ON NUMERICAL EQUIVALENCE FOR ALGEBRAIC COBORDISM

ANANDAM BANERJEE AND JINHYUN PARK

Abstract. We define and study the notion of numerical equivalence on algebraic cobordism cycles. We prove that algebraic cobordism modulo numerical equivalence of a smooth projective variety is a finitely generated module over the Lazard ring, and it reproduces the Chow group modulo numerical equivalence.

We compare it with homological equivalence and smash-equivalence for cobordism cycles, using Kimura finiteness on cobordism motives. We partially resolve the cobordism analogue of a conjecture by Voevodsky on smash-equivalence and numerical equivalence.

1. Introduction

In the theory of algebraic cycles, various adequate equivalences on them play essential roles (see [1, Sections 3.1–2]). Now, the theory of algebraic cobordism, as pioneered by Levine and Morel in [9] (see also [10] and Section 2) allows one to study algebraic cycles and motives from a more general perspective. It is an interesting question to ask whether various adequate equivalences for algebraic cycles can be lifted up to the level of algebraic cobordism cycles. One such attempt for algebraic equivalence was made in [7] so that one obtained the theory $\Omega^*_{\mathrm{alg}}(-)$ of algebraic cobordism modulo algebraic equivalence.

The objective of the present work is to define and study the notion of numerical equivalence for algebraic cobordism cycles. In the classical situation of algebraic cycles, it has a long history and was the ground of "enumerative geometry". In simple words, this is based on counting the numbers of intersection points of two varieties, with suitable multiplicities. For cobordism cycles, this naive notion of counting does not work and this requires a bit of care.

This trouble can be overcome by treating the graded ring $\mathbb{L}$ (called the Lazard ring; see Section 2), as a substitute for the ring $\mathbb{Z}$ of integers; instead of naively counting the number of intersection points, we "count" the $\mathbb{L}$-"intersection" values of two cobordism cycles. Notice that such an idea of using $\mathbb{L}$ systematically instead of $\mathbb{Z}$ is not new. In [9], $\mathbb{L}$ is the cobordism ring $\Omega^*(pt)$ of a point. In [7], it was proved that the Griffiths group is a finitely generated abelian group if and only if the cobordism Ω^*_{alg} is a finitely generated $\mathbb{L}$-module. Furthermore, the functor $-\otimes_{\mathbb{L}}\mathbb{Z}$ carries the cobordism groups to Chow groups with the respective adequate equivalences.

Using the product in cobordism rings with values in $\mathbb{L}$, we define numerical equivalence for algebraic cobordism cycles, and algebraic cobordism modulo numerical equivalence in Section 3. We first prove the following results:

Theorem 1.1. *Let X be a smooth projective variety over a field k of characteristic zero. Let $\Omega^*_{\mathrm{num}}(X)$ be the algebraic cobordism of X modulo numerical equivalence. Then,*

(1) *there is an isomorphism $\Omega^*_{\mathrm{num}}(X)\otimes_{\mathbb{L}}\mathbb{Z}\simeq \mathrm{CH}^*_{\mathrm{num}}(X)$, where $\mathrm{CH}^*_{\mathrm{num}}(X)$ is the Chow group of X modulo numerical equivalence, and*

2010 *Mathematics Subject Classification.* Primary 14F43; Secondary 55N22.

Key words and phrases. algebraic cobordism, algebraic cycle, numerical equivalence.

(2) $\Omega^*_{\mathrm{num}}(X)$ *is a finitely generated* $\mathbb{L}$*-module.*

In case of algebraic cycles, to each choice of a Weil cohomology theory, there corresponds a homological equivalence. Its independence from the choice of a Weil cohomology theory is conjectured, but not known. For cobordism cycles, a notion of homological equivalence was defined in [7] using complex cobordism when X is a smooth variety over $\mathbb{C}$. On the other hand, one can also use the étale cobordism of Quick [14] to give another notion of homological equivalence for varieties over more general base fields. In Section 4, we prove the following:

Theorem 1.2. *Let X be a smooth projective variety over k.*

(1) *Let k be a field with an embedding into $\mathbb{C}$. Considering the homological equivalence given by the complex cobordism* MU*, a homologically trivial cobordism cycle is also numerically trivial.*
(2) *Let k be any field of characteristic zero. Considering the homological equivalence given by the étale cobordism* $\mathrm{MU}_{ét}$ *of Quick, a homologically trivial cobordism cycle is also numerically trivial.*

The notion of smash nilpotence of algebraic cycles was considered by Voevodsky [21] and Voisin [22]. It was extended to cobordism cycles in [7, Section 10], where it was proved that algebraically trivial cobordism cycles are smash nilpotent. For algebraic cycles, Voevodsky [21, Conjecture 4.2] conjectures that an algebraic cycle is smash nilpotent if and only if it is numerically trivial. The forward direction is known ([21, Corollary 3.3] and [22, Lemma 2.3]). Recently, Sebastian [17] proved part of the Voevodsky conjecture in the backward direction for 1-cycles on smooth projective varieties dominated by products of curves. The question on the cobordism analogue of Voevodsky conjecture was first raised in [7, Remark 10.5]. We prove the forward direction in Section 5, and the cobordism analogue of the result of Sebastian is answered in Section 7. In the process, we develop the notion of Kimura finiteness on cobordism motives in Section 6. Here is a summary:

Theorem 1.3. *Let X be a smooth projective variety over k of characteristic* 0 *and let α be a cobordism cycle on X.*

(1) *If k has an embedding into $\mathbb{C}$ and α is smash nilpotent, then it is homologically trivial.*
(2) *If α is smash nilpotent, then it is numerically trivial.*
(3) *If $k = \bar{k}$ and X is dominated by a product of curves, then any numerically trivial cobordism* 1*-cycle α is smash nilpotent.*

2. Recollection of algebraic cobordism theories

We recall some definitions on algebraic cobordism from [9, Sections 2.1–4]. Let X be a k-scheme of finite type.

Definition 2.1 ([9, Definition 2.1.6])**.** A *cobordism cycle* over X is a family $(f : Y \to X, L_1, \ldots, L_r)$, where Y is smooth and integral, f is projective, and $(L_1, \ldots, L_r)$ is a finite sequence of $r \geq 0$ line bundles over Y. Its dimension is defined to be $\dim(Y) - r \in \mathbb{Z}$. An isomorphism Φ of cobordism cycles $(Y \to X, L_1, \ldots, L_r) \overset{\sim}{\to} (Y' \to X, L'_1, \ldots, L'_r)$ is a triple $\Phi = (\varphi : Y \to Y', \sigma, (\psi_1, \ldots, \psi_r))$ consisting of an isomorphism $\varphi : Y \to Y$ of X-schemes, a bijection $\sigma : \{1, \ldots, r\} \overset{\sim}{\to} \{1, \ldots, r\}$, and isomorphisms $\psi_i : L_i \overset{\sim}{\to} \varphi^* L'_{\sigma(i)}$ of lines bundles over Y for all i.

Let $\mathcal{Z}(X)$ be the free abelian group on the set of isomorphism classes of cobordism cycles over X. Grading by the dimension of cobordism cycles makes $\mathcal{Z}_*(X)$ into a graded abelian group. The image of a cobordism cycle $(Y \to X, L_1, \ldots, L_r)$ in $\mathcal{Z}_*(X)$ is denoted by $[Y \to X, L_1, \ldots, L_r]$. When X is smooth and equidimensional, the class $[\mathrm{Id}_X : X \to X] \in \mathcal{Z}_d(X)$ is denoted as 1_X.

Definition 2.2 ([9, Sections 2.1.2–3])**.**

(1) For a projective morphism $g : X \to X'$ in $\mathbf{Sch}_k$, composition with g defines the graded group homomorphism $g_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(X')$ given by $[f : Y \to X, L_1, \ldots, L_r] \mapsto [g \circ f : Y \to X', L_1, \ldots, L_r]$. g_* is called the *push-forward along* g.

(2) If $g : X \to X'$ is a smooth equidimensional morphism of relative dimension d, the *pull-back along* g is defined to be the homomorphism $g^* : \mathcal{Z}_*(X') \to \mathcal{Z}_{*+d}(X)$, $[f : Y \to X', L_1, \ldots, L_r] \mapsto [pr_2 : Y \times_{X'} X \to X, pr_1^*(L_1), \ldots, pr_1^*(L_r)]$.

(3) Let L be a line bundle on X. The homomorphism $\widetilde{c}_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X)$ defined by $[f : Y \to X, L_1, \ldots, L_r] \mapsto [f : Y \to X, L_1, \ldots, L_r, f^*(L)]$ is called the *first Chern class operator of* L. If X is smooth, the first Chern class $c_1(L)$ of L is defined to be the cobordism cycle $c_1(L) := \widetilde{c}_1(L)(1_X)$.

(4) The *external product* $\times : \mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \to \mathcal{Z}_*(X \times Y)$ on the functor $\mathcal{Z}_*$ is defined by $[f : X' \to X, L_1, \ldots, L_r] \times [g : Y' \to Y, M_1, \ldots, M_s] \mapsto [f \times g : X' \times Y' \to X \times Y, pr_1^*(L_1), \ldots, pr_1^*(L_r), pr_2^*(M_1), \ldots, pr_2^*(M_s)]$.

While for the Chow ring, $c_1(L \otimes M) = c_1(L) + c_1(M)$, this is not true in general for oriented cohomology theories (see [9, Definition 1.1.2]) and addition has to be replaced by a formal group law: $c_1(L \otimes M) = F(c_1(L), c_1(M))$ for some power series F in two variables. A commutative formal group law (R, F_R) of rank 1 consists of a ring R and $F_R \in R[[u, v]]$ satisfying conditions analogous to the operations in a group. In [8], Lazard showed that there exists a formal group law $(\mathbb{L}, F_{\mathbb{L}})$ of rank 1 which is universal: for any other law (R, F_R) there exists a unique morphism $\Phi_{(R,F_R)} : \mathbb{L} \to R$ which maps the coefficients of $F_{\mathbb{L}}$ onto those of F_R. The ring $\mathbb{L}$, called the *Lazard ring*, is isomorphic to the polynomial ring $\mathbb{Z}[a_i | i \geq 1]$, and can be made into a graded ring $\mathbb{L}_*$ by assigning $\deg a_i = i$. See [9, Section 1.1] for details.

Definition 2.3 ([9, Definitions 2.4.5, 2.4.10])**.** For $X \in \mathbf{Sch}_k$, algebraic cobordism $\Omega_*(X)$ is defined to be the quotient of $\mathcal{Z}_*(X) \otimes \mathbb{L}_*$ by the following three relations:
(Dim) If there is a smooth quasi-projective morphism $\pi : Y \to Z$ with line bundles $M_1, \ldots, M_{s>\dim Z}$ on Z with $L_i \xrightarrow{\sim} \pi^* M_i$ for $i = 1, \ldots, s \leq r$, then $[f : Y \to X, L_1, \ldots, L_r] = 0$.
(Sect) For a section $s : Y \to L$ of a line bundle L on Y with the associated smooth divisor $i : D \to Y$, we impose $[f : Y \to X, L_1, \ldots, L_r, L] = [f \circ i : D \to X, i^* L_1, \ldots, i^* L_r]$.
(FGL) For line bundles L and M on X, we impose the equality $F_{\mathbb{L}}(\widetilde{c}_1(L), \widetilde{c}_1(M))([f : Y \to X, L_1, \ldots, L_r]) = \widetilde{c}_1(L \otimes M)([f : Y \to X, L_1, \ldots, L_r])$. By the relation (Dim), the expression $F_{\mathbb{L}}(\widetilde{c}_1(L), \widetilde{c}_1(M))$ is a finite sum so that the operator is well-defined.

When X is smooth and equidimensional of dimension n, the codimension of a cobordism d-cycle is defined to be $n - d$. We set $\Omega^{n-d}(X) := \Omega_d(X)$, and $\Omega^*(X)$ is the direct sum of the groups over all codimensions. Levine and Morel showed that algebraic cobordism Ω^* is a universal oriented cohomology theory on $\mathbf{Sm}_k$ (see [9, Definition 1.1.2]):

Theorem 2.4 ([9, Theorem 1.2.6])**.** *Assume k has characteristic 0. Then, given any oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism $\nu_A : \Omega^* \to A^*$ of oriented cohomology theories.*

3. Numerical equivalence on cobordism cycles

Let X be a smooth projective variety over a field k of characteristic 0. Consider the composition of maps

$$\Omega_*(X) \otimes \Omega_*(X) \xrightarrow{\times} \Omega_*(X \times X) \xrightarrow{\Delta_X^*} \Omega_*(X) \xrightarrow{\pi_*} \Omega_*(k), \tag{3.1}$$

where $\times$ is the external product of cobordism cycles, Δ_X is the diagonal morphism, and π is the structure morphism $X \to \mathrm{Spec}\,(k)$. This gives a map of $\mathbb{L}$-modules $\Omega_*(X) \longrightarrow \mathrm{Hom}_{\mathbb{L}}(\Omega_*(X), \Omega_*(k))$.

Definition 3.1. We say that a cobordism cycle is *numerically equivalent to* 0 if it is in the kernel of this map, $\mathcal{N}_*(X) := \ker(\Omega_*(X) \to \mathrm{Hom}_{\mathbb{L}}(\Omega_*(X), \Omega_*(k)))$, and we let $\Omega_*^{\mathrm{num}}(X) := \Omega_*(X)/\mathcal{N}_*(X)$, which is algebraic cobordism modulo numerical equivalence.

If $\alpha \in \mathcal{N}_*(X)$, using the fact that φ commutes with pull-backs and push-forwards and respects the product on the Chow group, one easily checks that $\varphi : \Omega_*(X) \to \mathrm{CH}_*(X)$ maps α to an algebraic cycle numerically equivalent to 0. This gives a well-defined map $\varphi^{\mathrm{num}} : \Omega_*^{\mathrm{num}}(X) \longrightarrow \mathrm{CH}_*^{\mathrm{num}}(X)$.

Recall that φ factors through the canonical morphism $\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\overline{\varphi}} \mathrm{CH}_*(X)$, which is an isomorphism of Borel-Moore weak homology theories on $\mathbf{Sch}_k$ (see [9, Definition 4.1.9]). Thus, $\varphi = \overline{\varphi} \circ \tau$ where $\tau : \Omega_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z}$ is the extension of scalars via $\Phi_a : \mathbb{L} \to \mathbb{Z}$. This shows that φ is surjective.

Theorem 3.2. *Let X be a smooth projective variety over a field k of characteristic 0.*

1. *φ^{num} induces an isomorphism $\overline{\varphi}^{\mathrm{num}} : \Omega_*^{\mathrm{num}}(X) \otimes_{\mathbb{L}} \mathbb{Z} \simeq \mathrm{CH}_*^{\mathrm{num}}(X)$.*
2. *$\Omega_*^{\mathrm{num}}(X)$ is a finitely generated $\mathbb{L}$-module.*

Proof. (1) Let $\beta \in \mathrm{CH}_*^{\mathrm{num}}(X)$ and let β' be a lift in $\mathrm{CH}_*(X)$. Since φ is surjective, $\beta' = \varphi(\alpha)$ for some $\alpha \in \Omega_*(X)$ whose image $\overline{\alpha}$ in $\Omega_*^{\mathrm{num}}(X)$ maps to β under φ^{num}. Thus, φ^{num} is surjective. Now, we look at the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}_*(X) & \longrightarrow & \Omega_*(X) & \longrightarrow & \Omega_*^{\mathrm{num}}(X) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \varphi^{\mathrm{num}}} & & \\ 0 & \longrightarrow & \mathrm{Num}_*(X) & \longrightarrow & \mathrm{CH}_*(X) & \longrightarrow & \mathrm{CH}_*^{\mathrm{num}}(X) & \longrightarrow & 0, \end{array} \tag{3.2}$$

where $\mathrm{Num}_*(X)$ denotes the group of numerically trivial algebraic cycles. Now, we find the kernel of φ^{num}.

Let $\overline{\alpha} \in \Omega_*^{\mathrm{num}}(X)$ be the image of $\alpha \in \Omega_*(X)$. If $\varphi^{\mathrm{num}}(\overline{\alpha}) = 0$, then $\varphi(\alpha) \in \mathrm{Num}_*(X)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \Omega_*(X) & \xrightarrow{\pi_*} & \Omega_*(k) \\ {\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_a} \\ \mathrm{CH}_*(X) & \xrightarrow[\pi_*]{} & \mathrm{CH}_*(k). \end{array} \tag{3.3}$$

For any $\gamma \in \Omega_*(X)$,

$$\Phi_a(\pi_*(\alpha \cdot \gamma)) = \pi_*(\varphi(\alpha) \cdot \varphi(\gamma)) = 0 \Rightarrow \pi_*(\alpha \cdot \gamma) \in \mathbb{L}_{>0},$$

since φ is a ring homomorphism. $\mathcal{N}_*(X) + \mathbb{L}_{>0} \cdot \Omega_*(X)$ being an ideal in $\Omega_*(X)$, we may write $\alpha \in \alpha_1 + \mathcal{N}_*(X) + \mathbb{L}_{>0} \cdot \Omega_*(X)$. Clearly, for any $\gamma \in \Omega_*(X)$, $\pi_*(\alpha_1 \cdot \gamma) \in \mathbb{L}_{>0}$. We may assume α_1 is a homogenous element. Let $\alpha_1 \in \Omega_i(X)$. If $i \leq n = \dim_k X$, pick any non-zero $\gamma \in \Omega^i(X)$. Then, $\alpha_1 \cdot \gamma \in \Omega_0(X) \Rightarrow \pi_*(\alpha_1 \cdot \gamma) \in \Omega_0(k) \xrightarrow{\sim} \mathbb{L}_0 \Rightarrow \pi_*(\alpha_1 \cdot \gamma) =$

$0 \Rightarrow \alpha_1 \in \mathcal{N}_*(X) \Rightarrow \alpha_1 = 0$. If $i > n$, then by the generalized degree formula (see [9, Theorem 4.4.7]), $\alpha_1 \in \mathbb{L}_{>0} \cdot \Omega_*(X) \Rightarrow \alpha_1 = 0$. This shows that $\alpha \in \mathcal{N}_*(X) + \mathbb{L}_{>0} \cdot \Omega_*(X)$. Going modulo $\mathcal{N}_*(X)$, we have $\overline{\alpha} \in \mathbb{L}_{>0} \cdot \Omega_*^{\mathrm{num}}(X)$. Thus, $\ker(\varphi^{\mathrm{num}}) = \mathbb{L}_{>0} \cdot \Omega_*^{\mathrm{num}}(X)$.

Hence, φ^{num} induces an isomorphism $\overline{\varphi}^{\mathrm{num}} : \Omega_*^{\mathrm{num}}(X) \otimes_{\mathbb{L}} \mathbb{Z} \xrightarrow{\sim} \mathrm{CH}_*^{\mathrm{num}}(X)$, since $\Omega_*^{\mathrm{num}}(X)/\mathbb{L}_{>0} \cdot \Omega_*^{\mathrm{num}}(X) \xrightarrow{\sim} \Omega_*^{\mathrm{num}}(X) \otimes_{\mathbb{L}} \mathbb{Z}$.

(2) This follows easily from [7, Lemma 9.8] since $\mathrm{CH}^*_{\mathrm{num}}(X)$ is finitely generated. In fact, $\Omega^*_{\mathrm{num}}(X)$ is generated by any set of elements that map via φ to a set of generators of $\mathrm{CH}^*_{\mathrm{num}}(X)$. □

Let X be a smooth irreducible k-scheme. The degree of a cobordism cycle on X has been defined in [9, Definition 4.4.4] to be a homomorphism $\deg : \Omega_*(X) \to \Omega_{*-\dim_k X}(X)$.

Proposition 3.3. *The degree of a numerically trivial cobordism cycle on X is zero.*

Proof. Let $\alpha \in \mathcal{N}_*(X)$. By the generalized degree formula (see [9, Theorem 4.4.7]), for each closed integral subscheme $Z \subset X$, we have a projective birational morphism $\widetilde{Z} \to Z$ with $\widetilde{Z}$ in $\mathbf{Sm}_k$ and an element $\omega_Z \in \Omega_{*-\dim_k Z}(k)$, all but finitely many being zero, such that

$$\alpha = \deg(\alpha)[\mathrm{Id}_X] + \sum_{Z,\, \mathrm{codim}_X Z > 0} \omega_Z \cdot [\widetilde{Z} \to X].$$

By the definition of numerical triviality, for any $\gamma \in \Omega_*(X)$, $\pi_*(\alpha \cdot \gamma) = 0$. Thus, $\deg(\alpha)\pi_*(\gamma) + \sum_{Z,\, \mathrm{codim}_X Z > 0} \omega_Z \cdot \pi_*([\widetilde{Z} \to X] \cdot \gamma) = 0$. In particular, choose a $\gamma \in \Omega_0(X)$ such that $\pi_*(\gamma) \neq 0$. Then, $[\widetilde{Z} \to X] \cdot \gamma = 0$ for all Z with $\mathrm{codim}_X Z > 0$. Since $\pi_*(\gamma) \neq 0$, we must have $\deg(\alpha) = 0$. □

4. Homological equivalence

In this section, we show that homologically trivial cobordism cycles are numerically trivial. Here, homological equivalence is considered for the complex cobordism MU and the étale cobordism $\widehat{\mathrm{MU}}_{\text{ét}}$.

4.1. **Complex cobordism.** Let X be a smooth projective variety over k. In [15], Quillen defined a notion of complex oriented cohomology theories on the category of differentiable manifolds and showed that the complex cobordism theory $X \to \mathrm{MU}^*(X)$ can be interpreted as the universal complex oriented cohomology theory. For any embedding $\sigma : k \hookrightarrow \mathbb{C}$, we have a canonical morphism of graded rings $\Phi^{\mathrm{top}} : \Omega^*(X) \longrightarrow \mathrm{MU}^{2*}(X_\sigma(\mathbb{C}))$ by the universality of algebraic cobordism. In [7], $\ker(\Phi^{\mathrm{top}})$ is defined to be the group of cobordism cycles *homologically equivalent to* 0.

Theorem 4.1. *Let X be a smooth projective variety over a field k with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Under the above definition, a homologically trivial cobordism cycle is numerically trivial.*

Proof. Let $\pi : X \to \operatorname{Spec} k$ be the structure map. The map Φ^{top} is a ring homomorphism and commutes with push-forwards. Thus, we have the following commutative square:

$$\begin{tikzcd} \Omega^*(X) \arrow[r, "\pi_*"] \arrow[d, "\Phi^{\mathrm{top}}"'] & \Omega^{*+d}(k) \arrow[d, "\Phi^{\mathrm{top}}", "\wr"'] \\ \mathrm{MU}^{2*}(X_\sigma(\mathbb{C})) \arrow[r, "\pi_*"] & \mathrm{MU}^{2*+2d}(pt), \end{tikzcd}$$

where $d := \dim_k X$. By [9, Corollary 1.2.11.(1)], $\Phi^{\mathrm{top}} : \Omega^*(k) \to \mathrm{MU}^{2*}(pt)$ is an isomorphism. Now let $\alpha \in \Omega^*(X)$ be homologically trivial, *i.e.* $\Phi^{\mathrm{top}}(\alpha) = 0$. For any

$\gamma \in \Omega^*(X)$, we have $\Phi^{\mathrm{top}}(\alpha\cdot\gamma) = \Phi^{\mathrm{top}}(\alpha)\cdot\Phi^{\mathrm{top}}(\gamma) = 0$. Since the square above commutes, $\Phi^{\mathrm{top}}(\pi_*(\alpha\cdot\gamma)) = \pi_*(\Phi^{\mathrm{top}}(\alpha\cdot\gamma)) = 0$. $\Phi^{\mathrm{top}} : \Omega^*(k) \to \mathrm{MU}^{2*}(pt)$ being an isomorphism, $\pi_*(\alpha\cdot\gamma) = 0$, showing that α is numerically trivial. □

4.2. **Etale cobordism.** Let ℓ be a prime. Quick defined a notion of étale cobordism in [14, Definition 4.11] on $\mathbf{Sm}_k$ and showed that the étale cobordism $\mathrm{MU}^{2*}_{\text{ét}}(-;\mathbb{Z}/\ell)$ is an oriented cohomology theory in [14, Theorem 5.7]. Hence, there is a canonical morphism of oriented cohomology theories $\theta : \Omega^*(X) \to \Omega^*(X)\otimes_{\mathbb{Z}}\mathbb{Z}/\ell \to \mathrm{MU}^{2*}_{\text{ét}}(X;\mathbb{Z}/\ell)$. He also showed that for a separably closed field k of characteristic 0, the morphism $\Omega^*(k;\mathbb{Z}/\ell^\nu) \to \mathrm{MU}^{2*}_{\text{ét}}(k;\mathbb{Z}/\ell^\nu)$ is an isomorphism. This implies that we have an isomorphism $\Omega^*(k;\mathbb{Z}_\ell) \xrightarrow{\theta_\ell} \widehat{\mathrm{MU}}^{2*}_{\text{ét}}(k;\mathbb{Z}_\ell)$, where $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers. Thus, we have the following commutative diagram:

(4.1)
$$\begin{tikzcd}
\Omega^*(X) \arrow[r, "\pi_*"] \arrow[d] \arrow[dd, bend right=60, "\theta"'] & \Omega^{*+d}(k) \arrow[d, "\eta_\ell"] \arrow[dd, bend left=60, "\theta"] \\
\Omega^*(X)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \arrow[r, "\pi_*"] \arrow[d] & \Omega^{*+d}(k)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \arrow[d, "\theta_\ell", "\wr"'] \\
\widehat{\mathrm{MU}}^{2*}_{\text{ét}}(X;\mathbb{Z}_\ell) \arrow[r, "\pi_*"] & \widehat{\mathrm{MU}}^{2*+2d}_{\text{ét}}(k;\mathbb{Z}_\ell)
\end{tikzcd}$$

Note that the natural map η_ℓ is injective. We may use étale cobordism $\widehat{\mathrm{MU}}_{\text{ét}}$ to define another homological equivalence of cobordism cycles. A cobordism cycle is said to be homologically trivial with respect to $\widehat{\mathrm{MU}}_{\text{ét}}$ if it is in $\ker(\theta)$. As in the case of complex cobordism, we prove

Theorem 4.2. *Let X be a smooth projective variety over a field k of characteristic zero, and consider the homological equivalence defined by the étale cobordism $\widehat{\mathrm{MU}}_{\text{ét}}$. Then, a homologically trivial cobordism cycle is numerically trivial.*

Proof. Let $\alpha \in \Omega^*(X)$ be homologically trivial, *i.e.* $\theta(\alpha) = 0$. For any $\gamma \in \Omega^*(X)$, $\theta(\alpha\cdot\gamma) = \theta(\alpha)\cdot\theta(\gamma) = 0$. By the commutativity of (4.1), we have $\theta\circ\pi_*(\alpha\cdot\gamma) = \pi_*\circ\theta(\alpha\cdot\gamma) = 0$. But $\theta = \overline{\theta}\circ\eta_\ell$ and $\overline{\theta}$ is an isomorphism. Thus, $\eta_\ell(\pi_*(\alpha\cdot\gamma)) = 0$. Since η_ℓ is injective, $\pi_*(\alpha\cdot\gamma) = 0$, which means α is numerically trivial. □

5. Smash nilpotence

We recall the definition of smash nilpotence:

Definition 5.1 ([7, Definition 10.1])**.** Let $X \in \mathbf{Sch}_k$. A cobordism cycle $\alpha \in \Omega_*(X)$ is said to be *rationally smash nilpotent* if, for some positive integer N, $\alpha^{\boxtimes N} := \alpha\times\cdots\times\alpha = 0$ in $\Omega_*(X^N)_{\mathbb{Q}}$.

Since we will be working with algebraic cobordism with $\mathbb{Q}$-coefficients from now on, we would say "smash nilpotent" to mean "rationally smash nilpotent".

Theorem 5.2. *Let X be a smooth projective variety over a field k with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then, smash nilpotent cycles in $\Omega^*(X)_{\mathbb{Q}}$ are homologically trivial.*

Proof. In [19, p.471], it is shown that $\mathrm{MU}^*(X)_{\mathbb{Q}}$ is a free $\mathbb{L}_*\otimes\mathbb{Q}$-module generated by any set of elements that map to a basis of $H^*(X;\mathbb{Q})$. This, along with [3, Theorem 44.1] shows that for smooth projective varieties X, Y, the homomorphism $\chi : \mathrm{MU}^*(X)_{\mathbb{Q}}\otimes_{\mathbb{L}_*\otimes\mathbb{Q}}\mathrm{MU}^*(Y)_{\mathbb{Q}} \to \mathrm{MU}^*(X\times Y)_{\mathbb{Q}}$ is an isomorphism. Note also that, by definition, if $\alpha \in \Omega^*(X)$, then $\theta(\alpha^{\boxtimes N}) \in \mathrm{MU}^*(X^N)$ equals $\chi(\theta(\alpha)^{\otimes N})$. Thus, if α satisfies $\alpha^{\boxtimes N} = 0 \in$

$\Omega^*(X^N)_{\mathbb{Q}}$ for some N, then $\theta(\alpha)^{\otimes N} = 0 \in \mathrm{MU}^*(X)^{\otimes N}_{\mathbb{Q}}$. From [15, Section 6], we know that $\mathbb{L}_* \otimes \mathbb{Q}$ is isomorphic to a polynomial ring over $\mathbb{Q}$, and hence it has no non-zero nilpotent elements. Since $\mathrm{MU}^*(X)_{\mathbb{Q}}$ is a free $\mathbb{L}_* \otimes \mathbb{Q}$-module, $\theta(\alpha)^{\otimes N} = 0$ implies that $\theta(\alpha) = 0$. □

Theorem 5.3. *Let X be a smooth projective variety over a field k of characteristic 0. Then, smash nilpotent cycles in $\Omega^*(X)_{\mathbb{Q}}$ are numerically trivial.*

Proof. Note that if $\beta \in \Omega^*(k)_{\mathbb{Q}}$ is smash-nilpotent, then it is nilpotent. However, $\Omega^*(k)_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{L}_* \otimes \mathbb{Q}$, which is a polynomial ring over $\mathbb{Q}$, and thus has no nonzero nilpotent. Thus, $\beta = 0$. Let $\pi : X \to \operatorname{Spec} k$ be the structure map. If $\alpha \in \Omega^*(X)_{\mathbb{Q}}$ is smash nilpotent, then for any $\gamma \in \Omega^*(X)_{\mathbb{Q}}$, $\alpha \cdot \gamma$ is smash-nilpotent by [7, Lemma 10.2] since the push-forward and pull-back maps respect external products. This implies that $\pi_*(\alpha \cdot \gamma)$ is smash-nilpotent in $\Omega^*(k)_{\mathbb{Q}}$, hence $\pi_*(\alpha \cdot \gamma) = 0$, which means α is numerically trivial. □

We study the converse of Theorem 5.3 in Section 7. Specifically, we look at the cases, where the results of Kahn-Sebastian [5] and Sebastian [17] are proven for algebraic cycles. We develop some cobordism analogues of known results for algebraic cycles.

6. Kimura finiteness on cobordism motives

From now on, we will work on the category $\mathbf{SmProj}_k$ of smooth projective varieties over an algebraically closed field k of characteristic 0, and consider algebraic cobordism with $\mathbb{Q}$-coefficients.

6.1. **Cobordism motives.** In [13, Sections 5-6], for an oriented cohomology theory A^*, Nenashev and Zainoulline constructed the *A-motive* of a smooth projective variety X, following the ideas of [11]. We briefly recall its construction. Given A^*, we have the category of A-correspondences, denoted as Cor_A, where

- $Ob(\mathrm{Cor}_A) := Ob(\mathbf{SmProj}_k)$,
- $\mathrm{Hom}_{\mathrm{Cor}_A}(X, Y) := A^*(X \times Y)$ and
- for $\alpha \in A^*(X, Y)$ and $\beta \in A^*(Y, Z)$, we have $\beta \circ \alpha := (p_{XZ})_*(p^*_{XY}(\alpha) \cdot p^*_{YZ}(\beta))$ in $A^*(X \times Z)$.

We have a functor $c : \mathbf{SmProj}_k^{op} \to \mathrm{Cor}_A$ given by $c(X) = X$ and $c(f) = (\Gamma_f)_*(1_{A(X)}) \in A^*(Y \times X)$ for a morphism $f : X \to Y$, where $\Gamma_f = (f, \mathrm{Id}) : X \to Y \times X$ is the graph of f. The grading on A^* induces a grading on $\mathrm{Hom}_{\mathrm{Cor}_A}$ given by $\mathrm{Hom}^n_{\mathrm{Cor}_A}(X, Y) := \oplus_i A^{n+d_i}(X_i \times Y)$, where the X_i's are the irreducible components of X and $d_i = \dim X_i$, making $\mathrm{Hom}_{\mathrm{Cor}_A}$ into a graded algebra under composition.

Definition 6.1. Consider the category Cor^0_A with $\mathrm{Hom}_{\mathrm{Cor}^0_A}(X, Y) := \mathrm{Hom}^0_{\mathrm{Cor}_A}(X, Y)$. The pseudo-abelian completion of Cor^0_A is called the *category of effective A-motives*, denoted by $\mathcal{M}^{\mathrm{eff}}_A$. Thus, the objects in $\mathcal{M}^{\mathrm{eff}}_A$ are pairs (X, p) where $X \in Ob(\mathrm{Cor}_A)$ and $p \in \mathrm{Hom}_{\mathrm{Cor}^0_A}(X, X)$ is a projector, and

$$\mathrm{Hom}_{\mathcal{M}^{\mathrm{eff}}_A}((X, p), (Y, q)) = \frac{\{\alpha \in \mathrm{Hom}_{\mathrm{Cor}^0_A}(X, Y) | \alpha \circ p = q \circ \alpha\}}{\{\alpha \in \mathrm{Hom}_{\mathrm{Cor}^0_A}(X, Y) | \alpha \circ p = q \circ \alpha = 0\}}.$$

The *category of A-motives*, denoted by $\mathcal{M}_A$, has the triplets (X, p, m) as objects, where (X, p) is an object in $\mathcal{M}^{\mathrm{eff}}_A$ and $m \in \mathbb{Z}$. The morphisms are defined as:

$$\mathrm{Hom}_{\mathcal{M}_A}((X, p, m), (Y, q, n)) = \frac{\{\alpha \in \mathrm{Hom}^{n-m}_{\mathrm{Cor}_A}(X, Y) | \alpha \circ p = q \circ \alpha\}}{\{\alpha \in \mathrm{Hom}^{n-m}_{\mathrm{Cor}_A}(X, Y) | \alpha \circ p = q \circ \alpha = 0\}}.$$

Note that, this means $\mathrm{Id}_{(X,p,0)} = \mathrm{Id}_X = p \in \mathrm{Hom}_{\mathcal{M}_A}((X,p,0),(X,p,0))$. The motive $(X, \mathrm{Id}_X, 0)$ is called the *motive of* X and denoted by $h_A(X)$.

6.2. **Finite dimensionality of cobordism motives.** Following [6, Section 3], we define finite-dimensionality of $\Omega^*_{\mathbb{Q}}$-motives. Each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of an integer $n \geq 1$, determines an irreducible representation W_λ of Σ_n over $\mathbb{Q}$. The group S_n^{op} acts on the n-fold product X^n of a smooth projective variety X by $\sigma(x_1, \dots, x_n) := (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for $\sigma \in \Sigma_n$. Let $f_\sigma : X^n \to X^n$ be the morphism associated to the action of σ and define $d_\lambda \in \mathrm{Cor}_{\Omega^*_{\mathbb{Q}}}(X^n, X^n)$ to be $d_\lambda := (\dim W_\lambda / n!) \cdot \sum_{\sigma \in \Sigma_n} \chi_{W_\lambda}(\sigma) \cdot c(f_\sigma)^t$. Using the properties of W_λ and the fact that c is a functor, we get $\sum d_\lambda = c(\mathrm{Id}_{X^n})$, $d_\lambda \circ d_\lambda = d_\lambda$ and $d_\lambda \circ d_\mu = 0$ whenever $\lambda \neq \mu$. Thus, we have $h(X^n) \simeq \oplus_\lambda \mathbb{T}_\lambda X^n$ in $\mathcal{M}_{\Omega^*_{\mathbb{Q}}}$, where $\mathbb{T}_\lambda X^n := (X^n, d_\lambda, 0)$.

Definition 6.2. For a $\Omega^*_{\mathbb{Q}}$-motive $M = (X, p, m)$, we define $M^{\otimes n} := (X^n, p^{\boxtimes n}, mn)$. If $f : M \to N$ is a morphism of motives for $M = (X, p, m_1)$ and $N = (Y, q, m_2)$, we have the morphism $f^{\otimes n} : M^{\otimes n} \to N^{\otimes n}$ defined to be $f^{\otimes n} := f^{\boxtimes n} \in \Omega^*(X^n \times Y^n)_{\mathbb{Q}}$.

Let $M = (X, p, m)$ be a motive. It follows by direct computation that $c(f_\sigma)^t$ and $p^{\boxtimes n}$ commute with each other. This implies that $d_\lambda \circ p^{\boxtimes n} = p^{\boxtimes n} \circ d_\lambda$ is idempotent. Thus, we have the following definition:

Definition 6.3. For a motive $M = (X, p, m)$, we define $\mathbb{T}_\lambda M$ to be the motive $(X^n, d_\lambda \circ p^{\boxtimes n}, mn)$. When $\lambda = (n)$, we denote $\mathbb{T}_{(n)} M$ by $\mathrm{Sym}^n M$ and for $\lambda = (1, 1, \dots, 1)$, we denote $\mathbb{T}_{(1,1,\dots,1)} M$ by $\wedge^n M$.

A motive M is *evenly (resp. oddly) finite dimensional* if there exists a positive integer N, such that $\wedge^N M = 0$ (resp. $\mathrm{Sym}^N M = 0$). M is said to be *finite-dimensional* if it can be written as a direct sum $M = M_+ \oplus M_-$ such that M_+ is evenly finite dimensional and M_- is oddly finite dimensional.

Remark 6.4. Let $\boldsymbol{L} := (pt, \mathrm{Id}_{pt}, -1)$. This is called the *Lefschetz motive*. Note that, for any $i \geq 0$, $\wedge^2 \boldsymbol{L}^i = 0$ since Σ_2^{op} acts trivially on $\boldsymbol{L}^i$. Thus, $\boldsymbol{L}^i$ is evenly 1-dimensional.

The following result of [6] holds in the case of $\Omega^*_{\mathbb{Q}}$-motives by the same arguments as in the proof of [6, Proposition 6.1], since the proof uses formal properties of the construction above, rather than properties specific to the Chow ring.

Proposition 6.5. *Any morphism between motives of different parity is smash-nilpotent.*

Proposition 6.6. *Let $\widetilde{\varphi} : \mathcal{M}_\Omega \to \mathcal{M}_{\mathrm{CH}}$ be the map induced by the canonical morphism $\varphi : \Omega^* \to \mathrm{CH}^*$. If M is an $\Omega^*_{\mathbb{Q}}$-motive such that $\widetilde{\varphi}(M)$ is evenly (resp. oddly) finite-dimensional as a Chow motive, then, M is evenly (resp. oddly) finite-dimensional.*

Proof. Let $M = (X, p, m)$ be an $\Omega^*_{\mathbb{Q}}$-motive such that $\widetilde{\varphi}(M) = (X, \varphi(p), m)$ is oddly finite-dimensional as a Chow motive. That is, for some $N \geq 1$, $\mathrm{Sym}^N \widetilde{\varphi}(M) = (X^N, (1/N!) \cdot \sum_{\sigma \in \Sigma_N} c_{\mathrm{CH}}(f_\sigma)^t \circ \varphi(p)^{\boxtimes N}, mN) = 0$. But, $c_{\mathrm{CH}}(f_\sigma) = \varphi(c_\Omega(f_\sigma))$ and $\varphi(c_\Omega(f_\sigma)^t) \circ \varphi(p)^{\boxtimes N} = \varphi(c_\Omega(f_\sigma)^t \circ p^{\boxtimes N})$ since φ is a ring homomorphism and commutes with push-forwards and pullbacks. Thus, $\widetilde{\varphi}(\mathrm{Sym}^N M) = 0$. Vishik and Yagita showed in [20, Corollary 2.8] that any isomorphism of Chow motives can be lifted to an isomorphism of cobordism motives. This implies that $\mathrm{Sym}^N M = 0$. The proof is similar when $\widetilde{\varphi}(M)$ is evenly finite-dimensional. $\square$

By [2, Corollary 8.2], for an abelian variety A of dimension g, we get a canonical decomposition of the cobordism motive of A, $h_\Omega(A) = \bigoplus_{i=0}^{2g} h^i_\Omega(A)$, where $h_\Omega(A) = (A, \mathrm{Id}_A, 0)$ and $h^i_\Omega(A) = (A, \pi_i, 0)$.

Corollary 6.7. *The motive $h^{\mathrm{odd}}_{\Omega}(A) := \oplus_{r:odd} h^r_{\Omega}(A)$ of an abelian variety A is oddly finite-dimensional.*

Proof. In [18], Shermenev gave a decomposition of the Chow motive of an abelian variety A of dimension n as

$$h_{\mathrm{CH}}(A) = \bigoplus_{i=0}^{n} \mathrm{Sym}^i(X, a_+) \oplus \bigoplus_{i=0}^{n-1} \mathrm{Sym}^i(X, a_+) \otimes \boldsymbol{L}^{n-i}$$

for some curve X and a projector a_+ on X. It follows from [6, Theorem 4.2] that the motive (X, a_+) defined by Shermenev is oddly finite-dimensional. Since odd symmetric powers of an oddly finite-dimensional motive is oddly finite-dimensional, we get that $h^{\mathrm{odd}}_{\mathrm{CH}}(A)$ is oddly finite-dimensional. From [2, Corollary 8.2], we get $\widetilde{\varphi}(h^{\mathrm{odd}}_{\Omega}(A)) = h^{\mathrm{odd}}_{\mathrm{CH}}(A)$. Thus, Proposition 6.6 implies that $h^{\mathrm{odd}}_{\Omega}(A)$ is oddly finite-dimensional. □

7. Voevodsky's conjecture for cobordism cycles

Let $k = \bar{k}$. Let A be an abelian variety over k. Our objective is to prove Theorem 7.10 and discuss its consequences. Recall that $\beta \in \Omega^*(A)_{\mathbb{Q}}$ is called a *skew cycle* if $\langle -1\rangle^*\beta = -\beta$, where, $\langle n\rangle$ denotes the endomorphism $\times n$ on A for $n \in \mathbb{Z}$.

Proposition 7.1. *Any skew cycle on an abelian variety is smash-nilpotent.*

We follow the sketch of [5, Proposition 1].

Proof. Let $g = \dim A$. By [2, Corollary 8.2], we have the canonical decomposition, $h_{\Omega}(A) = \bigoplus_{i=0}^{2g} h^i_{\Omega}(A)$, where $h^i_{\Omega}(A) = (A, \pi_i, 0)$ such that $c(\langle n\rangle)\circ\pi_i = n^i\pi_i = \pi_i \circ c(\langle n\rangle)$. By Definition 6.1, we have $\Omega^d(X)_{\mathbb{Q}} = \mathrm{Hom}(\boldsymbol{L}^d, h_{\Omega}(X))$, where $\boldsymbol{L} = (pt, \mathrm{Id}_{pt}, -1)$ is the Lefschetz motive. It is easy to check that $\mathrm{Hom}(\boldsymbol{L}^d, h^r_{\Omega}(A)) \xrightarrow{\sim} \Omega^d_{2d-r}(A)_{\mathbb{Q}}$, where the latter is defined to be $\{\alpha \in \Omega^d(A)_{\mathbb{Q}} | \langle n\rangle^*\alpha = n^r\alpha, \forall n \in \mathbb{Z}\}$. Indeed, $\mathrm{Hom}(\boldsymbol{L}^d, h^r_{\Omega}(A)) = \pi_{r*}\Omega^d(A)_{\mathbb{Q}}$ and $\langle n\rangle^*\pi_{r*}\alpha = c(\langle n\rangle)\circ\pi_r\circ\alpha$.

Let $\beta \in \Omega^d(A)_{\mathbb{Q}}$ be a skew cycle. Viewing β as a morphism in $\mathrm{Hom}(\boldsymbol{L}^d, h_{\Omega}(A))$, we may write $-\pi_r\circ\beta = \pi_r\circ(\langle -1\rangle^*\beta) = \pi_r\circ c(\langle -1\rangle)\circ\beta = (-1)^r\pi_r\circ\beta$. Thus, $\pi_r\circ\beta = 0$ for even r. This implies that β factors through $h^{\mathrm{odd}}_{\Omega}(A)$ via a morphism $\beta' \in \mathrm{Hom}(\boldsymbol{L}^d, h^{\mathrm{odd}}_{\Omega}(A))$. Since $\boldsymbol{L}^d$ is evenly finite-dimensional by Remark 6.4, it follows from Corollary 6.7 and Proposition 6.5 that β is smash nilpotent. □

Now, we will show that numerically trivial cobordism 1-cycles on a product of curves is smash nilpotent. To achieve this, following the ideas in [17], we show that the 'modified diagonal' cobordism cycle Δ_c of Definition 7.5 is smash-nilpotent in Corollary 7.6. We project Δ_c to a smaller product of curves and apply induction to get our desired result.

Let Y be a smooth projective curve of genus g and let $\mathrm{Jac}(Y)$ denote its Jacobian. Fix $N \geq 3$ and $m > \max\{N, 2g+2\}$. By [17, Lemma 11], there is a collection $\{r_1, r_2, \ldots, r_m\}$ such that the following conditions are satisfied.

(S1) $\sum_{l=1}^m \binom{m}{l} l r_l = 0$.
(S2) For every even integer $0 \leq i \leq g-1$, $\sum_{l=1}^m \binom{m}{l} l^{2+i} r_l = 0$.
(S3) $\sum_{l=N}^m \binom{m-N}{l-N} r_l \neq 0$.

We may slightly modify (S2) to include $i = -2$;

(S2′) For every even integer $-2 \leq i \leq g-1$, $\sum_{l=1}^m \binom{m}{l} l^{2+i} r_l = 0$.

Let $S := \{1, 2, \ldots, m\}$ and let $p_i : Y^m \to Y$ denote the i-th projection. Choose a base point $y_0 \in Y$. For every non-empty subset $T \subset S$, define the morphism $\varphi_T : Y \to Y^m$ to be the unique one such that $p_i \circ \varphi_T(y) = y$ if $i \in T$ and $p_i \circ \varphi_T(y) = y_0$ if $i \notin T$. Let

Δ_T denote the cobordism cycle $[\varphi_T : Y \to Y^m]$. Let $f : Y^m \to \mathrm{Sym}^m Y$ be the natural quotient morphism. This is a projective morphism so that f_* exists on cobordism cycles. On the other hand, by [6, Proposition 4.1] and [16], $\mathrm{Sym}^m Y$ is smooth since $m \geq 2g-2$. Thus, f is also an l.c.i. morphism, which implies f^* exists for cobordism cycles by [9, Section 6.5.4].

Consider the cobordism cycle $\Delta_l := f_* \Delta_T$ for some $T \subset S$ with $\#T = l$. Note that Δ_l does not depend on the choice of T. In fact, $\Delta_l = \binom{m}{k}^{-1} \cdot \sum_{\#T=l} [Y \xrightarrow{f \circ \varphi_T} \mathrm{Sym}^m Y]$.

Lemma 7.2. *$f^* \Delta_l = l!(m-l)! \sum_{\#T=l} \Delta_T$ in $\Omega_1(Y^m)_{\mathbb{Q}}$.*

Proof. Clearly, $f^* \Delta_l = c \sum_{\#T=l} \Delta_T$ for some $c \in \mathbb{Z}$. Also, $f_* f^* = m!$. Thus, applying f_*, we get $m! \Delta_l = c \sum_{\#T=l} f_* \Delta_T = c\binom{m}{l} \Delta_l$, implying $c = l!(m-l)!$. □

Definition 7.3. Define $\Gamma := \sum_{l=1}^m \binom{m}{l} r_l \Delta_l$ in $\Omega_1(\mathrm{Sym}^m Y)_{\mathbb{Q}}$.

Lemma 7.4. *Γ is smash-nilpotent.*

Proof. Since $m > 2g-2$, the natural map $\mathrm{Sym}^m Y \xrightarrow{\pi} \mathrm{Jac}(Y)$ is the projective bundle associated to a locally free sheaf over $\mathrm{Jac}(Y)$ (see [6, Proposition 4.1] and [16]). Thus, by the definition of oriented cohomology theory (see [9, Definition 1.1.2.(PB)], we have $\Gamma = c_1(\mathcal{O}(1))^{m-g-1} \cdot \pi^* \beta_0 + c_1(\mathcal{O}(1))^{m-g} \cdot \pi^* \beta_1$, for some $\beta_i \in \Omega_i(\mathrm{Jac}(Y))_{\mathbb{Q}}$. We first check that $\pi_* \Gamma$ is smash-nilpotent. Let $\psi : Y \to \mathrm{Jac}(Y)$ be the embedding using the base-point y_0. By [2, Theorem 6.2], we have a Beauville decomposition of $\Omega^{g-1}(\mathrm{Jac}(Y))_{\mathbb{Q}}$, giving $\psi_*(1_Y) = \sum_{-2 \leqslant i \leqslant g-1} x_i$, such that $\langle n \rangle_* x_i = n^{2+i} x_i$, where $\langle n \rangle$ is the morphism $\times n$ on $\mathrm{Jac}(Y)$. Thus,

$$\pi_* \Gamma = \sum_{l=1}^m \binom{m}{l} r_l \pi_* \Delta_l = \sum_{l=1}^m \binom{m}{l} r_l \langle l \rangle_* \psi_*(1_Y) = \sum_{i=-2}^{g-1} \Big(\sum_{l=1}^m \binom{m}{l} r_l l^{2+i} \Big) x_i. \tag{7.1}$$

Since the r_l's satisfy (S2$'$), we have that $\sum_{l=1}^m \binom{m}{l} r_l l^{2+i} = 0$ for i even. Thus, Proposition 7.1 implies that $\pi_* \Gamma$ is smash-nilpotent. Now, by the projection formula, $\pi_* \Gamma = \pi_*(c_1(\mathcal{O}(1))^{m-g-1}) \cdot \beta_0 + \pi_*(c_1(\mathcal{O}(1))^{m-g}) \cdot \beta_1$. Let $\alpha_i = \pi_*(c_1(\mathcal{O}(1))^{m-g+i}) \in \Omega^i(\mathrm{Jac}(Y))_{\mathbb{Q}}$, so $\pi_* \Gamma = \alpha_{-1} \beta_0 + \alpha_0 \beta_1$. We also have $\pi_* \{\Gamma \cdot c_1(\mathcal{O}(1))\} = \alpha_0 \beta_0 + \alpha_1 \beta_1$. Note that, $\varphi[\pi_* \{\Gamma \cdot c_1(\mathcal{O}(1))\}] = \pi_*[\varphi(\Gamma) \cdot \varphi\{c_1(\mathcal{O}(1))\}]$, which is a 0-cycle of degree 0 by [17, Proposition 3]. Thus, by [9, Lemma 4.5.10], $\pi_* \{\Gamma \cdot c_1(\mathcal{O}(1))\}$ is of the form $\sum n_i [\{p_i\} \to \mathrm{Jac}(Y)]$, where $n_i \in \mathbb{Z}$ and $\sum n_i = 0$. But, on a smooth projective variety, such a cobordism cycle is algebraically trivial by [12, Lemma in p.56] and [7, Theorem 5.1]. Thus, $\pi_*(\Gamma \cdot c_1(\mathcal{O}(1)))$ is smash-nilpotent by [7, Theorem 10.3]. We now use the degree formula (see [9, Theorem 4.4.7]) to conclude that β_0 and β_1 (and hence Γ) are smash-nilpotent. The degree formula gives that

- $\beta_0 = \sum n_i [\{p_i\} \to \mathrm{Jac}(Y)]$, where $n_i \in \mathbb{Z}$ and p_i's points in $\mathrm{Jac}(Y)$.
- $\beta_1 = \sum m_j [\widetilde{C_j} \to \mathrm{Jac}(Y)] + \sum \gamma_s [\{q_s\} \to \mathrm{Jac}(Y)]$, where $m_j \in \mathbb{Z}$, $\gamma_s \in \mathbb{L}_1$, q_s's are points in $\mathrm{Jac}(Y)$ and $\widetilde{C_j}$'s are smooth curves with projective birational morphisms $\widetilde{C_j} \to C_j \subset \mathrm{Jac}(Y)$.
- $\alpha_i = \sum_{j=\max(0,-i)}^{\min(g,g-i)} \sum_{l \in K_i^j} \omega_{i,j}^l x_{i,j}^l$, where $\omega_{i,j}^l \in \mathbb{L}_j$, $x_{i,j}^l \in \Omega^{i+j}(\mathrm{Jac}(Y))$ and $|K_i^j| < \infty$.

For $j=-i$, we have $K_i^j=\{1\}$, $x_{i,j}^1=[\mathrm{Id}_X]$ and it follows by [4, Proposition 3.1(a)(i)] that $\omega_{0,0}^1=1$. We may write

$$\begin{aligned}\alpha_{-1}\beta_0+\alpha_0\beta_1 &= (\omega^1_{-1,1}[\mathrm{Id}_X]+\sum_{j=2}^{g}\sum_{l\in K^j_{-1}}\omega^l_{-1,j}x^l_{-1,j})\sum n_i[\{p_i\}\to J]+\\ &+([\mathrm{Id}_X]+\sum_{j=1}^{g}\sum_{l\in K^j_0}\omega^l_{0,j}x^l_{0,j})(\sum m_j[\widetilde{C_j}\to \mathrm{Jac}(Y)]+\sum \gamma_l[\{q_l\}\to\mathrm{Jac}(Y)])\\ &=\sum n_i\omega^1_{-1,1}[\{p_i\}\to J]+\sum\gamma_s[\{q_s\}\to\mathrm{Jac}(Y)]+\sum m_j[\widetilde{C_j}\to\mathrm{Jac}(Y)]+\\ &\qquad\sum_j\sum_{l\in K^1_0}m_j\omega^l_{0,1}x^l_{0,1}[\widetilde{C_j}\to\mathrm{Jac}(Y)].\end{aligned}$$

Now, by [17, Lemma 4], $\varphi(\beta_1)=\sum m_j[C_j]$ is smash-nilpotent. [9, Lemma 4.5.3] gives us that $\sum m_j[\widetilde{C_j}\to\mathrm{Jac}(Y)]+\sum\mu_t[\{r_t\}\to\mathrm{Jac}(Y)]$ is smash-nilpotent for some points r_l in $\mathrm{Jac}(Y)$ and $\mu_l\in\mathbb{L}_1$. Multiplying with $\omega^l_{0,1}x^l_{0,1}$, we get that $\sum m_j\omega^l_{0,1}x^l_{0,1}[\widetilde{C_j}\to\mathrm{Jac}(Y)]$ is smash-nilpotent by [7, Lemma 10.2(1)]. Hence, $\sum m_j\sum_{l\in K^1_0}\omega^l_{0,1}x^l_{0,1}[\widetilde{C_j}\to\mathrm{Jac}(Y)]$ is smash-nilpotent. Since $\alpha_{-1}\beta_0+\alpha_0\beta_1$ is smash-nilpotent, this gives $\omega^1_{-1,1}\beta_0+\beta_1$ is smash-nilpotent. Next, note that $\alpha_0\beta_0+\alpha_1\beta_1=\beta_0+\sum_{l\in K^0_1}\omega^l_{1,0}x^l_{1,0}\sum m_j[\widetilde{C_j}\to\mathrm{Jac}(Y)]$ is smash nilpotent. Similarly as above, $\sum_{l\in K^0_1}\omega^l_{1,0}x^l_{1,0}\sum m_j[\widetilde{C_j}\to\mathrm{Jac}(Y)]$ is smash-nilpotent. This implies that β_0, and hence β_1 are smash-nilpotent, which completes the proof. □

Definition 7.5. Define the modified diagonal cobordism cycle to be $\Delta_c:=(1/m!)\cdot f^*\Gamma$.

By Lemma 7.2, $\Delta_c=\sum_{l=1}^m r_l(\sum_{T\subset S,\#T=l}\Delta_T)$. Then, Lemma 7.4 shows that

Corollary 7.6. *Δ_c is smash-nilpotent.*

Now, let $X:=C_1\times C_2\times\cdots\times C_N$ be a product of N smooth projective curves. Let Y be a smooth projective curve with a morphism $j:Y\to X$. Let $q_i:X\to C_i$ denote the projection onto the i-th factor. Define a morphism $\psi:Y^m\to X$ as

$$Y^m\xrightarrow{pr}Y^N\xrightarrow{j^N}X^N\xrightarrow{q_1\times\cdots\times q_N}X$$

where pr is the projection to the first N coordinates. Let $S_0:=\{1,2,\ldots,N\}$. For a closed point $v=(v_1,\ldots,v_N)$ of X and a subset $T\subset S_0$, we define $\zeta^v_T:X\to X$ to be the unique morphism such that $q_i\circ\zeta^v_T(x)=q_i(x)$ if $i\in T$, and $q_i\circ\zeta^v_T(x)=v_i$ if $i\notin T$. Note that $\zeta^v_T=\mathrm{Id}$ if and only if $T=S_0$. It is also clear from the definition that, for $T\subset S$ we have $\psi\circ\varphi_T=\zeta^{j(y_0)}_{T\cap S_0}\circ j$.

Lemma 7.7. *For two closed points v and v' in X, and $\alpha\in\Omega_*(X)$, two cobordism cycles $\zeta^v_{T*}(\alpha)$ and $\zeta^{v'}_{T*}(\alpha)$ are algebraically equivalent.*

Proof. Let $X_T=\prod_{i\in T}C_i$ and $pr_T:X\to X_T$ be the projection. Let $l=m-|T|$ and $\{1,2,\ldots,m\}\setminus T=\{a_1,\ldots,a_l\}$. Then, $\zeta^v_T=\iota_l\circ\cdots\circ\iota_1\circ p$ where $\iota_j:X_{T\cup\{a_1,\ldots,a_{j-1}\}}\to X_{T\cup\{a_1,\ldots,a_j\}}$ be the inclusion of v_{a_j}. Similarly, $\zeta^{v'}_T=\iota'_l\circ\cdots\circ\iota'_1\circ p$ where ι'_j is the inclusion of v'_{a_j}.

By [9, Proposition 3.1.9], for X' in $\mathbf{Sm}_k$ and a smooth projective curve C with a closed point $\{p\}$, we have $[X'\times\{p\}\to X'\times C]=[\mathcal{O}_{X'\times C}(X'\times\{p\})]$. For another closed point $\{p'\}$ in C, the line bundles $\mathcal{O}_{X'\times C}(X'\times\{p\})$ and $\mathcal{O}_{X'\times C}(X'\times\{p'\})$ are algebraically equivalent so that $[X' \times \{p\} \to X' \times C] \sim_{\mathrm{alg}} [X' \times \{p'\} \to X' \times C]$ by the definition of algebraic equivalence as in [7]. Thus, for any cobordism cycle $\beta_j \in \Omega^*(X_{T\cup\{a_1,\dots,a_{j-1}\}})$, we have $\iota_{j*}(\beta) \sim_{\mathrm{alg}} \iota'_{j*}(\beta)$ for any j, thereby implying $\zeta^v_{T*}(\alpha) \sim_{\mathrm{alg}} \zeta^{v'}_{T*}(\alpha)$. $\square$

Definition 7.8. Define $\kappa := \sum_{l=N}^{m} \binom{m-N}{l-N} r_l$, which is nonzero as the r_l's satisfy (S3).

Lemma 7.9. *The cobordism* 1*-cycle* $\alpha = [j : Y \to X] \in \Omega_1(X)_{\mathbb{Q}}$ *is smash-equivalent to a cycle coming from a smaller product of curves.*

Proof. Let $v = j(y_0)$. Then,

$$\psi_*\Delta_c = \sum_{l=1}^{m} r_l \Big(\sum_{T\subset S, \#T=l} \psi_*\Delta_T \Big) = \sum_{l=1}^{m} r_l \Big(\sum_{T\subset S, \#T=l} \zeta^v_{T\cap S_0 *}\alpha \Big).$$

Note that if $T \supset S_0$, $\zeta^v_{T\cap S_0} = \mathrm{Id}$ and $\psi_*\Delta_T = 0$ if $T \cap S_0 = \varnothing$. Let $\mathcal{S}$ be the set of all subsets of S that intersect S_0 and let $\mathcal{U}$ be the set of all subsets of S that contain S_0. Then,

$$\begin{aligned} \psi_*\Delta_c &= \sum_{l=1}^{m} r_l \Big(\sum_{T\in\mathcal{U}, \#T=l} \alpha \Big) + \sum_{l=1}^{m} r_l \Big(\sum_{T\in\mathcal{S}\setminus\mathcal{U}, \#T=l} \zeta^v_{T\cap S_0 *}\alpha \Big) \\ &= \alpha \left(\sum_{l=N}^{m} r_l \binom{m-N}{l-N} \right) + \sum_{l=1}^{m} r_l \Big(\sum_{T\in\mathcal{S}\setminus\mathcal{U}, \#T=l} \zeta^v_{T\cap S_0 *}\alpha \Big) \end{aligned}$$

Thus, by Corollary 7.6, $\alpha + (1/\kappa)\cdot \sum_{l=1}^{m} r_l (\sum_{T\in\mathcal{S}\setminus\mathcal{U}, \#T=l} \zeta^v_{T\cap S_0 *}\alpha)$ is smash-nilpotent. Note that $\zeta^v_{T\cap S_0}$ is a projection to a smaller product of curves, followed by an inclusion to X. This proves the lemma. $\square$

Theorem 7.10. *Let* α *be a numerically trivial cobordism* 1*-cycle on* X*. Then,* α *is smash-nilpotent.*

Proof. By the degree formula [9, Theorem 4.4.7], we have $\alpha = \sum n_i [j_i : \widetilde{Y}_i \to X] + \sum \gamma_j [\{p_j\} \to X]$, where $n_i \in \mathbb{Z}$, $\gamma_j \in \mathbb{L}_1$, p_j's are points in X, $\widetilde{Y}_i$'s are smooth projective curves, and j_i is the composition of a birational morphism $\widetilde{Y}_i \to Y_i$ with the inclusion $Y_i \to X$ for a closed irreducible $Y_i \subset X$. Since α is numerically trivial, $\varphi(\alpha) = \sum n_i [Y_i]$ is numerically trivial in $\mathrm{CH}_*(X)$. By Theorem 3.2, $\sum n_i [\widetilde{Y'_i} \to X]$ is numerically trivial for some $\widetilde{Y'_i}$s in $\mathbf{Sm}_k$ with projective birational morphisms $\pi'_i : \widetilde{Y'_i} \to Y_i$. Using [9, Lemma 4.5.3], we have

$$\sum n_i [\widetilde{Y'_i} \to X] = \sum n_i [j_i : \widetilde{Y}_i \to X] + \sum \beta_l [\{p'_l\} \to X].$$

This implies that $\sum \gamma_j [\{p_j\} \to X] - \sum \beta_l [\{p'_l\} \to X]$ is numerically trivial. Note that $[\{q\} \to X] \sim_{\mathrm{alg}} [\{p\} \to X]$ for any two points p and q in X. Thus, going modulo algebraic equivalence, $\omega[\{p\} \to X]$ is numerically trivial for some $\omega \in \mathbb{L}_1$, implying that $[\{p\} \to X]$ is numerically trivial. But $\sum \gamma_j [\{p_j\} \to X] \sim_{\mathrm{alg}} \omega' [\{p\} \to X]$ for some $\omega' \in \mathbb{L}_1$. Thus, $\sum \gamma_j [\{p_j\} \to X]$ is numerically trivial modulo algebraic equivalence. As we have observed in the proof of Lemma 7.4, a numerically trivial cobordism 0-cycle is smash-nilpotent. Thus, we only need to show that $\sum n_i [j_i : \widetilde{Y}_i \to X]$ is smash-nilpotent. We already know that, modulo algebraic equivalence, $\sum n_i [j_i : \widetilde{Y}_i \to X]$ is numerically trivial.

We proceed by induction on N. Suppose any numerically trivial cobordism 1-cycle on a product of l curves is smash-nilpotent, for $l < N$. By Lemma 7.9, $\sum n_i[j_i : \widetilde{Y}_i \to X]$ is smash-equivalent to

$$\frac{1}{\kappa}\sum n_i \sum_{l=1}^{m} r_l\Big(\sum_{T\in\mathcal{S}\setminus\mathcal{U},\#T=l} \zeta^{v^i}_{T\cap S_0 *}[j_i : \widetilde{Y}_i \to X]\Big),$$

where $v^i = j_i(y^i_0)$, y^i_0 being a chosen base point of $\widetilde{Y}_i$. However, by Lemma 7.7, $\zeta^{v^i}_{T\cap S_0 *}[j_i : \widetilde{Y}_i \to X] \sim_{\mathrm{alg}} \zeta^{v^1}_{T\cap S_0 *}[j_i : \widetilde{Y}_i \to X]$. Thus, modulo algebraic equivalence,

$$\sum n_i[j_i : \widetilde{Y}_i \to X] \sim_{\mathrm{smash}} \frac{1}{\kappa}\sum_{l=1}^{m} r_l\Big(\sum_{T\in\mathcal{S}\setminus\mathcal{U},\#T=l} \zeta^{v^1}_{T\cap S_0 *}\big(\sum n_i[j_i : \widetilde{Y}_i \to X]\big)\Big).$$

Since $\zeta^{v^1}_{T\cap S_0 *}(\sum n_i[j_i : \widetilde{Y}_i \to X])$ is numerically trivial, it is smash-nilpotent by the induction hypothesis. Hence, $\sum n_i[j_i : \widetilde{Y}_i \to X]$ is smash-nilpotent. It remains to check the cases $N = 1, 2$. The case of a curve is trivial as algebraic and numerical equivalence coincide for cobordism 1-cycles on curves, by [7, Theorem 9.6.(1)]. Now consider the case where X is a surface. Let $\alpha = \sum n_i[j_i : \widetilde{Y}_i \to X] + \sum \gamma_j[\{p_j\} \to X]$ be a numerically trivial cobordism 1-cycle in $\Omega^{\mathrm{alg}}_1(X)_{\mathbb{Q}}$. Since $\mathrm{CH}^{\mathrm{alg}}_*(X)$ coincides with $\mathrm{CH}^{\mathrm{num}}_*(X)$, we have $\varphi(\alpha) = \sum n_i[Y_i] = 0$, whence [9, Lemma 4.5.3] implies, $\sum n_i[j_i : \widetilde{Y}_i \to X] = \sum \beta_l[\{p'_l\} \to X]$. Thus, $\alpha = \omega[\{p\} \to X] \in \Omega^{\mathrm{alg}}_1(X)_{\mathbb{Q}}$. This shows $[\{p\} \to X]$ is numerically trivial, hence smash-nilpotent by the argument above. Therefore, α is smash-nilpotent. □

Corollary 7.11. *Let Y be a smooth projective variety and let $h : X = C_1 \times C_2 \times \cdots \times C_N \to Y$ be a dominant morphism. Then, numerical equivalence and smash equivalence coincide for cobordism 1-cycles on Y.*

Proof. Let $\mathcal{L}$ be a relatively h-ample line bundle on X. Let $r := N - \dim(Y)$ be the relative dimension of h. Now, consider $h_*(c_1(\mathcal{L})^r) \in \Omega^0(Y)_{\mathbb{Q}}$. Note that since $\deg(h_*(c_1(\mathcal{L})^r)) \in \mathbb{Z}$, if $\deg(h_*(c_1(\mathcal{L})^r)) = 0$, then $\varphi(h_*(c_1(\mathcal{L})^r)) = 0 \in \mathrm{CH}^0(Y)$, which is not the case since $\mathcal{L}$ is relatively h-ample. Denote $d := \deg(h_*(c_1(\mathcal{L})^r))$. Thus, by the degree formula [9, Theorem 4.4.7],

$$h_*(c_1(\mathcal{L})^r) = d[\mathrm{Id}_Y] + \sum_{\substack{Z\subset Y\\ \mathrm{codim}_Y Z>0}} \omega_Z[\widetilde{Z} \to Y] \text{ with } \widetilde{Z} \text{ smooth and birational over } Z.$$

Now, let $\alpha \in \Omega_1(Y)_{\mathbb{Q}}$ be numerically equivalent to 0. Since X and Y are smooth, h is l.c.i. Thus, we may consider the pullback $h^*\alpha$. Note that by the projection formula,

$$h_*(c_1(\mathcal{L})^r \cdot h^*\alpha) = h_*(c_1(\mathcal{L})^r)\,\alpha = d\alpha + \sum_{\substack{\mathrm{codim}_Y Z=1\\ \omega_Z\in\mathbb{L}_1}} \omega_Z[\widetilde{Z} \to Y]\cdot\alpha.$$

But, $[\widetilde{Z} \to Y]\cdot\alpha \in \Omega_0(Y)_{\mathbb{Q}}$ and is numerically trivial. We observed in the proof of Lemma 7.4 that a numerically trivial cobordism 0-cycle on a smooth projective variety is smash-nilpotent. Also, $c_1(\mathcal{L})^r \cdot h^*\alpha$ being a numerically trivial cobordism 1-cycle on X, is smash-nilpotent by Theorem 7.10. Thus, $h_*(c_1(\mathcal{L})^r \cdot h^*\alpha)$ is smash-nilpotent, which implies $d\alpha$, and hence α, is smash-nilpotent since $d \neq 0$. □

Acknowledgments. During the work, the authors were supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIP) (No. 2013042157). During this work, JP was also partially supported by Korea Institute for Advanced Study (KIAS) grant funded by the Korean government (MSIP), and the TJ Park Junior Faculty Fellowship funded by POSCO TJ Park Foundation.

References

[1] Y. André, Une introduction aux motives (motifs purs, motifs mixtes, périodes), *Panoramas et Synthèses* 17 (Soc. Math. France, Paris, 2004). xii+261 pp.
[2] A. Banerjee and T. Hudson, Fourier-Mukai transformation on algebraic cobordism, preprint available at http://arxiv.org/abs/1311.4039, 2013.
[3] P. E. Conner and E. E. Floyd, Differentiable periodic maps, *Bull. Amer. Math. Soc.* 68 (1962) no. 2, 76–86.
[4] W. Fulton, Intersection theory, Second Edition, *Ergebnisse der Math. Grenzgebiete* 3 (Springer-Verlag, Berlin, 1998). xiv+470 pp.
[5] B. Kahn and R. Sebastian, Smash-nilpotent cycles on Abelian 3-folds, *Math. Res. Lett.* 16 (2009) no. 6, 1007–1010.
[6] S-I. Kimura, Chow groups are finite dimensional in some sense, *Math. Ann.* 331 (2005) 173–201.
[7] A. Krishna and J. Park, Algebraic cobordism theory attached to algebraic equivalence, *J. K-theory.* 11 (2013), 73–112.
[8] M. Lazard, Sur les groupes de Lie formels à un paramètre, *Bull. Soc. Math. France*, 83 (1955), 251–274.
[9] M. Levine and F. Morel, Algebraic Cobordism, *Springer Monographs Math.* (Springer, Berlin, 2007). xii+244 pp.
[10] M. Levine and R. Pandharipande, Algebraic cobordism revisited, *Invent. Math.* 176 (2009) 63–130.
[11] Ju. I. Manin, Correspondences, motifs and monoidal transformations, *Math. USSR-Sb.* 6 (1968), 439–470.
[12] D. Mumford, Abelian Varieties, Oxford University Press, 1974.
[13] A. Nenashev and K. Zainoulline, Oriented cohomology and motivic decompositions of relative cellular spaces, *J. Pure Appl. Algebra* 205 (2006), 323–340.
[14] G. Quick, Stable étale realization and étale cobordism, *Adv. in Math.* 214, no.2 (2007) 730–760.
[15] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, *Adv. Math.* 7 (1971), 29–56.
[16] R. L. E. Schwarzenberger, Jacobians and symmetric products, *Illinois J. Math.* 7 (1963), 257–268.
[17] R. Sebastian, Smash nilpotent cycles on varieties dominated by products of curves, *to appear in Compositio Mathematica*, doi:10.1112/S0010437X13007197.
[18] A. M. Shermenev, The motif of an abelian variety, *Funct. Anal. Appl.* 8 (1974), 47–53.
[19] B. Totaro, Torsion algebraic cycles and complex cobordism, *J. Amer. Math. Soc.* 10 (1997) 467–493.
[20] A. Vishik and N. Yagita, Algebraic cobordisms of a Pfister quadric, *J. London Math. Soc.* 76 (2007) no. 2, 586–604.
[21] V. Voevodsky, A nilpotence theorem for cycles algebraically equivalent to zero, *Internat. Math. Res. Notices* (1995) 187–198.
[22] C. Voisin, Remarks on zero-cycles of self-products of varieties, in Maruyama, Masaki (ed.), Moduli of vector bundles, *Lect. Notes in Pure Appl. Math.* 179 (Marcel Dekker, New York, 1996) 265–285.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 305-701, Republic of Korea (South)
E-mail address: anandamb@gmail.com

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 305-701, Republic of Korea (South)
E-mail address: jinhyun@mathsci.kaist.ac.kr; jinhyun@kaist.edu