On a generalization of Hirzebruch's theorem to Bott towers

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Abstract

The aim of this paper is to generalize a theorem of Hirzebruch for the complex 2-dimensional Bott manifolds, usually called Hirzebruch surfaces, to more general Bott towers of height n. To be more precise, in this paper we show that two Bott manifold $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ and $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha'_n)$ are diffeomorphic to each other, provided that both $\alpha_n \equiv \alpha'_n \mod 2$ and $\alpha_n^2 = (\alpha'_n)^2$ hold in the cohomology ring of $B_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$ over integer coefficients. We also give some partial affirmative results essentially saying that the converse is true under certain conditions.

Among other things, the fact that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes plays an important role in the proofs of our main results.

Keywords: Bott towers, Bott manifolds, Hirzebruch surfaces, toric varieties, Petrie's conjecture, strong cohomological rigidity conjecture.

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1 Introduction and Main Results

Our main concern of this paper is a family of compact complex manifolds, called a Bott tower introduced first by Bott and Samelson in [2], which have recently attracted a great amount of attention in toric topology world (refer to [7] and [3]). They form a very nice family of toric manifolds, and possess several extra structures such as iterated fibrations and certain distinguished sections.

In order to construct such a family of compact complex manifolds, let $\mathbb{L}_1$ be a holomorphic complex line bundle over $B_1 = \mathbb{CP}^1$. Then take its direct sum with the trivial complex line bundle $\underline{\mathbb{C}}$ and projectivize each fiber to obtain a complex manifold $B_2 = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_1)$. B_2 is a fiber bundle over B_1 with a fiber $\mathbb{CP}^1$, and is called a *Hirzebruch surface*. We can repeat this process n times, so that each B_j is a fiber bundle over B_{j-1} with a fiber $\mathbb{CP}^1$.

To be more precise, a *Bott tower* $\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^n$ *of height* n is inductively defined as a sequence of $\mathbb{CP}^1$-bundles starting from a point $*$:

$$B_n(\alpha_1, \cdots, \alpha_n) \xrightarrow{\pi_n} B_{n-1}(\alpha_1, \cdots, \alpha_{n-1}) \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1(\alpha_1) \xrightarrow{\pi_1} B_0 = \{*\},$$

where $B_j(\alpha_1, \alpha_2, \cdots, \alpha_j)$ denotes the projectivization $\mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_j)$ of the trivial complex line bundle $\underline{\mathbb{C}}$ and a complex line bundle $\mathbb{L}_j$ over $B_{j-1}(\alpha_1, \cdots, \alpha_{j-1})$, and

$$\pi_j : B_j(\alpha_1, \alpha_2, \cdots, \alpha_j) \to B_{j-1}(\alpha_1, \alpha_2, \cdots, \alpha_{j-1})$$

denotes the projection for each j with $1 \leq j \leq n$. By definition, each $B_j(\alpha_1, \cdots, \alpha_j)$ is a toric manifold which admits an effective algebraic action of the torus $(\mathbb{C} - \{0\})^n$ having it as an open dense orbit, and it is called a *j-step Bott manifold*, or just a *Bott manifold* (refer to [2] and [8] for more details). Note that by its construction α_1 always vanishes. In particular, if all the first Chern classes α_i used to construct a Bott tower of height n are zero, then $B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is diffeomorphic to $(\mathbb{CP}^1)^n$, and $B_2(\alpha_1, \alpha_2)$ is simply a Hirzebruch surface. Analogously, a generalized Bott manifold of height n can be inductively defined also as a sequence of $\mathbb{CP}^{n_i}$-bundles with $n_i \geq 1$ starting from a point.

From now on, for the sake of simplicity, we will very often use the notation B_n for $B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$, if there is no danger of any confusion. Moreover, we shall use the prime notation to denote any objects for the second Bott manifold $B_n(\alpha'_1, \alpha'_2, \cdots, \alpha'_n)$.

Let x_j denote the first Chern class of the tautological line bundle γ_j over B_j. it follows from a formula of Borel and Hirzebruch in [1] that $H^*(B_j; \mathbb{Z})$ is a free module over $H^*(B_{j-1}; \mathbb{Z})$ through the projection $\pi_j^* : H^*(B_{j-1}; \mathbb{Z}) \to H^*(B_j; \mathbb{Z})$ with two generators 1 and x_j of degree 0 and 2, respectively. Thus, when $H^*(B_j; \mathbb{Z})$ is regarded as a subring of $H^*(B_n; \mathbb{Z})$ by using the pullback of the composition

$$\pi_{j+1} \circ \pi_{j+2} \circ \cdots \circ \pi_n : B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_{j+2}} B_{j+1} \xrightarrow{\pi_{j+1}} B_j$$

of the projection maps $\pi_{j+1}, \pi_{j+2}, \cdots$, and π_n, it can be shown that the cohomology ring $H^*(B_n; \mathbb{Z})$ of $B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is given by

$$\mathbb{Z}[x_1, x_2, \cdots, x_n] / \langle x_i^2 = \alpha_i x_i \,|\, i = 1, 2, \cdots, n \rangle.$$

Since $\alpha_j \in H^2(B_{j-1}(\alpha_1, \cdots, \alpha_{j-1}); \mathbb{Z})$, it follows from [6], Proposition 3.1 that we may write

$$\alpha_j = \sum_{i=1}^{j-1} c_j^i x_i, \quad c_j^i \in \mathbb{Z}. \tag{1.1}$$

Note also that $H^2(B_j; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^j$ and that $\mathbb{Z}^j$ is bijective with the collection of isomorphism classes of holomorphic line bundles over B_j which are in turn classified by the first Chern classes ([8], Lemmas 2.11 and 2.14).

For the purposes of this paper, an *isomorphism* between two Bott towers is defined to be a collection $\{F_j\}_{j=1}^n$ of diffeomorphisms

$$F_j : B_j(\alpha_1, \alpha_2, \cdots, \alpha_{j-1}, \alpha_j) \to B_j(\alpha'_1, \alpha'_2, \cdots, \alpha'_{j-1}, \alpha'_j)$$

which commute with the projection maps π_j and π'_j in that the following diagram commutes:

$$\begin{array}{ccc} B_j(\alpha_1, \alpha_2, \cdots, \alpha_{j-1}, \alpha_j) & \xrightarrow{F_j} & B_j(\alpha'_1, \alpha'_2, \cdots, \alpha'_{j-1}, \alpha'_j) \\ \pi_j \downarrow & & \downarrow \pi'_j \\ B_{j-1}(\alpha_1, \alpha_2, \cdots, \alpha_{j-2}, \alpha_{j-1}) & \xrightarrow{F_{j-1}} & B_{j-1}(\alpha'_1, \alpha'_2, \cdots, \alpha'_{j-2}, \alpha'_{j-1}). \end{array}$$

Note that this definition is weaker than that given in [8], Definition 2.6.

It is well-known from a result of Hirzebruch in [9] that $B_2(\alpha_1, \alpha_2)$ is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, if $\alpha_2 \equiv 0 \bmod 2$, while $B_2(\alpha_1, \alpha_2)$ is diffeomorphic to $\mathbb{CP}^1 \# \overline{\mathbb{CP}^1}$, otherwise. That is, we have the following theorem (see, e.g., [13], p. 16 or [11], Example 2.2).

Theorem 1.1. *$B_2(\alpha_1, \alpha_2)$ is isomorphic (or diffeomorphic) to $B_2(\alpha_1, \alpha'_2)$ if and only if $\alpha_2 \equiv \alpha'_2 \bmod 2$ holds.*

In particular, this implies that $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ is graded ring isomorphic to $H^*(B_2(\alpha_1, \alpha'_2); \mathbb{Z})$ if and only if both α_2 and α'_2 are equal to $0 \bmod 2$, or equivalently $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ is graded ring isomorphic to $H^*(B_2(\alpha_1, \alpha'_2); \mathbb{Z})$ if and only if neither α_2 nor α'_2 are equal to $0 \bmod 2$.

Related to these observations, the following conjecture has been posed (refer to [11] and [12]).

Conjecture 1.2. *Let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \cdots, \alpha_n); \mathbb{Z}) \to H^*(B_n(\alpha'_1, \alpha'_2, \cdots, \alpha'_n); \mathbb{Z})$$

be a graded ring isomorphism over integers. Then ψ is actually given by the pullback of a diffeomorphism from $B_n(\alpha'_1, \alpha'_2, \cdots, \alpha'_n)$ to $B_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

This conjecture is usually called a *strong cohomological rigidity conjecture for Bott manifolds*. It is still open in its full generality, and seems to be a very difficult problem, even though there are some partial affirmative answers for certain special cases (refer to, e.g., [4] and [5]). Conjecture 1.2 is also closely related to the well-known Petrie's conjecture, and there are some recent results for Bott manifolds, related to the Petrie's conjecture (see, e.g., [14], [15], [10] and [5]).

Our aim of this paper is not to directly deal with Conjecture 1.2, but rather to generalize the theorem of Hirzebruch for the complex 2-dimensional Bott manifolds $B_2(\alpha_1, \alpha_2)$ to more general Bott towers of height n. In view of the very definitions of Bott towers or Bott manifolds, we think that this kind of generalization is more appropriate and more useful in understanding the Bott towers or Bott manifolds.

Theorem 1.3. *Let a Bott tower $\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^{n-1}$ be isomorphic to $\{(B_j(\alpha'_1, \cdots, \alpha'_j), \pi'_j)\}_{j=1}^{n-1}$ by a family $\{F_j\}_{j=1}^{n-1}$ of diffeomorphisms, and let α_n and α'_n be two elements of*

$$H^2(B_{n-1}(\alpha_1, \cdots, \alpha_{n-1}); \mathbb{Z}) \text{ and } H^2(B_{n-1}(\alpha'_1, \cdots, \alpha'_{n-1}); \mathbb{Z}),$$

respectively, such that

(a) $\alpha_n \equiv F^*_{n-1}(\alpha'_n) \bmod 2$*, and*

(b) $\alpha_n^2 = (F^*_{n-1}(\alpha'_n))^2$.

Then $(B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n), \pi_n)$ is diffeomorphic to $(B_n(\alpha'_1, \cdots, \alpha'_{n-1}, \alpha'_n), \pi'_n)$ by a diffeomorphism F_n which commutes with π_n and π'_n, so that two Bott towers $\{(B_j(\alpha_1, \alpha_2, \cdots, \alpha_j), \pi_j)\}_{j=1}^{n}$ and

$$\{(B_j(\alpha'_1, \alpha'_2, \cdots, \alpha'_j), \pi'_j)\}_{j=1}^{n-1} \cup \{(B_n(\alpha'_1, \alpha'_2, \cdots, \alpha'_{n-1}, \alpha'_n), \pi'_n)\}$$

are isomorphic to each other.

Remark 1.4. Since F^*_{n-1} is a graded ring isomorphism, the condition that $\alpha_n \equiv F^*_{n-1}(\alpha'_n) \bmod 2$ is equivalent to saying that either both of α_2 and α'_2 are equal to 0 mod 2 or neither of α_2 and α'_2 are equal to 0 mod 2. This fits with the condition in Theorem 1.1.

As an immediate consequence of Theorem 1.3, we have the following corollary which can be regarded as a higher dimensional analogue of a result of Hirzebruch in [9] (Theorem 1.1).

Corollary 1.5. *Given two Bott manifolds*

$$B_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n) \text{ and } B_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n'),$$

$B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ *is diffeomorphic to* $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n')$*, provided that both*

$$\alpha_n \equiv \alpha_n' \bmod 2, \quad \text{and} \quad \alpha_n^2 = (\alpha_n')^2 \tag{1.2}$$

hold.

Remark 1.6. (a) In case of Hirzebruch surfaces, every complex vector bundle of rank 2 over $B_1(\alpha_1) = \mathbb{CP}^1$ is classified by the *first* Chern class only, so we do not need the second condition $\alpha_2^2 = (\alpha_2')^2$ in Theorem 1.1, contrary to Theorem 1.3 or Corollary 1.5. In fact, in this case the extra condition $\alpha_2^2 = (\alpha_2')^2$ automatically holds, since they always vanish, due to the degree reason.

(b) In Section 2, Example 2.6, we will provide infinitely many non-trivial Bott manifolds which satisfy two conditions in (1.2).

We organize this paper, as follows. In Section 2, we give a detailed proof of Theorem 1.3 by the mathematical induction on n. In the same section, we prove an important result that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes (Theorem 2.1). It is a natural question to ask whether or not the converse of Theorem 1.3 or Corollary 1.5 is also true, at least up to some permutation in the symmetric group S_n on the index set $\{1, 2, \cdots, n\}$. In Section 3, we give some partial affirmative results for this converse question (see Section 3 for more discussion).

2 Proof of Theorem 1.3

The aim of this section is to give a proof of Theorem 1.3. To do so, it suffices to prove the following theorem.

Theorem 2.1. *Let a Bott tower* $\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^{n-1}$ *be isomorphic to itself by a family* $\{F_j\}_{j=1}^{n-1}$ *of diffeomorphisms, and let* α_n *and* α_n' *be two elements of* $H^2(B_{n-1}(\alpha_1, \cdots, \alpha_{n-1}); \mathbb{Z})$ *such that*

(a) $\alpha_n \equiv F_{n-1}^*(\alpha_n') \bmod 2$*, and*

(b) $\alpha_n^2 = (F_{n-1}^*(\alpha_n'))^2$.

Then $(B_n(\alpha_1,\cdots,\alpha_{n-1},\alpha_n),\pi_n)$ *is diffeomorphic to* $(B_n(\alpha_1,\cdots,\alpha_{n-1},\alpha'_n),\pi'_n)$ *by a diffeomorphism* F_n *which commutes with* π_n *and* π'_n*, so that two Bott towers* $\{(B_j(\alpha_1,\alpha_2,\cdots,\alpha_j),\pi_j)\}_{j=1}^n$ *and*

$$\{(B_j(\alpha_1,\alpha_2,\cdots,\alpha_j),\pi_j)\}_{j=1}^{n-1}\cup\{(B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\alpha'_n),\pi'_n)\}$$

are isomorphic to each other.

Proof. To prove it, we first assume that both $\alpha_n \equiv F^*_{n-1}(\alpha'_n) \bmod 2$ and $\alpha_n^2 = (F^*_{n-1}(\alpha'_n))^2$ hold in $H^2(B_{n-1}(\alpha_1,\cdots,\alpha_{n-1});\mathbb{Z})$, and then use the induction on n. Note that Theorem 2.1 holds for $n=2$ case, due to the validity of Theorem 1.1.

So suppose that Theorem 2.1 holds for any $n-1\geq 2$. Then consider the following commutative diagram:

$$\begin{array}{ccc} B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},F^*_{n-1}(\alpha'_n)) & \xrightarrow{\tilde{F}_n} & B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\alpha'_n) \\ \tilde{\pi}_n\big\downarrow & & \big\downarrow\pi'_n \\ B_{n-1}(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha_{n-1}) & \xrightarrow{F_{n-1}} & B_{n-1}(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha_{n-1}) \end{array}$$

Here the Bott manifold $B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},F^*_{n-1}(\alpha'_n))$ is nothing but the projectivization $\mathbb{P}(\underline{\mathbb{C}}\oplus\tilde{\mathbb{L}}_n)$ of the trivial complex line bundle $\underline{\mathbb{C}}$ and a complex line bundle $\tilde{\mathbb{L}}_n$ with the first Chern class $F^*_{n-1}(\alpha'_n)$ over $B_{n-1}(\alpha_1,\cdots,\alpha_{n-1})$, and $\tilde{F}_n$ is a bundle isomorphism between $B_n(\alpha_1,\cdots,\alpha_{n-1},F^*_{n-1}(\alpha'_n))$ and $B_n(\alpha_1,\cdots,\alpha_{n-1},\alpha'_n)$. Thus, in particular, $B_n(\alpha_1,\cdots,\alpha_{n-1},F^*_{n-1}(\alpha'_n))$ and $B_n(\alpha_1,\cdots,\alpha_{n-1},\alpha'_n)$ are diffeomorphic to each other by $\tilde{F}_n$.

Next we want to show that $B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},F^*_{n-1}(\alpha'_n))$ is actually diffeomorphic to $B_n(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\alpha_n)$ by a diffeomorphism G which commutes with π_n and $\tilde{\pi}_n$. To do so, note first that, since $F^*_{n-1}(\alpha'_n)\equiv\alpha_n$ mod 2, there are some integers b_n^j such that

$$F^*_{n-1}(\alpha'_n)=\alpha_n+2\sum_{j=1}^{n-1}b_n^j x_j. \tag{2.1}$$

We then recall the following well-known fact (refer to, e.g., [4], Lemma 2.1).

Lemma 2.2. *Let* $\pi:\mathbb{E}\to B$ *be a complex vector bundle over a smooth manifold* B *and let* $\mathbb{P}(\mathbb{E})$ *be the projectivization of* E*. Let* $\mathbb{L}$ *be a complex line bundle over* B*. We denote by* $\mathbb{E}^*$ *the complex vector bundle dual to* $\mathbb{E}$*. Then both* $\mathbb{P}(\mathbb{E}^*)$ *and* $\mathbb{P}(\mathbb{E}\otimes\mathbb{L})$ *are isomorphic to* $\mathbb{P}(\mathbb{E})$ *as fiber bundles, and, in particular, they are diffeomorphic to each other.*

Let γ be a complex line bundle over $B_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$ defined by

$$\gamma = \gamma_1^{-b_n^1} \otimes \gamma_2^{-b_n^2} \otimes \cdots \otimes \gamma_{n-1}^{-b_n^{n-1}},$$

where γ_j denotes the tautological line bundle over B_j, and here is regarded as a complex line bundle over the same Bott manifold B_{n-1} through the obvious pullback map.

Lemma 2.3. *The first Chern class $c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma)$ of a complex vector bundle $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ of rank 2 over $B_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$ is equal to α_n that is the first Chern class of a complex line bundle $\mathbb{L}_n$ over the same Bott manifold $B_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$.*

Proof. It is easy to see

$$\begin{aligned} c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) &= c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n \otimes \gamma) = 2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n) \\ &= -2\sum_{j=1}^{n-1} b_n^j x_j + F_{n-1}^*(\alpha_n') \overset{(2.1)}{=} \alpha_n, \end{aligned}$$

as desired. ☐

The following theorem also plays an important role in the proof of Theorem 2.1.

Theorem 2.4. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two complex vector bundles of rank 2 over a Bott manifold B_j $(j \geq 2)$ such that the total Chern class $c(\mathbb{E}_1)$ coincides with the total Chern class $c(\mathbb{E}_2)$. Then $\mathbb{E}_1$ is isomorphic to $\mathbb{E}_2$ as complex vector bundles.*

This theorem says that all complex vector bundles of rank 2 over a Bott manifold are classified by their total Chern classes.

Proof. To prove it, note first from [4], Lemma 3.4 that that every complex vector bundle of rank 2 over a Bott manifold B_2 is classified by the total Chern class. Thus we need to prove the theorem only for $j \geq 3$. To do so, consider the following exact sequence

$$[B_j, U/U(2)] \to [B_j, BU(2)] \xrightarrow{c} [B_j, U] \tag{2.2}$$

induced from the fibration

$$U/U(2) \to BU(2) \to BU.$$

It can be shown as in [4], Lemma 3.4 that

$$[B_1, U/U(2)] = [B_2, U/U(2)] = 0,$$

since B_1 and B_2 are of the real dimension at most 4, and $U/U(2)$ is 4-connected. Note also that there is an exact sequence

$$[B_{j-1}, U/U(2)] \to [B_j, U/U(2)] \to [B_1, U/U(2)] \tag{2.3}$$

induced from the fibration

$$B_1 = \mathbb{CP}^1 \hookrightarrow B_j \to B_{j-1}.$$

Since $[B_1, U/U(2)] = [B_2, U/U(2)] = 0$, it follows inductively from the exact sequence (2.3) that all $[B_j, U/U(2)] = 0$ for all $j \geq 3$. Thus, by the exact sequence (2.2) the Chern class map c is injective.

Since $H^{\text{odd}}(B_j; \mathbb{Z}) = 0$, $[B_j, BU]$ is torsion free. This together with the fact that c is injective implies that all elements of $[B_j, BU(2)]$ can be classified by their Chern classes. This completes the proof of Theorem 2.4. □

With the help of Theorem 2.4, we can prove the following corollary.

Corollary 2.5. *Two complex vector bundles* $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ *and* $\underline{\mathbb{C}} \oplus \mathbb{L}_n$ *of rank* 2 *over* $B_{n-1}(\alpha_1, \cdots, \alpha_{n-1})$ *is isomorphic to each other, as complex vector bundles of rank* 2.

Proof. To prove it, recall first that by assumption we have the following identities:

$$\alpha_n^2 = \left(F_{n-1}^*(\alpha_n')\right)^2 = c_1(\tilde{\mathbb{L}}_n)^2 \text{ in } H^*(B_{n-1}; \mathbb{Z}). \tag{2.4}$$

Since by Lemma 2.3 we have

$$\alpha_n = 2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n), \tag{2.5}$$

it follows from (2.4) that in $H^*(B_{n-1}; \mathbb{Z})$ we have

$$\alpha_n^2 = 4c_1(\gamma)^2 + 4c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) + c_1(\tilde{\mathbb{L}}_n)^2.$$

Thus we have

$$4(c_1(\gamma)^2 + c_1(\gamma)c_1(\tilde{\mathbb{L}}_n)) = \alpha_n^2 - c_1(\tilde{\mathbb{L}}_n)^2 = 0 \text{ in } H^*(B_{n-1}; \mathbb{Z}). \tag{2.6}$$

Since there is no torsion in $H^*(B_{n-1};\mathbb{Z})$, this implies that $c_1(\gamma)^2+c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) = 0$. It is also easy to show that in $H^*(B_{n-1};\mathbb{Z})$ we have

$$\begin{aligned} c((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) &= 1 + c_1((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) + c_2((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma) \\ &= c(\gamma)c(\tilde{\mathbb{L}}_n \otimes \gamma) = (1 + c_1(\gamma))(1 + c_1(\tilde{\mathbb{L}}_n) + c_1(\gamma)) \\ &= 1 + (2c_1(\gamma) + c_1(\tilde{\mathbb{L}}_n)) + (c_1(\gamma)c_1(\tilde{\mathbb{L}}_n) + c_1(\gamma)^2) \\ &= 1 + \alpha_n, \quad \text{by (2.5) and (2.6)} \\ &= c(\underline{\mathbb{C}} \oplus \mathbb{L}_n). \end{aligned}$$

Therefore, it follows from Theorem 2.4 that two vector bundles $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ and $\underline{\mathbb{C}} \oplus \mathbb{L}_n$ are isomorphic to each other, as claimed. □

Finally, we are ready to finish the proof of Theorem 1.3, as follows. Since $B_n(\alpha_1, \cdots, \alpha_{n-1}, F^*_{n-1}(\alpha'_n))$ is diffeomorphic to $\mathbb{P}(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n)$, it follows from Lemma 2.2 that it is also diffeomorphic to $\mathbb{P}((\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma)$. On the other hand, the total Chern class of $(\underline{\mathbb{C}} \oplus \tilde{\mathbb{L}}_n) \otimes \gamma$ coincides with that of $\underline{\mathbb{C}} \oplus \mathbb{L}_n$, and so they are isomorphic to each other by Corollary 2.5. Hence clearly their projectivizations $B_n(\alpha_1, \cdots, \alpha_{n-1}, F^*_{n-1}(\alpha'_n))$ and $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ should be diffeomorphic by a diffeomorphism G. Notice also that by its construction G commutes with the projection maps π_n and $\tilde{\pi}_n$, as follows.

$$\begin{array}{ccc} B_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n) & \xrightarrow{G} & B_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, F^*_{n-1}(\alpha'_n)) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_n} \\ B_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-2}, \alpha_{n-1}) & = & B_{n-1}(\alpha_1, \alpha_2, \cdots, \alpha_{n-2}, \alpha_{n-1}). \end{array}$$

It is now easy to see that the composition $F_n := \tilde{F}_n \circ G$ of two diffeomorphisms G and $\tilde{F}_n$ gives rise to a diffeomorphism between two Bott manifolds $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$ and $B_n(\alpha_1, \cdots, \alpha_{n-1}, \alpha'_n)$ which commutes with π_n and π'_n. Therefore, two Bott towers $\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^n$ and

$$\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^{n-1} \cup \{(B_n(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha'_n), \pi'_n)\},$$

are isomorphic to each other. This implies that by the induction on n Theorem 2.1 holds for any $n \geq 2$, which completes the proof. □

Finally, we close this section with infinitely many non-trivial examples which satisfy two conditions in Corollary 1.5.

Example 2.6. Consider the 2-step Bott manifold $B_2(\alpha_1, \alpha_2)$ with $c_2^1 = -(l+1)$ for any integer l. Let α_3 and α_3' be two elements of degree 2 of

$$H^*(B_2(\alpha_1,\alpha_2);\mathbb{Z}) = \mathbb{Z}[x_1,x_2]/\langle x_1^2 = 0, x_2^2 = \alpha_2 x_2\rangle, \quad \alpha_2 = c_2^1 x_1$$

such that

$$\alpha_3 = x_1 + l x_2, \quad \text{and} \quad \alpha_3' = (1+2l)x_1 + (l+2)x_2.$$

Then clearly $\alpha_3 \equiv \alpha_3' \bmod 2$, and it is also easy to show that

$$\alpha_3^2 = -l(l-1)(l+2)x_1x_2 = (\alpha_3')^2,$$

as required. Therefore, in this case two Bott manifolds

$$B_3(\alpha_1,\alpha_2,\alpha_3) \quad \text{and} \quad B_3(\alpha_1,\alpha_2,\alpha_3')$$

are diffeomprhic to each other by Corollary 1.5.

3 Further Results

As mentioned in Section 1, it is a natural question to ask whether or not the converses of Theorem 1.3 or Corollary 1.5 holds. The aim of this section is to discuss this question in more detail.

To do so, we first want to recall an alternative construction of Bott towers or Bott manifolds (see [8] for more details). It is easy to see that a one-step Bott tower $B_1(\alpha_1) = \mathbb{CP}^1$ can be obtained as a quotient of $(\mathbb{C}^2 - \{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ denotes the complex numbers $\mathbb{C}$ minus the origin, and $\mathbb{C}^*$ acts on $(\mathbb{C}^*)^2$ diagonally. Then take a complex line bundle $\mathbb{L}_1 = (\mathbb{C}^*)^2 \times_{\mathbb{C}^*} \mathbb{C}$ over B_1, where $g_1 \in \mathbb{C}^*$ acts on $\mathbb{C}$ by $g_1 \cdot v = g_1^{c_2^1} v$ for some $c_2^1 \in \mathbb{Z}$ and in $\mathbb{L}_1$ we have

$$[(z_1,w_1),v] = [(z_1g_1, w_1g_1), g_1^{-c_2^1} v].$$

Let $\alpha_2 = c_1(\mathbb{L}_1) \in H^2(B_1(\alpha_1);\mathbb{Z})$. Then the 2-step Bott tower $B_2(\alpha_1,\alpha_2) = \mathbb{P}(\underline{\mathbb{C}} \oplus \mathbb{L}_1)$ can be written as the quotient $(\mathbb{C}^2 - \{0\})^2/(\mathbb{C}^*)^2$, where $(g_1,g_2) \in (\mathbb{C}^*)^2$ acts on $(\mathbb{C}^2 - \{0\})^2$ by

$$(g_1,g_2)\cdot((z_1,w_1),(z_2,w_2)) = ((z_1g_1,w_1g_1),(z_2g_2, g_1^{-c_2^1} w_2g_2)).$$

We can continue this process to construct higher dimensional Bott towers

$$\{(B_j(\alpha_1,\cdots,\alpha_j),\pi_j)\}_{j=1}^n$$

as the quotient of $(\mathbb{C}^2 - \{0\})^n/(\mathbb{C}^*)^n$ by the free action of $(\mathbb{C}^*)^n$ given by

$$\begin{aligned}
&(g_1, g_2, \cdots, g_j, \cdots, g_n) \cdot ((z_1, w_1), (z_2, w_2), \cdots, (z_j, w_j), \cdots, (z_n, w_n)) \\
&= ((z_1 g_1, w_1 g_1), (z_2 g_2, g_1^{-c_2^1} w_2 g_2), \cdots, \\
&\qquad (z_j g_j, \left(\prod_{i=1}^{j-1} g_i^{-c_j^i}\right) w_j g_j), \cdots, (w_n g_n, \left(\prod_{i=1}^{n-1} g_i^{-c_n^i}\right) w_n g_n)),
\end{aligned}$$

where $\{c_j^i\}_{1 \le i < j \le n}$ denotes any collection of $n(n-1)/2$ integers. The obvious projections given by

$$\begin{aligned}
&((z_1, w_1), \cdots, (z_n, w_n)) \xrightarrow{\pi_n} ((z_1, w_1), \cdots, (z_{n-1}, w_{n-1})) \xrightarrow{\pi_{n-1}} \cdots \\
&\qquad \cdots \xrightarrow{\pi_3} ((z_1, w_1), (z_2, w_2)) \xrightarrow{\pi_2} (z_1, w_1) \xrightarrow{\pi_1} \{*\}
\end{aligned}$$

then induces the Bott tower. As mentioned in Section 1, (1.1), it is interesting to note that

$$\alpha_j = \sum_{i=1}^{j-1} c_j^i x_i, \quad c_j^i \in \mathbb{Z}.$$

Let $\mathcal{N}_n$ be the set of integral strictly upper triangular square matrices of size n. Then it has been shown in [8], Section 2.3 that the map from $\mathcal{N}_n$ to the collection of isomorphism classes of n-step Bott towers is bijective. In other words, the Bott tower is also determined by a matrix $C \in \mathcal{N}_n$, and so we may write the n-step Bott tower as $B_n(C)$ for some $C \in \mathcal{N}_n$. If C is the zero matrix in $\mathcal{N}_n$, then clearly $B_n(C)$ is diffeomorphic to $(\mathbb{CP}^1)^n$, and the converse is also true ([11]).

However, in general $B_n(C)$ and $B_n(D)$ may be diffeomorphic, even if C and D are different. In particular, let P denote the permutation matrix of a permutation σ in S_n on n letters $\{1, 2, \cdots, n\}$ whose (i,j)-entry is 1 if $\sigma(j) = i$, and 0 otherwise. When both C and PCP^{-1} are elements of $\mathcal{N}_n$, $B_n(C)$ and $B_n(PCP^{-1})$ are diffeomorphic (refer to [5], Lemma 6.1). In view of these observations, the converse of Corollary 1.5 might not be true in that form, but might be true at least up to some permutation of the index set $\{1, 2, \cdots, n\}$.

For the rest of this section, we will make this discussion more precise and give a partial converse of Theorem 1.3 and Corollary 1.5. To do so, we first need to collect some preliminary results. As in the paper [5], it will be useful to let

$$y_i = x_i - \frac{1}{2}\alpha_i, \quad 1 \le i \le n. \tag{3.1}$$

Then it is easy to see from (3.1) that the following lemma holds.

Lemma 3.1. *For each $1 \le i \le n$, we have*

$$x_i = \sum_{j=1}^{i} b_i^j y_j, \tag{3.2}$$

where all of b_i^j are elements of $\mathbb{Q}$ and $b_i^i = 1$.

Proof. To prove it, we will use the mathematical induction on n. For $n = 1$, clearly $x_1 = y_1$, since $\alpha_1 = 0$. Next suppose that the lemma holds for any $n - 1 \ge 1$. Then we have

$$\begin{aligned} x_n &= y_n + \frac{1}{2}\alpha_n = y_n + \frac{1}{2}\sum_{j=1}^{n-1} c_n^j x_j = y_n + \frac{1}{2}\sum_{j=1}^{n-1}\left(c_n^j\left(\sum_{k=1}^{j} b_j^k y_k\right)\right) \\ &= y_n + \frac{1}{2}\sum_{k=1}^{n-1}\left(\sum_{j=k}^{n-1} c_n^j b_j^k\right) y_k, \end{aligned}$$

which completes the proof of the identity of the equation (3.2) for any n, as desired. □

Note that Lemma 3.1 actually implies that $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ satisfy the following linear equation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & 0 & 0 & \cdots & 0 & 0 \\ * & * & 1 & 0 & \cdots & 0 & 0 \\ * & * & * & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & 1 & 0 \\ * & * & * & * & \cdots & * & 1 \end{pmatrix}_{n\times n} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}. \tag{3.3}$$

Since the coefficient matrix of the above linear equation (3.3) is lower triangular and unipotent, we can also express $y_1, y_2, \cdots, y_n$ in terms of $x_1, x_2, \cdots, x_n$ in such a way that we have

$$y_i = \sum_{j=1}^{i} \tilde{b}_i^j x_j,$$

where all of $\tilde{b}_i^j$ are elements of $\mathbb{Q}$ and $\tilde{b}_i^i = 1$.

The following lemma immediately follows from (3.1), so that its proof will be left to the reader.

Lemma 3.2. *If $x_i^2 = \alpha_i x_i$ holds, then $y_i^2 = \frac{1}{4}\alpha_i^2$ holds. Conversely, if $y_i^2 = \frac{1}{4}\alpha_i^2$ holds, then $x_i^2 = \alpha_i x_i$ holds.*

For the sake of notational simplicity, let

$$P := \mathbb{Q}[x_1, x_2, \cdots, x_n]/\langle x_i^2 = \alpha_i x_i \,|\, i = 1, 2, \cdots, n\rangle,$$
$$Q := \mathbb{Q}[y_1, y_2, \cdots, y_n]/\langle y_i^2 = \frac{1}{4}\alpha_i^2 \,|\, i = 1, 2, \cdots, n\rangle.$$

Then, it is straightforward to see from the identities (1.1) and (3.1), and Lemmas 3.1 and 3.2 that P is isomorphic to Q. It is often convenient to use Q instead of P in order to show some important properties of the cohomology ring $H^*(B_n; \mathbb{Q})$, although we do not use this fact very much in this paper.

We need the following lemma whose proof can be found in [4], Corollary 3.1.

Lemma 3.3. *Let ω be a primitive element of $H^2(B_n(\alpha_1, \alpha_2, \cdots, \alpha_n); \mathbb{Z})$ whose square is zero. Then ω is either one of the following forms: $\pm(x_i - \frac{1}{2}\alpha_i)$, if $\alpha_i \equiv 0 \mod 2$, and $\pm(2x_i - \alpha_i)$, otherwise.*

The following lemma from [5], Proposition 4.1 also plays an important role in the proof of Theorem 3.5.

Lemma 3.4. *Let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \cdots, \alpha_n); \mathbb{Z}) \to H^*(B_n(\alpha_1', \alpha_2', \cdots, \alpha_n'); \mathbb{Z})$$

be a graded ring isomorphism over integers. Then, for each $1 \leq i \leq n$ there is $r_i \in \mathbb{Q}$ such that $\psi(y_i) = r_i y_{\sigma(i)}'$, where σ is a permutation in S_n.

Proof. For the sake of reader's convenience, we will provide a proof of the lemma, essentially due to that of [5], Proposition 4.1.

For the proof, we shall use the induction on i with $1 \leq i \leq n$. For $i = 1$, note that

$$\psi(y_1)^2 = \psi(y_1^2) = \psi(0) = 0,$$

where we used the identities $y_i^2 = \frac{1}{4}\alpha_i y_i$ and $\alpha_1 = 0$. Then it follows from Lemma 3.3 that there is some r_1 in $\mathbb{Q}$ such that

$$\psi(y_1) = r_1 y_{\sigma(1)}',$$

where σ is an element of the symmetric group S_n. So, we are done with the case of $i = 1$.

Next, suppose that the lemma holds for any $i < k$, i.e.,

$$\psi(y_i) = r_i y'_{\sigma(i)}, \quad r_i, \in \mathbb{Q},\ 1 \le i < k,$$

where σ is an element of the symmetric group S_n. We then may assume without loss of generality that $\sigma(i) = i$ for any $1 \le i < k$. It is also easy to obtain

(3.4)

$$\begin{aligned}
\psi(y_k)^2 = \psi(y_k^2) = \psi\left(\frac{1}{4}\alpha_k^2\right) &\overset{(1.1)}{=} \psi\left(\frac{1}{4}\left(\sum_{i=1}^{k-1} c_k^i x_i\right)^2\right) \\
&\overset{\text{(Lem. 3.1)}}{=} \frac{1}{4}\left(\psi\left(\sum_{i=1}^{k-1}\sum_{l=1}^{i} c_k^i b_i^l y_l\right)\right)^2 = \frac{1}{4}\left(\psi\left(\sum_{l=1}^{k-1}\left(\sum_{i=l}^{k-1} c_k^i b_i^l\right) y_l\right)\right)^2 \\
&= \frac{1}{4}\left(\psi\left(\sum_{l=1}^{k-1} d_l y_l\right)\right)^2 = \frac{1}{4}\sum_{t,s=1}^{k-1} d_t d_s \psi(y_t)\psi(y_s) \\
&= \frac{1}{4}\sum_{t,s=1}^{k-1} (d_t r_t)(d_s r_s) y'_t y'_s, \quad \psi(y_t) = r_t y'_t,\ \psi(y_s) = r_s y'_s,
\end{aligned}$$

where $d_l = \sum_{l=l}^{k-1} c_k^i b_i^l$.

Now, let

$$\psi(y_k) = \sum_{m=1}^{n} e_m y'_m, \quad e_m \in \mathbb{Q}.$$

It is not difficult to see from (3.4) that $e_m = 0$ at least for all $m \ge k+1$. Thus we have

$$\psi(y_k) = \sum_{m=1}^{k} e_m y'_m.$$

We then claim that all of e_m are zero for $1 \le m \le k-1$. To see it, suppose that there is some non-zero coefficient e_m for $1 \le m \le k-1$, then there should be a term of the form $A y'_m y'_k$ ($A \in \mathbb{Q}$) in the expression of $\psi(y_k)^2$. But this contradicts the equation (3.4). Therefore, we have $\psi(y_k) = e_k y'_k$, which implies that, in general, for a permutation $\sigma \in S_n$ we have

$$\psi(y_k) = e_k y'_{\sigma(k)}, \quad 1 \le k \le n.$$

This completes the proof of Lemma 3.4. □

Note that r_i in Lemma 3.4 is either $\pm\frac{1}{2}$, or ± 2, or ± 1 (see, e.g., [5], Lemma 4.1). With this understood, we give a partial converse of Corollary 1.5, as follows.

Theorem 3.5. *Let $B_n(\alpha_1, \cdots, \alpha_n)$ be diffeomorphic to $B_n(\alpha'_1, \cdots, \alpha'_n)$ by a diffeomorphism F, and let*

$$\psi : H^*(B_n(\alpha_1, \alpha_2, \cdots, \alpha_n); \mathbb{Z}) \to H^*(B_n(\alpha'_1, \alpha'_2, \cdots, \alpha'_n); \mathbb{Z})$$

be the graded ring isomorphism over integers given by the pullback of F^{-1}. If all of r_i $(1 \leq i \leq n)$ are equal to ± 1, then there is some permutation $\sigma \in S_n$ such that we have

$$\psi(\alpha_i) \equiv \alpha'_{\sigma(i)} \mod 2$$

for all $1 \leq i \leq n$.

Proof. To prove it, recall first that $\alpha_1 = \alpha'_1 = 0$ by definition. So the theorem trivially holds for $n = 1$.

For the case of $n = 2$, by assumption two Bott manifolds $B_2(\alpha_1, \alpha_2)$ and $B_2(\alpha'_1, \alpha'_2)$ are diffeomorphic. Thus there is a graded ring isomorphism ψ from $H^*(B_2(\alpha_1, \alpha_2); \mathbb{Z})$ and $H^*(B_2(\alpha'_1, \alpha'_2); \mathbb{Z})$. It then follows from Lemma 3.4 that there is a permutation $\sigma \in S_2$ such that

$$\psi(y_1) = \pm y'_{\sigma(1)}, \text{ and } \psi(y_2) = \pm y'_{\sigma(2)}.$$

In other words, we have

$$\psi(2x_1 - \alpha_1) = \pm(2x'_{\sigma(1)} - \alpha'_{\sigma(1)}), \text{ and } \psi(2x_2 - \alpha_2) = \pm(2x'_{\sigma(2)} - \alpha'_{\sigma(2)}).$$

This implies that

$$\psi(\alpha_1) \equiv \alpha'_{\sigma(1)}, \text{ and } \psi(\alpha_2) \equiv \alpha'_{\sigma(2)} \mod 2,$$

as claimed. It should be now clear that exactly same arguments apply to any general $n \geq 3$. So we are done. □

In case of Bott towers, we can give a much stronger partial converse of Theorem 1.3, as follows.

Theorem 3.6. *Suppose that two Bott towers*

$$\{(B_j(\alpha_1, \cdots, \alpha_j), \pi_j)\}_{j=1}^n \text{ and } \{(B_j(\alpha'_1, \cdots, \alpha'_j), \pi'_j)\}_{j=1}^n,$$

are isomorphic to each other. Let ψ be a graded ring isomorphism as in Theorem 3.5. If all of r_{i_j} $(1 \leq i_j \leq j)$ are equal to ± 1 for each $1 \leq j \leq n$, then we have

$$\psi(\alpha_i) \equiv \alpha'_i \mod 2$$

for all $1 \leq i \leq n$.

Proof. By the definition of an isomorphism between two Bott towers, any two j-step Bott manifolds $B_j(\alpha_1, \cdots, \alpha_j)$ and $B_j(\alpha'_1, \cdots, \alpha'_j)$ are diffeomorphic. Thus, by Theorem 3.5 there is a permutation $\sigma_j \in S_j$ such that

$$\psi(\alpha_i) \equiv \alpha'_{\sigma_j(i)} \mod 2 \tag{3.5}$$

for all $1 \leq i \leq j$. In particular, if we set $j = 1$, then it follows from (3.5) that we have $\psi(\alpha_1) \equiv \alpha'_1 \mod 2$.

Next, assume that the theorem holds for any positive integer less than k. So we have

$$\psi(\alpha_i) \equiv \alpha'_i \mod 2 \tag{3.6}$$

for all $1 \leq i \leq k-1$. Since two k-step Bott manifolds $B_k(\alpha_1, \cdots, \alpha_k)$ and $B_k(\alpha'_1, \cdots, \alpha'_k)$ are diffeomorphic by assumption, it follows from (3.5) that there is a permutation $\sigma_k \in S_k$ such that

$$\psi(\alpha_i) \equiv \alpha'_{\sigma_k(i)} \mod 2 \tag{3.7}$$

for all $1 \leq i \leq k$. If $\sigma_k(k) = k$, then by (3.7) we have $\psi(\alpha_k) \equiv \alpha'_k \mod 2$. So we are done. On the other hand, if $\sigma_k(k) = l$ for $1 \leq l < k$, then there is some minimal positive integer m such that $\sigma_k^m(l) = k$. By applying the identities (3.6) and (3.7) repeatedly, we then have

$$\underbrace{\psi(\alpha_k) \equiv \alpha'_{\sigma_k(k)} \equiv \alpha'_l}_{(3.7)} \overset{(3.6)}{\equiv} \underbrace{\psi(\alpha_l) \equiv \alpha'_{\sigma_k(l)}}_{(3.7)} \overset{(3.6)}{\equiv} \cdots \overset{(3.6)}{\equiv} \underbrace{\psi(\alpha_{\sigma_k^{m-1}(l)}) \equiv \alpha'_k}_{(3.7)} \mod 2,$$

which implies that the theorem also holds for k. This completes the proof of Theorem 3.6. □

References

[1] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458–538.

[2] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029.

[3] V. Buchstaber and T. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, Vol. **24**, Amer. Math. Soc., Providence, Rhode Island, 2002.

[4] S. Choi and M. Masuda, *Classification of $\mathbb{Q}$-trivial Bott towers*, J. Symp. Geom. **10** (2012), 447–462.

[5] S. Choi, M. Masuda, and S. Murai, *Invariance of Pontrjagin classes for Bott manifolds*, preprint (2014); arXiv:1401.0893v1.

[6] Y. Civan and N. Ray, *Homotopy decompositions and K-theory of Bott towers*, K-theory **34** (2005), 1–33.

[7] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds, and torus actions*, Duke Math. J. **62** (1991), 417–451.

[8] M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the exterior character of representations*, Duke Math. J. **76** (1994), 23–58.

[9] F. Hirzebruch, *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Math. Ann. **124** (1951), 201–216.

[10] J.-H. Kim, *On a generalized Petrie's conjecture via index theorems*, Topology and Appl. **160** (2013), 1353–1363.

[11] M. Masuda and T. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, Sbornik Math. **199** (2008), 1201–1223.

[12] M. Masuda and D. Y. Suh, *Classification problems of toric manifolds via topology*, Toric Topology, Contemp. Math., Vol. 60, Amer. Math. Soc., Providence, Rhode island, 2008, 273–286.

[13] J. Morrow and K. Kodaira, *Complex manifolds*, AMS Chelsea Publishing, Amer. Math. Soc., Providence, Rhode Island, 1971.

[14] T. Petrie, *Smooth S^1 actions on homotopy complex projective spaces and related topics*, Bull. Amer. Math. Soc. **78** (1972), 105–153.

[15] T. Petrie, *Torus actions on homotopy complex projective spaces*, Invent. Math. **20** (1973), 139–146.

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