QUANTITATIVE QUANTUM ERGODICITY AND THE NODAL DOMAINS OF MAASS-HECKE CUSP FORMS

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Abstract. We prove a quantitative statement of the quantum ergodicity for Maass-Hecke cusp forms on $SL(2,\mathbb{Z})\backslash\mathbb{H}$. As an application of our result, we obtain a sharp lower bound for the L^2-norm of the restriction of even Maass-Hecke cusp form f's to any fixed compact geodesic segment in $\{iy \mid y>0\} \subset \mathbb{H}$, with a possible exceptional set which is polynomially smaller in the size than the set of all f. We also improve L^∞ estimate for Maass-Hecke cusp forms given by Iwaniec and Sarnak, for almost all Maass-Hecke cusp forms. We then deduce that the number of nodal domains of f which intersect a fixed geodesic segment increases with the eigenvalue, with a small number of exceptional f's. In the recent work of Ghosh, Reznikov, and Sarnak, they prove the same statement for all f without exception, assuming the Lindelof Hypothesis and that the geodesic segment is long enough. For almost all Maass-Hecke cusp forms, we give better lower bound of number of nodal domains.

1. Introduction

Let $\mathbb{X} = SL_2(\mathbb{Z})\backslash\mathbb{H}$ and let ϕ be an L^2-normalized Maass-Hecke cusp form on the modular surface $\mathbb{X}$. In other words, ϕ is a function on $\mathbb{H}$ such that:

1. $\int_{\mathbb{X}} |\phi(z)|^2 dA(z) = 1$,
2. $\phi(\gamma z) = \phi(z)$ for all $\gamma \in SL_2(\mathbb{Z})$,
3. $-\Delta_{\mathbb{H}}\phi = (\frac{1}{4}+t_\phi^2)\phi$, and
4. $T_n\phi = \lambda_\phi(n)\phi$ for some $\lambda_\phi(n)$ for all $n>0$, where T_n is the normalized n-th Hecke operator:

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{b\ (d)\\ ad=n}} f\left(\frac{az+b}{d}\right).$$

Such ϕ has a Fourier expansion of the type

$$\phi(z) = \sqrt{\cosh(\pi t_\phi)} \sum_{n\neq 0} \rho_\phi(n)\sqrt{y}K_{it_\phi}(2\pi|n|y)e(nx).$$

We would like to thank Peter Sarnak for introducing his recent paper with Ghosh and Reznikov, and suggesting this problem as a part of our thesis. We also appreciate Peter Sarnak and Nicolas Templier for providing much encouragement and many helpful comments. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP)(No. 2013042157). The author was also partially supported by TJ Park Post-doc Fellowship funded by POSCO TJ Park Foundation.

The coefficients satisfy $\rho_\phi(\pm n) = \rho_\phi(\pm 1)\lambda_\phi(n)$ for $n > 0$, where we have the following estimate for the first Fourier coefficient

$$t_\phi^{-\epsilon} \ll_\epsilon |\rho_\phi(1)| \ll_\epsilon t_\phi^{\epsilon}\,{}^{1} \tag{1.1}$$

for any $\epsilon > 0$ (see [Iwa90] and [HL94]). The Hecke eigenvalues $\lambda_\phi(n)$ satisfy the following recurrence relation:

$$\lambda_\phi(nm) = \sum_{d|(n,m)} \lambda_\phi\left(\frac{nm}{d^2}\right) \tag{1.2}$$

and this is the main arithmetic input in our work.

If we assume further that

5. ϕ is an eigenfunction of σ, where $\sigma : \mathbb{X} \to \mathbb{X}$ is an orientation reversing isometry induced from $x + iy \mapsto -x + iy$ on $\mathbb{H}$,

then such ϕ's form an orthonormal basis of the cuspidal subspace $L^2_{cusp}(\mathbb{X})$ of $L^2(\mathbb{X})$. We say ϕ is *even* (resp. *odd*) if $\sigma\phi = \phi$ (resp. $\sigma\phi = -\phi$).

Now we define a measure μ_ϕ on $\mathbb{X}$ by

$$\mu_\phi = |\phi(z)|^2 \frac{dxdy}{y^2}.$$

Then the Arithmetic Quantum Unique Ergodicity (QUE) Theorem of Lindenstrauss [Lin06] and Soundararajan [Sou10] asserts that

$$\mu_\phi \underset{w}{\to} dA(z) \tag{1.3}$$

as $t_\phi \to \infty$.

In terms of the Fourier coefficients $\rho_\phi(n)$ of ϕ, it is known that

$$\frac{1}{t_\phi}\sum_n \rho_\phi(n+m)\rho_\phi(n)\psi\left(\frac{\pi|n|}{t_\phi}\right) \to \frac{8}{\pi}\delta_{0,m}\int_0^\infty \psi(y)dy \tag{1.4}$$

for any $\psi \in C_0^\infty(0,\infty)$ implies the arithmetic QUE theorem, whereas the converse is only known for a certain class of ψ (see Appendix A in [GRS12]).

Both forms (1.3) and (1.4) of QUE can be quantified with rates, which is called the Quantitative QUE (QQUE). We state the strong form as follows:

Conjecture 1.1 ((Strong) QQUE)**.** *There exist $\nu > 0$ and $k < \infty$ such that for any $\psi \in C_0^\infty(0,\infty)$,*

$$\left|\frac{1}{t_\phi}\sum_n \rho_\phi(n+m)\rho_\phi(n)\psi\left(\frac{\pi|n|}{t_\phi}\right) - \frac{8}{\pi}\delta_{0,m}\int_0^\infty \psi(y)dy\right| \ll_m t_\phi^{-\nu}||\psi||_{W^{k,\infty}(0,\infty)}. \tag{1.5}$$

Note that subconvexity estimate for the triple product L-function $L(s, \phi \times \phi \times \phi_0)$ with any fixed Maass form ϕ_0 is equivalent to the QQUE, if the implied constant depends polynomially on the derivatives of ϕ_0 ([Wat02]).

[1] Here and elsewhere, $A \ll_\omega B$ means $|A| < CB$ for some constant C depending only on ω.

Also note that the Lindelof Hypothesis for the triple product L-function allows one to take any $0 < \nu < 1/2$ for a certain class of $\psi \in C_0^\infty(0,\infty)$.

We investigate an average version (1.5) and prove:

Theorem 1.1. *Let $1/3 < \theta < 1$ be a fixed constant and let $G = T^\theta$. Assume that $\psi \in C_0^\infty(0,\infty)$ is supported in $(0,l) \subset (0,\infty)$.*

(i) *Let $0 < \delta < 1$ be a fixed constant. Then there exists $A > 0$ depending only on θ and ϵ such that*

$$\sum_{|t_\phi - T| < G} \left| \sum_n \rho_\phi(n+m)\rho_\phi(n)\psi(\frac{\pi n}{X}) \right|^2 \ll_{\epsilon,l} mXGT^{1+\epsilon} ||\psi||^2_{W^{A,\infty}} \tag{1.6}$$

holds uniformly in $1 \le m < X^{1-\delta}$. One can take, for example, $A = 100/\min\{3\theta - 1, \epsilon\}$.

(ii) *There exists $A > 0$ depending only on θ and ϵ such that*

$$\sum_{|t_\phi - T| < G} \left| \sum_{n \neq 0} \rho_\phi(n)^2 \psi\left(\frac{\pi|n|}{X}\right) - \frac{8X}{\pi}\int_0^\infty \psi(y)dy \right|^2 \ll_{\epsilon,l} XGT^{1+\epsilon} ||\psi||^2_{W^{A,\infty}}. \tag{1.7}$$

Such short average of the quantitative quantum ergodicity for holomorphic Hecke eigenforms is first studied in [LS03], and Theorem 1.1 is the generalization to Maass-Hecke eigenforms.

According to Weyl's law, there are asymptotically $\sim \frac{1}{12}T$ Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T+1\}$.[2] Hence Theorem 1.1 implies that

$$\left| \frac{1}{t_\phi} \sum_n \rho_\phi(n+m)\rho_\phi(n)\psi(\frac{\pi|n|}{t_\phi}) - \frac{8}{\pi}\delta_{0,m}\int_0^\infty \psi(y)dy \right| \ll_\epsilon t_\phi^{-1/2+\epsilon}$$

holds on average for any fixed $m \ge 0$ and $\psi \in C_0^\infty(0,\infty)$. As noted above, this implies that Lindelof Hypothesis holds for the triple product L-functions on this shorter range compared to longer range established in [LS95].

In a quantitative form, we have the best result towards QQUE conjecture:

Corollary 1.2. *Let $\psi \in C_0^\infty(0,\infty)$ and let δ and ν be fixed positive constants. All but $O_\epsilon(T^{1/3+\delta+2\nu+\epsilon})$ forms in $\{\phi \mid T < t_\phi < T+1\}$ satisfy*

$$\left| \frac{1}{t_\phi} \sum_n \rho_\phi(n+m)\rho_\phi(n)\psi(\frac{\pi|n|}{t_\phi}) - \frac{8}{\pi}\delta_{0,m}\int_0^\infty \psi(y)dy \right| < t_\phi^{-\nu}||\psi||_{W^{A,\infty}}$$

uniformly in $0 \le m < T^\delta$. Here $A > 0$ is a sufficiently large constant depending only on $\epsilon > 0$.

Now let Z_ϕ be the zero set of ϕ, which in turn is a finite union of real analytic curves. For any subset $\mathbb{K} \subseteq \mathbb{X}$, let $N^{\mathbb{K}}(\phi)$ be the number

[2] One can show using the Selberg's trace formula that asymptotically half of the forms within the set are even.

of connected components (the nodal domains) in $\mathbb{X}\backslash Z_\phi$ which meets $\mathbb{K}$. Let $N(\phi) = N^{\mathbb{X}}(\phi)$. Then the Bogomolny-Schmit Conjecture states that there exists a global constant $C > 0$ such that

$$N(\phi) = C\lambda_\phi + o(\lambda_\phi). \tag{1.8}$$

Note that it is not true for a general Riemannian surfaces that the number of nodal domains of an eigenfunction must increase with the eigenvalue. In [GRS12], the authors study nodal domains crossing $\delta = \{iy \mid y > 0\}$ and prove

$$t_\phi \ll N^\delta(\phi) \ll t_\phi \log t_\phi \tag{1.9}$$

for the even Maass-Hecke cusp forms ϕ. Assuming the conjecture (1.8), this estimate in particular implies that almost all nodal domains do not touch δ![3]

Note that most of the nodal domains they capture in (1.9) are in the region near the cusp determined by $y > t_\phi/100$. So far no unconditional lower bound for the number of nodal domains crossing a fixed compact geodesic segment is known. However one can find such lower bound assuming the Lindelof Hypothesis.

Theorem 1.3 ([GRS12]). *Let $\beta \subset \delta$ be a fixed compact geodesic segment which is sufficiently long. Assume the Lindelof Hypothesis for the L-functions $L(s, \phi)$. Then*

$$N^\beta(\phi) \gg_\epsilon t_\phi^{\frac{1}{12}-\epsilon}.$$

1.1. L^2 **restriction.** The assumption of β being sufficiently long is necessary in order to deduce the lower bound for L^2 norm for the restriction to β

$$\int_\beta |\phi(z)|^2 ds \gg_\beta 1$$

from the QUE theorem. The QQUE conjecture implies the same estimate for any fixed compact geodesic segment $\beta \subset \delta$ ([GRS12]), and therefore as an application of Corollary 1.2, we get:

Corollary 1.4. *Let $\beta \subset \delta$ be any fixed compact geodesic segment. Then*

$$\int_\beta |\phi(z)|^2 ds \gg_\beta 1$$

for all but $O_\epsilon(T^{1/3+\epsilon})$ forms within the set of even Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T+1\}$.

Such lower bound for the restriction is first proved in [HZ04]. In particular, the authors show that

[3] In [GRS12], such nodal domains are called "split."

Theorem 1.5 ([HZ04]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded piecewise smooth manifold with ergodic billiard map. Let $\{u_j\}$ be a sequence of interior eigenfunctions such that*

$$\begin{aligned} -\Delta u_j = \lambda_j^2 u_j \ \textit{in}\ \Omega,\ < u_j, u_k >_{L^2(\Omega)} = \delta_{jk} \\ \partial_\nu u_j = 0 \ \textit{on}\ \partial\Omega \\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots . \end{aligned}$$

Then there exist a positive constant c and a density one subset S of positive integers such that

$$\lim_{j\to\infty,\ j\in S} \int_{\partial\Omega} f|u_j|^2 d\sigma = c\int_{\partial\Omega} f d\sigma$$

for any smooth function f on $\partial\Omega$.

Observe that an even Maass-Hecke cusp form ϕ is an eigenfunction which satisfies the Neumann boundary condition on the domain

$$\Omega = \{z \in \mathbb{H} \mid |z| > 1,\ 0 < \mathrm{Re}(z) < 1/2\}.$$

Although this domain is non-compact, one can expect from Theorem 1.5 that there exists a positive constant c such that

$$\int_\beta |\phi(z)|^2 ds \to c\int_\beta ds$$

for any $\beta \subset \delta$ along a density one subset of even Maass-Hecke cusp forms. Corollary 1.4 hints the existence of such c with a possible exceptional set which is polynomially smaller in the size than the set of all even Maass-Hecke cusp forms.

1.2. L^∞ estimate.

In [IS95], using Selberg's trace formula and amplification method, a nontrivial improvement of L^∞-norm of Maass-Hecke cusp forms is achieved.

Theorem 1.6 ([IS95]). *Let ϕ be a Maass-Hecke cusp form on $\mathbb{X}$. Then for any fixed compact $C \subset \mathbb{X}$, we have*

$$\sup_{z\in C} |\phi(z)| \ll_{C,\epsilon} t_\phi^{\frac{5}{12}+\epsilon}.$$

The key inequality for this estimate is the following(Equation A.12 [IS95]):

$$\sum_{T<t_\phi<T+1} |\phi(z)|^2 \left|\sum_{n\leq N} \alpha_n \rho_\phi(n)\right|^2 \ll_\epsilon N^\epsilon T^\epsilon \left(T\sum |\alpha_n|^2 + (N + N^{1/2}y)T^{1/2}(\sum |\alpha_n|)^2\right)$$

Choosing $\alpha_n = \rho_\phi(n)$ and $N = T^{1/4}$ yields

$$|\phi(z)|^2 \left(\sum_{n \le T^{1/4}} |\rho_\phi(n)|^2 \right)^2 \ll_\epsilon T^{\frac{5}{4}+\epsilon}$$

provided that $z \in C$ for some compact $C \subset \mathbb{X}$. Therefore for ϕ satisfying

$$\sum_{n \le T^{1/4}} |\rho_\phi(n)|^2 \gg T^{1/4}, \tag{1.10}$$

we have

$$\sup_{z \in C} |\phi(z)| \ll_{C,\epsilon} t_\phi^{\frac{3}{8}+\epsilon}.$$

Note that Theorem 1.1 implies that most of Maass-Hecke cusp forms satisfy (1.10).

Corollary 1.7. *Let $C \subset \mathbb{X}$ be a compact subset. All but $O_\epsilon(T^{\frac{13}{12}+\epsilon})$ Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T + T^{1/3}\}$ satisfy*

$$\sup_{z \in C} |\phi(z)| \ll_{C,\epsilon} t_\phi^{\frac{3}{8}+\epsilon}.$$

Note that there are $\sim T^{4/3}$ Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T + T^{1/3}\}$.

1.3. **Application to the number of nodal domains.** Now we give lower bounds for $N^\beta(\phi)$ for almost all ϕ's without the assumptions in Theorem 1.3:

Theorem 1.8. *Let $\beta \subset \delta$ be any fixed compact geodesic segment. Fix $\epsilon > 0$. Then within the set of even Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T+1\}$, all but $O_\epsilon(T^{\frac{5}{6}+\epsilon})$ forms satisfy $N^\beta(\phi) > t_\phi^{\frac{1}{2}\epsilon}$.*

As an application of Corollary 1.7, we improve the lower bound of number of nodal domains given in Theorem 1.3 for almost all forms.

Theorem 1.9. *Let $\beta \subset \delta$ be any fixed compact geodesic segment. Fix $\epsilon > 0$. Then almost all forms within the set of even Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T + T^{1/3}\}$ satisfy*

$$N^\beta(\phi) > t_\phi^{\frac{1}{8}-\epsilon}.$$

2. Quantitative quantum ergodicity on average

We first prove the first case $m \ge 1$ of Theorem 1.1 assuming that l is fixed, for simplicity.

Let $h(y) = e^{-y^2}$ and let $h_{T,G}(y) = h((y-T)/G) + h(-(y+T)/G)$. By (1.1), the sum (1.6) is

$$\ll t_\phi^\epsilon \sum_\phi \frac{h_{T,G}(t_\phi)}{\rho_\phi(1)^2} \left| \sum_n \rho_\phi(n+m)\rho_\phi(n)\psi(\frac{\pi n}{X}) \right|^2 .$$

From the Hecke relation (1.2),

$$\begin{aligned} &\sum_\phi \frac{h_{T,G}(t_\phi)}{\rho_\phi(1)^2} \left| \sum_n \rho_\phi(n+m)\rho_\phi(n)\psi(\frac{\pi n}{X}) \right|^2 \\ =&\sum_f h_{T,G}(t_\phi) \left| \sum_{d|m} \sum_n \rho_f(n(n+d))\psi(\frac{\pi m n}{dX}) \right|^2 \\ \leq&\tau(m) \sum_{d|m} \sum_f h_{T,G}(t_\phi) \left| \sum_n \rho_f(n(n+d))\psi(\frac{\pi m n}{dX}) \right|^2 \end{aligned}$$

where τ is the divisor function. Now expanding the square and then applying the Kuznetsov trace formula, we obtain an identity of the form:

$$\begin{aligned} &\tau(m) \sum_{d|m} \sum_f h_{T,G}(t_\phi) \left| \sum_n \rho_f(n(n+d))\psi(\frac{\pi m n}{dX}) \right|^2 + \{\text{continuous}\} \\ =&\frac{\tau(m)}{\pi^2} \sum_{d|m} \sum_{r_1,r_2} \delta_{r_1(r_1+d),r_2(r_2+d)} \psi(\frac{\pi m r_1}{dX})\psi(\frac{\pi m r_2}{dX}) \int_{\mathbb{R}} \tanh(\pi y) h_{T,G}(y) y dy \\ +&\frac{2i\tau(m)}{\pi} \sum_{d|m} \sum_{c=1}^\infty \sum_{r_1,r_2} \frac{S(r_1(r_1+d), r_2(r_2+d), c)}{c} \psi(\frac{\pi m r_1}{dX})\psi(\frac{\pi m r_2}{dX}) \\ &\times g\left(\frac{4\pi}{c} \sqrt{r_1(r_1+d)r_2(r_2+d)} \right) \end{aligned}$$

where

$$g(x) = \int_{-\infty}^\infty J_{2iy}(x) \frac{h_{T,G}(y)y}{\cosh \pi y} dy.$$

Note that the contribution coming from the continuous spectrum is non-negative, and the diagonal contribution (the second line) is $O(XGT^{1+\epsilon})$. For the non-diagonal contribution (the sum involving Kloosterman sums,) we prove the following:

Lemma 2.1. *Let $\psi \in C_0^\infty(0,\infty)$, $1 \ll R \ll T$ and $0 < d \ll R^{1-\delta}$ for some fixed $\delta > 0$. Then there exists $A > 0$ depending only on θ and ϵ such that*

$$\sum_{c \geq 1} \sum_{r_1,r_2} \psi(\frac{r_1}{R})\psi(\frac{r_2}{R}) \frac{S(r_1(r_1+d), r_2(r_2+d), c)}{c} g\left(\frac{4\pi\sqrt{r_1 r_2 (r_1+d)(r_2+d)}}{c} \right)$$
$$\ll_\epsilon dGRT^{1+\epsilon} ||\psi||^2_{W^{A,\infty}}. \tag{2.1}$$

2.1. **Bessel transform.** To estimate the non-diagonal contribution, we first analyze

$$g(x) = \int_{-\infty}^{\infty} J_{2iy}(x) \frac{h_{T,G}(y)y}{\cosh \pi y} dy.$$

Lemma 2.2. *For any $\epsilon > 0$ and $A > 0$,*

$$g(x) \ll_{\epsilon,A} T^{-A}$$

holds uniformly in $0 < x < GT^{1-\epsilon}$.

Proof. From the identity

$$\left(\frac{J_{2iu}(x) - J_{-2iu}(x)}{\sinh \pi u} \right)^{\wedge} (y) = -i \cos\left(x \cosh(\pi y)\right)$$

and using the Plancherel theorem, we obtain

$$\begin{aligned} g(x) &= \int_{\mathbb{R}} \frac{J_{2iy}(x) - J_{-2iy}(x)}{\sinh \pi y} h_{T,G}(y) y \tanh \pi y dy \\ &= \int_{\mathbb{R}} \frac{J_{2iy}(x) - J_{-2iy}(x)}{\sinh \pi y} h_{T,G}(y)|y| dy + O_A(T^{-A}) \\ &= -i \int_{\mathbb{R}} \cos\left(x \cosh(\pi y)\right) \left(h_{T,G}(u)|u|\right)^{\wedge} (y) dy + O_A(T^{-A}). \end{aligned}$$

Note that

$$\begin{aligned} &\left(h(u)_{T,G}|u|\right)^{\wedge} (y) \\ =&\int_{\mathbb{R}} \left(h(\frac{u-T}{G}) + h(-\frac{u+T}{G}) \right) |u| e(yu) du \\ =&\int_{\mathbb{R}} h(\frac{u-T}{G}) u e(yu) du - \int_{\mathbb{R}} h(-\frac{u+T}{G}) u e(yu) du + O_A(T^{-A}(1+y^2)^{-1}) \\ =&\int_{\mathbb{R}} h(\frac{u-T}{G}) u e(yu) du + \int_{\mathbb{R}} h(\frac{u-T}{G}) u e(-yu) du + O_A(T^{-A}(1+y^2)^{-1}) \\ =&Ge(Ty) \left(h(u)(Gu+T)\right)^{\wedge} (Gy) + Ge(-Ty) \left(h(u)(Gu+T)\right)^{\wedge} (-Gy) \\ &+ O_A(T^{-A}(1+y^2)^{-1}). \end{aligned}$$

Therefore, we have

$$g(x) = -i \int_{\mathbb{R}} \cos\left(\frac{2\pi i T y}{G} + ix \cosh(\frac{\pi y}{G}) \right) \left(h(u)(Gu+T)\right)^{\wedge} (y) dy + O_A(T^{-A}).$$

Because $(h(u)(Gu+T))^{\wedge}(y)$ is rapidly decaying, we can (smoothly) truncate the range of y to $-T^{\epsilon/2} < y < T^{\epsilon/2}$. In such range, since $x < GT^{1-\epsilon}$,

$$\begin{aligned}\frac{d}{dy}\left(\frac{2\pi Ty}{G} + x\cosh(\frac{\pi y}{G})\right) &= \frac{2\pi T}{G} + x\frac{\pi}{G}\sinh(\frac{\pi y}{G}) \\ &> \frac{2\pi T}{G} - \frac{\pi^2 T^{1-\epsilon/2}}{G} \\ &> \frac{T}{G}\end{aligned}$$

for all sufficiently large T and

$$\frac{\partial^k}{\partial y^k}\left(\frac{2\pi Ty}{G} + x\cosh(\frac{\pi y}{G})\right) \ll T^{1-\epsilon/2}G^{-k-\lfloor\frac{k-1}{2}\rfloor} \qquad (k \geq 2)$$

Therefore successive integration by parts yields:

$$g(x) \ll_A G\max\{\frac{G}{T}, \frac{1}{G}\}^A.$$

□

Now let

$$\tilde{g}(x) = \int_0^\infty J_{2iy}(x)\frac{h_{T,G}(y)y}{\cosh \pi y}dy.$$

Lemma 2.3. *Assume that $GT^{1-\epsilon} < x$ with $0 < \epsilon < \theta/2$. For any $A > 0$, there exists $N > 0$ such that, $\tilde{g}(x)$ is a linear sum of*

$$\int_0^\infty g_{k,N}(y,x)yh_{T,G}(y)dy \qquad (k = 0, 1, \cdots, N)$$

plus $O(T^{-A})$, where

$$g_{k,N}(y,x) = (4y^2+x^2)^{-k/2-1/4}\exp(ix + i\sum_{m=1}^{N-1} c_m\frac{y^{2m}}{x^{2m-1}}).$$

Proof. For $x, y > 0$

$$\begin{aligned}J_{2iy}(x) = b(4y^2+x^2)^{-1/4}&\exp\left(i\sqrt{4y^2+x^2} - 2iy\sinh^{-1}(2y/x)\right) \\ &\times \cosh(\pi y)\left(\sum_{m=0}^{N-1} b_m(4y^2+x^2)^{-m/2} + O(x^{-N})\right),\end{aligned}$$

with some explicit constants $b, b_1, b_2, \cdots$. Assuming $y \sim T$ and $x > GT^{1-\epsilon} > T^{1+\frac{\theta}{2}}$, we may expand the exponent to get

$$\begin{aligned}&\exp\left(ix\sqrt{1+4y^2/x^2} - 2iy\sinh^{-1}(2y/x)\right) \\ &= \exp(ix + i\sum_{m=1}^{N-1} c_m\frac{y^{2m}}{x^{2m-1}}) + O(y^{2N}x^{-2N+1}).\end{aligned}$$

with some explicit constants c_m (here $c_1 = -2 \neq 0$). □

Lemma 2.4. *For* $0 < x < 1$,

$$g(x) \ll Gx^2.$$

Proof. From

$$J_{2iy+2}(x) = (x/2)^{2iy+2} \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^{2k}}{k!\Gamma(k+2+2iy)}$$

and the Sterling's formula, we get

$$J_{2iy+2}(x) \ll x^2(|y|+1)^{-3/2} \cosh \pi y.$$

Therefore by shifting the contour,

$$\begin{aligned} g(x) &= -\int_{-\infty}^{\infty} J_{2iy+2}(x) \frac{(y-i)h_{T,G}(y-i)}{\cosh \pi y} dy \\ &\ll x^2 \int_{-\infty}^{\infty} |h_{T,G}(y-i)| dy \\ &\ll Gx^2. \end{aligned}$$

□

Remark: In Lemma 2.2 and 2.3, one only needs the fact that $h(y)$ is a rapidly decreasing function, but for Lemma 2.4, analyticity is required.

2.2. **Reduction.** By Lemma 2.4 and the Weil's bound

$$|S(n,m,c)| \leq (n,m,c)^{1/2} c^{1/2} \tau(c),$$

we can assume that the sum in (2.1) is taken over $c \ll T^A$ for some large $A > 0$. If $R^2 \ll GT^{1-\epsilon}$ for some $\epsilon > 0$, then by Lemma 2.2, we have Lemma 2.1. Hence we may assume $GT^{1-\epsilon_1} \ll R^2$ with fixed ϵ_1 such that

$$\min\{\frac{\theta}{2}, \frac{3\theta-1}{2}\} > \epsilon_1 > 0$$

and that the sum is taken over $c \ll R^2 G^{-1} T^{-1+\epsilon_1}$.

Observe that $g(x)$ is the imaginary part of $\tilde{g}(x)$, ψ is real, and the Kloosterman sums are real. Therefore we may replace $g(x)$ with $\tilde{g}(x)$ in the sum.

Now applying Lemma 2.3, it is sufficient to prove that there exists $A > 0$ such that

$$\begin{aligned} \sum_{c \ll R^2 G^{-1} T^{-1+\epsilon_1}} \sum_{r_1, r_2} \psi(\frac{r_1}{R}) \psi(\frac{r_2}{R}) \frac{S(r_1(r_1+d), r_2(r_2+d), c)}{c} \\ \times g_{k,N}(y, \frac{4\pi\sqrt{r_1 r_2 (r_1+d)(r_2+d)}}{c}) \ll_\epsilon dR^{1+\epsilon} ||\psi||^2_{W^{A,\infty}} \end{aligned} \tag{2.2}$$

for fixed $k \geq 0$ and $y \sim T$. Note that for $X \ll R^{1-\epsilon}$, $|g_{k,N}(y,x)| \ll x^{-1/2}$ and the Weil's bound yield

$$\sum_{c \ll X} \sum_{r_1,r_2} \left|\psi(\frac{r_1}{R})\psi(\frac{r_2}{R})\right| \frac{|S(r_1(r_1+d), r_2(r_2+d), c)|}{c} \times \left|g_{k,N}(y, \frac{4\pi\sqrt{r_1r_2(r_1+d)(r_2+d)}}{c})\right| \ll_\epsilon XR^{1+\epsilon}||\psi||^2_{L^\infty},$$

which is worse than $R^{1+\epsilon}$ in R aspect unless $X \ll R^\epsilon$. Because $R^2G^{-1}T^{-1+\epsilon_1}$ can get as large as $T^{1+\epsilon_1-\theta}$, we have to capture the cancellation coming from the sign changes in the summation to get a right bound. In this article, we investigate cancellation coming from the sum over r_1 and r_2, as in [LS03].

To this end, firstly observe that, for $y \sim T$ and $x \gg T^{1+\epsilon}$, the oscillation of $g_{k,N}(y,x)$ is dictated by e^{ix}. In other words, we have

$$\frac{\partial^m}{\partial x^m}\left(e^{-ix}g_{k,N}(y,x)\right) \ll x^{-\frac{1}{2}}T^{-m\epsilon}.$$

for any $m > 0$. Also, the main oscillating factor of

$$\exp\left(\frac{4\pi i\sqrt{r_1r_2(r_1+d)(r_2+d)}}{c}\right)$$

with respect to $r_1 \sim R$ and $r_2 \sim R$ is

$$e_c(2r_1r_2 + dr_1 + dr_2),$$

where $e_c(x) = \exp(2\pi ix/c)$.

From these observations we define $f_c(r_1, r_2)$ by

$$e_c(2r_1r_2+dr_1+dr_2)f_c(r_1,r_2) = \psi(\frac{r_1}{R})\psi(\frac{r_2}{R})g_{k,N}(y, \frac{4\pi\sqrt{r_1r_2(r_1+d)(r_2+d)}}{c})$$

and for each c rearrange the sum modulo c:

$$\begin{aligned}&\sum_{r_1,r_2} S(r_1(r_1+d), r_2(r_2+d), c)e_c(2r_1r_2+dr_1+dr_2)f_c(r_1,r_2)\\ &= \sum_{a,b\ (c)} S(a(a+d), b(b+d), c)e_c(2ab+da+db) \sum_{\substack{r_1\equiv a\ (c)\\ r_2\equiv b\ (c)}} f_c(r_1,r_2)\\ &= \frac{1}{c^2}\sum_{u\ (c)}\sum_{v\ (c)}\left(\sum_{a,b\ (c)} S(a(a+d), b(b+d), c)e_c(2ab+(d+u)a+(d+v)b)\right)\\ &\quad\times \sum_{r_1,r_2} f_c(r_1,r_2)e_c(-ur_1-vr_2).\end{aligned}$$

We assume here that $|u|, |v| \leq \frac{c}{2}$. Note that as we expect $f_c(r_1,r_2)$ is mildly oscillating, the sum

$$\sum_{r_1,r_2} f_c(r_1,r_2)e_c(-ur_1-vr_2)$$

is going to be negligible unless both u and v are relatively smaller than c, which will be determined by the oscillation of $f_c(r_1, r_2)$. We quantify this and then estimate the sum via Poisson summation formula in next two sections.

For the rest two sections, we give an estimation of

$$\sum_{a,b\ (c)} S(a(a+d), b(b+d), c) e_c(2ab + (d+u)a + (d+v)b)$$

from [LS03] and prove Lemma 2.1.

2.3. **Estimating** $f_c(r_1, r_2)$**.** Recall that $d \ll R^{1-\delta}$, $y \sim T$, $GT^{1-\epsilon_1} \ll R^2$, and $c \ll R^2 G^{-1} T^{-1+\epsilon_1}$. For this and the next section, we further assume that $dR^{\epsilon_2} \ll c$ for some $\epsilon_2 > 0$. Let

$$\Delta(r_1, r_2) = \sqrt{r_1 r_2 (r_1 + d)(r_2 + d)}$$
$$\alpha(x) = \sum_{m=1}^{N-1} c_m \frac{y^{2m}}{x^{2m-1}}$$
$$\varphi(r_1, r_2) = \alpha(\frac{4\pi\Delta}{c}) + \frac{4\pi}{c}(\Delta - r_1 r_2 - \frac{dr_1}{2} - \frac{dr_2}{2})$$

and

$$g_c(r_1, r_2) = \psi\left(\frac{r_1}{R}\right)\psi\left(\frac{r_2}{R}\right)\left(4y^2 + \frac{16\pi^2 r_1 r_2 (r_1+d)(r_2+d)}{c^2}\right)^{-k/2-1/4}$$

Then

$$f_c(r_1, r_2) = g_c(r_1, r_2) \exp\left(i\varphi(r_1, r_2)\right).$$

Firstly,

$$\frac{\partial^{k_1+k_2} g_c}{\partial {r_1}^{k_1} \partial {r_2}^{k_2}} \ll c^{1/2} R^{-1-k_1-k_2} ||\psi||^2_{W^{k_1+k_2,\infty}}. \tag{2.3}$$

For $r_1 \sim r_2 \sim R$, we have

$$\Delta = r_1 r_2 + O(dR)$$
$$\Delta_{r_i} = r_{3-i} + O(d)$$
$$\Delta_{r_i r_i} = O(d^2 R^{-2})$$
$$\Delta_{r_1 r_2} = 1 + O(d^2 R^{-2})$$
$$\frac{\partial^{k_1+k_2}\Delta}{\partial {r_1}^{k_1} \partial {r_2}^{k_2}} = O(d^2 R^{-k_1-k_2}) \qquad (k_1 + k_2 \geq 3)$$
$$\frac{\partial^{k_1+k_2}}{\partial {r_1}^{k_1} \partial {r_2}^{k_2}}(\Delta - r_1 r_2 - \frac{dr_1}{2} - \frac{dr_2}{2}) = O(d^2 R^{-k_1-k_2}) \qquad (k_1 + k_2 \geq 1)$$

and

$$\alpha_x = -c_1 \frac{y^2}{x^2} + O(T^4 x^{-4})$$
$$\alpha_{xx} = 2c_1 \frac{y^2}{x^3} + O(T^4 x^{-5})$$
$$\frac{\partial^k \alpha}{\partial x^k} = O(T^2 x^{-1-k}).$$

Hence

$$\frac{\partial^{k_1+k_2}\varphi}{\partial {r_1}^{k_1} \partial {r_2}^{k_2}} \ll cT^2 R^{-2-k_1-k_2} \tag{2.4}$$

which implies that

$$\frac{\partial^{k_1+k_2} f_c}{\partial {r_1}^{k_1} \partial {r_2}^{k_2}} \ll \frac{c^{1/2}}{R} \left(\frac{cT^2}{R^3}\right)^{k_1+k_2} ||\psi||^2_{W^{k_1+k_2,\infty}}. \tag{2.5}$$

Note that

$$\frac{cT^2}{R^3} \ll \frac{T^{1+\epsilon_1}}{GR} \ll \left(\frac{T^{1+3\epsilon_1}}{G^3}\right)^{1/2} \ll T^{(1+3\epsilon_1-3\theta)/2} \ll T^{-(3\theta-1)/4}.$$

From $c \gg dR^{\epsilon_2}$, we also get

$$\varphi_{r_1} \sim \varphi_{r_2} \sim \frac{cT^2}{R^3} \tag{2.6}$$

and

$$\begin{aligned}
\varphi_{r_i r_j} &= \frac{16\pi^2 \alpha_{xx}(4\pi\Delta/c)}{c^2} \Delta_{r_i}\Delta_{r_j} + \frac{4\pi\alpha_x(4\pi\Delta/c)}{c}\Delta_{r_i r_j} + O(\frac{d^2}{cR^2}) \\
&= \frac{8\pi^2\alpha_{xx}(4\pi\Delta/c)}{c^2}(2\Delta_{r_i}\Delta_{r_j} - \Delta\Delta_{r_i r_j}) + O(\frac{\Delta_{r_i r_j} T^4 c^3}{R^8} + \frac{c}{R^{2+2\epsilon_2}}) \\
&= \frac{8\pi^2\alpha_{xx}(4\pi\Delta/c)}{c^2}((2-\delta_{i,j}) r_{3-i} r_{3-j} + O(dR)) + O(\frac{T^4 c^3}{R^8} + \frac{c}{R^{2+2\epsilon_2}}),
\end{aligned}$$

which implies

$$\varphi_{r_i r_j} \sim \frac{cT^2}{R^4} \quad \text{and} \quad |\varphi_{r_1 r_1}\varphi_{r_2 r_2} - \varphi^2_{r_1 r_2}| \gg \frac{c^2 T^4}{R^8}. \tag{2.7}$$

Lemma 2.5. *Let $f(x_1, x_2)$ be a real and algebraic function defined in a rectangle $D = [a, b] \times [c, d] \subset \mathbb{R}^2$. Assume throughout D that*

$$|f_{x_i x_i}| \sim \lambda \ \text{for } i = 1, 2, \ |f_{x_1 x_2}| \ll \lambda, \ \text{and} \ \left|\frac{\partial(f_{x_1}, f_{x_2})}{\partial(x_1, x_2)}\right| \gg \lambda^2;$$

then

$$\iint_D e^{if(x_1,x_2)} dx_1 dx_2 \ll \frac{1 + |\log(b-a)| + |\log(d-c)| + |\log\lambda|}{\lambda}.$$

2.4. **Poisson summation.** Applying the Poisson summation formula for the sum in r_1 and r_2, we get

$$\sum_{r_1,r_2} f_c(r_1,r_2)e_c(-ur_1-vr_2)=\sum_{j,k}B(j,k)$$

where

$$\begin{aligned}B(j,k)&=\iint f_c(r_1,r_2)e_c(-ur_1-vr_2)e(jr_1+kr_2)dr_1dr_2\\&=\iint f_c(r_1,r_2)e((j-\frac{u}{c})r_1+(k-\frac{v}{c})r_2)dr_1dr_2.\end{aligned}$$

By (2.5), integrating by parts shows that

$$B(j,k)=O\left(c^{1/2}R(\max\{|j|,|k|\}-\frac{1}{2})^{-A}\left(T^{(3\theta-1)/4}\right)^{-A}||\psi||^2_{W^{A,\infty}}\right)$$

Therefore

$$\sum_{j,k}B(j,k)=B(0,0)+O\left(c^{1/2}R\left(T^{(3\theta-1)/5}\right)^{-A}||\psi||^2_{W^{A,\infty}}\right)$$

for any $A>0$. Now for $B(0,0)$, we apply integration by parts to get

$$\begin{aligned}&\iint f_c(r_1,r_2)e_c(-ur_1-vr_2)dr_1dr_2\\ \ll&\left|\iint_D e^{i\varphi(r_1,r_2)}e_c(-ur_1-vr_2)dr_1dr_2\right|\iint|g_c(r_1,r_2)_{r_1r_2}|dr_1dr_2\\ \ll&||\psi||^2_{W^{1,\infty}}\frac{c^{1/2}}{R}\left|\iint_D e^{i\varphi(r_1,r_2)}e_c(-ur_1-vr_2)dr_1dr_2\right|\end{aligned}$$

where we can assume D is a rectangle such that $(r_1,r_2)\in D$ implies $r_1\sim R$ and $r_2\sim R$.

Observe that by (2.6),

$$\iint_D e^{i\varphi(r_1,r_2)}e_c(-ur_1-vr_2)dr_1dr_2$$

has a stationary phase only when

$$u\sim v\sim\frac{c^2T^2}{R^3}\tag{2.8}$$

is satisfied.

If any of u or v does not satisfy this, then from (2.3), (2.4), and (2.6), integrating by parts yields

$$\begin{aligned}\iint g_c(r_1,r_2)e^{i\varphi(r_1,r_2)-\frac{2\pi i}{c}(ur_1+vr_2)}dr_1dr_2&\ll c^{\frac{1}{2}}R(\frac{R^2}{cT^2})^A||\psi||^2_{W^{A,\infty}}\\&\ll c^{\frac{1}{2}}RT^{-\epsilon_2A}||\psi||^2_{W^{A,\infty}}\end{aligned}$$

for any $A>0$, from the assumption $c\gg dT^{\epsilon_2}$.

Now for u and v which satisfy (2.8), we use (2.7) and Lemma 2.5 to get

$$\iint_D e^{i\varphi(r_1,r_2)} e_c(-ur_1 - vr_2) dr_1 dr_2 \ll \frac{R^4}{cT^2} \log R$$

and therefore

$$\sum_{r_1,r_2} f_c(r_1,r_2) e_c(-ur_1 - vr_2) \ll ||\psi||^2_{W^{1,\infty}} \frac{R^3}{c^{1/2}T^2} \log R. \tag{2.9}$$

2.5. **Kloosterman sums.** In this section we give a bound of

$$\sum_{a,b\ (c)} S(a(a+d), b(b+d), c) e_c(2ab + (d+u)a + (d+v)b)$$

that is given in [LS03]. For fixed d, u, v, and integer γ, let

$$S_c(\gamma) = \sum_{a,b\ (c)} S\left(a\left(\gamma a + d\right), b\left(\gamma b + d\right), c\right) e_c(2\gamma ab + (d+u)a + (d+v)b).$$

Then for $(c_1, c_2) = 1$, we have $S_{c_1c_2}(\gamma) = S_{c_1}(\gamma c_2) S_{c_2}(\gamma c_1)$.

For $(c, 2\gamma) = 1$, note that

$$\begin{aligned} &a(\gamma a + d)x + 2\gamma ab + (d+u)a \\ \equiv&\gamma x\left(a^2 + 2(\bar{x}b + \bar{2}(d+u)\bar{x}\bar{\gamma} + \bar{2}d\bar{\gamma})a\right) \pmod{c} \end{aligned}$$

and

$$\begin{aligned} &-\gamma x(\bar{x}b + \bar{2}(d+u)\bar{x}\bar{\gamma} + \bar{2}d\bar{\gamma})^2 + b\left(\gamma b + d\right)\bar{x} + (d+v)b \\ \equiv&(v - \bar{x}u)b - \overline{4\gamma}x((d+u)\bar{x} + d)^2. \pmod{c} \end{aligned}$$

Therefore

$$\begin{aligned} S_c(\gamma) = &\sum_{\substack{x\ (c) \\ (x,c)=1}} \sum_{a,b\ (c)} e_c\left(a\left(\gamma a + d\right)x + b\left(\gamma b + d\right)\bar{x}\right) \\ &\times e_c(2\gamma ab + (d+u)a + (d+v)b) \\ = &\sum_{a\ (c)} e_c(\gamma x a^2) \sum_{\substack{x\ (c) \\ (x,c)=1}} e_c\left(-\overline{4\gamma}x((d+u)\bar{x} + d)^2\right) \sum_{b\ (c)} e_c((v - \bar{x}u)b) \end{aligned}$$

and by the evaluation of the Gauss sum, this is $\ll c^{3/2}(v,c)$ if $(v,c) = (u,c)$ and 0 otherwise. Writing $c = c_1c_2$ with $(2, c_1) = 1$ and $c_2 | 2^\infty$, we infer that

$$S_c(1) = S_{c_1}(c_2) S_{c_2}(c_1) = \begin{cases} O((v,c_1)c_1^{3/2}c_2^{5/2+\epsilon}) & \text{if } (u,c_1) = (v,c_1), \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

where we have bounded $S_{c_2}(c_1)$ by $c_2^{5/2+\epsilon}$ using the Weil's bound.

2.6. **Proof of the theorem.** Assume first that $dR^{\epsilon_2} \ll c$. Then by (2.9) and (2.10), we have

$$\begin{aligned}\sum_{u\ (c)}\sum_{v\ (c)}&\left(\sum_{a,b\ (c)} S(a(a+d),b(b+d),c)e_c(2ab+(d+u)a+(d+v)b)\right)\\ &\times\sum_{r_1,r_2} f_c(r_1,r_2)e_c(-ur_1-vr_2)\\ &\ll \sum_{\substack{u\sim v\sim\frac{c^2T^2}{R^3}\\(u,c)=(v,c)}}(v,c_1)c_1c_2^{2+\epsilon/2}\frac{R^3}{T^2}\log R||\psi||^2_{W^{A,\infty}}\\ &\ll c_1c_2^2T^2R^{-3}\log R||\psi||^2_{W^{A,\infty}}\end{aligned}$$

provided that, for instance, $\min\{3\theta-1,\epsilon_2\}A>100$.

Therefore the left hand side of (2.2) is

$$\begin{aligned}&=\sum_{c<dR^{\epsilon_2}}+\sum_{dR^{\epsilon_2}<c\ll R^2G^{-1}T^{-1+\epsilon_1}}\\ &\ll_\epsilon dR^{1+\epsilon_2+\epsilon}||\psi||^2_{L^\infty}+T^2R^{-3+\epsilon}\left(\sum_{\substack{c_2\ll R^2G^{-1}T^{-1+\epsilon_1}\\c_2|2^\infty}}c_2^3\sum_{c_1\ll R^2G^{-1}T^{-1+\epsilon_1}/c_2}c_1^2\right)||\psi||^2_{W^{A,\infty}}\\ &\ll_{\epsilon_2}\left(dR^{1+2\epsilon_2}+\frac{R^{3+2\epsilon_2}}{T^2}T^{1+3\epsilon_1-3\theta}\right)||\psi||^2_{W^{A,\infty}}\\ &\ll_{\epsilon_2} dR^{1+2\epsilon_2}||\psi||^2_{W^{A,\infty}}\end{aligned}$$

since $3\theta-1>3\epsilon_1>0$. This establishes (2.2) which implies Lemma 2.1, hence Theorem 1.1.

2.7. **The case $m=0$.** Let $G(s)$ be the Mellin transform of ψ:

$$G(s)=\int_0^\infty\psi(y)y^{s-1}dy.$$

Then from the Mellin inversion transform,

$$\begin{aligned}\sum_{n\neq0}\rho_\phi(n)^2\psi\left(\frac{\pi|n|}{X}\right)&=\frac{1}{\pi i}\int_{(2)}\sum_{n\geq1}\frac{\rho_\phi(n)^2}{n^s}\left(\frac{X}{\pi}\right)^sG(s)ds\\ &=\frac{\rho_\phi(1)^2}{\pi i}\int_{(2)}\frac{\zeta(s)}{\zeta(2s)}L(s,\mathrm{sym}^2\phi)\left(\frac{X}{\pi}\right)^sG(s)ds\\ &=\frac{\rho_\phi(1)^2}{\pi i}\int_{(1/2)}\frac{\zeta(s)}{\zeta(2s)}L(s,\mathrm{sym}^2\phi)\left(\frac{X}{\pi}\right)^sG(s)ds\\ &+\frac{2X}{\pi}\rho_\phi(1)^2L(1,\mathrm{sym}^2\phi)\int_0^\infty\psi(y)dy.\end{aligned}$$

Now that $\rho_\phi(1)^2 L(1, \mathrm{sym}^2\phi) = 4$, we have

$$\begin{aligned}&\sum_{n\neq 0}\rho_\phi(n)^2\psi\left(\frac{\pi|n|}{X}\right) - \frac{8X}{\pi}\int_0^\infty \psi(y)dy\\ =&\frac{\rho_\phi(1)^2}{\pi i}\int_{(1/2)}\frac{\zeta(s)}{\zeta(2s)}L(s,\mathrm{sym}^2\phi)\left(\frac{X}{\pi}\right)^s G(s)ds.\end{aligned}$$

Using the approximate functional equation, we can represent $L(s, \mathrm{sym}^2\phi)$ as a smooth sum of $\lambda_\phi(n^2)n^{-s}$ of length at most $t_\phi^{1+\epsilon}$. By smoothly summing over ϕ, and following the proof of the case when $m \geq 1$, we conclude the proof of (1.7).

3. Lower bound for the number of sign changes on a compact geodesic segment

Let

$$M_1(\phi) = \sup_{a<\alpha<\beta<b}\left|\int_\alpha^\beta \phi(iy)\frac{dy}{y}\right|.$$

Then

$$\begin{aligned}&\sum_{T<t_\phi<T+1} M_1(\phi)^2\\ \ll&\sum_{T<t_\phi<T+1} t_\phi^{-1/2}\left(\int_0^{2t_\phi}\left|L(\frac{1}{2}+it,\phi)\right|(1+|t-t_\phi|)^{-1/4}\min\{1,\frac{1}{t}\}dt\right)^2\\ \leq&\sum_{T<t_\phi<T+1} t_\phi^{-1/2}\left(\int_0^{2t_\phi}\left|L(\frac{1}{2}+it,\phi)\right|^2(1+|t-t_\phi|)^{-1/4}\min\{1,\frac{1}{t}\}dt\right)\\ &\times\left(\int_0^{2t_\phi}(1+|t-t_\phi|)^{-1/4}\min\{1,\frac{1}{t}\}dt\right)\\ \ll& T^\epsilon\end{aligned}$$

where we used

$$\sum_{T<t_\phi<T+1}\left|L(\frac{1}{2}+it,\phi)\right|^2 \ll (T+t^{2/3})^{1+\epsilon} \qquad \text{([Jut04])}$$

in the last inequality. Therefore among even Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T+1\}$, all but $O(T^{\frac{5}{6}+\epsilon})$ forms satisfy $M_1(\phi) < t_\phi^{-\frac{5}{12}-\frac{1}{3}\epsilon}$. Note that for any function f on $[a,b]$, denoting the number of sign changes of f by $S(f)$, we have

$$S(f)M_1(f) \geq ||f||_{L^1} \geq \frac{||f||_{L^2}^2}{||f||_{L^\infty}}. \qquad (3.1)$$

Therefore from Corollary 1.4 and $\sup|\phi(z)| \ll_\epsilon t_\phi^{5/12+\epsilon}$ ([IS95]), we get Theorem 1.8, since $S(\phi) \ll N^\beta(\phi)$ [GRS12].

Above estimate also shows that almost all Maass-Hecke cusp forms in $\{\phi \mid T < t_\phi < T + T^{1/3}\}$ satisfy $M_1(\phi) < t_\phi^{-\frac{1}{2}+\epsilon}$. Therefore if we apply Corollary 1.7 instead of the L^∞ estimate given in [IS95], we obtain Theorem 1.9.

Remark 3.1. *From the Hölder's inequality, we have*

$$||f||_{L^p}^p ||f||_{L^1}^{p-2} \geq ||f||_{L^2}^{2(p-1)}$$

for any $p > 2$. If we assume L^∞ conjecture for Maass forms, then

$$\int_a^b |\phi(iy)|^p dy \ll_{p,\epsilon} t_\phi^\epsilon.$$

Therefore for a sufficiently long β, a sharp upper bound for $||\phi||_{L^p(\beta)}$ any $p > 2$ yields $N^\beta(\phi) \gg_\epsilon t_\phi^{1/2-\epsilon}$ under the Lindelof Hypothesis, or $N^\beta(\phi) \gg_\epsilon t_\phi^{1/6-\epsilon}$ unconditionally.

References

[GRS12] A. Ghosh, A. Reznikov, and P. Sarnak. Nodal domains of hecke-maass cusp forms on the modular surface. *arXiv:1207.6625 [math.NT]*, 2012.

[HL94] J. Hoffstein and P. Lockhart. Coefficients of maass forms and the siegel zero. *Ann. of Math.*, 140:161–181, 1994.

[HZ04] A. Hassell and S. Zelditch. Quantum ergodicity of boundary values of eigenfunctions. *Communications in Mathematical Physics*, 248:119–168, 2004.

[IS95] H. Iwaniec and P. Sarnak. L^∞ norms of eigenfunctions of arithmetic surfaces. *Annals of Mathematics*, 141:301–320, 1995.

[Iwa90] H. Iwaniec. Small eigenvalues of laplacian for $\Gamma_0(N)$. *Acta Arith.*, 16:65–82, 1990.

[Jut04] M. Jutila. The spectral mean square of hecke L-functions on the critical line. *Publications De L'institut Mathematique*, 76:41–55, 2004.

[Lin06] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math.*, 163:165–219, 2006.

[LS95] W. Luo and P. Sarnak. Quantum ergodicity of eigenfunctions on $PSL_2\mathbb{Z}\backslash\mathbb{H}^2$. *Inst. Hautes tudes Sci. Publ. Math.*, 81:207–237, 1995.

[LS03] W. Luo and P. Sarnak. Mass equidistribution for hecke eigenforms. *Communications on Pure and Applied Mathematics*, LVI:0874–0891, 2003.

[Sou10] K. Soundararajan. Quantum unique ergodicity for $SL(2,\mathbb{Z})\backslash\mathbb{H}$. *Ann. of Math.*, 172:1529–1538, 2010.

[Wat02] T. C. Watson. Rankin triple products and quantum chaos. *Ph.D. Thesis, Princeton University (eprint available at arxiv.org:math/0810.0425v3 [math.NT]), to appear in Ann. of Math.*, 2002.