ON THE TATE MODULE OF A NUMBER FIELD

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Abstract. Following Kuz'min, we investigate the Tate module of a number field k in terms of certain arithmetic properties of the p-units via class field theory. This gives rise to a relation of the Tate module of k with the conjectures of Gross and Coates-Lichtenbaum as well as a question of Kato. We reduce the Gross conjecture to the study of the universal norm subgroup of the p-units over the cyclotomic Z_p-extension and prove the Gross conjecture when the number of primes of k dividing p is at most 2.

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1. Introduction

One of equivalent forms of the Gross conjecture can be described as a finiteness of the Galois invariant $T_p(k)^\Gamma$ of the Tate module $T_p(k)$ of k over $k_\infty^{\mathrm{cyc}}/k$. Kuz'min interpreted $T_p(k)^\Gamma$ in terms of a deep arithmetic properties of the p-units.

The purpose of this paper is to extend further this point of view in a clear way using Hasse's local-global norm theorem of the p-units of k and to prove the Gross conjecture when the number of primes of k dividing p is at most 2.

For a number field k, let $k_\infty = \bigcup_n k_n$ be a $\mathbb{Z}_p$-extension of k with k_n the unique subfield of k_∞ of degree p^n over k. Let $N_{m,n} = N_{k_m/k_n}$ denote the norm map from k_m to k_n and let $N_m = N_{m,0}$ denote the norm map from k_m to the ground field $k_0 = k$.

For a subgroup $\mathcal{C}_n$ of $k_n^\times$, we define the norm compatible subgroup $\mathcal{C}_n^{\mathrm{comp}}$ and the universal norm subgroup $\mathcal{C}_n^{\mathrm{univ}}$ as follows

$$\mathcal{C}_n^{\mathrm{univ}} := \bigcap_{m\geq n} N_{m,n}\mathcal{C}_m, \;\; \mathcal{C}_n^{\mathrm{comp}} := \pi(\varprojlim_{m\geq n} \mathcal{C}_m)$$

where the inverse limits are taken with respect to the norm maps and $\pi = \pi_n$ denotes the natural projection from $\varprojlim_{m\geq n} \mathcal{C}_m$ to $\mathcal{C}_n$ defined as

$$\pi((b_m)_{m\geq n}) = b_n.$$

By taking ramifications over a $\mathbb{Z}_p$-extension into account, we notice that

$$\mathcal{C}_n^{\mathrm{univ}} = \mathcal{C}_n^{\mathrm{univ}} \cap U_n(p), \;\; \mathcal{C}_n^{\mathrm{comp}} = (\mathcal{C}_n \cap U_n(p))^{\mathrm{comp}}$$

where $U_n(p)$ denotes the p-units of k_n and

$$(\mathcal{C}_n \cap U_n(p))^{\mathrm{comp}} = \pi(\varprojlim_{m\geq n} (\mathcal{C}_m \cap U_n(p))).$$

For $\{\mathcal{C}_n \otimes \mathbb{Z}_p\}_{m\geq n}$, let π also denote the natural projection

$$\pi : \varprojlim_{m\geq n} (\mathcal{C}_m \otimes \mathbb{Z}_p) \to \mathcal{C}_n \otimes \mathbb{Z}_p.$$

The norm compatible subgroup $(\mathcal{C}_n \otimes \mathbb{Z}_p)^{\mathrm{comp}}$ and the universal norm subgroup $(\mathcal{C}_n \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ of $\mathcal{C}_n \otimes \mathbb{Z}_p$ are defined as

$$(\mathcal{C}_n \otimes \mathbb{Z}_p)^{\mathrm{univ}} := \bigcap_{m\geq n} N_{m,n}(\mathcal{C}_m \otimes \mathbb{Z}_p), \quad (\mathcal{C}_n \otimes \mathbb{Z}_p)^{\mathrm{comp}} := \pi(\varprojlim_{m\geq n} (\mathcal{C}_m \otimes \mathbb{Z}_p)).$$

For a finite cyclic extension K/k and for a finite set S of primes of k, let $U_k(S)$ be the global S-units of k. Let S' be the set of primes of K lying over each prime $v \in S$,

$$S' = \{w|v \ ; \ v \in S\}.$$

We also let $U_K(S) := U_K(S')$ denote the global S'-units of K. Let

$$J_{K,S} := \prod_{w\notin S'} U_v \times \prod_{w\in S'} k_v^{\times}$$

be the S-idele group where we identify $U_K(S)$ with a subgroup of $J_{K,S}$ via the the diagonal imbedding $\phi_{K,S} : U_K(S) \longrightarrow J_{K,S}$.

For $K = k_n$ and $S = \{v|p\}$, we write

$$U_n(p) := U_{k_n}(S), \quad \phi_n := \phi_{k_n,S}.$$

We have the following exact sequence

$$1 \longrightarrow \ker(\phi_n) \longrightarrow U_k(p) \xrightarrow{\phi_n} \widehat{H}^0(G_n, J_{k_n,S})$$

where $G_n := G(k_n/k)$ denotes the Galois group of k_n/k.

Write

$$\mathfrak{N}_n(p) := \ker(\phi_n).$$

Then Hasse's theorem for $k^{\times}$ implies that

$$N_n U_n(p) \subset \mathfrak{N}_n(p) \subset U_k(p) \cap N_n k_n^{\times}.$$

Let

$$\mathcal{E}_n(p) := \{a \in k_n^{\times} \mid N_n(a) \in \mathfrak{N}_n(p)\}$$

be the inverse image $N_n^{-1}(\mathfrak{N}_n(p))$ of $\mathfrak{N}_n(p)$ inside $k_n^{\times}$. By tensoring with $\mathbb{Z}_p$, the exact sequence induces

$$1 \longrightarrow \ker(\overline{\phi}_n) = \mathfrak{N}_n(p) \otimes \mathbb{Z}_p \longrightarrow U_k(p) \otimes \mathbb{Z}_p \xrightarrow{\overline{\phi}_n} \widehat{H}^0(G_n, J_{k_n,S})$$

together with

$$N_n(U_n(p) \otimes \mathbb{Z}_p) \subset \mathfrak{N}_n(p) \otimes \mathbb{Z}_p = N_n(\mathcal{E}_n(p) \otimes \mathbb{Z}_p) \subset (U_k(p) \cap N_n k_n^{\times}) \otimes \mathbb{Z}_p.$$

Then

$$\ker(\overline{\phi}_\infty) := \bigcap_{n\geq 0} \ker(\overline{\phi}_n) = \bigcap_{n\geq 0} (\mathfrak{N}_n(p) \otimes \mathbb{Z}_p) = (\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

Now let $k_\infty = k_\infty^{\mathrm{cyc}} = \bigcup_{n\geq 0} k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of k. Let Γ denote the procyclic group $G(k_\infty/k)$ and for each $n \geq 0$, let $\Gamma_n = G(k_\infty/k_n)$ be the unique subgroup of Γ with index p^n.

Let $S := S_n = \{v|p\}$ be the set of primes of k_n dividing p. Let $Cl(k_n)$ be the ideal class group of k_n and let

$$Cl_S(k_n) := Cl(k_n)/\langle cl(v)\rangle_{v\in S}$$

be the S-ideal class group of k_n where $cl(v) \in Cl_n$ is the ideal class containing v.

We define the Tate module $T_p(k)$ of k following Kuz'min as the inverse limit of $Cl_S(k_n)$ with respect to the norm maps,

$$T_p(k) := \varprojlim_n (Cl_S(k_n) \otimes \mathbb{Z}_p).$$

Using infinite class field theory we will prove the following result of Kuz'min(see Proposition 7.5 of [18]). The statement and the proof of the result of Kuz'min will then be more clear due to $\overline{\phi}_\infty$.

Proposition 1.1. *Let k be a number field. Then we have the following isomorphism*

$$T_p(k)^\Gamma = \frac{\ker(\overline{\phi}_\infty)}{(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}} = \frac{(\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}.$$

For a finitely generated $\mathbb{Z}$-module M, let $M/\mathrm{tor}(M)$ be the quotient of M by its torsion $\mathrm{tor}(M)$ and let

$$\mathrm{rank}_{\mathbb{Z}}\ M := \mathrm{rank}_{\mathbb{Z}}\ (M/\mathrm{tor}(M))$$

be the $\mathbb{Z}$-rank of $M/\mathrm{tor}(M)$. Similarly, we define $\mathrm{rank}_{\mathbb{Z}_p} M$ for a finitely generated $\mathbb{Z}_p$-module M.

Corollary 1.2. *If k satisfies the Gross conjecture, then*

$$\mathrm{rank}_{\mathbb{Z}}\ \mathcal{E}_k(p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\ U_k(p)^{\mathrm{univ}}.$$

Finally we prove the Gross conjecture when the number r of primes of k dividing p is at most 2.

Let k^{gal} denote the Galois closure of k in a fixed algebraic closure k^{alg} of k and for each $n > 0$, let μ_n denote the group of nth roots of unity in k^{alg}.

Theorem 1.3. *If k is a number field such that for $K := k^{\mathrm{gal}}(\mu_p)$, there are at most two primes dividing p in $K_\infty^{\mathrm{cyc}} = k^{\mathrm{gal}}(\mu_{p^\infty})$, then the Gross conjecture is true for k_n for all $n \geq 0$.*

Remark. We remark here that a great deal of progress has been made in recent years towards proving (refined, equivariant forms of) Gross's conjecture not only its order of vanishing part, which is the main point of concern of this paper, but also its leading term part, which has deep arithmetic consequences.

Refined versions of Gross's conjecture have been showed to be consequences of the Equivariant Tamagawa Number Conjecture and Equivariant Versions of the Main Conjecture and the latter have been proved under the hypothesis that a certain mu-invariant vanishes in the context of abelian CM extensions of totally real number fields.

For details, we refer, for instance, to a series of papers (cf. [4], [5], [6]) of David Burns.

2. The Tate module and the Gross conjecture

For the local field k_v which is the completion of k at a finite place v, let $k_{v,\infty}$ be the corresponding $\mathbb{Z}_p$-extension of k_v. Let

$$k^{\mathrm{loc}} = \{\alpha \in k^\times \mid \text{there is } (\alpha_{v,n}) \in \varprojlim_n k_{v,n} \ \text{ such that } \alpha_{v,0} = \alpha \text{ for all } \ v\}$$

be be the set of all elements which are locally norm comparable at all finite palces.

We need the following well known lemmas and we give here brief proofs for the convenience of readers.

Lemma 2.1. *Let Θ_n be a subgroup of $U_n(p)$ such that $N_{m,n} : \Theta_m \to \Theta_n$. Then we have $(\Theta \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{univ}} = (\Theta \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}}$.*

Proof. We use the fact that the p-adic completion $\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p = \varprojlim_r \Theta_n/(\Theta_n)^{p^r}$ of Θ_n is compact. For $r \geq m > 0$ and for $\alpha \in \bigcap_n N_n(\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, let $X_r(\alpha) = N_{r,m}\Theta_r \otimes_{\mathbb{Z}} \mathbb{Z}_p \bigcap N_m^{-1}\alpha$ where $N_m^{-1}\alpha = \{b \in U_m(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p | N_m(b) = \alpha\}$. Since $\alpha \in \bigcap_n N_n(\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, $X_r(\alpha)$ is non empty and compact.

The family $X_r(\alpha)$ has the finite intersection property as $r \geq m$ varies because for a finite set of numbers $n_g > ... > n_1 > m$, $X_{n_i}(\alpha)$ is a non empty decreasing chain. It follows that there is $\beta_m \in \bigcap_{r \geq m} N_{r,m}(\Theta_r \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ such that $N_m\beta_m = \alpha$.

In this way, one can construct a norm comparable sequence whose first term is α. This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *Let k be a number field. Then, we have $k^{\mathrm{loc}} = k^{\mathrm{univ}} = U_k(p) \cap k^{\mathrm{univ}}$.*

Proof. It follows immediately from the Hasse's local-global norm theorem. $\square$

Let
$$\Delta' : k^\times \longrightarrow \prod_{v \neq \infty} \widehat{k}_v^\times,\ \alpha \mapsto (\alpha, \cdots, \alpha, \cdots)$$
denote the diagonal imbedding where $\widehat{M} = M^\wedge := \varprojlim_n M/M^{p^n}$ is the p-adic completion of any abelian group M and let
$$\widehat{\Delta'} : U_k(p) \otimes \mathbb{Z}_p \longrightarrow \prod_{v|p} \widehat{k}_v^\times \times \prod_{v \nmid p\infty} \widehat{U}_v \subset \prod_{v \neq \infty} \widehat{k}_v^\times = (\prod_{v \neq \infty} k_v^\times)^\wedge$$
be the induced map from Δ'. Note that the p-adic completion $\widehat{k}_v^\times$ of $k_v^\times$ is compact and hence by the Tychonoff Theorem, $\prod_{v \neq \infty} \widehat{k}_v^\times$ is compact. The topological closure $\overline{\Delta'(U_k(p))}$ of the image $\Delta'(U_k(p))$ of $U_k(p)$ under Δ' in the group $\prod_{v|p} \widehat{k}_v^\times \times \prod_{v \nmid p\infty} \widehat{U}_v$ is compact and similarly, $\overline{\Delta'(k^\times)}$ is compact in $\prod_{v \neq \infty} \widehat{k}_v^\times$. For $v \nmid p$, $\widehat{U}_v$ is the p-torsion part of U_v and $\widehat{k}_v^\times$ is the direct sum of $\mathbb{Z}_p$ and $\widehat{U}_v$.

Since $U_k(p)$ is a $\mathbb{Z}$-module of finte type, the topological closure $\overline{\Delta'(U_k(p))}$ is equal to the image $\widehat{\Delta'}(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ of
$$U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\widehat{\Delta'}} \overline{\Delta'(U_k(p))} \longrightarrow 1.$$
It is well known that $\widehat{\Delta'}$ is injective for the global units U_k which is a stronger statement than the Leopoldt conjecture (see Theorem 4.4 and Remark 4.4.8 of [9])
$$\widehat{\Delta'} : U_k \otimes \mathbb{Z}_p \cong \overline{\Delta'(U_k)}.$$

For the reader's convenience, we will sketch the proof of the isomorphism for the global p-units $U_k(p)$ following Theorem 4.4 of *loc.cit.* We fix a $\mathbb{Z}$-basis $\{e_i\}$ for $U_k(p) = \sum_i e_i\mathbb{Z}$. Suppose that $\varepsilon = \prod_i e_i \otimes \alpha_i \in \mathrm{Ker}(\widehat{\Delta'}) \subset U_k(p) \otimes \mathbb{Z}_p$ for some $\alpha_i \in \mathbb{Z}_p$. Write $\alpha_i = a_{i,n} + p^n b_{i,n}$ for some $a_{i,n} \in \mathbb{Z}$, $b_{i,n} \in \mathbb{Z}_p$. We have
$$\begin{aligned}\varepsilon = \prod_i e_i \otimes \alpha_i &= \prod_i e_i \otimes (a_{i,n} + p^n b_{i,n}) \\ &= \prod_i (e_i \otimes a_{i,n}) \prod_i (e_i^{p^n} \otimes b_{i,n}) \\ &= \prod_i (e_i^{a_{i,n}} \otimes 1) \prod_i (e_i^{p^n} \otimes b_{i,n}).\end{aligned}$$

For all tame primes $v \nmid p$, Hensel's lemma shows that $\widehat{\Delta'}(\varepsilon) = 1$ implies that

$$e_i^{a_{i,n}} \in k_v^{\times p^n}.$$

It follows from the local-global principle for powers(see II.6.3.3 of *loc.cit*) that

$$e_i^{a_{i,n}} \in U_k(p) \cap k^{\times p^n} = U_k(p)^{p^n}$$

and hence, for all $n \geq 1$,

$$\varepsilon \in (U_k(p) \otimes \mathbb{Z}_p)^{p^n}.$$

Since $\mathcal{U}_k := U_k(p) \otimes \mathbb{Z}_p$ is a $\mathbb{Z}_p$-module of finite type, $\mathcal{U}_k$ is Hausdorff(see III.2.3.3 Exercise (ii) and its solution of *loc.cit*) which shows that

$$\varepsilon \in \bigcap_n \mathcal{U}_k^{p^n} = 1.$$

This completes the proof that $\widehat{\Delta'}$ is injective.

2.1. **Proof of Proposition 1.1.** We define the Tate module $T_p(k)$ as

$$T_p(k) = \varprojlim_n G(L_n/k_n)$$

the inverse limit of $G(L_n/k_n)$ where L_n is the maximal abelian p-extension of k_n which is unramified and all primes dividing p split completely over k_n. By Class field theory, this is equal to the one defined in the introduction.

Let X be the Galois group $G(F/k_\infty^{\mathrm{cyc}})$ of the maximal abelian p-extension F of $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$. Then X is a regular Γ-module, i.e., for all $n \geq 0$, $X^{\Gamma_n} = 0$, by Kuz'min (Lemma 7.4 of [18]).

Let W denote the subgroup of X generated by the inertia groups for $\mathfrak{p} \nmid p$ and the decomposition groups for $\mathfrak{p}|p$. The exact sequence $1 \longrightarrow W \longrightarrow X \longrightarrow T_p(k) \longrightarrow 1$ results in the following exact sequence

$$1 = X^\Gamma \longrightarrow T_p(k)^\Gamma \longrightarrow W_\Gamma \xrightarrow{\beta} X_\Gamma \longrightarrow T_p(k)_\Gamma \longrightarrow 1.$$

If F_0 denotes the maximal abelian subextension of F over k, then

$$X_\Gamma = G(F_0/k_\infty^{\mathrm{cyc}}).$$

If L_0 denotes the maximal subextension of F_0 such that $L_0/k_\infty^{\mathrm{cyc}}$ is unramified at all primes $\mathfrak{p} \nmid p$ and $L_{0,\mathfrak{P}} = k_{\infty,\mathfrak{p}}^{\mathrm{cyc}}$ over primes $\mathfrak{P}$ dividing primes $\mathfrak{p}$ of k_∞^{cyc}, then

$$T_p(k)_\Gamma = G(L_0/k_\infty^{\mathrm{cyc}}).$$

For $v|p$, let $\widetilde{U}_v$ denote the norm compatible elements $\pi(\varprojlim k_{v,n}) \subset k_v^\times$ over the cyclotomic $\mathbb{Z}_p$-extension $k_{v,\infty}^{\mathrm{cyc}} = \bigcup_n k_{v,n}$ of k_v. For the basic facts on infinite class field theory which will be used here, we refer to §3.7 of [9].

Let J_k denote the idele group of a number k and let

$$\widetilde{k}_v = \bigcap_n N_n k_{v,n}^\times$$

be the universal norm elements of $k_{v,\infty}^{\mathrm{cyc}} = \bigcup_n k_{v,n}$ over k_v.

Using class field theory (cf. the second paragraph of page 280 of [18]), we will prove the following claim.

$$G(L_0/k) \cong \text{the } p\text{-part of } J_k/\overline{k^\times U_\infty \prod_{v \nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}.$$

Proof of the claim. The field L_0' corresponds to the maximal extension of k which is unramified at all primes prime to p and $(L_0')_v = (k_\infty^{\mathrm{cyc}})_v$ at all v dividing p. Since an extension Ω/k which is unramified at all primes prime to p and $(L_0)_v = (k_\infty^{\mathrm{cyc}})_v$ at all v dividing p corresponds to an idele subgroup containing $\prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v$, it is obvious that the idele subgroup corresponding to L_0' is the closure

$$\overline{k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v}$$

in the idele topology of $k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v$.

Let $U_\infty = \prod_{v|\infty} U_v = (\mathbb{R}_+^\times)^{r_1} \times \mathbb{C}^{\times r_2}$ and let $\overline{k^\times U_\infty}$ denotes the closure of $k^\times U_\infty$ in the idele topology. It follows that

$$\overline{k^\times U_\infty} = \bigcap_{\mathfrak{m}} k^\times U_{\mathfrak{m}} U_\infty$$

where $\mathfrak{m} = \prod p_v^{m_v}$ runs over all finite products of finite primes of k and $U_{\mathfrak{m}} = \prod (1 + \mathfrak{p}_v^{m_v}) = \prod U_v^{m_v}$.

Let

$$\widetilde{U}_v = \bigcap_n N_n U_{v,n}$$

be the universal local units where $U_{v,n}$ denotes the local units of $k_{v,n}^\times = \pi_{v,n}^{\mathbb{Z}} \oplus U_{v,n}$ for a fixed uniformizer $\pi_{v,n}$ of $k_{v,n}$. Since $U_{v,n}$ is compact and norm map is continuous, $\widetilde{U}_v$ is compact.

Since each prime $v|p$ is totally ramified over $k_{v,n}$ for a sufficiently large n, we have

$$\widetilde{k}_v = \langle \pi_v^{f_v} u_v \rangle \oplus \widetilde{U}_v$$

where π_v is a fixed uniformizer of k_v and u_v is an element of U_v.

Note that since $\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{U}_v$ is compact, $\overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{U}_v$ is closed by a well known fact (see Lemma 1 of page 77 of [1]). Then the quotient space

$$\mathcal{J}_k = J_k / \overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{U}_v$$

is a topological group which is again locally compact (cf. §19 of [21]).

Since the subgroup

$$\mathcal{Q} = \overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v / \overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{U}_v \subset \mathcal{J}_k$$

is a discrete subgroup of $\mathcal{J}_k$ generated by $\{\pi_v^{f_v} u_v\}_{v|p}$, it is a closed subgroup of $\mathcal{J}_k$. Since the natural map $p : J_k \to J_k / \overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{U}_v$ is continuous, its inverse $p^{-1}(\mathcal{Q}) = \overline{k^\times U_\infty} \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v$ of $\mathcal{Q}$ is a closed subgroup of J_k.

This shows that

$$\begin{aligned} \overline{k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v} &= \overline{k^\times U_\infty} \cdot \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v \\ &= \overline{k^\times U_\infty} \cdot \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}. \end{aligned}$$

This completes the proof of the claim. In fact, since $k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v$ is locally compact, $k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v$ is closed in J_k (see Appendix).

We have the following isomorphisms

$$\begin{aligned}
\overline{k^\times U_\infty \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}/\overline{k^\times U_\infty} &= \overline{k^\times U_\infty} \cdot \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}/\overline{k^\times U_\infty} \\
&\cong \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}/(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \cap \overline{k^\times U_\infty}) \\
&\cong \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}/(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \cap \bigcap_{\mathfrak{m}} k^\times U_{\mathfrak{m}}) \\
&\cong \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}/(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \cap \bigcap_{\mathfrak{m}} U_k(p) U_{\mathfrak{m}})
\end{aligned}$$

where the first isomorphism follows from (G) of §20 of [21] since J_k is locally compact which can be represented as set theoretic union of a countable number of compact subsets.

The p-part of $(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \cap \bigcap_{\mathfrak{m}} U_k(p)U_{\mathfrak{m}})$ can be identified with $\ker(\overline{\phi}_\infty) := \bigcap_n \ker(\overline{\phi}_n)$ which is the intersection of the closure $U_k(p) \otimes \mathbb{Z}_p$ of $U_k(p)$ with the p-part $(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v})^\wedge$ inside $\prod_{v\neq\infty} \widehat{k}_v^\times$.

Hence we have (cf. equation 7.11 of Kuz'min [18])

$$\begin{aligned}
\mathrm{Im}(\beta) &\cong \text{the } p\text{-part of } \Big(\frac{\overline{k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v}}{\overline{k^\times U_\infty}}\Big) \\
&\cong \text{the } p\text{-part of } \Big(\frac{\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v}}{(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \cap \bigcap_{\mathfrak{m}} U_k(p) U_{\mathfrak{m}}}\Big) \\
&\cong \frac{(\overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v})^\wedge}{\ker(\overline{\phi}_\infty)}.
\end{aligned}$$

Write

$$\omega_{m,n} = \frac{\gamma^{p^m} - 1}{\gamma^{p^n} - 1}.$$

Then, for $m \geq n$, $\omega_{m,n}$ induces

$$\omega_{m,n} : T_p(k_m)^{\Gamma_m} \longrightarrow T_p(k_n)^{\Gamma_n}$$

such that the corresponding inverse limit becomes trivial

$$\varprojlim T_p(k_n)^{\Gamma_n} = 1$$

using finiteness of class number of k_n.

Then by replacing the ground field k by k_n, we have

$$\begin{aligned}
W = \varprojlim_n W_{\Gamma_n} &= \varprojlim_n ((\overline{\prod_{v\nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^\wedge / \ker(\overline{\phi}_{n,\infty}) \\
&= \varprojlim_n (\overline{\prod_{v\nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^\wedge / \varprojlim_n \ker(\overline{\phi}_{n,\infty}).
\end{aligned}$$

Since

$$\varprojlim_n \ker(\overline{\phi}_{n,\infty}) = \varprojlim_n (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$$

the expression of W above leads to

$$1 \longrightarrow \varprojlim_n (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow \varprojlim_n (\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge} \longrightarrow W \longrightarrow 1.$$

The short exact sequence above yields

$$\longrightarrow (\varprojlim_n (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}})_{\Gamma} \xrightarrow{\alpha} (\varprojlim_n (\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge})_{\Gamma} \longrightarrow W_{\Gamma} \longrightarrow 1.$$

Since $\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}}$ is cohomologically trivial by Kuz'min (see page 302 of [18]), it reduces via the natural projections, i.e., the norm maps to

$$\pi((\varprojlim_n (\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge})_{\Gamma}) = (\overline{\prod_{v \nmid p\infty} U_{v} \prod_{v|p} \widetilde{k}_{v}})^{\wedge}.$$

In fact, $U_{n,v}$ is cohomologically trivial $G(k_n/k)$-module for $v \nmid p$ since it is unramified at $v \nmid p$. For $v|p$, $\widetilde{k}_{n,v}$ is cohomologically trivial $G(k_n/k)$-module by Lemma 7.1 of [18]. Thus, the p-part $(\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v})^{\wedge}$ is also cohomologically trivial $G(k_n/k)$-module.

By taking the limits to the following short exact sequence

$$1 \longrightarrow (\gamma-1)(\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge} \longrightarrow (\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge} \longrightarrow (\overline{\prod_{v \nmid p\infty} U_{v} \prod_{v|p} \widetilde{k}_{v}})^{\wedge} \longrightarrow 1$$

we have

$$\pi : (\varprojlim_n (\overline{\prod_{v \nmid p\infty} U_{n,v} \prod_{v|p} \widetilde{k}_{n,v}})^{\wedge})_{\Gamma} \xrightarrow{\sim} (\overline{\prod_{v \nmid p\infty} U_{v} \prod_{v|p} \widetilde{k}_{v}})^{\wedge}.$$

Under the identification of above, the image of α corresponds to

$$\mathrm{Im}(\alpha) = \pi(\varprojlim_n (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}) = (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{comp}}.$$

Hence

$$W_{\Gamma} = (\overline{\prod_{v \nmid p\infty} U_{v} \prod_{v|p} \widetilde{k}_{v}})^{\wedge} / (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{comp}}.$$

Now the image of β reads off

$$\mathrm{Im}(\beta) = W_{\Gamma}/\mathrm{Ker}(\beta) = (\overline{\prod_{v \nmid p\infty} U_{v} \prod_{v|p} \widetilde{k}_{v}})^{\wedge} / \ker(\overline{\phi}_{\infty}).$$

We have

$$T_p(k)^{\Gamma} = \mathrm{Ker}(\beta) = \frac{(\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

since $\ker(\overline{\phi}_{\infty}) = (\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$. This completes the proof of Proposition 1.1.

We have the following corollary which is Corollary 1.2 of the introduction.

Corollary 2.3. *If k satisfies the Gross conjecture, then*

$$\mathrm{rank}_{\mathbb{Z}}\, \mathcal{E}_k(p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\, U_k(p)^{\mathrm{univ}}.$$

Proof. Write $\mathcal{A}(n) := N_n\mathcal{E}_n(p) \cap U_k(p)$. Then the inverse limit of $\mathcal{A}(n)$ with respect to the inclusion maps is $\mathcal{E}_k(p)^{\mathrm{univ}}$ and the inverse limit of $\mathcal{A}(n) \otimes \mathbb{Z}_p$ with respect to the inclusion maps is $(\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ since

$$\mathcal{A}(n) \otimes \mathbb{Z}_p = N_n(\mathcal{E}_n(p) \otimes \mathbb{Z}_p) \cap (U_k(p) \otimes \mathbb{Z}_p).$$

By taking inverse limits with respect to the inclusions, the exact sequence

$$1 \longrightarrow N_n(U_n(p) \otimes \mathbb{Z}_p) \longrightarrow \mathcal{A}(n) \otimes \mathbb{Z}_p \longrightarrow (\mathcal{A}(n)/N_n(U_n(p)) \otimes \mathbb{Z}_p \longrightarrow 1$$

leads to

$$1 \longrightarrow (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow (\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow \varprojlim(\frac{\mathcal{A}(n)}{N_n(U_n(p))} \otimes \mathbb{Z}_p) \longrightarrow 1$$

where the inverse limit is exact since $\mathcal{A}(n) \otimes \mathbb{Z}_p$ is a compact topological group. By the assumption and Proposition 1.1, we have

$$\varprojlim(\frac{\mathcal{A}(n)}{N_n(U_n(p))} \otimes \mathbb{Z}_p) = \frac{(\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}} \cong T_p(k)^{\Gamma} < \infty. \tag{1}$$

It follows from $U_k(p) \supset \mathcal{A}(n) \supset N_n(U_n(p)) \supset U_k(p)^{p^n}$ that $\mathcal{A}(n)/N_n(U_n(p))$ is a p-group and there is an isomorphism

$$\varprojlim(\frac{\mathcal{A}(n)}{N_n(U_n(p))}) \cong \varprojlim(\frac{\mathcal{A}(n)}{N_n(U_n(p))} \otimes \mathbb{Z}_p). \tag{2}$$

By taking inverse limits to

$$1 \longrightarrow N_n(U_n(p)) \longrightarrow \mathcal{A}(n) \longrightarrow \frac{\mathcal{A}(n)}{N_n(U_n(p))} \longrightarrow 1$$

we have the following left exact sequence

$$1 \longrightarrow U_k(p)^{\mathrm{univ}} \longrightarrow \mathcal{E}_k(p)^{\mathrm{univ}} \longrightarrow \varprojlim(\frac{\mathcal{A}(n)}{N_n(U_n(p))}).$$

From (1) and (2), this leads to the following identity

$$\mathrm{rank}_{\mathbb{Z}}\, U_k(p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\, \mathcal{E}_k(p)^{\mathrm{univ}}.$$

□

2.2. **The Gross conjecture.** We briefly introduce following Iwasawa (see §4 of [14]) and Kolster (see §1 of [17]) equivalent forms of the Gross conjecture using the cohomologies over $\Gamma := G(k_\infty^{\mathrm{cyc}}/k)$ from the original one which concerns the minus part of the p-units of a CM field to the generalized form of an arbitrary number field.

From the seven term exact sequence of Auslander-Brumer and Chase-Harrison-Rosenberg, there exists a map

$$\psi_S : H^2(\Gamma, U_\infty(S)) \longrightarrow B_S := \ker(H^2(\Gamma, k_\infty^{\mathrm{cyc}\times}) \to H^2(\Gamma, I_\infty))$$

where $U_\infty(S)$ and I_∞ are the group of the $S := \{v|p\}$-units and the group of fractional ideals of k_∞^{cyc} respectively. Then it can be shown that the Gross conjecture for k is equivalent to one of the following equivalent statements;
i) ψ_S is surjective,
ii) $Cl_\infty(S)^\Gamma < \infty$,
iii) $H^1(\Gamma, Cl_\infty(S)) = 0$,
where $Cl_\infty(S)$ is the p-part of the S-ideal class group of k_∞^{cyc},
iv) $T_p(k)_\Gamma < \infty$,
v) $T_p(k)^\Gamma < \infty$.

The equivalence of i), ii) and iii) is explained in §4 of [14] and the equivalence of ii), iv) and v) is explained in Theorem 1.14 of [17].

If k is a totally real field, the Leopoldt conjecture, i.e., the uniqueness of a $\mathbb{Z}_p$-extension of k, implies ψ_S is surjective.

When k is a CM field and p is odd, there exists a decomposition of ± 1-eigenspaces via the complex conjugation

$$\psi_S^{\pm} : H^2(\Gamma, U_\infty(S))^{\pm} \longrightarrow B_S^{\pm}.$$

Note that the surjectivity of the plus part ψ_S^+ is equivalent to Leopoldt's original conjecture and the surjectivity of the minus part ψ_S^- is equivalent to Gross' original conjecture.

2.2.1. *Second proof of Corollary 1.2.* In this subsection we prove Corollary 1.2 without using Proposition 1.1. Let K/k be a finite cyclic extension and for a finite set S of primes of k containing all ramified primes in K/k, let $U_k(S)$ be the global S-units of k. Let

$$S' = \{w|v \; ; \; v \in S\}$$

be the set of primes of K lying over each prime $v \in S$. We also let $U_K(S) := U_K(S')$ denote the global S'-units of K. Let

$$J_{K,S} := \prod_{w \notin S'} U_v \times \prod_{w \in S'} k_v^\times$$

be the S-idele group and let

$$C_{K,S} := J_{K,S}/U_K(S)$$

be the S-idele class group where we identify $U_K(S)$ with a subgroup of $J_{K,S}$ via the the diagonal imbedding $\phi_{K,S} : U_K(S) \longrightarrow J_{K,S}$.

Since $w \notin S$ is unramified in K/k, the local units U_w is cohomologically trivial for $w \notin S$. Hence by Shapiro's lemma, we have

$$\widehat{H}^i(G_{K/k}, J_{K,S}) \cong \prod_{w \in S} \widehat{H}^i(G_{K_w/k_v}, K_w^\times).$$

Using the isomorphism above and Hilbert's theorem 90, the exact sequence $1 \longrightarrow U_K(S) \overset{\phi_{K,S}}{\longrightarrow} J_{K,S} \longrightarrow C_{K,S} \longrightarrow 1$ leads to

$$(3) \;\; 1 \longrightarrow \widehat{H}^{-1}(G_{K/k}, C_{K,S}) \longrightarrow \frac{U_k(S)}{N_{K/k}U_K(S)} \overset{\tilde{\phi}_{K/k,S}}{\longrightarrow} \prod_{w \in S} \widehat{H}^0(G_{K_w/k_v}, K_w^\times)$$

where $N_{K/k}$ is the norm map of K/k and $\tilde{\phi}_{K/k,S}$ is the map induced from $\phi_{K,S}$. It follows that

$$\widehat{H}^{-1}(G_{K/k}, C_{K,S}) \cong \ker(\tilde{\phi}_{K/k,S})$$

We let

$$\phi_{K/k,S} : U_k(S) \longrightarrow \prod_{w \in S} \widehat{H}^0(G_{K_w/k_v}, K_w^\times) \cong \prod_{w \in S} k_v^\times / N_{K_w/k_v} K_w^\times$$

denote the composition of the natural projection $\pi : U_k(S) \longrightarrow U_k(S)/N_{K/k}U_K(S)$ and $\tilde{\phi}_{K/k,S} : U_k(S)/N_{K/k}U_K(S) \longrightarrow \prod_{w \in S} \widehat{H}^i(G_{K_w/k_v}, K_w^\times)$.

Hasse's theorem for $k^\times$ implies that

$$N_{K/k}U_K(S) \subset \ker(\phi_{K/k,S}) \subset U_k(S) \cap N_{K/k}K^\times.$$

For the cyclotomic $\mathbb{Z}_p$-extension $k_\infty^{\mathrm{cyc}}/k$ and for the set S of primes lying over p, we write $\phi_n := \phi_{k_n/k,S}$ and

$$\ker(\phi_\infty) := \ker(\phi_{k_\infty,S}) = \bigcap_{n\geq 0} \ker(\phi_n).$$

Lemma 2.4. *If $\#(\widehat{H}^0(G_n, Cl_S(k_n)))$ is bounded independently of n, then*

$$\mathrm{rank}_{\mathbb{Z}}\, \mathcal{E}_k(p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\, U_k(p)^{\mathrm{univ}}.$$

Proof. Let $U_{k_n,S}$ be the compact subgroup

$$U_{k_n,S} := \prod_{v|p}\{1\} \times \prod_{v\nmid p} U_v$$

of the J_{k_n} and let

$$C_S(k_n) := J_{k_n}/k_n^\times U_{k_n,S} \cong C_{k_n}/U_{k_n,S}.$$

We have the following exact sequence (see Proposition 8.3.5 of [20])

$$1 \longrightarrow C_{k_n,S} \longrightarrow C_S(k_n) \longrightarrow Cl_S(k_n) \longrightarrow 1 \tag{4}$$

and the following isomorphism (see Proposition 8.3.7 of [20])

$$\widehat{H}^i(G_n, C_S(k_n)) \cong \widehat{H}^i(G_n, C_{k_n}) \tag{5}$$

where

$$C_{k_n} := J_{k_n}/k_n^\times$$

is the idele class group of k_n. By class field theory, we have

$$\widehat{H}^{-1}(G_n, C_S(k_n)) \cong \widehat{H}^{-1}(G_n, C_{k_n}) = 1.$$

Hence the exact hexagon induced from (4) leads to

$$\widehat{H}^0(G_n, C_S(k_n)) \longrightarrow \widehat{H}^0(G_n, Cl_S(k_n)) \longrightarrow \widehat{H}^{-1}(G_n, C_{k_n,S}) \longrightarrow 1 \tag{6}$$

with

$$\widehat{H}^{-1}(G_n, C_{k_n,S}) \cong \ker(\phi_n)/N_n U_n(p).$$

By the sequence (3) with $K = k_n$, there exists a constant c such that

$$(\ker(\phi_n) : N_n U_n(p)) < c.$$

If $\alpha \in \ker(\phi_\infty) = \bigcap_n \ker(\phi_n)$, then $\alpha^c \in \bigcap_n N_n U_n(p) = U_k(p)^{\mathrm{univ}}$.

Since $\ker(\phi_\infty) = \mathcal{E}_k(p)^{\mathrm{univ}}$ is a finitely generated $\mathbb{Z}$-module and $\mathcal{E}_k(p)^{\mathrm{univ}}/U_k(p)^{\mathrm{univ}}$ is p-group, we have

$$(\mathcal{E}_k(p)^{\mathrm{univ}} : U_k(p)^{\mathrm{univ}}) < c^r$$

where $r = \mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{E}_k(p)^{\mathrm{univ}}/p \cdot \mathcal{E}_k(p)^{\mathrm{univ}})$.

This completes the proof of Lemma 2.4. □

Since there is $n \geq 0$ such that the primes dividing p are totally ramified over $k_\infty^{\mathrm{cyc}}/k_n$, we may assume that $k_\infty^{\mathrm{cyc}}/k$ is totally ramified at primes dividing p by replacing the base field with k_n.

Let $N_n : Cl_n \longrightarrow Cl_0 = Cl_k$ be the norm map over the ideal class groups and $j_n : Cl_0 \longrightarrow Cl_n$ be the natural map. Then we have

$$j_n \circ N_n = tr_n := \sum_{\sigma \in G_n} \sigma$$

and

$$\widehat{H}^0(G_n, Cl_S(k_n)) = \frac{Cl_S(k_n)^{G_n}}{tr_n Cl_S(k_n)}.$$

Let

$$Cl_S(k_n)_{N_n} := \{a \in Cl_S(k_n) | N_n(a) = 1\}$$

be the kernel of the norm map N_n. By Proposition 1.1, the Gross conjecture ii) and Proposition 1.4 of [17], we have

$$\#(Cl_S(k_n)^{G_n}) < \infty$$

independently of n. This leads to

$$\#(\widehat{H}^0(G_n, Cl_S(k_n))) < \infty.$$

By Lemma 2.4, we complete the proof of Corollary 1.2.

2.2.2. *Examples of* $(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \neq (k^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}$. Proposition 1.1 leads to the following corollary.

Corollary 2.5. *Let k be a number field.*

$$(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} = (k^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}} \implies T_p(k)^\Gamma = 1.$$

Using Corollary 2.5, we can find examples k such that

$$(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \neq (k^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

This is obtained by finding examples k such that $T_p(k)^\Gamma \neq 1$. Write $A'_n := Cl_S(k_n) \otimes \mathbb{Z}_p$. Let Ψ_0 denote the kernel of the natural projection from the coinvariant $T_p(k)_\Gamma$ of $T_p(k) = \varprojlim_{n \geq 0} A'_n$ to A'_0. This defines the following exact sequence

$$1 \longrightarrow \Psi_0 \longrightarrow T_p(k)_\Gamma \longrightarrow A'_0.$$

In Table 2 of [8], there are examples such that Ψ_0 is nontrivial for real quadratic fields and $p = 3$. Since Greenberg's conjecture holds for such fields, $T_p(k)$ is finite and hence the triviality of $T_p(k)^\Gamma$ is equivalent to that of $T_p(k)_\Gamma$. Thus examples of $\Psi_0 \neq 1$ lead to examples of $T_p(k)^\Gamma \neq 1$.

It will be interesting to determine whether the following identities on rank hold.

$$\mathrm{rank}_{\mathbb{Z}_p} (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}_p} (k^\times \otimes \mathbb{Z}_p)^{\mathrm{univ}}$$

$$\mathrm{rank}_{\mathbb{Z}}\, U_k(p)^{\mathrm{univ}} = \mathrm{rank}_{\mathbb{Z}}\, k^{\mathrm{univ}}.$$

2.3. **The structure of the Tate module.** We state the structure of the Tate module of k_n as a $\mathbb{Z}_p[G(k_n/k)]$-module. For a $\mathbb{Z}_p$-module M, we write

$$F(M) := M/\mathrm{tor}(M)$$

for the quotient of M by its torsion $\mathrm{tor}(M)$. Let

$$\overline{U}_\infty(p) = \varprojlim F(U_n(p) \otimes \mathbb{Z}_p)$$

denote the inverse limit of $U_n(p) \otimes \mathbb{Z}_p$ with respect to the norm maps.

The rank of $\overline{U}_\infty(p)$ as a Γ-module is computed in the following theorem which is Theorem 7.2 of [18].

Lemma 2.6. *Let k be a number field and let r_1 and r_2 be the number of real and complex places of k. Then $\overline{U}_\infty(p)$ is a free Γ-module of rank $r_1 + r_2$.*

Proof. See the proof of Theorem 7.2 of [18]. Since the weak Leopoldt conjecture holds for the cyclotomic $\mathbb{Z}_p$-extension of a number field, we can find an another proof from Corollary 10.3.24 and Theorem 11.3.11 of [20].

In fact, the statement (ii) of Theorem 11.3.11 of *loc.cit* holds with $S'_\infty = S^{cd} = \varnothing$ for the cyclotomic $\mathbb{Z}_p$-extension over any number field (cf. Remark and Corollary 11.3.12 of *loc.cit*). □

In Theorem 7.3 of [18], Kuz'min proved an isomorphism

$$\pi : (\overline{U}_\infty(p))_\Gamma \xrightarrow{\cong} F(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$$

which could lead to a proof to the following theorem. We will give here a simple proof using a result of Iwasawa together with an argument of compactness without using the Leopoldt conjecture. The proof is different from Kuz'min's proof.

Lemma 2.7. $(\overline{U}_\infty(p))_\Gamma = F(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$.

Proof. Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module and commutes with Galois actions, we have

$$N_n(k_n^\times) \otimes_{\mathbb{Z}} \mathbb{Z}_p = N_n(k_n^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p)$$

$$(k_n^\times/(\gamma-1)k_n^\times) \otimes \mathbb{Z}_p = (k_n^\times \otimes \mathbb{Z}_p)/(\gamma-1)(k_n^\times \otimes \mathbb{Z}_p).$$

By tensoring $\mathbb{Z}_p$, the exact sequence

$$1 \longrightarrow H^{-1}(G(k_n/k), k_n^\times) \longrightarrow k_n^\times/(\gamma-1)k_n^\times \xrightarrow{N_n} N_n(k_n^\times) \longrightarrow 1$$

leads to

$$1 \longrightarrow H^{-1}(G(k_n/k), k_n^\times) \otimes \mathbb{Z}_p \longrightarrow k_n^\times \otimes \mathbb{Z}_p/(\gamma-1)(k_n^\times \otimes \mathbb{Z}_p) \xrightarrow{N_n \otimes 1} N_n(k_n^\times \otimes \mathbb{Z}_p) \longrightarrow 1.$$

This shows the following isomorphism

$$H^{-1}(G(k_n/k), k_n^\times) \otimes \mathbb{Z}_p \cong H^{-1}(G(k_n/k), k_n^\times \otimes \mathbb{Z}_p).$$

Using Hilbert's Theorem 90 for $k_n^\times$, the above isomorphism leads to

$$\mathrm{Ker}(N_n \otimes 1) = (k_n^\times \otimes \mathbb{Z}_p)^{\gamma-1}.$$

Write

$$V_n = ((\gamma-1)(k_n^\times \otimes \mathbb{Z}_p)) \cap (U_n(p) \otimes \mathbb{Z}_p).$$

By taking inverse limits with respect to the norm maps and using exactness of inverse limit over compact groups, the exact sequence

$$1 \longrightarrow V_n \longrightarrow U_n(p) \otimes \mathbb{Z}_p \longrightarrow U_k(p) \otimes \mathbb{Z}_p \longrightarrow 1$$

leads to

$$1 \longrightarrow \varprojlim_n V_n \longrightarrow \overline{U}_\infty(p) \longrightarrow (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{comp}} \longrightarrow 1. \tag{7}$$

We need the following result of Iwasawa.

Lemma 2.8. *The order* $\#(H^1(G(k_n/k), U_n(p)))$ *is bounded independently of* n.

Proof. This follows immediately from [13]. More precisely, since every primes outside p are unramified over any $\mathbb{Z}_p$-extension k_∞/k, Proposition 3 of [13] and the five term exact sequence of Hochschild-Serre spectral sequence lead to the proof (see Corollary 2.4.2 of [20] and Proposition 13.2 of [23]). □

Notice that from Hilbert's Theorem 90,

$$H^1(G(k_n/k), U_n(p)) \cong H^{-1}(G(k_n/k), U_n(p)) = \frac{((\gamma-1)k_n^\times) \cap U_n(p)}{(\gamma-1)U_n(p)}.$$

For $s \geq n$, one can prove that the identity map induces the following injection

$$1 \longrightarrow \frac{((\gamma-1)k_n^\times) \cap U_n(p)}{(\gamma-1)U_n(p)} \longrightarrow \frac{((\gamma-1)k_s^\times) \cap U_s(p)}{(\gamma-1)U_s(p)}.$$

By Lemma 2.8, we have that for all sufficiently large $s \geq n \gg 0$, the inclusion map induces an isomorphism

$$\frac{((\gamma-1)k_n^\times) \cap U_n(p)}{(\gamma-1)U_n(p)} \xrightarrow{\cong} \frac{((\gamma-1)k_s^\times) \cap U_s(p)}{(\gamma-1)U_s(p)} < \infty.$$

Since $H^{-1}(G(k_n/k), U_n(p)) = (\gamma-1)k_n^\times \cap U_n(p)/(\gamma-1)U_n(p)$ is a p-group and the norm map $N_{s,n}$ is the p^{s-n} power map for all sufficiently large $s \geq n$, we have

$$\varprojlim\left(\left(\frac{(\gamma-1)k_n^\times \cap U_n(p)}{(\gamma-1)U_n(p)}\right) \otimes \mathbb{Z}_p\right) = \varprojlim\left(\frac{(\gamma-1)k_n^\times \cap U_n(p)}{(\gamma-1)U_n(p)}\right) = 1.$$

This results in the following identities

$$\varprojlim((\gamma-1)k_n^\times \cap U_n(p)) = \varprojlim((\gamma-1)U_n(p))$$

and

$$\varprojlim V_n = \varprojlim(((\gamma-1)k_n^\times \cap U_n(p)) \otimes \mathbb{Z}_p) = \varprojlim((\gamma-1)U_n(p) \otimes \mathbb{Z}_p). \tag{8}$$

By taking inverse limits over compact groups

$$1 \longrightarrow U_k(p) \otimes \mathbb{Z}_p \longrightarrow U_n(p) \otimes \mathbb{Z}_p \xrightarrow{\gamma-1} (\gamma-1)U_n(p) \otimes \mathbb{Z}_p \longrightarrow 1$$

we have from (8),

$$(\gamma-1)\overline{U}_\infty(p) \cong \varprojlim((\gamma-1)U_n(p) \otimes \mathbb{Z}_p) = \varprojlim V_n.$$

From (7), we have

$$1 \longrightarrow (\gamma-1)\overline{U}_\infty(p) \longrightarrow \overline{U}_\infty(p) \longrightarrow (U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{comp}} \longrightarrow 1.$$

This completes the proof of Lemma 2.7. □

From Lemmas 2.6 and 2.7, we recover the following result of Kuz'min (cf. [18]).

Corollary 2.9. $(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \cong \mathbb{Z}_p[G(k_n/k)]^{r_1+r_2} \oplus \mathrm{tor}_p(U_n(p))$ *where* $\mathrm{tor}_p(U_n(p))$ *denotes the p-power roots of unity in* $U_n(p)$.

For all $m \geq n$, there exists a natural embedding

$$i_{n,m} : F(U_n(p) \otimes \mathbb{Z}_p)/F(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \to F(U_m(p) \otimes \mathbb{Z}_p)/F(U_m(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

Note that if we choose n_0 such that k_∞/k_{n_0} is totally ramified, then for all $n \geq n_0$,

$$\mathrm{rank}(F(U_n(p) \otimes \mathbb{Z}_p)/F(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}) = r - 1$$

where $r = r_{n_0}$ is the number of primes of k_n dividing p.

In the following, we define $(\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ as the same way as $(\mathcal{E}_k(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ by replacing the base field k with k_n. Write

$$\Omega_n := F(\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}/F(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

By Corollary 2.9, it follows that $i_{n,m}$ is injective and the induced map

$$i_{n,m} : \Omega_n \to \Omega_m$$

is also injective and

$$\mathrm{rank}(\Omega_n) \leq r-1.$$

Hence it follows from Proposition 1.1 that the torsion subgroups $\mathrm{tor}(\Omega_n) = \mathrm{tor}(T_p(k_n)^{\Gamma_n}) = \mathrm{tor}(T_p(k_n))^{\Gamma_n}$ are isomorphic for all sufficiently large n.

Since $F(U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$ is a cohomologically trival $\mathbb{Z}_p[G(k_n/k)]$-module by Corollary 2.8, it follows that the image $\mathrm{im}(i_{n,m})$ of $i_{n,m}$ can be identified with

$$\mathrm{im}(i_{n,m}) \cong H^0(G(k_m/k_n), \Omega_m).$$

We choose a constant p^t such that $\Omega_n^{p^t}$ is torsion free for all $n \geq 0$. Now Γ_n acts trivially on

$$\Omega_m^{p^t} \cap \mathrm{im}(i_{n,m})$$

and hence Γ_n acts trivially on $\Omega_m^{p^t}$ since

$$(\Omega_m^{p^t} : \Omega_m^{p^t} \cap \mathrm{im}(i_{n,m})) < \infty$$

and $\Omega_m^{p^t}$ is a torsion free module. Since

$$\mathrm{cok}(i_{n,m}) = (\Omega_m : \Omega_m^{\Gamma_n}) \leq (\Omega_m : \Omega_m^{p^t}) \leq p^t(r-1)$$

the cokernel $\mathrm{cok}(i_{n,m})$ of $i_{n,m}$ is bounded by a constant for all sufficiently large $m \geq n$.

We hence conclude that for all sufficiently large $m \geq n$, the cokernel of $i_{n,m}$ is trivial and hence $i_{n,m}$ is isomorphic. We obtain the following corollary which is due to Kuz'min (cf. [18]).

Corollary 2.10. *For all sufficiently large $m \geq n$, the imbedding $i_{n,m}$ induces an isomorphism*

$$i_{n,m} : \frac{(\mathcal{E}_{k_n}(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}}{(U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}} \cong \frac{(\mathcal{E}_{k_m}(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}}{(U_m(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}}.$$

Since $\mathcal{E}_{k_n}(p)^{\mathrm{univ}}\otimes\mathbb{Z}_p \subset (\mathcal{E}_{k_n}(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$ and $U_n(p)^{\mathrm{univ}}\otimes\mathbb{Z}_p \subset (U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$, it follows that

$$\mathrm{rank}_{\mathbb{Z}}\frac{U_n(p)}{U_n(p)^{\mathrm{univ}}} = \mathrm{rank}_{\mathbb{Z}_p}(\frac{U_n(p)}{U_n(p)^{\mathrm{univ}}}\otimes\mathbb{Z}_p) \geq \mathrm{rank}_{\mathbb{Z}_p}\frac{U_n(p)\otimes\mathbb{Z}_p}{(U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}}$$

$$\mathrm{rank}_{\mathbb{Z}}\frac{U_n(p)}{\mathcal{E}_{k_n}(p)^{\mathrm{univ}}} \geq \mathrm{rank}_{\mathbb{Z}_p}\frac{U_n(p)\otimes\mathbb{Z}_p}{(\mathcal{E}_{k_n}(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}}.$$

Write $\mathcal{A}_n(m) := N_{m,n}\, U_m(p)$. Then the inverse limit of $\mathcal{A}_n(m)$ with respect to the inclusion maps is equal to

$$\varprojlim_{m\geq n} \mathcal{A}_n(m) = U_n(p)^{\mathrm{univ}}.$$

Similarly, since $\mathcal{A}_n(m)\otimes\mathbb{Z}_p = (N_{m,n}\, U_m(p))\otimes\mathbb{Z}_p = N_{m,n}\,(U_m(p)\otimes\mathbb{Z}_p)$, the inverse limit of $\mathcal{A}_n(m)\otimes\mathbb{Z}_p$ with respect to the inclusion maps is equal to

$$\varprojlim_{m\geq n} (\mathcal{A}_n(m)\otimes\mathbb{Z}_p) = (U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}.$$

By taking inverse limits with respect to the inclusions, the exact sequence

$$1 \longrightarrow \mathcal{A}_n(m)\otimes\mathbb{Z}_p \longrightarrow U_n(p)\otimes\mathbb{Z}_p \longrightarrow (\frac{U_n(p)}{\mathcal{A}_n(m)})\otimes\mathbb{Z}_p \longrightarrow 1$$

leads to

$$1 \longrightarrow (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow U_n(p) \otimes \mathbb{Z}_p \longrightarrow \varprojlim_{m \geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)} \otimes \mathbb{Z}_p) \longrightarrow 1$$

where the inverse limit is exact since $U_n(p) \otimes \mathbb{Z}_p$ is compact.

Since $U_n(p)/\mathcal{A}_n(m)$ is a p-group, the above identity will lead to

$$\varprojlim_{m \geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)}) \cong \varprojlim_{m \geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)} \otimes \mathbb{Z}_p).$$

By taking inverse limits to

$$1 \longrightarrow \mathcal{A}_n(m) \longrightarrow U_n(p) \longrightarrow \frac{U_n(p)}{\mathcal{A}_n(m)} \longrightarrow 1$$

we have the left exact sequence

$$1 \longrightarrow U_n(p)^{\mathrm{univ}} \longrightarrow U_n(p) \longrightarrow \varprojlim_{m \geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)}).$$

Hence it follows from Corollary 2.10 that the following inclusion

$$1 \longrightarrow \frac{U_n(p)}{U_n(p)^{\mathrm{univ}}} \longrightarrow \frac{U_n(p) \otimes \mathbb{Z}_p}{(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

leads to

$$1 \longrightarrow \frac{U_n(p)}{U_n(p)^{\mathrm{univ}}} \longrightarrow \frac{U_m(p)}{U_m(p)^{\mathrm{univ}}}.$$

We proved the following lemma.

Lemma 2.11. *$H^0(G_{m,n}, U_m(p)^{\mathrm{univ}}) = U_n(p)^{\mathrm{univ}}$.*

For $N_{m,n}\mathcal{E}_m(p)^\circ := N_{m,n}\mathcal{E}_m(p) \cap U_n(p)$, we have

$$\varprojlim_{m \geq n} N_{m,n}\mathcal{E}_m(p)^\circ = \varprojlim_{m \geq n} N_{m,n}\mathcal{E}_m(p) = \mathcal{E}_{k_n}(p)^{\mathrm{univ}}$$

$$\varprojlim_{m \geq n} N_{m,n}(\mathcal{E}_m(p)^\circ \otimes \mathbb{Z}_p) = \varprojlim_{m \geq n} N_{m,n}(\mathcal{E}_m(p) \otimes \mathbb{Z}_p) = (\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}.$$

By taking the inverse limits to the following exact sequence

$$1 \longrightarrow N_{m,n}U_m(p) \longrightarrow N_{m,n}\mathcal{E}_m(p)^\circ \longrightarrow \frac{N_{m,n}\mathcal{E}_m(p)^\circ}{N_{m,n}U_m(p)} \otimes \mathbb{Z}_p \longrightarrow 1$$

we have

$$1 \longrightarrow U_n(p)^{\mathrm{univ}} \longrightarrow \mathcal{E}_{k_n}(p)^{\mathrm{univ}} \longrightarrow \varprojlim_{m \geq n} (\frac{N_{m,n}\mathcal{E}_m(p)^\circ}{N_{m,n}U_m(p)} \otimes \mathbb{Z}_p).$$

The last term is isomorphic to

$$\varprojlim_{m \geq n} (\frac{N_{m,n}\mathcal{E}_m(p)^\circ}{N_{m,n}U_m(p)} \otimes \mathbb{Z}_p) \cong \frac{(\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

by taking the inverse limits to the following exact sequence

$$1 \longrightarrow N_{m,n}U_m(p) \otimes \mathbb{Z}_p \longrightarrow N_{m,n}\mathcal{E}_m(p)^\circ \otimes \mathbb{Z}_p \longrightarrow \frac{N_{m,n}\mathcal{E}_m(p)^\circ}{N_{m,n}U_m(p)} \otimes \mathbb{Z}_p \longrightarrow 1.$$

Hence it follows that there is an injective map

$$1 \longrightarrow \frac{\mathcal{E}_{k_n}(p)^{\mathrm{univ}}}{U_n(p)^{\mathrm{univ}}} \longrightarrow \frac{(\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}{(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}}$$

and from Lemma 2.11 that the inclusion induces an injective map

$$1 \longrightarrow \frac{\mathcal{E}_{k_n}(p)^{\mathrm{univ}}}{U_n(p)^{\mathrm{univ}}} \longrightarrow \frac{\mathcal{E}_{k_m}(p)^{\mathrm{univ}}}{U_m(p)^{\mathrm{univ}}}.$$

By Proposition 1.1, this is summarized as

Proposition 2.12. *There is an injective map*

$$1 \longrightarrow \frac{\mathcal{E}_{k}(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{univ}}} \longrightarrow T_p(k)^{\Gamma}.$$

If the Gross conjecture holds for $\{k_n\}_n$, then for all sufficiently large $m \geq n$, the inclusion induces an isomorphism

$$i_{n,m} : \frac{\mathcal{E}_{k_n}(p)^{\mathrm{univ}}}{U_n(p)^{\mathrm{univ}}} \cong \frac{\mathcal{E}_{k_m}(p)^{\mathrm{univ}}}{U_m(p)^{\mathrm{univ}}} < \infty.$$

Remark. As in the proof of Proposition 2.12, we obtain

$$U_n(p) \cap (U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} = U_n(p)^{\mathrm{univ}}$$

$$U_n(p) \cap (\mathcal{E}_{k_n}(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}} = \mathcal{E}_{k_n}(p)^{\mathrm{univ}}.$$

2.4. **Proof of Theorem 1.2.** It is well known that Theorem 1.2 is true for the case $r = 1$. In fact it follows from Corollary 2.9 that the $\mathbb{Z}_p$-ranks of $U_n(p) \otimes \mathbb{Z}_p$ and $(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{univ}}$ are the same. Hence Theorem 1.2 follows immediately from Proposition 1.1 in this case.

One can also prove this without using Proposition 1.1 as follows. Since the Gross conjecture descents, we assume that $k_\infty^{\mathrm{cyc}}/k$ is totally ramified at primes dividing p by replacing k by k_n for sufficiently large n.

Since k_n/k is totally ramified at primes over p, N_n is surjective from class field theory. Hence we have the following exact sequences

$$1 \longrightarrow Cl_S(k_n)^{G_n} \longrightarrow Cl_S(k_n) \xrightarrow{\gamma-1} (\gamma-1)Cl_S(k_n) \longrightarrow 1$$

$$1 \longrightarrow Cl_S(k_n)_{N_n} \longrightarrow Cl_S(k_n) \xrightarrow{N_n} Cl_S(k) \longrightarrow 1.$$

Since $(\gamma - 1)Cl_S(k_n) \subset Cl_S(k_n)_{N_n}$, we have

$$\begin{aligned} \#(Cl_S(k_n)^{G_n}) &= (Cl_S(k_n) : (\gamma-1)Cl_S(k_n)) \\ &= (Cl_S(k_n) : Cl_S(k_n)_{N_n})(Cl_S(k_n)_{N_n} : (\gamma-1)Cl_S(k_n)) \\ &= \#(Cl_S(k)) \cdot (Cl_S(k_n)_{N_n} : (\gamma-1)Cl_S(k_n)). \end{aligned}$$

Let M_n be the maximal unramified abelian extension of k_n such that every primes dividing p split completely in M_n/k_n. Since M_0 and k_n are linearly disjoint over k, we have

$$G(M_0/k) \cong G(k_n M_0/k_n).$$

By class field theory we have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Cl_S(k_n)_{N_n} & \longrightarrow & Cl_S(k_n) & \xrightarrow{N_n} & Cl_S(k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
1 & \longrightarrow & G(M_n/k_nM_0) & \longrightarrow & G(M_n/k_n) & \xrightarrow{rest} & G(M_0/k) & \longrightarrow & 1
\end{array}$$

where the vertical map is the Artin map of class field theory. Hence we have

$$Cl_S(k_n)_{N_n} \cong G(M_n/k_nM_0).$$

On the other hand, we have

$$(\gamma-1)\cdot Cl_S(k_n) \cong (\gamma-1)\cdot G(M_n/k_n) \cong G(M_n/k)'$$

where $G(M_n/k)'$ is the commutator subgroup of $G(M_n/k)$. Let $M_\infty = \bigcup_n M_n$. By taking the inertia group of $G(M_\infty/k)$ into account, we have

$$G(M_\infty/k)' \cong G(M_\infty/k_\infty M_0) \cong (\gamma-1)\cdot G(M_\infty/k_\infty).$$

This proves that for all $n \geq 0$,

$$Cl_S(k_n)_{N_n} \cong (\gamma-1)\cdot Cl_S(k_n)$$

and

$$\#(Cl_S(k_n)^{G_n}) = \#(Cl_S(k)).$$

Hence we have

$$\begin{aligned}
\#(\widehat{H}^0(G_n, Cl_S(k_n))) &= \#Cl_S(k_n)^{G_n}/\#(tr_n Cl_S(k_n)) \\
&= \#(Cl_S(k))/\#(j_n Cl_S(k)) \\
&= \#\ker(j_n : Cl_S(k) \to Cl_S(k_n)).
\end{aligned}$$

This shows that the zero-dimensional Tate cohomology of the S-ideal class group is bounded independently of n. By ii) of §2.1, we complete the proof for the case $r = 1$.

For the case $r = 2$, we begin with a basic lemma which is due to Kuz'min (see Proposition 8.2 of *loc.cit*).

Lemma 2.13. *$U_k(p)\otimes\mathbb{Z}_p/(\mathcal{E}_k(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$ is a torsion free module.*

Proof. Note that $U_k(p)\otimes\mathbb{Z}_p$ imbeds into $J_{k,S}\otimes\mathbb{Z}_p$ via diagonal map. Let α be an element of $U_k(p)\otimes\mathbb{Z}_p$ such that $\alpha^{p^s} \in (\mathcal{E}_k(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$.

If we identify $U_k(p)\otimes\mathbb{Z}_p$ with its image via the injective map of

$$\widehat{\Delta'} : U_k(p)\otimes\mathbb{Z}_p \longrightarrow \prod_{v|p}\widehat{k}_v^\times \times \prod_{v\nmid p\infty}\widehat{U}_v \subset \prod_{v\neq\infty}\widehat{k}_v^\times = (\prod_{v\neq\infty}k_v^\times)^\wedge$$

then it follows that $\beta \in (\mathcal{E}_k(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$ if and only if $\beta \in \widehat{U}_v$ for all $v \nmid p$ and $\beta \in \widehat{S}_v \subset \widehat{k}_v^\times$ for all $v|p$ where S_v is the group of universal norm elements of $k_v^\times$. Since

$$\widehat{k}_v^\times/\widehat{S}_v \cong k_v^\times/S_v \cong \mathbb{Z}_p$$

is torsion free module, it follows that $\alpha^{p^s} \in (\mathcal{E}_k(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$ if and only if $\alpha \in (\mathcal{E}_k(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}}$. □

From now on, we assume that $r = 2$, i.e., there are two primes of k dividing p over the cyclotomic $\mathbb{Z}_p$-extension of k. As in the assumption of Theorem 1.2, we assume further that k is a Galois extension containing μ_p such that $k_\infty^{\mathrm{cyc}}/k$ is totally ramified at primes dividing p.

It follows from the previous subsections that

$$\mathrm{rank}_{\mathbb{Z}_p}(U_n(p)\otimes\mathbb{Z}_p) = r_2(n)+1, \ \ \mathrm{rank}_{\mathbb{Z}_p}(U_n(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}} = r_2(n)$$

where $r_2(n)$ denotes the number of complex places of k_n.

If we negate the Gross conjecture for k, then since the Gross conjecture descents, Lemma 2.13 shows that for all $n \geq 0$,

$$U_n(p)\otimes\mathbb{Z}_p = (\mathcal{E}_{k_n}(p)\otimes\mathbb{Z}_p)^{\mathrm{univ}} = (k_n^\times\otimes\mathbb{Z}_p)^{\mathrm{univ}} \tag{9}$$

which is of rank $r_2(n)+1$.

The following theorem is due to Bertrandias and Payan (see Theorem 2 of [3]).

Theorem 2.14. *If K is a number field containing μ_p such that $K/\mathbb{Q}$ is Galois and the primes of K dividing p have the same decomposition field $Dec_p(K)$, then*

$$\mathrm{rank}_{\mathbb{Z}}\, K^{\mathrm{univ}} = r_2 - r + 1 \ \ \textit{or} \ \ r_2 - r/2 + 1$$

according as $Dec_p(K)$ is real or imaginary respectively and $K^{\mathrm{univ}}/\mathrm{tor}(U_K(p))$ is a direct factor of $U_K(p)/\mathrm{tor}(U_K(p))$.

Note that for $r = 2$, the hypotheses of Theorem 2.14 are automatically satisfied for a Galois extension $K/\mathbb{Q}$ since $Dec_p(K)$ is a quadratic extension (see Remark (b) following Theorem 2 of *loc.cit*).

Write $\mathcal{A}_n(m) := N_{m,n}\, k_m^\times \cap U_n(p)$. Then the inverse limit of $\mathcal{A}_n(m)$ with respect to the inclusion maps is equal to

$$\varprojlim_{m\geq n} \mathcal{A}_n(m) = k_n^{\mathrm{univ}}.$$

Similarly, since $\mathcal{A}_n(m)\otimes\mathbb{Z}_p = (N_{m,n}\, k_m^\times \cap U_n(p))\otimes\mathbb{Z}_p = (N_{m,n}\,(k_m^\times\otimes\mathbb{Z}_p))\cap(U_n(p)\otimes\mathbb{Z}_p)$, the inverse limit of $\mathcal{A}_n(m)\otimes\mathbb{Z}_p$ with respect to the inclusion maps is equal to

$$\varprojlim_{m\geq n} (\mathcal{A}_n(m)\otimes\mathbb{Z}_p) = (k_n^\times\otimes\mathbb{Z}_p)^{\mathrm{univ}}.$$

By taking inverse limits with respect to the inclusions, the exact sequence

$$1 \longrightarrow \mathcal{A}_n(m)\otimes\mathbb{Z}_p \longrightarrow U_n(p)\otimes\mathbb{Z}_p \longrightarrow (\frac{U_n(p)}{\mathcal{A}_n(m)})\otimes\mathbb{Z}_p \longrightarrow 1$$

leads to

$$1 \longrightarrow (k_n^\times\otimes\mathbb{Z}_p)^{\mathrm{univ}} \longrightarrow U_n(p)\otimes\mathbb{Z}_p \longrightarrow \varprojlim_{m\geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)}\otimes\mathbb{Z}_p) \longrightarrow 1 \tag{10}$$

since $U_n(p)\otimes\mathbb{Z}_p$ is compact.

It follows from (9) and (10) that

$$\varprojlim_{m\geq n} (\frac{U_n(p)}{\mathcal{A}_n(m)}\otimes\mathbb{Z}_p) = 1.$$

It follows from $U_n(p) \supset \mathcal{A}_n(m) \supset U_n(p)^{p^{m-n}}$ that $U_n(p)/\mathcal{A}_n(m)$ is a p-group.

The above identity will lead to

$$\varprojlim_{m\geq n}\,(\frac{U_n(p)}{\mathcal{A}_n(m)}) \cong \varprojlim_{m\geq n}\,(\frac{U_n(p)}{\mathcal{A}_n(m)}\otimes \mathbb{Z}_p) = 1.$$

Hence by taking inverse limits to

$$1 \longrightarrow \mathcal{A}_n(m) \longrightarrow U_n(p) \longrightarrow \frac{U_n(p)}{\mathcal{A}_n(m)} \longrightarrow 1$$

we have the left exact sequence

$$1 \longrightarrow k_n^{\text{univ}} \longrightarrow U_n(p) \longrightarrow \varprojlim_{m\geq n}\,(\frac{U_n(p)}{\mathcal{A}_n(m)}).$$

Note that since the $\mathbb{Z}$-rank of $\mathbb{Z}_p$ is infinite, we can not compare the $\mathbb{Z}$-ranks in the above left exact sequence unless the last term which is a $\mathbb{Z}_p$-module is finite.

In our setting, since the last term is trivial, we have the following identity

$$k_n^{\text{univ}} = U_n(p).$$

It follows from Theorem 2.14 that the identity above leads to the following contradiction

$$r_2(n) \ \geq \ \text{rank}_{\mathbb{Z}}\, k_n^{\text{univ}} = \text{rank}_{\mathbb{Z}}\, U_n(p) = r_2(n) + 1.$$

This completes the proof of Theorem 1.2.

Corollary 2.15. *Under the same condition of Theorem 1.2, we have for all $n \geq 0$,*

$$\text{rank}_{\mathbb{Z}}\, \mathcal{E}_{k_n}^{\text{univ}} = \text{rank}_{\mathbb{Z}}\, U_n(p)^{\text{univ}}.$$

3. Appendix

Let G be a locally compact Hausdorff topological group. Let H be a closed normal subgroup of G. Then H and G/H are locally compact (see Theorem 7 and Theorem 11 of §§ 23, 24 of [12]). We state basic lemmas of topological group which is well known. The following lemma is Corollary 2 of § 3.3, Ch III of [3].

Lemma 3.1. *If H is any locally compact subgroup of a Hausdorff topological group G then H is closed in G.*

The following lemma is (G) of § 20 of [21].

Lemma 3.2. *Let G be a locally compact topological group and let G be represented as the set theoretic union of a countable number of compact subsets. Let A be a closed subgroup and B be a closed normal subgroup of G. If AB is a closed subgroup of G, then we have the following isomorphism*

$$AB/A \cong B/B\cap A$$

The idele group J_k is a locally compact Hausdorff topological group. By Lemma 3.1, $k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v$ is closed and hence

$$\begin{aligned} k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v = \overline{k^\times \prod_{v\nmid p} U_v \prod_{v|p} \widetilde{k}_v} &= \overline{k^\times U_\infty}\cdot \overline{\prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v} \\ &= \overline{k^\times U_\infty}\cdot \prod_{v\nmid p\infty} U_v \prod_{v|p} \widetilde{k}_v. \end{aligned}$$

By Lemma 3.2, we have the following isomorphisms which was used in the proof of Proposition 1.1

$$\frac{\overline{k^\times U_\infty}\cdot\overline{\prod_{v\nmid p\infty}U_v\prod_{v|p}\widetilde{k}_v}}{\overline{k^\times U_\infty}}\cong\frac{\overline{\prod_{v\nmid p\infty}U_v\prod_{v|p}\widetilde{k}_v}}{\overline{\prod_{v\nmid p\infty}U_v\prod_{v|p}\widetilde{k}_v}\cap\overline{k^\times U_\infty}}.$$

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