On the free rank of abelian groups acting freely on products of spheres

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Abstract

The aim of this paper is to affirmatively resolve a long-standing conjecture in algebraic topology, called the *rank conjecture*, related to the free p-rank of abelian groups acting freely on products of spheres. To be precise, let $(\mathbb{Z}/p)^r$ act freely on $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ for a prime p and let l_o denote the number of odd dimensional spheres in the product of spheres. In this paper we show that, if p is odd prime, then r is less than or equal to l_o and the free p-rank of $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ is equal to l_o. On the other hand, if p is equal to 2, it is shown that r is less than or equal to l and the free p-rank of $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ is equal to l.

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1 Introduction and Main Results

Our main concern of this paper is to find the *free p-rank*, or simply *free rank*,

$$\max\{r \in \mathbb{Z}_{\geq 0} \,|\, (\mathbb{Z}/p)^r \text{ acts topologically and freely on } X\}$$

defined for any topological space X and any prime number p.

It is well known by a work [14] of Smith that if a group acts freely on a sphere S^n, then its abelian subgroups should be cyclic. Thus in this case the free rank is less than or equal to 1. Moreover, the following fact has been known from the classical Smith theory (refer to [9], Theorem 1.1).

Theorem 1.1. *The free p-rank of S^n is equal to* 1 *for odd n and all primes p, and even n and $p = 2$, and is equal to* 0 *for even n and all primes $p > 2$.*

A natural generalization of this result to products of spheres has been established by Adem and Browder in their paper [2], Theorem 4.2:

Theorem 1.2. *Let $(\mathbb{Z}/p)^r$ act freely on $(S^n)^l := \underbrace{S^n \times S^n \times \cdots \times S^n}_{l \text{ times}}$ for odd prime p. Then r should be less than or equal to l.*

In fact, even for $p = 2$ they proved an analogous result in the same paper [2], when $n \neq 1, 3, 7$. Later, in his paper [15] Yalçin affirmatively resolved the case for $p = 2$ and $n = 1$, while other two cases (i.e., $p = 2$ and $n = 3$, and $p = 2$ and $n = 7$) have remained as an open question, until now.

In view of the above results, it is more natural to consider a corresponding result for a product of different spheres, not just for a product of the same spheres, and the following long-standing conjecture or question, called the *rank conjecture*, has been proposed in the literature ([1], Conjecture 2.1, [2], Question 7.2, and [12], Problem 809).

Conjecture 1.3. *Let $(\mathbb{Z}/p)^r$ act freely on $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ for a prime p. Then r should be less than or equal to l.*

Various partial results to Conjecture 1.3 have been established with various different techniques by many authors (refer to, e.g., [10], [7], [8], and [9]).

Let l_o denote the number of odd dimensional spheres in the product $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$. In view of Theorem 1.1, it is more reasonable to conjecture that r is less than or equal to l_o, when p is odd prime. Our main result of this paper is to affirmatively prove Conjecture 1.3 in this sharp form, as follows:

Theorem 1.4. *Let $(\mathbb{Z}/p)^r$ act freely on $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ for a prime p. Then the following statements hold:*

(a) *If p is odd prime, then r is less than or equal to l_o, and the free p-rank of $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ is equal to l_o.*

(b) *If p is equal to 2, then r is less than or equal to l, and the free p-rank of $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_l}$ is equal to l.*

The second statement of Theorem 1.4 (a) (resp. Theorem 1.4 (b)) follows from the fact that $(\mathbb{Z}/p)^{l_o}$ (resp. $(\mathbb{Z}/2)^l$) can be made to act freely on a product of l_o odd dimensional (resp. l even or odd dimensional) spheres. One of the key ingredients for the proof of Theorem 1.4 is to use the ideas in the paper [6] employed by Cao and Lü for the proof of the Halperin-Carlsson conjecture in the category of moment-angle complexes.

This paper is organized, as follows. In Section 2, we first review the construction of a moment-angle complex associated to an abstract simplicial complex on vertex set, and then introduce the notion of a generalized moment-angle complex associated to a product of abstract simplicial complexes. In Section 3, we explain how to calculate the cohomology of moment-angle complexes in terms of Stanley-Reisner face ring and its Betti numbers. In Section 4, following the paper [6] of Cao and Lü we introduce the Möbius transforms and give some results which estimate the support of a $\mathbb{Z}_2$-valued Möbius transform. In the same section, we will present some results which clearly show a very close connection between the notion of a moment-angle complex and Conjecture 1.3. Note that, in Sections 3 and 4, we provide more than the material that is necessary simply for the proof of Theorem 1.4. This is because, we hope, this might be more useful for some future work related to this topic. Finally, Sections 5 and 6 are devoted to proving our main Theorem 1.4.

2 Moment-angle complexes and their generalizations

The goal of this section is to review the construction of a moment-angle complex associated to an abstract simplicial complex on vertex set, and then introduce the notion of a generalized moment-angle complex associated to a product of abstract simplicial complexes.

To do so, let m be a positive integer and let us denote by $[m]$ the set $\{1,2,\dots,m\}$. Let K be an abstract simplicial complex on vertex set $[m]$. For each simplex $\sigma \in K$, we set

$$B_\sigma(D^2,S^1) = \prod_{i=1}^{m} A_i,$$

where $D^2 = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$, $S^1 = \partial D^2$, and

$$A_i = \begin{cases} D^2, & i \in \sigma, \\ S^1, & i \in [m] \backslash \sigma. \end{cases}$$

Then the moment-angle complex $\mathscr{Z}_K$ on K is defined to be a subspace of $(D^2)^m$, as follows:

$$\mathscr{Z}_K := \bigcup_{\sigma \in K} B_\sigma(D^2,S^1) \subset (D^2)^m.$$

When $K = 2^{[m]}$, it is easy to see that $\mathscr{Z}_K = (D^2)^m$. On the other hand, when $K = 2^{[m]} \backslash \{[m]\}$ where $2^{[m]}$ denotes the power set of $[m]$, it can be easily shown that $\mathscr{Z}_K = S^{2m-1}$ (refer to, e.g., [3], Example 2.4).

Since $(D^2)^m$ as a subspace of $\mathbb{C}^m$ is invariant under the standard action of T^m on $\mathbb{C}^m$ given by

$$((g_1,g_2,\dots,g_m),(z_1,z_2,\dots,z_m)) \mapsto (g_1z_1,g_2z_2,\dots,g_mz_m),$$

$(D^2)^m$ inherits a natural T^m-action whose orbit space is the unit cube $I^m := [0,1]^m \subset \mathbb{R}^m_{\geq 0}$. This T^m-action on $(D^2)^m$ then induces a canonical T^m-action on the moment-angle complex $\mathscr{Z}_K$.

For our purposes of this paper, we next want to generalize the construction of the moment-angle complex for a single abstract simplicial complex to that for products of abstract simplicial complexes whose construction is rather straightforward. For the sake of simplicity, we shall explain how to construct the generalized moment-angle complex just for two abstract simplicial complexes, since the construction for more general cases is completely analogous.

To do so, let K_1 and K_2 be two abstract simplicial complexes on vertex sets $[m_1]$ and $[m_2]$, respectively. For each $\sigma^1 \times \sigma^2 \in K_1 \times K_2$ where σ^1 (resp. σ^2) is a simplex in K_1 (resp. K_2), we define

$$B_{\sigma^1\times\sigma^2}(D^2,S^1) := \left(\prod_{i=1}^{m_1} A_i^1\right) \times \left(\prod_{i=1}^{m_2} A_i^2\right),$$

where

$$A_i^j = \begin{cases} D^2, & i \in \sigma^j, \\ S^1, & i \in [m_j]\backslash\sigma^j \end{cases}$$

for each $j = 1,2$. Then the *generalized moment-angle complex* $\mathscr{Z}_{K_1\times K_2}$ on $K_1 \times K_2$ is defined to be

$$\mathscr{Z}_{K_1\times K_2} := \bigcup_{\sigma^1\times\sigma^2\in K_1\times K_2} B_{\sigma^1\times\sigma^2}(D^2,S^1).$$

It follows from its construction that we have

$$\mathscr{Z}_{K_1\times K_2} = \mathscr{Z}_{K_1} \times \mathscr{Z}_{K_2}.$$

For more abstract simplicial complexes $K_1,K_2,\dots,K_l$ on vertex sets $[m_1],[m_2],\dots,[m_l]$, respectively, it is also easy to see that the following holds:

$$\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l} = \mathscr{Z}_{K_1} \times \mathscr{Z}_{K_2} \times \cdots \times \mathscr{Z}_{K_l}.$$

In particular, when $K_i = 2^{[m_i]}\backslash\{[m_i]\}$ for each $1 \leq i \leq l$, we have

$$\begin{aligned} \mathscr{Z}_{K_1\times K_2\times\cdots\times K_l} &= \mathscr{Z}_{K_1} \times \mathscr{Z}_{K_2} \times \cdots \times \mathscr{Z}_{K_l} \\ &= S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1}. \end{aligned} \tag{2.1}$$

Moreover, since there exists a canonical T^{m_i}-action on the moment-angle complex $\mathscr{Z}_{K_i}$, there exists a canonical $T^{m_1+m_2+\dots+m_l}$-action Φ on the generalized moment-angle complex $\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}$.

On the other hand, if we apply the above procedure to obtain the moment-angle complex $\mathscr{Z}_K$ to the pair $(D^1,S^0)\subset(\mathbb{R},S^0)$ instead of the pair (D^2,S^1), we can obtain the *real moment-angle complex* $\mathbb{R}\mathscr{Z}_K\subset(D^1)^m$ on K. Since $(D^1)^m$ as a subspace of $\mathbb{R}^m$ is invariant under the standard action of $(\mathbb{Z}_2)^m=(\{-1,1\})^m$ on $\mathbb{R}^m$ given by

$$((g_1,g_2,\dots,g_m),(x_1,x_2,\dots,x_m))\mapsto(g_1x_1,g_2x_2,\dots,g_mx_m),$$

$(D^1)^m$ inherits a natural $(\mathbb{Z}_2)^m$-action whose orbit space is again the unit cube $I^m:=[0,1]^m\subset\mathbb{R}^m_{\geq 0}$. This $(\mathbb{Z}_2)^m$-action on $(D^1)^m$ then induces a canonical $(\mathbb{Z}_2)^m$-action on the real moment-angle complex $\mathbb{R}\mathscr{Z}_K$.

Similarly, for l abstract simplicial complexes $K_1,K_2,\dots,K_l$ on vertex sets $[m_1+1],[m_2+1],\dots,[m_l+1]$, respectively, we can construct the generalized real moment-angle complex $\mathbb{R}\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}$ as above, and the following identity holds:

$$\mathbb{R}\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}=\mathbb{R}\mathscr{Z}_{K_1}\times\mathbb{R}\mathscr{Z}_{K_2}\times\cdots\times\mathbb{R}\mathscr{Z}_{K_l}.$$

As a special case, when $K_i=2^{[m_i+1]}\backslash\{[m_i+1]\}$ for each $1\leq i\leq l$, we have

$$\begin{aligned}\mathbb{R}\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}&=\mathbb{R}\mathscr{Z}_{K_1}\times\mathbb{R}\mathscr{Z}_{K_2}\times\cdots\times\mathbb{R}\mathscr{Z}_{K_l}\\&=S^{m_1}\times S^{m_2}\times\cdots\times S^{m_l}.\end{aligned}\tag{2.2}$$

Moreover, since there exists a canonical $(\mathbb{Z}_2)^{m_i+1}$-action on the real moment-angle complex $\mathbb{R}\mathscr{Z}_{K_i}$, there exists a canonical $(\mathbb{Z}_2)^{\sum_{i=1}^{l}(m_i+1)}$-action $\Phi_{\mathbb{R}}$ on the generalized real moment-angle complex $\mathbb{R}\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}$.

Finally, we recall the construction of the join of two simplicial complexes. As above, let K_1 and K_2 be two abstract simplicial complexes on vertex sets $[m_1]$ and $[m_2]$, respectively. Then the *join* K_1*K_2 of K_1 and K_2 is the simplicial complex on vertex set $[m_1]\cup[m_2]$, defined as follows.

$$K_1*K_2=\{\sigma\subset[m_1]\cup[m_2]\mid\sigma=\sigma_1\cup\sigma_2,\sigma_1\in K_1,\sigma_2\in K_2\}.$$

It then will be important to observe the following lemma which enables us to reduce all the considerations about the generalized moment-angle complexes to just moment-angle complexes (see, e.g., [13], Proposition 7.6).

Lemma 2.1. *Let K_1, K_2, …, K_{l-1}, and K_l be abstract simplicial complexes on vertex sets $[m_1]$, $[m_2]$, …, $[m_{l-1}]$, and $[m_l]$, respectively. Then we have*

$$\mathscr{Z}_{K_1*K_2*\cdots*K_l}=\mathscr{Z}_{K_1}\times\mathscr{Z}_{K_2}\times\cdots\times\mathscr{Z}_{K_l}=\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}.$$

Proof. For the proof, it suffices to prove it only for $l=2$, and the proof follows immediately from the definition of moment-angle complexes. Indeed, we have

$$\begin{aligned}\mathscr{Z}_{K_1*K_2} &= \bigcup_{\sigma_1\in K_1,\sigma_2\in K_2} B_{\sigma_1\cup\sigma_2}(D^2,S^1) = \bigcup_{\sigma_1\in K_1,\sigma_2\in K_2} B_{\sigma_1}(D^2,S^1)\times B_{\sigma_2}(D^2,S^1)\\ &= \left(\bigcup_{\sigma_1\in K_1} B_{\sigma_1}(D^2,S^1)\right)\times\left(\bigcup_{\sigma_2\in K_2} B_{\sigma_2}(D^2,S^1)\right)\\ &= \mathscr{Z}_{K_1}\times\mathscr{Z}_{K_2},\end{aligned}$$

as desired. □

3 Cohomology of moment-angle complexes

In this section, we briefly explain the way to calculate the cohomology of moment-angle complexes in terms of Stanley-Reisner face ring and its Betti numbers. The material of this section is largely taken from the paper [6], Section 2, so that a reader may refer to the paper for more details.

Let $\mathbf{k}$ denote a field of arbitrary characteristic, and $\mathbf{k}[\mathbf{v}]=\mathbf{k}[v_1,v_2,\dots,v_m]$ be the polynomial algebra over $\mathbf{k}$ in m indeterminates $v_1,v_2,\dots,v_m$ with degree of v_i equal to 2. For an abstract simplicial complex K on vertex set $[m]$, the *Stanley-Reisner ideal* of K is defined as

$$I_K := \langle \mathbf{v}^\tau \mid \tau\notin K \text{ for } \tau\in 2^{[m]}\rangle,$$

where $\mathbf{v}^\tau=\prod_{i\in\tau} v_i$. The quotient ring

$$\mathbf{k}(K)=\mathbf{k}[\mathbf{v}]/I_K$$

is called the *Stanley-Reisner face ring of* K. For instance, if $K=2^{[m]}$, then $\mathbf{k}(K)=\mathbf{k}[\mathbf{v}]$, and if $K=2^{[m]}\backslash\{[m]\}$, then $\mathbf{k}(K)=\mathbf{k}[\mathbf{v}]/\langle\mathbf{v}^{[m]}\rangle$.

Buchstaber and Panov showed how to compute the cohomology of the moment-angle complex $\mathscr{Z}$ in terms of the Stanlet-Reisner face ring of K. To be precise, they proved the following theorem in [5], Theorem 7.6:

Theorem 3.1. *As* $\mathbf{k}$*-algebras,*

$$H^*(\mathscr{Z}_K;\mathbf{k})\cong \operatorname{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k}).$$

As an immediate consequence, we have the following

Corollary 3.2. *As* $\mathbf{k}$*-algebras,*

$$H^*(\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l};\mathbf{k})\cong \otimes_{i=1}^{l}\operatorname{Tor}^{\mathbf{k}[\mathbf{v}^{(i)}]}(\mathbf{k}(K_i),\mathbf{k}).$$

Moreover, it is known that the total Betti number of the moment-angle complex $\mathscr{Z}_K$ is given by

$$\sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_K;\mathbf{k}) = \sum_{i=0}^{h} \dim_{\mathbf{k}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}^{(i)}]}(\mathbf{k}(K),\mathbf{k}) = \sum_{i=0}^{h} \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}, \tag{3.1}$$

where h is the length of the minimal free resolution of $\mathbf{k}(K)$ and $\beta_{i,a}^{\mathbf{k}(K)}$ denotes the (i,a)-th Betti number of $\mathbf{k}(K)$ (refer to [6], Subsection 2.3 for more details). Hence it is straightforward from (2.1) and (3.1) to obtain the following corollary.

Corollary 3.3. *The following identity holds:*

$$\sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l};\mathbf{k}) = \prod_{i=1}^{l}\left(\sum_{j_i=0}^{h_i}\sum_{a_i\in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_i)}\right).$$

Finally, we remark that, according to [6], Theorem 4.2, our Theorem 3.1, and Corollaries 3.2 and 3.3 hold to be true even for the generalized real moment-angle complexes $\mathbb{R}\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}$ as graded $\mathbf{k}$-modules.

4 Möbius transforms and their supports

The goal of of this section is to give the lower and upper bounds for the support of a Möbius transform of a certain $\mathbb{Z}_2$-valued function on a product of the power sets of $[m]$'s. This mildly generalizes the results in [6], Section 3 under our new settings.

From now on, we set

$$2^{[m]^*} := \{f \,|\, f : 2^{[m]} \to \mathbb{Z}_2 = \{0,1\}\}.$$

Then $2^{[m]^*}$ forms an algebra over $\mathbb{Z}_2$ under the pointwise addition and multiplication. Given a function $f \in 2^{[m]^*}$, the *support* of f is defined to be the set

$$\mathrm{supp}(f) := f^{-1}(\{1\}),$$

and f is said to be *nice* if the support $\mathrm{supp}(f)$ of f forms an abstract simplicial complex on the vertex set $\cup_{a\in\mathrm{supp}(f)} a \subset [m]$. Then it follows from the definition of an abstract simplicial complex that f is nice if and only if for each $a \in \mathrm{supp}(f)$ any subset b of a satisfies $f(b) = 1$.

Similarly, for positive integers $m_1, m_2, \dots, m_l$ we set

$$W^*_{[m_1],[m_2],\dots,[m_l]}$$
$$:= \{f = \bigotimes_{i=1}^{l} f_i \in 2^{[m_1]^*} \otimes 2^{[m_2]^*} \otimes \cdots \otimes 2^{[m_l]^*} \,|\, f_i : 2^{[m_i]} \to \mathbb{Z}_2, \text{ i.e., } f_i \in 2^{[m_1]^*}\}.$$

Note that

$$2^{[m_1]^*} \otimes 2^{[m_2]^*} \otimes \cdots \otimes 2^{[m_l]^*}$$

is generated by $W^*_{m_1,m_2,\dots,m_l}$, i.e.,

$$2^{[m_1]^*} \otimes 2^{[m_2]^*} \otimes \cdots \otimes 2^{[m_l]^*} = \langle W^*_{m_1,m_2,\dots,m_l} \rangle.$$

Then $2^{[m_1]^*} \otimes 2^{[m_2]^*} \otimes \cdots \otimes 2^{[m_l]^*}$ clearly forms an algebra over $\mathbb{Z}_2$ under the pointwise addition and multiplication defined by

$$\left(\bigotimes_{i=1}^{l} f_i + \bigotimes_{i=1}^{l} g_i\right)(a_1, a_2, \dots, a_l) = \bigotimes_{i=1}^{l} f_i(a_1, a_2, \dots, a_l) + \bigotimes_{i=1}^{l} g_i(a_1, a_2, \dots, a_l).$$
$$\left(\bigotimes_{i=1}^{l} f_i \cdot \bigotimes_{i=1}^{l} g_i\right)(a_1, a_2, \dots, a_l) = \bigotimes_{i=1}^{l} f_i(a_1, a_2, \dots, a_l) \cdot \bigotimes_{i=1}^{l} g_i(a_1, a_2, \dots, a_l).$$

However, $W^*_{[m_1],[m_2],\dots,[m_l]}$ does not necessarily form an algebra over $\mathbb{Z}_2$.

The support of $f = \bigotimes_{i=1}^{l} f_i \in W^*_{[m_1],[m_2],\dots,[m_l]}$ is given by

$$\begin{aligned} \operatorname{supp}(f) &= f_1^{-1}(\{1\}) \times f_2^{-1}(\{1\}) \times \cdots \times f_l^{-1}(\{1\}) \\ &= \{a_1 \times a_2 \times \cdots \times a_l \in 2^{[m_1]} \times 2^{[m_2]} \times \cdots \times 2^{[m_l]} \,|\, f_i(a_i) = 1, i = 1, 2, \dots, l\}. \end{aligned}$$

As in the case of $2^{[m]^*}$, for each $a_1 \times a_2 \times \cdots \times a_l \in 2^{[m_1]} \times 2^{[m_2]} \times \cdots \times 2^{[m_l]}$ we define $\delta_{a_1,a_2,\dots,a_l} \in W^*_{[m_1],[m_2],\dots,[m_l]}$ by

$$\delta_{a_1,a_2,\dots,a_l}(b_1, b_2, \dots, b_l) = \begin{cases} 1, & b_i = a_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

That is, it is easy to see

$$\delta_{a_1,a_2,\dots,a_l}(b_1, b_2, \dots, b_l) = \bigotimes_{i=1}^{l} \delta_{a_i}(b_1, b_2, \dots, b_l) = \prod_{i=1}^{l} \delta_{a_i}(b_i),$$

where

$$\delta_{a_i}(b_i) = \begin{cases} 1, & b_i = a_i \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{\delta_{a_i} \,|\, a_i \in 2^{[m_i]}\}$ forms a basis for $2^{[m_i]^*}$ for each i, the set

$$\mathscr{B} := \{\delta_{a_1,a_2,\ldots,a_l} \,|\, a_1 \times a_2 \times \cdots \times a_l \in 2^{[m_1]} \times 2^{[m_2]} \times \cdots \times 2^{[m_l]}\}$$

also forms a basis for $2^{[m_1]^*} \otimes 2^{[m_2]^*} \otimes \cdots \otimes 2^{[m_l]^*}$, so that every element of

$$W^*_{[m_1],[m_2],\ldots,[m_l]}$$

can be written uniquely as a linear combination of the set $\mathscr{B}$.

Next we are ready to give a definition of the $\mathbb{Z}_2$-valued Möbius transform on $W^*_{[m_1],[m_2],\ldots,[m_l]}$ which is a straightforward generalization of the $\mathbb{Z}_2$-valued Möbius transform $\mathscr{M}$ on $2^{[m]^*}$ given in [6], Section 2.

The map $\mathscr{M} : W^*_{[m_1],[m_2],\ldots,[m_l]} \to W^*_{[m_1],[m_2],\ldots,[m_l]}$ given by

$$\mathscr{M}(f)(a_1,a_2,\ldots,a_l) = \sum_{b_i \subset a_i} f(b_1,b_2,\ldots,b_l)$$

for all $f \in W^*_{[m_1],[m_2],\ldots,[m_l]}$ and $a_1 \times a_2 \times \cdots \times a_l \in 2^{[m_1]} \times 2^{[m_2]} \times \cdots \times 2^{[m_l]}$ will be called the *$\mathbb{Z}_2$-valued Möbius transform* on $W^*_{[m_1],[m_2],\ldots,[m_l]}$. In fact, it follows from the definition of $W^*_{[m_1],[m_2],\ldots,[m_l]}$ that for $f = \bigotimes_{i=1}^{l} f_i \in W^*_{[m_1],[m_2],\ldots,[m_l]}$, we have

$$\begin{aligned}
\mathscr{M}(f)(a_1,a_2,\ldots,a_l) &= \sum_{b_i \subset a_i} f(b_1,b_2,\ldots,b_l) = \sum_{b_i \subset a_i} f_1(b_1)\cdot f_2(b_2)\cdots f_l(b_l) \\
&= \left(\sum_{b_1 \subset a_1} f_1(b_1)\right)\cdots\left(\sum_{b_l \subset a_l} f_1(b_l)\right) \\
&= \mathscr{M}(f_1)(a_1)\cdot\mathscr{M}(f_2)(a_2)\cdots\mathscr{M}(f_l)(a_l), \text{ i.e.,}
\end{aligned}$$

$$\mathscr{M}(f) = \bigotimes_{i=1}^{l} \mathscr{M}(f_i). \tag{4.1}$$

Let $\underline{1}$ be the constant function on $2^{[m]}$, and x_i be the i-th coordinate function given by the condition that, for every $a \in 2^{[m]}$, $x_i(a) = 1$ if and only if $i \in a$. Recall then from [6], Section 2 that for each $a \in 2^{[m]}$, μ_a is given by

$$\mu_a = \begin{cases} \prod_{i\in a} x_i, & a \neq \emptyset, \\ \underline{1}, & a = \emptyset. \end{cases}$$

Since $\mathscr{M}^2$ is the identity on $2^{[m]^*}$ and $\mathscr{M}(\delta_{a_i}) = \mu_{a_i}$, it is also easy to check out from (4.1) that $\mathscr{M}^2$ is the identity on $W^*_{[m_1],[m_2],\ldots,[m_l]}$ and

$$\mathscr{M}(\delta_{a_1,a_2,\ldots,a_l}) = \bigotimes_{i=1}^{l} \mu_{a_i} =: \mu_{a_1,a_2,\ldots,a_l}.$$

Since the Möbius transform $\mathscr{M}$ is the involution, obviously it is true that

$$\mathscr{M}(\mu_{a_1,a_2,\ldots,a_l}) = \delta_{a_1,a_2,\ldots,a_l}.$$

With these preliminaries on the Möbius transforms in place, we are now ready to give our main results of this section. To do so, we start with the following simple lemma.

Lemma 4.1. *For $f = \bigotimes_{i=1}^{l} f_i \in W^*_{[m_1],[m_2],\ldots,[m_l]}$, assume that each $f_i \in 2^{[m_i]^*}$ is a nice function such that $K_{f_i} := \mathrm{supp}(f_i)$ is an abstract simplicial complex on vertex set $[m_i]$. Then we have*

$$\mathscr{M}(f) = \bigotimes_{i=1}^{l} \left(\sum_{j_i=0}^{h_j} \sum_{a_i \in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})} \delta_{a_i} \right). \tag{4.2}$$

Proof. Since we have

$$\mathscr{M}(f) = \bigotimes_{i=1}^{l} \mathscr{M}(f_i) \text{ (see (4.1))},$$

and

$$\mathscr{M}(f_i) = \sum_{j_i=0}^{h_j} \sum_{a_i \in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})} \delta_{a_i} \text{ (see [6], Theorem 3.1)},$$

it is immediate to obtain the equality (4.2), as desired. □

As a consequence of Lemma 4.1, we can show the following

Theorem 4.2. *For $f = \bigotimes_{i=1}^{l} f_i \in W^*_{[m_1],[m_2],\ldots,[m_l]}$, assume that f is a nice function in that each $f_i \in 2^{[m_i]^*}$ is a nice function so that $K_{f_i} := \mathrm{supp}(f_i)$ is an abstract simplicial complex on vertex set $[m_i]$. Then the support $\mathrm{supp}(\mathscr{M}(f))$ of f satisfies*

$$|\mathrm{supp}(\mathscr{M}(f))| \leq \prod_{i=1}^{l} \left(\sum_{j_i=0}^{h_j} \sum_{a_i \in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})} \right) = \sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}; \mathbf{k}). \tag{4.3}$$

Proof. Note first that it follows from (4.2) that

$$\begin{aligned} \mathscr{M}(f) &= \bigotimes_{i=1}^{l} \left(\sum_{j_i=0}^{h_j} \sum_{a_i \in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})} \delta_{a_i} \right) \\ &= \sum_{a_1 \in 2^{[m_1]}, a_2 \in 2^{[m_2]}, \ldots, a_l \in 2^{[m_l]}} \prod_{i=1}^{l} \left(\sum_{j_i=0}^{h_j} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})} \right) \delta_{a_1,a_2,\ldots,a_l}. \end{aligned} \tag{4.4}$$

This implies that, for $a_1 \times a_2 \times \cdots \times a_l \in \mathrm{supp}(\mathscr{M}(f))$, the coefficient $\prod_{i=1}^{l}\left(\sum_{j_i=0}^{h_j} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})}\right)$ in the equation (4.4) should be a non-negative and odd integer greater than or equal to 1. Thus, we have

$$\begin{aligned}
|\mathrm{supp}(\mathscr{M}(f))| &= \sum_{a_1 \times a_2 \times \cdots \times a_l \in \mathrm{supp}(\mathscr{M}(f))} 1 \\
&\leq \sum_{a_1 \times a_2 \times \cdots \times a_l \in \mathrm{supp}(\mathscr{M}(f))} \left(\prod_{i=1}^{l}\left(\sum_{j_i=0}^{h_j} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})}\right)\right) \\
&\leq \sum_{a_1 \in 2^{[m_1]}, a_2 \in 2^{[m_2]}, \ldots, a_l \in 2^{[m_l]}} \left(\prod_{i=1}^{l}\left(\sum_{j_i=0}^{h_j} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})}\right)\right) \\
&= \prod_{i=1}^{l}\left(\sum_{j_i=0}^{h_j} \sum_{a_i \in 2^{[m_i]}} \beta_{j_i,a_i}^{\mathbf{k}(K_{f_i})}\right),
\end{aligned}$$

as required. □

Let $\mathscr{F}_{[m_1],[m_2],\ldots,[m_l]}$ be the collection of all $f \in W^*_{[m_1],[m_2],\ldots,[m_l]}$ such that each $f_i \in 2^{[m_i]^*}$ is nice, i.e., $f_i \in \mathscr{F}_{[m_i]}$. Note that all functions of $\mathscr{F}_{[m_i]}$ correspond bijectively to all abstract simplicial complexes on vertex set $\subset [m_i]$ in terms of the supports ([6], Proposition 2.1). Thus, for any function $f = \bigotimes_{i=1}^{l} f_i \in W^*_{[m_1],[m_2],\ldots,[m_l]}$ the supports $\mathrm{supp}(f_i)$ of $f_i \in \mathscr{F}_{[m_i]}$ are abstract simplicial complexes on vertex set $\subset [m_i]$. Moreover, we have the following

Theorem 4.3. *For any $f = \bigotimes_{i=1}^{l} f_i \in \mathscr{F}_{[m_1],[m_2],\ldots,[m_l]}$, there exists some $a_1 \times a_2 \times \cdots \times a_l \in \mathrm{supp}(f)$ such that*

$$|\mathrm{supp}(\mathscr{M}(f))| \geq 2^{\sum_{i=1}^{l} m_i - \sum_{i=1}^{l} |a_i|}.$$

Proof. It follows from the definition of $f \in \mathscr{F}_{[m_1],[m_2],\ldots,[m_l]}$ and [6], Theorem 3.5 that there exists some $a_i \in \mathrm{supp}(f_i)$ such that

$$|\mathrm{supp}(\mathscr{M}(f_i))| \geq 2^{m_i - |a_i|}.$$

Since each a_i lies in the $\mathrm{supp}(f_i)$ of f_i, clearly $a_1 \times a_2 \times \cdots \times a_l$ lies in the support $\mathrm{supp}(f)$ of f. Hence we have

$$\begin{aligned}
|\mathrm{supp}(\mathscr{M}(f))| &= \prod_{i=1}^{l} |\mathrm{supp}(\mathscr{M}(f_i))| \\
&\geq 2^{\sum_{i=1}^{l} m_i - \sum_{i=1}^{l} |a_i|},
\end{aligned}$$

as required. □

5 Proof of Theorem 1.4 (b)

The goal of this section is to give a proof of Theorem 1.4 (b). Note that every action throughout this paper is assumed to be effective, as always.

The following theorem can be regarded as one of our main results of this section.

Theorem 5.1. *Let H be a torus $T^r := (S^1)^r$ of rank r, and let H act freely on $\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l}$ in such a way that the action is not necessarily given by the restriction of the action Φ. Then we have*

$$2^r \leq \sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l};\mathbf{k}).$$

In particular, if $K_i = 2^{[m_i]}\backslash\{[m_i]\}$ for each i, then we have

$$\mathscr{Z}_{K_1\times K_2\times\cdots\times K_l} = S^{2m_1-1}\times S^{2m_2-1}\times\cdots\times S^{2m_l-1},$$

and so r is less than or equal to l.

Remark 5.2. It is straightforward to check that the conclusion of Theorem 5.1 also holds for a torus $H = (\mathbb{Z}/p)^r$ of rank r without much modification, where p is prime.

In order to prove Theorem 5.1, we will need the following lemma which is analogous to [6], Lemma 5.1. But one crucial difference is that we dot *not* require the action to be given by the restriction of Φ.

Lemma 5.3. *Let L be an abstract simplicial complex on vertex set $[m]$, and let H be a torus T^r of rank r such that H acts freely on $\mathscr{Z}_L$. Then r satisfies*

$$r \leq m - \dim_{\mathbf{k}} L - 1.$$

Proof. To prove it, we first take an element $a\in L$ such that

$$|a| = \dim_{\mathbf{k}} L + 1.$$

Note that we may assume without loss of generality that each a is of the form $\{1,2,\ldots,|a|\}$. Thus $\mathscr{Z}_L$ should contain an element z of the form

$$z = (0,\ldots,0,z_{|a|+1},\ldots,z_m)\in B_a(D^2,S^1).$$

By using the linear representation of T^r at the point z, we may also assume that the action of $t=(t_1,t_2,\ldots,t_r)\in T^r$ to z is given by

$$\begin{aligned}(5.1)\qquad t\cdot z = (0,\ldots,0,t_1^{n_{(|a|+1)1}}t_2^{n_{(|a|+1)2}}\cdots t_r^{n_{(|a|+1)r}}z_{|a|+1},\ldots,\\ \ldots,t_1^{n_{j1}}t_2^{n_{j2}}\cdots t_r^{n_{jr}}z_j,\ldots,t_1^{n_{m1}}t_2^{n_{m2}}\cdots t_r^{n_{mr}}z_m),\end{aligned}$$

where n_{jk}, called the *weight*, denotes an integer for $|a|+1 \leq j \leq m$ and $1 \leq k \leq r$. Since each z_j is non-zero, it follows from (5.1) that we can obtain a system of $m-|a|$ equations with r unknowns $\theta_1, \ldots \theta_{r-1}$, and θ_r, as follows.

$$n_{j1}\theta_1 + n_{j2}\theta_2 + \cdots + n_{jr}\theta_r = 0, \quad |a|+1 \leq j \leq m, \tag{5.2}$$

where θ_i denotes a real number in $[0,1)$ for $1 \leq i \leq r$.

On the other hand, since the action of T^r on $\mathscr{Z}_L$ is assumed to be free, by (5.2) we should have only one solution

$$(\theta_1, \ldots, \theta_r) = (0, \ldots, 0).$$

It is easy to see that it is possible only when the rank of the $(m-|a|) \times r$-matrix (n_{jk}) is r. This implies that $m-|a|$ should be greater than or equal to r. That is, we have

$$r \leq m - |a| = m - \dim_{\mathbf{k}} L - 1.$$

This completes the proof of Lemma 5.3. □

Next we are ready to give a proof of Theorem 5.1.

Proof of Theorem 5.1. For the proof of the first statement, let us denote by L (resp. $[m]$) the join $K_1 * K_2 * \cdots * K_l$ of $K_1, K_2, \ldots, K_{l-1}$, and K_l (resp. the disjoint union $[m_1] \cup \ldots \cup [m_l]$ of l vertex sets $[m_1], [m_2], \ldots, [m_l]$). Then we can choose an element $f \in \mathscr{F}_{[m]}$ such that $\mathrm{supp}(f) = L$, by [6], Proposition 2.1. Since the action of H on $\mathscr{Z}_L$ is free, f cannot be the constant function $\underline{1}$. Thus we may assume that by [6], Theorem 3.5 there exists $a \in 2^{[m]}$ with $a \neq [m]$ such that $a \in \mathrm{supp}(f) = L$ and

$$|\mathrm{supp}(\mathscr{M}(f))| \geq 2^{m-|a|}. \tag{5.3}$$

Since $a \in L$, clearly $|a| \leq \dim_{\mathbf{k}} L + 1$, and so by Lemma 5.3 we have

$$\dim_{\mathbf{k}} L \leq m - r - 1.$$

As a consequence, we can obtain

$$|a| \leq m - r, \text{ i.e., } r \leq m - |a|.$$

Hence, it follows from (5.3), Corollary 3.3, Theorem 4.2, and Theorem 4.3 that

$$\begin{aligned} 2^r &\leq 2^{m-|a|} \\ &\leq |\mathrm{supp}(\mathscr{M}(f))| \leq \sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_L; \mathbf{k}) \\ &= \sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}; \mathbf{k}), \end{aligned}$$

as asserted.

For the proof of the second statement, notice first that, by the choice of $K_i = 2^{[m_i]} \backslash \{[m_i]\}$ and the identity (2.1), we have

$$\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l} = S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1}.$$

On the other hand, it follows from Lemma 2.1 that

$$\mathscr{Z}_L = \mathscr{Z}_{K_1 * K_2 * \cdots * K_l} = \mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}.$$

Hence it follows from the first statement of the theorem and the Künneth formula that

$$2^r \leq \sum_i \dim_{\mathbf{k}} H^i(\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}; \mathbf{k}) = \sum_{k=0}^{l} {}_lC_k = 2^l.$$

Thus r is less than or equal to l, as asserted. □

It is easy to see that we can apply the preceding arguments to the generalized real moment-angle complex $\mathbb{R}\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}$ without any serious modification. So we have the following theorem which is completely analogous to Theorem 5.1 and so whose proof will be left to the reader.

Theorem 5.4. *Let H be a torus $(\mathbb{Z}/2)^r$ of rank r, and let H act freely on $\mathbb{R}\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}$ in such a way that the action is is not necessarily given by the restriction of the action $\Phi_{\mathbb{R}}$. Then we have*

$$2^r \leq \sum_i \dim_{\mathbf{k}} H^i(\mathbb{R}\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}; \mathbf{k}).$$

In particular, if $K_i = 2^{[m_i+1]} \backslash \{[m_i+1]\}$ for each i, then we have

$$\mathbb{R}\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l} = S^{m_1} \times S^{m_2} \times \cdots \times S^{m_l},$$

and so r is less than or equal to l.

We now close this section with a few remarks. Let p be an odd prime, and, for example, let $(\mathbb{Z}/p)^r$ act freely on

$$S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1}.$$

by the restriction of the action map Φ. Then it can be easily shown that r is less than or equal to l, and the free p-rank of $S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1}$ is equal to l. Indeed, by applying the Smith theory (see, e.g., [11], Theorem 5.35) we may assume that $(\mathbb{Z}/p)^r$ always acts on any sphere S^{n_i} with an isotropy subgroup

H_i with index p. Hence, since $(\mathbb{Z}/p)^r$ acts freely on the product $S^{n_1} \times \cdots \times S^{n_l}$, $\cap_{i=1}^{l} H_i$ should be trivial. This implies that r is less than or equal to l, since each H_i is a hyperplane in $(\mathbb{Z}/p)^r$.

However, note that there exists a free $(\mathbb{Z}/p)^r$-action on $\mathscr{Z}_K$ that cannot be extended to the natural torus action. Indeed, in his paper [4] Browder constructed free actions of $(\mathbb{Z}/p)^r$-actions on $(S^m)^k$ for each odd $m \geq 3$ and $r \geq 3$, $k \geq 4$, $k \geq r$, $p > km/2$, called *homologically exotic*, such that they cannot be extended to T^r-actions.

6 Proof of Theorem 1.4 (a)

The goal of this section is to give a proof of Theorem 1.4 (a). To do so, we begin with the following theorem.

Theorem 6.1. *Let $(\mathbb{Z}/p)^r$ act freely on $S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1}$ for a prime p. Then r is less than or equal to l.*

Proof. Note first that, if $K_i = 2^{[m_i]} \backslash \{[m_i]\}$ for each $1 \leq i \leq l$, then we have

$$S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_l-1} = \mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}.$$

Thus $(\mathbb{Z}/p)^r$ act freely on the generalized moment-angle complex $\mathscr{Z}_{K_1 \times K_2 \times \cdots \times K_l}$. It follows from Remark 5.2 that the r is less than or equal to l, as desired. Note that this proves Theorem 1.4 (a) in case of $l = l_o$. $\square$

In order to deal with even dimensional spheres in the products of spheres, we will need the following lemma.

Lemma 6.2. *Let p be odd prime, and let $G := (\mathbb{Z}/p)^r$ act freely on $S^{2m_1-1} \times S^{2m_2}$ by the action map Ψ. Then the $(\mathbb{Z}/p)^r$-action Ψ is split in that*

$$\Psi(g,(t,s)) = (\chi_s(g,t),s), \quad g \in (\mathbb{Z}/p)^r,\ (t,s) \in S^{2m_1-1} \times S^{2m_2},$$

where χ_s induces a free $(\mathbb{Z}/p)^r$-action on S^{2m_1-1} for each $s \in S^{2m_2}$.

Proof. Note first that there is a commutative diagram for some continuous map $\chi_s : G \times S^{2m_1-1} \to S^{2m_1-1}$, depending on the parameter $s \in S^{2m_2}$:

$$\begin{array}{ccc} G \times \left(S^{2m_1-1} \times S^{2m_2}\right) & \xrightarrow{\ \Psi\ } & S^{2m_1-1} \times S^{2m_2} \\ \Big\downarrow{\scriptstyle id \times p_1} & & \Big\downarrow{\scriptstyle p_1} \\ G \times S^{2m_1-1} & \xrightarrow{\ \chi_s\ } & S^{2m_1-1}, \end{array}$$

where $p_1 : S^{2m_1-1} \times S^{2m_2} \to S^{2m_1-1}$ denotes the projection onto the first factor. In fact, we have

$$\chi_s(g,t) = p_1(\Psi(g,(t,s))).$$

For the sake of simplicity, let us write $\Psi(g,(t,s))$ as $g\cdot(t,s)$. Let $p_2 : S^{2m_1-1} \times S^{2m_2} \to S^{2m_2}$ denote the projection onto the second factor. Then we have

$$\begin{aligned}&(g_2g_1)\cdot(t,s) = g_2\cdot(g_1\cdot(t,s)) = g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\\ &= (p_1\left(g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\right), p_2\left(g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\right)).\end{aligned}$$

Thus we obtain

$$p_2(g_2g_1\cdot(t,s)) = p_2\left(g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\right). \tag{6.1}$$

Using the second factor of $\Psi(g,(t,s))$, we next define a continuous map

$$\rho : G\times S^{2m_2} \to S^{2m_2}, \quad (g,s) \mapsto p_2(\Psi(g,(t,s))),$$

for some $t\in S^{2m_1-1}$ satisfying the additional condition that if $\rho(g_1,s) = p_2(g_1\cdot(t,s))$ for $g_1\in G$ and some $t\in S^{2m_1-1}$, then for any other $g_2\in G$

$$\begin{aligned}\rho(g_2g_1,s) &= p_2(g_2g_1\cdot(t,s)),\\ \rho(g_2,p_2(g_1\cdot(t,s))) &= p_2\left(g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\right).\end{aligned} \tag{6.2}$$

The meaning of the extra condition in (6.2) goes as follows: once we start to use $t\in S^{2m_1-1}$ in order to define $\rho(g,s)$ in terms of $\Psi(g,(t,s))$, we use the same t to define $\rho(g',s)$ for all $g'\in G$. On the other hand, we will use $p_1(g\cdot(t,s))\in S^{2m_1-1}$ in order to define $\rho(g',p_2(g\cdot(t,s)))$ for all $g'\in G$ and $p_2(g\cdot(t,s))\in S^{2m_2}$.

Then we can show that ρ actually defines a group action on S^{2m_2}. Indeed, obviously $\rho(e,s) = s$ for all $s\in S^{2m_2}$. Moreover, it follows from (6.1) and (6.2) that we have

$$\begin{aligned}\rho(g_2g_1,s) &= p_2(g_2g_1\cdot(t,s))\\ &= p_2\left(g_2\cdot(p_1(g_1\cdot(t,s)), p_2(g_1\cdot(t,s)))\right),\\ &= \rho(g_2, p_2(g_1\cdot(t,s))),\\ &= \rho(g_2,\rho(g_1,s)),\end{aligned}$$

as required.

However, by Theorem 1.1 there is no non-trivial group action of G on S^{2m_2} for any odd prime p. This implies that

$$s = \rho(g,s) = p_2(\Psi(g,(t,s))$$

for all $s \in S^{2m_2}$ and some $t \in S^{2m_1-1}$. But notice that we can start and repeat our arguments with any t for a given $(g,s) \in G \times S^{2m_2}$, so we can conclude that $p_2(\Psi(g,(t,s)) = s$ for any $g \in G$ and any $(t,s) \in S^{2m_1-1} \times S^{2m_2}$. Hence we have

$$\Psi(g,(t,s)) = (\chi_s(g,t),s), \quad g \in (\mathbb{Z}/p)^r, \ (t,s) \in S^{2m_1-1} \times S^{2m_2}.$$

This also says that $\chi_s : G \times S^{2m_1-1} \to S^{2m_1-1}$ actually defines a free group action. This completes the proof of Lemma 6.2. □

More generally, it is easy to see that the following lemma also holds, whose proof is completely analogous to that of Lemma 6.2 and so will be left to the reader:

Lemma 6.3. *Let p be odd prime, and let $G := (\mathbb{Z}/p)^r$ act freely on*

$$S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_{l_o}-1} \times S^{2m_{l_o+1}} \times \cdots \times S^{2m_l}$$

by the action map Ψ. Then the $(\mathbb{Z}/p)^r$-action Ψ is split in that

$$\Psi(g,(t_1,\ldots t_{l_o}, s_1,\ldots,s_{l-l_o})) = (\chi_{s_1,\ldots,s_{l-l_o}}(g,(t_1,\ldots,t_{l_o})), s_1,\ldots,s_{l-l_o}),$$

where $\chi_{s_1,\ldots,s_{l-l_o}}$ induces a free $(\mathbb{Z}/p)^r$-action on $S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_{l_0}-1}$.

We can now finish the proof of Theorem 1.4 (a), as follows.

Theorem 6.4. *Let p be odd prime, and let $G := (\mathbb{Z}/p)^r$ act freely on*

$$S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_{l_o}-1} \times S^{2m_{l_o+1}} \times \cdots \times S^{2m_l}$$

by the action map Ψ. Then r is less than or equal to l_o.

Proof. By Lemma 6.2, the free G-action Ψ induces a free G-action χ on $S^{2m_1-1} \times S^{2m_2-1} \times \cdots \times S^{2m_{l_o}-1}$. Then the theorem follows immediately from Theorem 6.1. This proves Theorem 1.4 (a) in case of $l_o \leq l$. □

Finally, note that Theorems 5.4 and 6.4 complete the proof of Theorem 1.4.

References

[1] A. Adem, *Constructing and deconstructing group actions*, Contemp. Math. **346** (2004), 1–8.

[2] A. Adem and W. Browder, *The free rank of symmetry of* $(S^n)^k$, Invent. Math. **92** (1988), 431–440.

[3] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: A method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. in Math. **225** (2010), 1634–1668.

[4] W. Browder, *Homologically exotic free actions on poducts of S^m*, Topology, Geometry, and Algebra: Interactions and new directions, Contemporary Mathematics, Vol. **279**, 57–72, Amer. Math. Soc., Providence, 2001.

[5] V. M. Buchstaber and T. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, Vol. **24**, Amer. Math. Soc., Providence, RI, 2002.

[6] X. Cao and Z. Lü, *Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture*, J. Algebraic Combin. **35** (2012), 121–140; arXiv:0908.3174v2.

[7] G. Carlsson, *On the rank of abelian groups acting freely on* $(S^n)^k$, Invent. Math. **69** (1982), 393–400.

[8] G. Carlsson, $(\mathbb{Z}/2)^3$*-actions on finite complexes*, Algebraic toplogy and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud. **113** (1987), 332–344.

[9] B. Hanke, *The stable free rank of symmetry of products of spheres*, Invent. Math. **178** (2009), 265–298.

[10] A. Heller, *A note on spaces with operators*, Illinois J. Math. **3** (1959), 98–100.

[11] K. Kawakubo, *The theory of transformation groups*, Oxford University Press, 1987.

[12] J. van Mill and G. M. Reed, *Open problems in topology*, Elsevier, Amsterdam (1990).

[13] T. Panov, *Geometric structures on moment-angle manifolds*, preprint (2013); arXiv:1302.2563v1.

[14] P. A. Smith, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. **30** (1944), 105–108.

[15] E. Yalçin, *Group actions and group extensions*, Trans. Amer. Math. Soc. **352** (2000), 2689–2700.

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