.$$

These wave front elements correspond to the points $(0,\tau,\tau\nu_q,\tau\nu_q) \in T_0^*\mathbb{R} \times T_{q,in}^*M \times T_{q,in}^*M$, i.e. where both covectors are co-normal to ∂M. Indeed, as in (1.6) of [Zel92] the wave front set of $S_f(t)$ is the set

$$\{(t,\tau) \in T^*\mathbb{R} : \exists (x,\xi,y,\eta) \in C_t' \cap N^*(\partial\Omega) \times N^*\partial\Omega\}$$

in the support of the symbol. However, due to the tangential cutoff $((I - \chi(q, D_t, D_q))E_B^b(t,q,q')$ has no such co-normal vectors in its wave front set. Thus, we may neglect the tangential part of $E_B^b(t,q,q')$ in determining the asymptotics of $S_f(\rho,\lambda)$. But then it follows from [Mel75, Tay] that the non-tangential part has a geometric optics Fourier integral representation, i.e. $S_f(t)$ is classical co-normal at $t = 0$.

The non-tangential part $\chi(q, D_t, D_q))E_B^b(t,q,q')$ may be expressed in terms of the "free wave kernel" or ambient wave kernel $E_X(t,x,y)$ of (X,g). Given $q \in \partial M$, we may separate M into an illuminated region (from a source at q) and a shadow region. The hyperplane $T_q\partial M \subset T_qX$ divides the full tangent space into two halfspaces. By concavity, geodesics with initial direction ξ in the lower half-space lie in M for $|t| < \varepsilon$. We call the image of the unit tangent vectors in the lower half space under the geodesic flow up to time ε the 'illuminated region' for a point source at q. The complement of the illuminated region in $T_\varepsilon M \cap M$ is the 'shadow region'. Geodesics with ξ in the upper half plane exponentiate to $X\backslash M$ for at least a short time. If we cutout all ξ whose angle to $T_q\partial M$ is $\leq \delta$, then geodesics in the upper half space remain in $X\backslash M$ for a uniform length of time, which may may assume with loss of generality is $> \varepsilon$. We write this set as $T_\delta X$. For $\xi \in T_\delta$, we define the q-dependent reflection map $(\exp_q \xi)^* := \exp_q \xi^*$. The normal reflection map equals the q-dependent reflection map only in the normal direction.

We then cut off the ambient cosine wave kernel $E_X(t, x, q)$ in the shadow region, removing the singularities of the ambient kernel due to geodesics that leave M and travel through $X \backslash M$.

Lemma 5.4. *For $|t| < \varepsilon$, the non-tangential part $\chi(q, D_t, D_q))E_B^b(t, q, q')$ of $E_B^b(t, q, q')$ can be expressed as $A(q, D_q)\gamma_q^B \gamma_{q'}^B E_X(t, q, q')\chi(q, D_t, D_q))$ where $E_X(t, x, y)$ is the cosine wave kernel of (X, g) and A is a pseudo-differential operator on ∂M of order zero.*

By the wave front calculations above,

$$E_B^b(t, x, q)\chi(q, D_t, D_q)), \text{ resp. } E_X^b(t, x, q)\chi(q, D_t, D_q))$$

are Fourier integral operators with the same wave front set equal to a pull-back of the diagonal at $t = 0$, and therefore by the calculus of Fourier integral operators, there exists a pseudo-differential operator A whose composition with the cutoff free wave kernel agrees to any given order with the the E_B^b kernel.

Therefore, to prove Proposition 5.3 it is sufficient to consider the intgral

$$\begin{array}{l} \int_{\mathbb{R}} \int_{(q,q,'):d(q,q')<\varepsilon} \hat{\rho}(t)\chi_q(q, D_t, D_q)e^{it\lambda} \\ \\ \gamma_B^q \gamma_B^{q'}\ E_X(t, x, q)f(q)f(q')dS(q)dS(q')dt. \end{array} \tag{5.7}$$

We use a Hormander style small time parametrix for $E_X(t, x, q)$, i.e. there exists an amplitude A so that modulo smoothing operators,

$$\gamma_B^b E_X(t, x, q) = \int_{T_q^* X} A(t, x, q, \xi) \exp(i\langle Exp_q^{-1}(x), \xi\rangle - t|\xi|)d\xi.$$

The amplitude has order zero. We then take the boundary trace and apply the cutoff operator $\chi(q, D_t, D_q)$, which modifies the amplitidue of (5.7) as a sum of terms with the same support as $\chi_q(\tau, \xi)$.

Changing variables $\xi \to \lambda\xi$, (5.7) may be expressed in the form,

$$\begin{array}{l} \lambda^2 \int_{\mathbb{R}} \int_{\partial M} \int_{\partial M} \int_{T_q^* X} \hat{\rho}(t)e^{it\lambda}\chi_q(\xi) \\ \\ A(t, q', q, \lambda\xi) \exp(i\lambda \left(\langle Exp_q^{-1}(q'), \xi\rangle - t|\xi|\right))f(q)f(q;)d\xi dt dS(q)dS(q'). \end{array}$$

We now compute the asymptotics by the stationary phase method.

We already know that for small t, the phase

$$t + \langle Exp_q^{-1}(q'), \xi\rangle - t|\xi|$$

is stationary only at $t = 0$ and $q = q'$. We calculate the expansion by putting the integral over $T_q^* X$ in polar coordinates,

$$\begin{array}{l} \int_{\mathbb{R}} \int_0^{\infty} \int_{\partial M} \int_{\partial M} \int_{S_q^* X} \hat{\rho}(t)e^{it\lambda}\chi_q(\xi)A(t, q', q) \\ \\ \exp(i\lambda\rho \left(\langle Exp_q^{-1}(q'), \omega\rangle - t\right))f(q)f(q;)\rho^{n-1}d\rho d\omega dt dS(q)dS(q'), \end{array}$$

and in these coordinates the phase becomes,

$$\Psi(q, \rho, t, \omega, q') := t + \rho\langle Exp_q^{-1}(q'), \omega\rangle - t\rho.$$

We get a non-degenerate critical point in the variables (t, ρ) when $\rho = 1, t = \langle Exp_q^{-1}(q'), \omega\rangle$. Eliminating these variables by stationary phase , we get

$$\frac{1}{\lambda}\int_{\partial M}\int_{\partial M}\int_{S_q^* X} e^{i\lambda\langle Exp_q^{-1}(q'),\omega\rangle}\chi_q(\omega)\tilde{A}(t, q', q, \rho\omega)f(q)f(q')d\omega dS(q)dS(q'),$$

for another amplitude $\tilde{A}$. Now the phase is

$$\Psi_q(q', \omega) = \langle Exp_q^{-1}(q'), \omega\rangle.$$

We fix q and view the phase as a function on $\partial M \times S_q^*$. We claim that the phase has a critical point if and only if $q' = q$ and $\omega \perp T_q\partial M$. Moreover, the stationary phase point is non-degenerate. Since we are working very close to the diagonal, we may approximate the phase by

$$\langle q - q', \omega\rangle + O(|q - q'|^3),$$

in local normal coordinates at q. Then

$$\nabla_{q'}\langle q - q', \omega\rangle = \omega^T,$$

the tangential projection of ω and we see (as discussed more abstractly above) that $\omega \perp T_q\partial M$ at the stationary phase point. Moreover,

$$\nabla_\omega\langle q' - q, \omega\rangle = (q' - q)^T.$$

The right side vanishes for q' near q if and only if $q' = q$ for a concave curve. The Hessian in (q', ω) has the form,

$$\begin{pmatrix} & q' & \omega \\ q' & -II_q & I_q \\ \omega & I_q & & 0 \end{pmatrix},$$

whose determinant is $(\det II_q)$, where II_q is the second fundamental form at q. In the two dimensional case, we obtain another factor of λ^{-1} from the stationary phase expansion. Therefore (5.7) is asymptotic to a multiple of

$$\int_{\partial M} f^2(q)dS(q),$$

as stated in Proposition 5.5.

To determine the multiple, or more precisely to show that it is positive, we need to find the principal symbol of the pseudo-differential operator in (5.4). In fact, it is a constant equal to 2 when pulled back to ∂M. This follows from the calculations in [HZ12] Proposition 4 and in [XH], which prove:

Lemma 5.5. *Suppose that $\hat{\rho}$ is supported in $[-\varepsilon, \varepsilon]$ and equal to 1 in a neighbourhood of 0.Then, for sufficiently small ε (depending on δ),*

(1) *the kernels of*

$$\hat{\rho}(t)\chi(y, D_t, D_y) \circ R_y R_{y'} d_{n_y} d_{n_{y'}} \cos(t\sqrt{\Delta_D}),$$
$$\hat{\rho}(t) R_y R_{y'} d_{n_y} d_{n_{y'}} \cos(t\sqrt{\Delta_D}) \circ \chi(y, D_t, D_y)$$

are distributions conormal to $\{y = y', t = 0\}$ *with principal symbol*

$$2\chi(y, \tau, \eta)\left(1 - \frac{|\eta|^2_{\tilde{g}}}{\tau^2}\right)^{\frac{1}{2}}; \tag{5.8}$$

(2) *the kernels of*

$$\hat{\rho}(t)\chi(y, D_t, D_y) \circ R_y R_{y'} \cos(t\sqrt{\Delta_N}),$$
$$\hat{\rho}(t) R_y R_{y'} \cos(t\sqrt{\Delta_N}) \circ \chi(y, D_t, D_y)$$

are distributions conormal to $\{y = y', t = 0\}$ *with principal symbol*

$$2\chi(y, \tau, \eta)\left(1 - \frac{|\eta|^2_{\tilde{g}}}{\tau^2}\right)^{-\frac{1}{2}}. \tag{5.9}$$

This completes the proof of Theorem 1.7. □

6. Sup norm bounds on Cauchy-data: Remarks on Theorem 1.2

We now outline the proof of Theorem 1.2 and explain the background to the result. The 'standard' sup norm bound

$$||\varphi_j||_{L^\infty} \leq \lambda_j^{\frac{n-1}{2}} \tag{6.1}$$

on n-dimensional Riemannian manifolds without boundary was extended to manifolds with boundary in [Gri02, Sog02] But this bound is rarely achieved, and in the articles [SZ02, STZ11] there are successively stronger constraints on Riemannian manifolds without boundary for which (6.1) is achieved. We refer to them as '(M, g) with maximum eigenfunction growth' and express the condition as

$$||\varphi_j||_{L^\infty} = \Omega(\lambda_j^{\frac{n-1}{2}}), \tag{6.2}$$

where Ω is the negation of "little oh", i.e. it indicates that there exists some subsequence of eigenfunctions φ_j with $\lambda_j \to \infty$ such that the standard bound is achieved. In the boundary-less case of [SZ02], it is proved that a necessary condition for maximal growth (6.2) is that there exists a 'self-focal point' or 'partial blow down point' p; in [STZ11] it is proved that a necessary and almost sufficient condition is that the first return map preserve an L^1 density on $S^*_p M$.

Theorem 1.2 is the analogous result for Cauchy data of eigenfunctions on non-positively curved manifolds with concave boundary. In fact, much less is needed and it is expected that it is sufficient to rule out self-focal points lying on the boundary. That is, we make the

Conjecture 6.1. *Let (M, g) be a compact Riemannian manifold with smooth boundary, and with no partial self-focal points on ∂M. Then*

$$||\varphi_j^b||_{L^\infty(\partial M)} = o(\lambda_j^{\frac{n-1}{2}}).$$

Thus, the Neumann eigenfunctions satisfy,

$$\sup_{q\in\partial M} |\varphi_j(q)| = o(\lambda_j^{\frac{n-1}{2}}),$$

and the Dirichlet eigenfunctions satisfy

$$\sup_{q\in\partial M} \lambda_j^{-\frac{1}{2}} |\partial_\nu \varphi_j(q)| = o(\lambda_j^{\frac{1}{2}}).$$

A good example to keep in mind for such universal bounds is the case of $S^n_r := S^n\backslash B_r(p)$, the complement of a 'polar cap' of radius r on the standard sphere of radius one. For $r < \pi/2$, S^n_r is a domain with concave boundary in S^n and the Cauchy data should have sub-maximal growth. When $r = \frac{\pi}{2}$, S^n_r becomes an upper hemisphere and its eigenfunctions saturate the remainder bounds and sup norm bounds above.

To prove Theorem 1.2 we adapt the methods of [SZ02, STZ11] on manifolds without boundary.

Remark 6.1. *In Theorem 1.1. In* [JZ13] *we used that* $||\varphi_j||_{L^\infty} = O(\lambda_j^{\frac{1}{2}}/\log\lambda)$ *for negatively curved surfaces. We conjecture that such logarithmic improvement on the standard sup norm bound are also true for non-positively curved Sinai billiards.*

Theorem 1.2 and Conjecture 6.1 are the only elements of the proof of Theorem 1.1 which are not yet known for domains or surfaces with corners. In future work we plan to investigate the generalizations of the results of [SZ02, SZa] to manifolds with boundary and to domains with corners, as well as to study logarithmic improvements in the case of smooth Sinai billiards.

The first step in the proof is to determine the singularity at $t = 0$ of

$$S_q(t) : \quad = \quad E^b_B(t, q, q) = \textstyle\sum_j \cos t\lambda_j \left|\varphi_j^b(q)\right|^2. \tag{6.3}$$

The wave front set of $E^b_B(t, q, q)$ in $T^*\mathbb{R} \times T^*\partial M$ for small $|t|$ equals $(\tau, \eta) \in T^*_0\mathbb{R}\backslash\{0\} \times T^*_q\partial M$ and in particular includes the 'glancing set' where $\tau^2 = |\eta|^2$. Hence we need a parametrix which is valid in the glancing directions. In the case of concave boundary, parametrices have been constructed by Melrose and Taylor. Melrose used the parametrix on a manifold with concave boundary to prove a two term Weyl law. We may use the same diffractive parametrix construction to obtain the pointwise Weyl law of the Cauchy data on the boundary. But it requires verification that no integration over M is needed to obtain the normality of the singularity at $t = 0$ when the variables are on the diagonal of the boundary. The glancing directions thus cannot be ignored as in the Kuznecov sum formula.

7. Topological arguments

7.1. **Euler inequality.** As done in [JZ13], we can give a graph structure (i.e. the structure of a one-dimensional CW complex) to Z_{φ_λ} as follows.

(1) For each embedded circle which does not intersect γ, we add a vertex.
(2) Each singular point is a vertex.
(3) Each intersection point in $\partial M \cap \left(\overline{Z_{\varphi_\lambda} \backslash \partial M}\right)$ is a vertex.
(4) Edges are the arcs of $Z_{\varphi_\lambda} \cup \partial M$ which join the vertices listed above.

This way, we obtain a graph embedded into the surface M. We recall that an embedded graph G in a surface M is a finite set $V(G)$ of vertices and a finite set $E(G)$ of edges which are simple (non-self-intersecting) curves in M such that any two distinct edges have at most one endpoint and no interior points in common. The *faces* f of G are the connected components of $M \backslash V(G) \cup \bigcup_{e \in E(G)} e$. The set of faces is denoted $F(G)$. An edge $e \in E(G)$ is *incident* to f if the boundary of f contains an interior point of e. Every edge is incident to at least one and to at most two faces; if e is incident to f then $e \subset \partial f$. The faces are not assumed to be cells and the sets $V(G), E(G), F(G)$ are not assumed to form a CW complex. Indeed the faces of the nodal graph of eigenfunctions are nodal domains, which do not have to be simply connected.

Remark 7.1. *Every vertex has degree ≥ 2, since any interior singular point is locally an intersection of simple curves, as follows from the local Bers expansion around a zero* [Cha73].

Let $\iota : M \hookrightarrow \tilde{M}$ be an embedding into a closed surface. We assume that $\tilde{M} \backslash \iota(M)$ is a disjoint union of disks, and we denote by h_M the number of connected components of $\tilde{M} \backslash \iota(M)$. (Such pair $\iota, \tilde{M}$ can be constructed, for example, by mapping cone.)

Let $v(\varphi_\lambda)$ (resp. $\tilde{v}(\varphi_\lambda)$) be the number of vertices, $e(\varphi_\lambda)$ (resp. $\tilde{e}(\varphi_\lambda)$) be the number of edges, $f(\varphi_\lambda)$ (resp. $\tilde{f}(\varphi_\lambda)$) be the number of faces, and $m(\varphi_\lambda)$ (resp. $\tilde{m}(\varphi_\lambda)$) be the number of connected components of the graph G inside M (resp. $\iota(G)$ inside $\tilde{M}$).

Then we have

$$v(\varphi_\lambda) = \tilde{v}(\varphi_\lambda)$$
$$e(\varphi_\lambda) = \tilde{e}(\varphi_\lambda)$$
$$m(\varphi_\lambda) = \tilde{m}(\varphi_\lambda)$$

while

$$f(\varphi_\lambda) + h_M = \tilde{f}(\varphi_\lambda).$$

Now by Euler's formula (Appendix F, [Gro12]),

$$v(\varphi_\lambda) - e(\varphi_\lambda) + f(\varphi_\lambda) - m(\varphi_\lambda) + h_M \tag{7.1}$$
$$= \tilde{v}(\varphi_\lambda) - \tilde{e}(\varphi_\lambda) + \tilde{f}(\varphi_\lambda) - \tilde{m}(\varphi_\lambda) \geq 1 - 2g_{\tilde{M}} \tag{7.2}$$

where $g_{\tilde{M}}$ is the genus of the surface $\tilde{M}$.

Theorem 7.2. *Let*

$$n(\varphi_j) = \begin{cases} \#Z_{\varphi_j} \cap \partial M & \textit{(Neumann case)} \\ \#\Sigma_{\varphi_j} \cap \partial M & \textit{(Dirichlet case)} \end{cases}$$

Then we have:

$$N(\varphi_j) \geq \frac{1}{2} n(\varphi_j) + 2 - 2g_{\tilde{M}} - h_M.$$

Proof. Since faces of G on M are nodal domains of φ_j, $f(\varphi_j) = N(\varphi_j)$. Observe that, in Neumann case, points in $Z_{\varphi_j} \cap \partial M$ ($\Sigma_{\varphi_j} \cap \partial M$, in Dirichlet case) correspond to vertices having degree at least 3 on the graph. Also, every vertex has degree ≥ 2 (Remark 7.1). Therefore,

$$\begin{aligned} 0 &= \sum_{x:vertices} \deg(x) - 2e(\varphi_j) \\ &\geq 2\left(v(\varphi_j) - n(\varphi_j)\right) + 3n(\varphi_j) - 2e(\varphi_j), \end{aligned}$$

i.e.

$$e(\varphi_j) - v(\varphi_j) \geq \frac{1}{2} n(\varphi_j).$$

Plugging into (7.1) with $m(\varphi_j) \geq 1$, we obtain

$$N(\varphi_j) \geq \frac{1}{2} n(\varphi_j) + 2 - 2g_{\tilde{M}} - h_M.$$

□

8. Proof of Theorem 1.1

In this section, we give a proof of Proposition 1.6 and Theorem 1.1 for Neumann eigenfunctions. The argument for Dirichlet eigenfunctions is exactly the same.

8.1. **Proof of Proposition 1.6.** Firstly let $\beta \subset \partial M$ be an interval and let $f \in C_0^\infty(\partial M)$ be a function such that

$$\begin{aligned} f(x) &\geq 0 \quad x \in \partial M \\ f(x) &= 0 \quad x \notin \beta \\ f(x) &> 0 \quad x \in \beta \end{aligned}$$

Denote by $N(\lambda)$ the number of eigenfunctions in $\{j \mid \lambda < \lambda_j < 2\lambda\}$. We have by Theorem 1.7 and Chebyshev's inequality,

$$\frac{1}{N(\lambda)} |\{j \mid \lambda < \lambda_j < 2\lambda, \left| \int_{\gamma_i} f\varphi_j ds \right|^2 \geq \lambda_j^{-1} M\}| = O_f(\frac{1}{M}).$$

Corollary 1.8 follows immediately.

Note that

$$\int_{\partial M} f|\varphi_j|^2 ds \leq \int_{\partial M} f|\varphi_j| ds \sup_{x \in \partial M} |\varphi_j(x)|.$$

For a density 1 subsequence $\{\varphi_j\}_{j\in A}$ which satisfies (1.2), we have

$$\int_{\partial M} f|\varphi_j|^2 ds \gg_f 1.$$

Therefore from the third assumption in Theorem 1.1,

$$\int_{\partial M} f|\varphi_j|ds > 2M\lambda_j^{-\frac{1}{2}}$$

is satisfied for all sufficiently large $j \in A$. Combining with Corollary 1.8, this proves the existence of a subsequence of density $\geq 1 - \frac{c}{M}$ which satisfies

$$\int_{\partial M} f|\varphi_j|ds > \left|\int_{\partial M} f\varphi_j ds\right|.$$

Putting

$$A_\beta = \{j \ : \ \int_{\partial M} f|\varphi_j|ds > \left|\int_{\partial M} f\varphi_j ds\right|\},$$

we obtain

$$\liminf_{N\to\infty} \frac{1}{N}|\{j < N \ : \ j \in A_\beta\}| \geq 1 - \frac{c}{M}.$$

Since the left quantity does not depend on M, this proves Proposition 1.6 for Neumann eigenfunctions.

8.2. **Proof of Theorem 1.1.** Note that because f is positive on β, a function φ_j has a sign change on β if and only if $j \in A_\beta$. Let $R \in \mathbb{N}$ be fixed, and let $\beta_1, \cdots, \beta_R \subset \partial M$ be disjoint segments in ∂M. Then by Proposition 1.6, each A_{β_k} $(1 \leq k \leq R)$ is a natural density 1 subset of $\mathbb{N}$. Therefore $A(R) = \cap_{k=1}^R A_{\beta_k}$ is a density 1 subset of $\mathbb{N}$, and any φ_j with $j \in A(R)$ has at least R sign changes along ∂M. We apply the following lemma to conclude Theorem 1.5 for Neumann eigenfunctions.

Lemma 8.1. *Let a_n be a sequence of real numbers such that for any fixed $R > 0$, $a_n > R$ is satisfied for almost all n. Then there exists a density* 1 *subsequence $\{a_n\}_{n\in A}$ such that*

$$\lim_{\substack{n\to\infty\\ n\in A}} a_n = +\infty.$$

Proof. Let n_k be the least number such that for any $n \geq n_k$,

$$\frac{1}{n}|\{j \leq n \mid a_j > k\}| > 1 - \frac{1}{2^k}.$$

Note that n_k is nondecreasing, and $\lim_{k\to\infty} n_k = +\infty$.

Define $A_k \subset \mathbb{N}$ by

$$A_k = \{n_k \leq j < n_{k+1} \mid a_j > k\}.$$

Then for any $n_k \leq m < n_{k+1}$,

$$\{j \leq m \mid a_j > k\} \subset \bigcup_{l=1}^{k} A_l \cap [1, m],$$

which implies by the choice of n_k that

$$\frac{1}{m}|\bigcup_{l=1}^{k} A_l \cap [1,m]| > 1 - \frac{1}{2^k}.$$

This proves

$$A = \bigcup_{k=1}^{\infty} A_k$$

is a density 1 subset of $\mathbb{N}$, and by the construction we have

$$\lim_{\substack{n\to\infty \\ n\in A}} a_n = +\infty.$$

$\square$

Now Theorem 1.1 for Neumann eigenfunctions is an immediate consequence of Theorem 1.5 and the topological argument, Theorem 7.2.

Appendix A. Appendix on Density one

Define the natural density of a set $A \in \mathbb{N}$ by

$$\lim_{X\to\infty} \frac{1}{X}|\{x \in A \mid x < X\}|$$

whenever the limit exists. We say "almost all" when corresponding set $A \in \mathbb{N}$ has the natural density 1. Note that intersection of finitely many density 1 set is a density 1 set. When the limit does not exist we refer to the lim sup as the upper density and the lim inf as the lower density.

A.1. **Diagonal argument.**

Let $\{f_n\} \subset C^\infty(H)$ be a countable dense subset of $C^0(H)$ with respect to the sup norm. For each n, we have a family $\Lambda_n(\lambda)$ of subsets for which

$$\frac{1}{N(\lambda)}\#\Lambda_n(\lambda) \to 1$$

and such that

$$\begin{cases} \int_H f\varphi_j^2 ds \to \int_H f d\nu, \text{ as } \lambda \to \infty \text{ with } \ \lambda_j \in \Lambda_n(\lambda) \\ \\ \lambda_j^{-1/2}\int_H f\varphi_j ds \to 0. \end{cases} \tag{A.1}$$

We may assume that $\Lambda_{n+1}(\lambda) \subset \Lambda_n(\lambda)$. For each n let Λ_n be large enough so that

$$\frac{1}{N(\lambda)}\#\Lambda_n(\lambda) \geq 1 - \frac{1}{n}, \ \ \lambda \geq \Lambda_n.$$

Define

$$\Lambda_\infty(\lambda) : \Lambda_n(\lambda), \ \ \Lambda_k \leq \lambda \leq \Lambda_{k+1}.$$

Then

$$\frac{1}{N(\lambda)}\#\Lambda_\infty(\lambda) \geq 1 - \frac{1}{n}, \ \ \lambda \geq \Lambda_n,$$

so $D^*(\Lambda_\infty) = D_*(\Lambda_\infty) = 1$ and and (A.1) is valid for the sequence Λ_∞.

Lemma A.1 (From [HZ04]). *Let (a_n) be a sequence of complex numbers. Suppose that for every $\delta > 0$, there is a set of integers S_δ of density at least $1-\delta$ such that, along it, the oscillation of the sequence $(a_n)_{n\in S_\delta}$ is at most δ. Then there is a set T of density 1 such that the sequence $(a_n)_{n\in T}$ converges.*

Remark. The oscillation of a sequence (b_n) is defined to be

$$\lim_{N\to\infty} \sup_{m,n>N} |b_m - b_n|.$$

Proof. By hypothesis, for each n there is a set of integers S_n of density at least $1 - 2^{-n}$ such that the oscillation of the corresponding subsequence is at most 2^{-n}. By replacing S_n by $S_n \cup \{1, 2, \ldots, K\}$ for suitable K, we can ensure that

$$\frac{\#S \cap \{1, 2, \ldots M\}}{M} > 1 - 2^{-n+1} \quad \text{for all } M.$$

Consequently, $T_n = S_n \cap S_{n+1} \cap \ldots$ has density at least $1 - 2^{-n+2}$, and the sequence $(a_k)_{k\in T_n}$ has zero oscillation, that is, is a Cauchy sequence, hence converges to some number A_n. Since $T_n \cap T_m$ has positive density, for $m > n \geq 3$, we conclude that the A_n are all equal to some fixed A.

By replacing T_n by $T_n \setminus \{1, 2, \ldots K\}$ for suitable K, we can ensure that $|a_k - A| \leq 2^{-n}$ for all $k \in T_n$. Assuming this condition, then, the set $T = \cup_n T_n$ is density one, and $(a_n)_{n\in T}$ converges. To prove this, let n be given. Then choose L_i so that

$$\sup |a_k - A| \leq 2^{-n} \quad \text{for } k \in T_i,\ k > L_i, \quad 1 \leq i \leq n.$$

Let $L = \max_{i\leq n} L_i$; then

$$\sup |a_k - A| \leq 2^{-n} \quad \text{for } k \in V,\ k > L, \quad 1 \leq i \leq n.$$

This proves that the sequence (a_n) converges to A along T. $\square$

References

[Bia13] M. Bialy. Hopf rigidity for convex billiards on the hemisphere and hyperbolic plane. *Discrete Contin. Dyn. Syst*, 33:3903–3913, 2013.

[BSC90] L. A. Bunimovich, Ya. G. Sinaĭ, and N. I. Chernov. Markov partitions for two-dimensional hyperbolic billiards. *Uspekhi Mat. Nauk*, 45(3(273)):97–134, 221, 1990.

[Bur05] N. Burq. Quantum ergodicity of boundary values of eigenfunctions: a control theory approach. *Canad. Math. Bull.*, 48(1):3–15, 2005.

[CFS82] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskiĭ.

[Cha73] Jacques Chazarain. Construction de la paramétrix du problème mixte hyperbolique pour l'équation des ondes. *C. R. Acad. Sci. Paris Sér. A-B*, 276:A1213–A1215, 1973.

[CS87] N.I. Chernov and Ya. G. Sinaĭ. *Ergodic properties of some systems of two-dimensional disks and three-dimensional balls*, volume 42. 1987.

[CTZ13] H. Christianson, J. A. Toth, and S. Zelditch. Quantum ergodic restriction for cauchy data: Interior QUE and restricted QUE. *MRL Vol. 20 (3), 1-11 (arXiv:1205.0286)*, 2013.

[Gri02] D. Grieser. Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary. *Comm. Partial Differential Equations*, 27(7-8):1283–1299, 2002.

[Gro12] Martin Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. *J. ACM*, 59(5):Art. 27, 64, 2012.

[GRS13] Amit Ghosh, Andre Reznikov, and Peter Sarnak. Nodal Domains of Maass Forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013.

[Hör90] Lars Hörmander. *The analysis of linear partial differential operators. I-IV*. Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

[HZ04] Andrew Hassell and Steve Zelditch. Quantum ergodicity of boundary values of eigenfunctions. *Comm. Math. Phys.*, 248(1):119–168, 2004.

[HZ12] Hamid Hezari and Steve Zelditch. C^∞ spectral rigidity of the ellipse. *Anal. PDE*, 5(5):1105–1132, 2012.

[JZ13] J. Jung and S. Zelditch. Number of nodal domains and singular points of eigenfunctions of negatively curved surfaces with an isometric involution. *arXiv:1310.2919*, 2013.

[Kra89] Simanyi N. Szasz D. Kramli, A. Dispersing billiards without focal points on surfaces are ergodic. *Comm. Math. Phys..*, 125(3):439–457, 1989.

[Mel75] R. B Melrose. Microlocal parametrices for diffractive boundary value problems. *Duke Math. J. .*, 42(4):605–635, 1975.

[MS78] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.

[Sin] Ya.G. Sinai.

[Sog02] Christopher D. Sogge. Eigenfunction and Bochner Riesz estimates on manifolds with boundary. *Math. Res. Lett.*, 9(2-3):205–216, 2002.

[Sto89] Luchezar Stoyanov. An estimate from above of the number of periodic orbits for semi-dispersed billiards. *CMP*, 124(2):217–227, 1989.

[Sto99] Luchezar Stoyanov. Exponential instability for a class of dispersing billiards. *ETDS*, 19(1):201–226, 1999.

[STZ11] Christopher D. Sogge, John A. Toth, and Steve Zelditch. About the blowup of quasimodes on Riemannian manifolds. *J. Geom. Anal.*, 21(1):150–173, 2011.

[SZa] Christopher D. Sogge and Steve Zelditch. Focal points and sup-norms of eigenfunctions. *arXiv:1311.3999*.

[SZb] Christopher D. Sogge and Steve Zelditch. Sup norms of cauchy data of eigenfunctions on manifolds with boundary.

[SZ02] Christopher D. Sogge and Steve Zelditch. Riemannian manifolds with maximal eigenfunction growth. *Duke Math. J.*, 114(3):387–437, 2002.

[Tay] M.E. Taylor. Grazing rays and reflection of singularities of solutions to wave equations. *Comm. Pure Appl. Math.*

[TZ09] John A. Toth and Steve Zelditch. Counting nodal lines which touch the boundary of an analytic domain. *J. Differential Geom.*, 81(3):649–686, 2009.

[Woj94] M Wojtkowski. Two applications of jacobi fields to the billiard ball problem. *J. Differential Geom.*, 40(1):155–164, 1994.

[XH] H. Hezari S. Zelditch X. Han, A. Hassell.

[Zel92] Steven Zelditch. Kuznecov sum formulae and Szegő limit formulae on manifolds. *Comm. Partial Differential Equations*, 17(1-2):221–260, 1992.

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