

# ON THE REGULARIZED IMAGINARY DOI-NAGANUMA LIFTING

SEOKHO JIN\* AND SUBONG LIM

ABSTRACT. Asai and Friedberg studied the imaginary Doi-Naganuma lifting which sends elliptic modular forms to automorphic forms over an imaginary quadratic field. In this paper we extend this lifting to weak Maass forms by using regularized integral. We construct an automorphic object with singularities on the quadratic upper half-plane  $\mathfrak{H}^1$  by the regularized theta lifting of a weak Maass form. We also give the convergence region and describe its singularity type. Finally we compute the Fourier coefficients of the lifted form explicitly and present the case of Poincaré series as an example.

## 1. INTRODUCTION

A theta lifting gives a correspondence between automorphic forms associated to two groups by the integration with a suitably constructed theta kernel of two variables. A theta lifting has been an important tool in studying automorphic forms by forming connections between different types of automorphic forms. For example, Katok and Sarnak [16] used theta liftings to prove the relation between the central values of the  $L$ -series of Maass cusp forms to the Fourier coefficients of counterparts under the Shimura correspondence. Theta liftings were also used to study generating series in arithmetic and differential geometry (see [17, 18, 19]).

These classical theta liftings were mainly focused on cusp forms. But there are many interesting researches about weak forms which have possible singularities at cusps (for example, see [3, 24, 25]). Therefore it is natural to consider and study the theta lifting of weak forms including weak Maass forms. Borchers [2] used a regularized theta lifting to get a remarkable product expansions of automorphic forms for  $O(p, 2)$ . This regularization, which was introduced by Harvey and Moore [13], allows us to overcome the convergence issue and extend theta liftings to weak forms. Following Borchers' regularization, Bruinier and Funke generalized a famous result of Zagier concerning the traces of singular moduli to traces of CM values of modular functions on modular curves of arbitrary genus. In addition, Bruinier proved that for a large class of groups  $\Gamma$  all linear relations of Heegner divisors on  $\mathfrak{H}/\Gamma$  in the class group are obtained via Borchers liftings from meromorphic modular forms on  $\mathfrak{H}$ , and Bruinier and Ono showed that the Fourier coefficients of weight  $\frac{1}{2}$  harmonic weak Maass forms can be realized as the values

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\*Corresponding Author.

and central derivatives of quadratic twists of weight 2 modular  $L$ -functions. For related works, see [5, 6, 7, 23].

Our goal is to extend the imaginary Doi-Naganuma lifting to weak Maass forms using the regularized integral. The imaginary Doi-Naganuma lifting is a map from elliptic modular forms to automorphic forms over an imaginary quadratic field, introduced by Asai [1] and Friedberg [10, 11]. This kind of lifting was first studied by Doi and Naganuma as a precursor of the base change lifting (see [9, 20]). In this paper we construct an automorphic object with singularities on the quadratic upper half-plane  $\mathfrak{H}^1$  by the regularized theta lifting of a weak Maass form. To obtain this result, we study the convergence of the regularized theta lifting and show that the lifted form is an eigenfunction of the Casimir operator. We also give the convergence region and describe its singularity type. Finally we compute the Fourier coefficients of the lifted form explicitly and present the case of Poincaré series as an example.

From now on we try to explain the results more precisely. Let  $k$  be a natural number and  $f, g$  functions on the upper half plane  $\mathfrak{H}$  such that  $(f|_k\sigma)(z) = f(z)$ ,  $(g|_k\sigma)(z) = g(z)$  for all  $\sigma \in \Gamma(N)$ . We define

$$\int_{\Gamma(N)\backslash\mathfrak{H}}^{reg} f(z)\overline{g(z)}y^{k-2}dxdy := \int_{SL_2(\mathbb{Z})\backslash\mathfrak{H}}^{reg} \sum_{\gamma \in \Gamma(N)\backslash SL_2(\mathbb{Z})} (f|_k\gamma)(z)\overline{(g|_k\gamma)(z)}y^{k-2}dxdy,$$

where  $\int_{SL_2(\mathbb{Z})\backslash\mathfrak{H}}^{reg}$  is the regularized integral in the sense of Borchers (see [2, section 6] for the definition) and  $z = x + iy \in \mathfrak{H}$ . In this paper we use this to define the regularized Doi-Naganuma lifting. We are given a certain quadratic lattice  $\Lambda$  of signature  $(3,1)$  and a vector  $V \in \Lambda^*$  (for definitions, see section 2). The regularized lifting of a weak Maass form  $f$  of weight  $k$  associated to  $\Lambda$  and  $V$  for  $|\alpha| \leq k - 1$  is given by  $F^{reg}(g, V, \Lambda; f) = {}^t(F_{1-k}^{reg}, \dots, F_{k-1}^{reg})$ , where  $F_{\alpha}^{reg}(g, V, \Lambda; f)$  is defined by

$$F_{\alpha}^{reg}(g, V, \Lambda; f) := \int_{\Gamma(N)\backslash\mathfrak{H}}^{reg} \theta_{k,\alpha}(z, V, g, \Lambda)\overline{f(z)}y^{k-2}dxdy.$$

Here,  ${}^tA$  denotes the transpose of  $A$  and  $\theta_{k,\alpha}(z, V, g, \Lambda)$  is a theta function of variables  $z \in \mathfrak{H}$  and  $g \in \mathbf{G}(= \mathrm{SL}_2(\mathbb{C}))$  given by (4). We also use the notation  $\langle \theta_{k,\alpha}, f \rangle^{reg}$  for  $F_{\alpha}^{reg}(g, V, \Lambda; f)$ .

Note that this lift is the complex conjugate of the lift in [2], in the sense that it is anti-linear rather than linear.

Let  $g(\mu, \nu) = \begin{pmatrix} \nu^{1/2} & \mu\nu^{-1/2} \\ 0 & \nu^{-1/2} \end{pmatrix} \in \mathbf{G}$ , where  $\mu \in \mathbb{C}$  and  $\nu > 0$ . This element indeed corresponds to  $\mu + \nu\mathbf{k} \in \mathfrak{H}^1 = \{\mu + \nu\mathbf{k} \in \mathcal{H}(-1, -1) | \mu \in \mathbb{C}, \nu > 0\}$ , where  $\mathcal{H}(-1, -1)$  is the quaternion algebra with the standard basis  $1, \mathbf{i}, \mathbf{j}$ , and  $\mathbf{k}$ . For convenience we let  $F_{\alpha}^{reg}(g(\mu, \nu); f) := F_{\alpha}^{reg}(g(\mu, \nu), V, \Lambda; f)$ . Then we prove that the lifted forms  $F^{reg}(g(\mu, \nu); f)$  on  $\mathfrak{H}^1$  are weak Maass forms for  $K$  of weight  $2k - 2$  and eigenvalue  $(k^2 - 2k)/4 + \lambda$  for the group  $G(\Lambda) := \{\gamma \in \mathbf{G} | \Lambda^{\gamma} = \Lambda\}$  with singularities.

As we see in [5], it is also important to compute the Fourier expansions of the regularized theta lifting and study its singularities. We compute the singularity type in Theorem 4.1 and the Fourier expansion in Theorem 4.2 and Theorem 4.3.

In [10] Friedberg gave an interesting example showing that the lifted form of the non-holomorphic Eisenstein series of weight  $k > 1$  is a Poincaré type series. The following example shows that the same phenomenon happens in the case where  $f$  is a weakly holomorphic Poincaré series. Let  $\psi$  be a negative integer. Then it is known that (see [15]) the Poincaré series

$$g_\psi(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(N)} \frac{e^{2\pi i \psi \gamma z}}{(c\tau + d)^k}$$

vanishes at all cusps of  $\Gamma(N)$  except possibly at  $i\infty$ , and at  $i\infty$  it has a Fourier expansion of the form

$$g_\psi(z) = e^{2\pi i \psi z} + \sum_{m>0} a_m(\psi) e^{2\pi i m z}.$$

Then the regularized lifting  $F^{reg}(g(\mu, \nu); g_\psi)$  is equal to

$$\sum_{\substack{X \in \Lambda \\ Q[X-V]=2\psi}} \frac{N\Gamma(k-1/2)}{\pi(Rg(\mu, \nu)[X-V] + \psi)^{k-1/2}}.$$

Since the result is absolutely convergent for  $k \geq 2$ , it can be regarded as a type of Poincaré series.

The remainder of the paper is organized as follows. In section 2, we deal with preliminary notions. We define weak Maass forms on  $\mathfrak{H}$  and  $\mathfrak{H}^1$  respectively and introduce the theta functions  $\theta_{k,\alpha}$  and their properties. This enables us to compute the Fourier expansion of  $\theta_{k,\alpha}$  at arbitrary cusps. In section 3 we prove several properties of the regularized theta lifting to prove Theorem 3.1. We compute the convergence region and study the modularity and the behavior of  $F_\alpha^{reg}(g(\mu, \nu); f)$  under the Casimir operators. Finally in section 4 we compute its singularity type and the Fourier expansion.

## 2. PRELIMINARIES

In this section we set several notations and define basic objects, following [10]. For a function  $f(z)$  on  $\mathfrak{H}$  and a matrix  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ , we define the slash operator of integral weight  $k$  by  $(f|_k \gamma)(z) = (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z)$ , where  $\gamma z = \frac{az+b}{cz+d}$ .

**2.1. Weak Maass Forms.** We start with the definition of a weak Maass form. Let  $\Delta_k$  be the hyperbolic Laplacian of weight  $k$  given by

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + ik y \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

**DEFINITION 2.1.** A *weak Maass form* of weight  $k \in \mathbb{Z}$  on the congruence subgroup  $\Gamma \subset SL_2(\mathbb{Z})$  and eigenvalue  $-\lambda \in \mathbb{C}$  is a smooth function  $f : \mathfrak{H} \rightarrow \mathbb{C}$  such that

- (1)  $f|_k \gamma = f$  for all  $\gamma \in \Gamma$ ,

- (2)  $\Delta_k f = -\lambda f$ ,  
(3)  $f(z)$  has at most linear exponential growth at all of the cusps, i.e.,  $(f|_k \gamma)(z) = O(e^{\delta y})$  for all  $\gamma \in SL_2(\mathbb{Z})$ , for some  $\delta > 0$ .

We denote the space of weak Maass forms by  $WMF_{k,-\lambda}(\Gamma)$ .

Let  $\gamma \in SL_2(\mathbb{Z})$ . It is known that the Fourier expansion for a weak Maass form  $(f|_k \gamma^{-1})(z) \in WMF_{k,-\lambda}(\Gamma(N))$  at the cusp corresponding to  $\gamma$  is given by

$$(1) \quad (f|_k \gamma^{-1})(z) = \sum_{\substack{n=-n_\gamma \\ n \neq 0}}^{l_\gamma} \tilde{a}_\gamma(n, s) |4\pi n y / N|^{-\frac{k}{2}} W_{-sgn(n)\frac{k}{2}, s-\frac{1}{2}}(-4\pi |n| y / N) e^{2\pi i n x / N} \\ + \sum_{n \in \mathbb{Z} \setminus \{0\}} a_\gamma(n, s) |4\pi n y / N|^{-\frac{k}{2}} W_{sgn(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi |n| y / N) e^{2\pi i n x / N} \\ + a_\gamma(0, s) y^{s-\frac{k}{2}} + \tilde{a}_\gamma(0, s) y^{1-s-\frac{k}{2}},$$

where  $n_\gamma, l_\gamma \in \mathbb{N}$ ,  $W_{k,m}(y)$  and  $W_{-k,m}(-y)$  are the standard W-Whittaker's functions and  $-\lambda = s(1-s) + \frac{k^2-2k}{4}$  (for the details see [4]). It is also known that  $a_\gamma(n, s) = O(e^{C\sqrt{n}})$  for some  $C > 0$  (see [5, 6]). We define a subspace  $WMF_{k,-\lambda}^+(\Gamma(N))$  by

$$WMF_{k,-\lambda}^+(\Gamma(N)) = \{f \in WMF_{k,-\lambda}(\Gamma(N)) \mid \tilde{a}_\gamma(n, s) = 0 \text{ for } n > 0\}.$$

**2.2. Weak Maass forms for  $K$ .** Now we introduce the space of Hilbert modular forms on the quaternionic upper half space  $\mathfrak{H}^1$ . Recall that  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G} = SL_2(\mathbb{C})$  acts on  $z_1$  in  $\mathfrak{H}^1$  by  $\sigma z_1 = (az_1 + b)(cz_1 + d)^{-1}$  and that the stabilizer of  $\mathbf{k}$  is  $SU(2, \mathbb{C})$ , the special unitary group. Note that  $SU(2, \mathbb{C})$  is not abelian so we need to consider multi-dimensional representations of  $SU(2, \mathbb{C})$ . For  $n \in \mathbb{Z}_{\geq 0}$  let  $(a \ b)_n := {}^t(a^n, a^{n-1}b, \dots, ab^{n-1}, b^n)$  and define the  $n$ -fold symmetric representation  $\rho_n(g)$ , for  $g \in SL_2(\mathbb{C})$ , by  $\rho_n(g)(a \ b)_n := ((a \ b)^t g)_n$ . Then  $\rho_n$  gives an irreducible  $(n+1)$ -dimensional representation of  $SU(2, \mathbb{C})$ .

Let  $\rho(z_1 = \mu + \nu \mathbf{k}) = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}$  for  $\mu, \nu \in \mathbb{C}$ . For  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{G}$  and  $z_1 \in \mathfrak{H}^1$  set  $J(g, z_1) = \rho(cz_1 + d)$  and  $\rho_n^1(g) = {}^t \rho_n(tg)$ . Then we can define a slash operator for a function  $F^1 : \mathfrak{H}^1 \rightarrow \mathbb{C}^{n+1}$  by

$$(F^1 || [g]_n)(z_1) := \rho_n^1(J(g, z_1))^{-1} F^1(gz_1)$$

for all  $g \in \mathbf{G}$ . We can also extend the slash operator to  $A \in GL_2(\mathbb{C})$  by putting

$$F^1 || [A]_n = F^1 || [(\det A)^{-\frac{1}{2}} A]_n$$

(as stated in [10, p.491], the ambiguity of sign caused by the choice of square root will not cause any harm in the applications).

**DEFINITION 2.2.** Let  $\Xi$  a congruence subgroup of  $SL_2(\mathcal{O}_K)$ . A *weak Maass form* for  $K$  of weight  $n$  on  $\Xi$  and eigenvalues  $(\lambda, \lambda')$  is a real analytic function  $F^1 : \mathfrak{H}^1 \rightarrow \mathbb{C}^{n+1}$  with singularities satisfying

- (1)  $\Psi^n F^1 = \lambda F^1$  and  $\tilde{\Psi}^n F^1 = \lambda' F^1$ ,

- (2)  $F^1|[\sigma]_n = F^1$  for all  $\sigma$  in  $\Xi$ ,
- (3)  $(F^1|[[A]_n)(z_1) = O(\nu^\delta)$  as  $\nu \rightarrow \infty$ , uniformly in  $x$  for all  $A \in GL_2(\mathcal{O}_K)$ , for some fixed real  $\delta$ .

Here  $\Psi^n, \tilde{\Psi}^n$  are Casimir operators of weight  $n$  on  $\mathfrak{H}^1$  which are given by

$$(2) \quad \begin{aligned} \Psi^n &= y^2(\partial_{x\bar{x}}^2 + \partial_{yy}^2/4)E_{n+1} + y\partial_x \begin{pmatrix} 0 & 1 & & & \\ & \ddots & 2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & n \\ & & & & 0 \end{pmatrix} \\ &+ y\partial_y \cdot \frac{1}{4} \begin{pmatrix} 2n-1 & & & & \\ & 2n-3 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} n(n-2) & & & & \\ & (n-1)(n-3) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \end{aligned}$$

$$(3) \quad \begin{aligned} \tilde{\Psi}^n &= y^2(\partial_{x\bar{x}}^2 + \partial_{yy}^2/4)E_{n+1} - y\partial_x \begin{pmatrix} 0 & & & & \\ n & \ddots & & & \\ & n-1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \\ &+ y\partial_y \cdot \frac{1}{4} \begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & 2n-3 & & \\ & & & \ddots & \\ & & & & 2n-1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & (n-1)(n-3) & & \\ & & & \ddots & \\ & & & & n(n-2) \end{pmatrix}, \end{aligned}$$

where  $E_{n+1}$  is the identity matrix of size  $n+1$ .

Let  $\{\{\Xi, n, \{\lambda, \lambda'\}\}\}$  be the space of all these forms. If  $F^1$  is defined without singularities, then it is said to be a *modular form over  $K$* , and moreover if  $F^1$  satisfies a better growth condition at infinity, namely for all  $A \in GL_2(\mathcal{O}_K)$ ,  $F^1|[[A]_n(z_1) = O(e^{-\delta\nu})$  as  $\nu \rightarrow \infty$  uniformly in  $\mu$ , for some fixed  $\delta > 0$ ,  $F^1$  is said to be *cuspidal* or a *cusp form*. If  $\lambda = \lambda'$ , we say simply that  $F^1$  has eigenvalue  $\lambda$ .

PROPOSITION 2.3. [10, Proposition 2.2] *There is a one-to-one correspondence between*

$$\{F : \mathbf{G} \rightarrow \mathbb{C}^{n+1} | F(\gamma g \kappa) = {}^t \rho_n(\kappa) F(g) \text{ for all } \gamma \in \Xi, \kappa \in SU(2, \mathbb{C})\}$$

and

$$\{F^1 : \mathfrak{H}^1 \rightarrow \mathbb{C}^{n+1} | F^1(\gamma z_1) = \rho_n^1(J(\gamma, z_1)) F^1(z_1) \text{ for all } \gamma \in \Xi\}$$

given by  $F^1(z_1) = \rho_n^1(J(g, \mathbf{k})) F(g)$  and  $F(g) = \rho_n^1(J(g, \mathbf{k}))^{-1} F^1(g\mathbf{k})$ , where  $z_1 = g\mathbf{k}$ .

This correspondence is also compatible with the action of Casimir operators (see [10, section 2]).

**2.3. Theta function.** We introduce a quadratic vector space  $\mathcal{V} = \{X \in M_{2 \times 2}(\mathbb{C}) | {}^t X = \bar{X}\}$  with a symmetric bilinear form  $Q(X, Y) := -\text{tr}(X \text{adj} Y)$  and a quadratic form  $Q[X] := Q(X, X) = -2 \det X$ . Here  $\text{adj} Y$  is the complex conjugate of  $\tilde{Y}$ , where  $\tilde{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Note that  $\tilde{Y}$  was used in [1, 10], but indeed using  $\text{adj} Y$  is correct. For

$X \in \mathcal{V}$  and  $g \in \mathbf{G}$  let  $X^g := {}^t\bar{g}Xg$ , and  $R^g[X] := \text{tr}((X^g)^2)$  its majorant corresponding to  $g$ . Let  $K$  be an imaginary quadratic number field with discriminant  $-D$ . Given  $M, P \in \mathbb{Z}_{\neq 0}$  and an ideal  $I$ , we introduce the lattice

$$\Lambda_i := \left\{ \begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix} \in \mathcal{V} \mid m \in M\mathbb{Z}, p \in P\mathbb{Z}, r \in J_i \right\},$$

where  $J_i = I\mathfrak{E}_i\mathfrak{e}_i^{-1}$ ,  $\mathfrak{E}_1 = \mathcal{O}_K, \dots, \mathfrak{E}_{h(K)}$  is a set of ideal class representatives, and take  $\mathfrak{e}_i$  ideal numbers for these ideals in the sense of Hecke (see [14]). Let  $N_1 = \text{l.c.m.}(MP, \mathbf{N}(I)D)$ .

For a fixed  $\Lambda = \Lambda_i$  let  $\Lambda^*$  be the dual lattice of  $\Lambda$  given by  $\Lambda^* = \Lambda_i^* = \left\{ \begin{pmatrix} m & r \\ \bar{r} & p \end{pmatrix} \in \mathcal{V} \mid Pm \in \mathbb{Z}, Mp \in \mathbb{Z}, r \in \bar{J}_i^{-1}\mathcal{D}_K^{-1} \right\}$ , where  $\mathcal{D}_K^{-1}$  is the inverse different. Then for a vector  $V$  in the dual lattice  $\Lambda^*$ , there is a theta series

$$(4) \quad \theta_{k,\alpha}(z, V, g, \Lambda) := y^{1/2} \sum_{X \in \Lambda} \eta_{k,\alpha}((X - V)^g) e((xQ + iyR^g)[X - V])$$

for  $z = x + iy \in \mathfrak{H}$ ,  $g \in \mathbf{G}$  and  $k \in \mathbb{Z}_{\geq 1}$  (see [1]), where  $e(x) = e^{\pi ix}$ . Here,

$$\eta_{k,\alpha}(X) := (k-1)! \sum_{\beta, \gamma} \frac{r^\alpha L_\beta^{(\alpha)}(2r\bar{r}) H_\gamma(m-p)}{(\alpha + \beta)! \gamma! 2^\beta},$$

where  $\sum_{\beta, \gamma}$  denote the sum over  $\beta$  and  $\gamma$  nonnegative integers such that  $2\beta + \gamma = k - 1 - \alpha$  and  $\alpha + \beta \geq 0$  and  $L_n^{(\alpha)}$  and  $H_n$  are Laguerre and Hermite polynomials given by

$$L_n^{(\alpha)}(y) = e^y (n! y^\alpha)^{-1} \frac{d^n}{dy^n} (e^{-y} y^{n+\alpha}), \quad H_n(y) = (-1)^n e^{y^2/2} \frac{d^n}{dy^n} (e^{-y^2/2}).$$

Note that  $\eta_{k,\alpha}(X)$  forms a  $\mathbb{C}$ -basis of spherical harmonic polynomials of degree  $k - 1$ .

PROPOSITION 2.4. [10, Proposition 3.1] *For all  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1)$*

$$(\theta_{k,\alpha}|_k \sigma)(z, dV, g, \Lambda) = \chi_{-D}(d) \theta_{k,\alpha}(z, V, g, \Lambda),$$

where  $\chi_{-D}(n) = \left(\frac{-D}{n}\right)$  and  $(-)$  is the Kronecker symbol. In particular, for  $\sigma \in \Gamma(N_1)$ ,

$$(\theta_{k,\alpha}|_k \sigma)(z, V, g, \Lambda) = \theta_{k,\alpha}(z, V, g, \Lambda).$$

We write  $\text{Tr}_{\mathbb{R}}^{\mathbb{C}}$  for the trace from  $\mathbb{C}$  to  $\mathbb{R}$  and  $\mathbf{N} = \text{Norm}(\mathbb{C}/\mathbb{R})$ . Let

$$\phi_{\alpha,\beta}(z, u, v) := \sum_{r \in J} y^{-\beta} L_\beta^{(\alpha)}(4\pi y \mathbf{N}(r - v)) (r - v)^\alpha e(2\mathbf{N}(r - v)z + 2\text{Tr}_{\mathbb{R}}^{\mathbb{C}}(r\bar{u}) - \text{Tr}_{\mathbb{R}}^{\mathbb{C}}(u\bar{v})),$$

where  $L_\beta^{(\alpha)}(y)$  is the Laguerre polynomial defined in Section 2.3. Then

$$\phi_{\alpha,\beta}(\sigma z, au + bv, cu + dv) = \chi_{-D}(d) (cz + d)^{\alpha+2\beta+1} \phi_{\alpha,\beta}(z, u, v)$$

for  $\alpha, \beta \geq 0, z \in \mathfrak{H}, u, v \in \mathbb{C}$  and  $\sigma \in \Gamma_0(DN(I))$ . We can see this by applying Theorem 1.2 of [11] with the two-dimensional lattice  $J = J_i$ ,  $\omega = {}^t(\frac{1}{2}, \frac{i}{2})$  and quadratic form  $(u, v) := \text{Tr}_{\mathbb{R}}^{\mathbb{C}}(u\bar{v})$ . The definition of  $\phi_{\alpha,\beta}$  is motivated by the fact that  $\theta_{k,\alpha}$  is the sum of products of functions of  $r$  and  $\bar{r}$  and functions of  $m$  and  $p$ , with  $\phi_{\alpha,\beta}$  being precisely the former (see [11, Lemma 3.6]).

**2.4. Fourier expansion of  $(\theta_{k,\alpha}|_k\gamma)(z, V, g, \Lambda)$ .** In this subsection we briefly explain that  $\theta_{k,\alpha}$  can be realized as the theta function with the Weil representation and find the Fourier expansion of  $(\theta_{k,\alpha}|_k\gamma)(z)$ . We follow the notations given in [21].

We briefly recall the first spherical property (for the details see [8]). Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and  $l \in \mathbb{Z}$ . Then  $\varphi$  is said to be of *first spherical property* for weight  $\frac{l}{2}$  if  $r(\kappa(\phi), Q)\varphi = \varepsilon(\kappa(\phi))^{p-q}\sqrt{e^{-i\phi}{}^{-l}}\varphi$  for  $\kappa(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}$ , where

$$\varepsilon(\sigma) = \begin{cases} \sqrt{i} & c > 0 \\ i^{(1-\text{sgn } d)/2} & c = 0, \\ \sqrt{i}^{-1} & c < 0 \end{cases}$$

for  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ . Now we introduce the theta series coming from the Weil representation. If we let

$$\varphi(X) = \eta_{k,\alpha}(X^g)e^{-\pi R^g[X]}$$

for  $X \in \mathcal{V}$ , then this function clearly is a Schwartz function. In the following we show that this  $\varphi$  satisfies the first spherical property of weight  $k$ .

Let  $\eta_{(k)}(X) := (\eta_{k,1-k}(X), \dots, \eta_{k,k-1}(X))$  and  $P_{k-1}(X) := \eta_{(k)}(X) {}^t(a \ b)_{2k-2}$ . Then we see that  $P_{k-1}(X) = Q(X, A_1)^{k-1}$ , where  $(a \ b)_{2k-2} = (a^{2k-2}, a^{2k-3}b, \dots, b^{2k-2})$  and  $A_1 = {}^t(a \ b)(a \ b) \begin{pmatrix} & \\ & 1 \end{pmatrix}^{-1}$  (see [11, section 1.3]). We let  $F_z(X) := P_{k-1}(X)e^{\pi i Q_z(X)}$ , where  $Q_z(X) = xQ[X] + iyR[X]$  and  $z = x + iy \in \mathfrak{H}$ . Furthermore, since  $Q(A, B) = Q(A^g, B^g)$  for all  $g \in SL_2(\mathbb{C})$ , we see  $\eta_{(k)}(X^g) {}^t(a \ b)_{2k-2} = Q(X^g, A_1)^{k-1} = Q(X, A_1^{g^{-1}})^{k-1} = P_{k-1}(X^g)$ . Then

$$F_i(X^g) = P_{k-1}(X^g)e^{-\pi R(X^g)} = Q(X, A_1^{g^{-1}})^{k-1}e^{-\pi R^g(X)}.$$

Therefore we can see that  $F_i(X^g)$  satisfies the first spherical property for weight  $k$  by Lemma 1.2 of [21]. Finally, since the Weil representation acts independently of  $a, b$ , the function  $\eta_{k,\alpha}(X^g)e^{-\pi R^g(X)}$  is also of the first spherical property for weight  $k$ . Then  $\theta_{k,\alpha}$  coincides with the theta function in [8]:  $\theta(z, \varphi, -V) = \theta_{k,\alpha}(z, V, g, \Lambda)$ .

From this interpretation we can compute the Fourier expansion of  $\theta_{k,\alpha}|_k\gamma$  by using Theorem 1.5 of [8]. Note that the signature is  $(3, 1)$  and the weight is  $k$  for  $\theta_{k,\alpha}$ . For  $\gamma \in \Gamma$  we get

$$(\theta_{k,\alpha}|_k\gamma)(z, V, g, \Lambda) = i^{-\text{sgn } c} \sum_{X \in \Lambda^*} c(-V, X)_\gamma y^{1/2} \eta_{k,\alpha}(X^g) e^{\pi i(xQ + iyR^g)[X]},$$

where  $v(\Lambda) = \int_{\mathcal{V}/\Lambda} dx$  and

$$c(h, k)_\gamma = \begin{cases} \delta_{h,ak} e^{\pi i ab Q(h,k)} & \text{if } c = 0 \\ |\det Q|^{1/2} v(\Lambda)^{-1} |c|^{-n/2} \sum_{r \in \Lambda/c\Lambda} e^{2\pi i \left( \frac{aQ[h+r] - 2Q(k, h+r) + dQ[k]}{2c} \right)} & \text{if } c \neq 0. \end{cases}$$

### 3. Main result

**THEOREM 3.1.** *The lifted forms  $F^{reg}(g(\mu, \nu); f)$  on  $\mathfrak{H}^1$  are weak Maass forms for  $K$  of weight  $2k - 2$  and eigenvalue  $(k^2 - 2k)/4 + \lambda$  for the group  $G(\Lambda) := \{\gamma \in \mathbf{G} \mid \Lambda^\gamma = \Lambda\}$  with singularities on*

$$\bigcup_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} \bigcup_{\substack{X \in \Lambda^* \\ -n_\gamma \leq \frac{NQ[X]}{2} \leq -1}} \{g = g(\mu, \nu) \mid (Q + R^g)[X] = 0\}$$

where  $n_\gamma$  is the index coming from the Fourier expansion (1) of  $f$ .

In this section we prove Theorem 3.1 saying that  $F^{reg}(g(\mu, \nu); f)$  is a weak Maass form for  $K$ . To show the convergence of  $F_\alpha^{reg}(g, V, \Lambda; f)$ , we need the convergence of the function

$$\int_1^t \int_{-1/2}^{1/2} \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z) \overline{(f|_{k\gamma})(z)} y^{k-2-w} dx dy$$

as  $t \rightarrow \infty$ , for  $Re(w) \gg 0$ . Since  $\sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z) \overline{(f|_{k\gamma})(z)} y^k$  is  $SL_2(\mathbb{Z})$ -invariant, we may write  $\sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z) \overline{(f|_{k\gamma})(z)} y^k = \sum_{n \in \mathbb{Z}} a(n, y, V, g, \Lambda) e^{2\pi i n x}$ . Then

$$\begin{aligned} & \int_1^t \int_{-1/2}^{1/2} \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z) \overline{(f|_{k\gamma})(z)} y^{k-2-w} dx dy \\ &= \int_1^t \int_{-1/2}^{1/2} \sum_{n \in \mathbb{Z}} a(n, y, V, g, \Lambda) e^{2\pi i n x} y^{-2-w} dx dy = \int_1^t a(0, y, V, g, \Lambda) y^{-2-w} dy. \end{aligned}$$

Hence to ensure the convergence of the regularized theta lifting, we need to know the growth condition of the ‘constant term’ of  $\sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z) \overline{(f|_{k\gamma})(z)}$ . Let  $(f|_{k\gamma^{-1}})(z) = \sum_{n \in \mathbb{Z}} b_\gamma(n, y, s) e^{2\pi i n x/N}$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ . Then  $(\theta_{k,\alpha} |_{k\gamma})(z, V, g, \Lambda) \overline{(f|_{k\gamma})(z)}$  is equal to

$$i^{-sgn} c y^{\frac{1}{2}} \sum_{X \in \Lambda^*} \sum_{n \in \mathbb{Z}} c(-V, X)_\gamma \eta_{k,\alpha}(X^g) \overline{b_{\gamma^{-1}}(n, y, s)} e^{-\pi y R^g[X]} e^{\pi i x Q[X]} e^{-2\pi i n x/N}.$$

Since  $\sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta_{k,\alpha} |_{k\gamma})(z, V, g, \Lambda) \overline{(f|_{k\gamma})(z)} y^k$  is  $SL_2(\mathbb{Z})$ -invariant and has the Fourier expansion of the form  $\sum_{n \in \mathbb{Z}} a(n, y, V, g, \Lambda, s) e^{2\pi i n x}$ , summing all the expansions  $(\theta_{k,\alpha} |_{k\gamma})(z, V, g, \Lambda) \overline{(f|_{k\gamma})(z)} y^k$  over  $\Gamma(N) \backslash SL_2(\mathbb{Z})$  we have only the terms involving  $e^{2\pi i n x}$  for  $n$  integral, although each series  $(\theta_{k,\alpha} |_{k\gamma})(z, V, g, \Lambda) \overline{(f|_{k\gamma})(z)} y^k$  has the terms involving  $e^{2\pi i n x/N}$  in the expansion. Consequently we need to consider the convergence of the following integral

$$\int_1^t \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} i^{-sgn} c \sum_{X \in \Lambda^*} c(-V, X)_\gamma \eta_{k,\alpha}(X^g) \overline{b_{\gamma^{-1}}\left(\frac{NQ[X]}{2}, y, s\right)} e^{-\pi y R^g[X]} y^{k-\frac{3}{2}-w} dy$$

as  $t \rightarrow \infty$ .



**3.1. Convergence Region.** In this subsection we complete the proof for the convergence of the regularized theta lifting and describe the convergence region of  $F^{reg}(g(\mu, \nu); f)$ . Let  $g(\mu, \nu)$  be a point in the complement of

$$\bigcup_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} \bigcup_{\substack{X \in \Lambda^* \\ -n\gamma \leq \frac{NQ[X]}{2} \leq -1}} \{g = g(\mu, \nu) | (Q + R^g)[X] = 0\}.$$

Inputting the Fourier expansion of  $(f|_k \gamma)(z)$  into the expression of  $F_\alpha^{reg}(g(\mu, \nu); f)$  above, the estimates  $a_{\gamma^{-1}}(n, s) = O(e^{\sqrt{n}})$  and  $W_{\nu, \mu}(y) \sim e^{-y/2} y^\nu$  as  $|y| \rightarrow \infty$  give the convergence for the part concerning  $W_{\pm \frac{k}{2}, s - \frac{1}{2}}(2\pi|Q[X]|y)$ . For the constant term we have the integral of the form  $\int_1^t y^{\frac{k}{2} - \frac{1}{2} - s - w} dy$ , therefore it converges for  $Re(w) \gg 0$ , as  $t \rightarrow \infty$ . It remains to consider the part concerning  $W_{\frac{k}{2}, s - \frac{1}{2}}(-2\pi|Q[X]|y)$ . It is a fact that for any  $C \geq 0$  and any compact subset  $B \subset \mathfrak{H}^1$  the set

$$\{\lambda \in \Lambda^* | Q[\lambda] = m, \exists g = g(\mu, \nu) \in B \text{ with } (Q + R^g)[\lambda] \leq C\}$$

is finite. By assumption, there exists  $\varepsilon > 0$  such that  $(Q + R^g)[\lambda] > \varepsilon$  for all  $\lambda \in \Lambda^*$  with  $Q[\lambda] = m$  for a fixed  $g = g(\mu, \nu)$ . It follows that

$$\begin{aligned} & \sum_{\substack{X \in \Lambda^* \\ Q[X] = m < 0}} |c(-V, X)_\gamma \eta_{k, \alpha}(X^g) \overline{W_{\frac{k}{2}, s - \frac{1}{2}}(2\pi m y)} e^{-\pi y R^g[X]} y^{\frac{1}{2}k - \frac{3}{2} - w}| \\ & \ll \sum_{\substack{X \in \Lambda^* \\ Q[X] = m < 0}} |\eta_{k, \alpha}(X^g)| e^{\pi(-m + Q[X])y} e^{-\pi y(Q + R^g)[X]} y^{k - \frac{3}{2} - Re(w)} \\ & \ll e^{-\frac{\pi \varepsilon y}{2}} y^{k - \frac{3}{2} - Re(w)} \sum_{\substack{X \in \Lambda^* \\ Q[X] = m < 0}} |\eta_{k, \alpha}(X^g)| e^{-\pi(m + R^g[X])/2}, \end{aligned}$$

which implies the uniform convergence for  $y \in [1, \infty)$ . This implies the desired convergence statement.

**3.2. Modularity.** In this subsection we show that  $F^{reg}(g(\mu, \nu); f)$  satisfies the modularity as in Definition 2.2. Denote by  $\iota$  the main involution of  $M(2, \mathbb{C}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ . We define an involution  $\iota_n$  on the space of modular forms over  $K$  of weight  $n$  by  $\iota_n F^1(z) = \rho_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^1(-\bar{z})$ . Note that each  $G(\Lambda_i)$  is a congruence subgroup of  $SL_2(\mathcal{O}_K)_i$  (see [10]). The basic properties of the lifted form  $F^{reg}(g(\mu, \nu); f)$  are as follows.

**THEOREM 3.2.** *The lifted form  $F^{reg}(g(\mu, \nu); f)$  satisfies the following properties.*

- (1) For  $\kappa$  in  $SU(2, \mathbb{C})$  and  $\gamma$  in  $G(\Lambda)$ ,  $F^{reg}(\gamma g \kappa, V) = {}^t \rho_{2k-2}(\kappa) F^{reg}(g, V^\gamma)$ .
- (2)  $F^{reg}(g, V, \Lambda) = (-1)^{k-1} \rho_{2k-2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F^{reg}(\bar{g}, \bar{V}, \bar{\Lambda})$ .
- (3)  $F^{reg}({}^t g^{-1}, V, \Lambda) = (-1)^{k-1} F^{reg}(\bar{g}, V^\iota, \Lambda^\iota)$ .
- (4)  $F^{reg}(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, V) = (-1)^{k-1} \rho_{2k-2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F^{reg}\left(g, V \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ .
- (5) For  $z \in \mathfrak{H}^1$ ,  $\iota_{2k-2}(F^{reg})^1(z, V, \Lambda) = (F^{reg})^1\left(z, \bar{V} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \bar{\Lambda}\right)$ .

This is the generalization of Theorem 3.3 in [10]. Since we can prove by the almost same argument as in [10, Theorem 3.3] using the properties of  $\theta_{k,\alpha}$ , we omit the proof of Theorem 3.2. In particular, by Proposition 2.3 and Theorem 3.2 (1), the function  $(F^{reg}(g(\mu, \nu); f))^1$  on  $\mathfrak{H}^1$  corresponding to  $F^{reg}(g(\mu, \nu); f)$  is modular of weight  $2k - 2$ .

**3.3. Casimir operators.** In this subsection we describe the behavior of  $F^{reg}(g(\mu, \nu); f)$  under the Casimir operators and prove that  $F_\alpha^{reg}(g(\mu, \nu); f)$  is compatible with Maass differential operators and the Laplacian. This was already illustrated in the paper of Friedberg [10, 12] when  $f$  is a Maass cusp form. From now on we generalize it to weak Maass forms. We only consider the behavior under the Casimir operators, but the same result holds for Maass operators. Recall that  $\Delta_k = L_{k+2}R_k - k = R_{k-2}L_k$ , where  $R_k$  and  $L_k$  are Maass operators given by  $R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y}$  and  $L_k = 2iy^2\frac{\partial}{\partial \bar{z}}$ . Let  $\mathcal{F}_u = \{z = x + iy \in \mathfrak{H} \mid |z| \geq 1, |x| \leq \frac{1}{2}, y \leq u\}$  denote the truncated fundamental domain for  $\Gamma(1)$ . We recall the following

**LEMMA 3.3.** ([5, Lemma 4.2]) *Let  $f$  and  $g$  be smooth functions such that  $f|_{k+2}\gamma = f$  and  $g|_k\gamma = g$ . Then for all  $\gamma \in \Gamma(N)$*

$$\begin{aligned} & \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[ \sum_{\gamma} y^k \overline{(f|_{k+2}\gamma)(z)} (g|_k\gamma)(z) \right]_{y=u} dx \\ &= \int_{\mathcal{F}_u} y^{k-2} \sum_{\gamma} \overline{L_{k+2}(f|_{k+2}\gamma)(z)} g|_k\gamma(z) dx dy - \int_{\mathcal{F}_u} y^k \sum_{\gamma} \overline{(f|_{k+2}\gamma)(z)} R_k(g|_k\gamma)(z) dx dy. \end{aligned}$$

Recall that  $(\theta_{k,\alpha}|_k\gamma)(z, V, g, \Lambda) = \theta_{k,\alpha}(z, V, g, \Lambda)$  for all  $\gamma \in \Gamma(N)$  and  $N_1|N$ . Hence we can apply Lemma 3.3 to a pair of  $\theta_{k,\alpha}(z, V, g, \Lambda)$  and  $f(z)$ . Then the key fact is the following theorem in [10] describing the behavior of the theta kernel  $\theta_{k,\alpha}(z, V, g, \Lambda)$  under Casimir operators corresponding to  $SL_2(\mathbb{R})$  and  $SL_2(\mathbb{C})$ .

**THEOREM 3.4.** ([10, Theorem 3.4])  $\Psi\theta_{k,\alpha}(z, g) = \tilde{\Psi}\theta_{k,\alpha}(z, g) = (-\Delta_k + \frac{k^2-2k}{4})\theta_{k,\alpha}(z, g)$ .

For simplicity, we write  $F_\alpha^{reg}(g) := F_\alpha^{reg}(g, V, \Lambda; f)$ . By Theorem 3.4

$$\begin{aligned} (\Psi F_\alpha^{reg})(g) &= (\tilde{\Psi} F_\alpha^{reg})(g) = \langle \Psi\theta_{k,\alpha}(z, g), f(z) \rangle^{reg} \\ &= \left\langle \left( -\Delta_k + \frac{k^2-2k}{4} \right) \theta_{k,\alpha}(z, g), f(z) \right\rangle^{reg}. \end{aligned}$$

Applying Lemma 3.3 twice and following the argument of [5] we get

$$\begin{aligned}
& \left\langle \left( -\Delta_k + \frac{k^2 - 2k}{4} \right) \theta_{k,\alpha}(z, g), f(z) \right\rangle^{reg} \\
&= - \lim_{u \rightarrow \infty} \int_{\mathcal{F}_u} \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} (\theta|_k \gamma)(z) \overline{\left( \Delta_k - \frac{k^2 - 2k}{4} \right) (f|_k \gamma)(z) y^{k-2} dx dy} \\
&= - \left\langle \theta_{k,\alpha}(z, g), \left( \Delta_k - \frac{k^2 - 2k}{4} \right) f(z) \right\rangle^{reg} = - \left\langle \theta_{k,\alpha}(z, g), \left( -\lambda - \frac{k^2 - 2k}{4} \right) f(z) \right\rangle^{reg} \\
&= \left( \frac{k^2 - 2k}{4} + \lambda \right) F_\alpha^{reg}(g),
\end{aligned}$$

and in conclusion

$$(\Psi F_\alpha^{reg})(g) = (\tilde{\Psi} F_\alpha^{reg})(g) = \left( \frac{k^2 - 2k}{4} + \lambda \right) F_\alpha^{reg}(g).$$

#### 4. Singularity type and the Fourier expansion of $F_\alpha^{reg}$

In this section we compute the singularity type of  $F_\alpha^{reg}(g(\mu, \nu); f)$  and its Fourier expansion.

**4.1. Singularity type.** First we consider the singularity type. Let  $U \subset \mathfrak{H}^1$  be an open subset and  $f, g$  functions on a dense open subset of  $U$ . Then we write  $f \approx g$  if  $f - g$  can be continued to a real analytic function on  $U$ . In this case we say that  $f$  has a singularity of type  $g$ . For a subset  $U$  of  $\mathfrak{H}^1$  with compact closure, we define

$$\mathcal{S}(m, U) := \left\{ \lambda \in \Lambda^* \mid \frac{NQ[\lambda]}{2} = m, (Q + R^g)[\lambda] = 0, \exists g \in U \right\}.$$

Note that  $\mathcal{S}(m, U)$  is a finite set.

**THEOREM 4.1.** *Let  $U \subset \mathfrak{H}^1$  be an open subset with compact closure. On  $U$  the function  $F_\alpha^{reg}(g(\mu, \nu); f)$  has a singularity of type*

$$\begin{aligned}
& (-1)^{\frac{k}{2}} \sum_{n=0}^{k-1} \left\{ 1 + \frac{\{(\bar{s} - \frac{1}{2})^2 - (\frac{k}{2} - \frac{1}{2})^2\} \cdots \{(\bar{s} - \frac{1}{2})^2 - (\frac{k}{2} - n + \frac{1}{2})^2\}}{n!} \right\} \\
& \times \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} i^{-sgn} c \sum_{m < 0} \frac{\overline{\tilde{a}_{\gamma^{-1}(\frac{m}{2}, s)}}}{(2\pi m)^n} \sum_{X \in \mathcal{S}(m, U)} c(-V, X)_\gamma \eta_{k,\alpha}(X^{g(\mu, \nu)}) \\
& \times \Gamma\left(k - n - \frac{3}{2}\right) \sqrt{\pi(m + R^{g(\mu, \nu)}[X])}^{2n-2k+3}.
\end{aligned}$$

Here  $\overline{\tilde{a}_{\gamma^{-1}(\frac{m}{2}, s)}}$  are the Fourier coefficients of  $f|_k \gamma$ ,  $c(-V, X)_\gamma$  are the factors appearing in the Fourier expansion of  $\theta_{k,\alpha}|_k \gamma$  and  $\eta_{k,\alpha}(X^{g(\mu, \nu)})$  is the spherical harmonic polynomials given in the definition of  $\theta_{k,\alpha}$  (for the details, see section 2).

Again put the Fourier expansion of  $(f|_k\gamma)(z)$  inside as above. Then the part concerning  $W_{\pm\frac{k}{2},s-\frac{1}{2}}(2\pi|Q[\lambda]|y)$  and the constant term do not contribute to the singularity. In the remaining part concerning  $W_{\frac{k}{2},s-\frac{1}{2}}(-2\pi|Q[X]|y)$ , the integrals become singular if  $(Q + R^g)[\lambda] = 0$ , hence we have the singularity type

$$\begin{aligned} & \sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} i^{-sgn\ c} \sum_{m < 0} \overline{\tilde{a}_{\gamma^{-1}}\left(\frac{m}{2}, s\right)} \sum_{X \in \mathcal{S}(m, U)} |2\pi m|^{-\frac{k}{2}} c(-V, X)_{\gamma} \eta_{k, \alpha}(X^g) \\ & \times \int_1^{\infty} \overline{W_{\frac{k}{2}, s-\frac{1}{2}}(2\pi m y)} e^{-\pi y R^g[X]} y^{\frac{k}{2} - \frac{3}{2}} dy. \end{aligned}$$

Note that the integral is evaluated near the singular point but not at it. It is known that  $W_{p,q}(z)$  has the following asymptotic expansion (see [22, p.345])

$$W_{p,q}(z) = e^{-\frac{z}{2}} z^p \left\{ 1 + \sum_{L=1}^n \frac{\{q^2 - (p - \frac{1}{2})^2\} \{q^2 - (p - \frac{3}{2})^2\} \cdots \{q^2 - (p - L + \frac{1}{2})^2\}}{L! z^L} + O\left(\frac{1}{z^{n+1}}\right) \right\}$$

as  $|z| \rightarrow \infty, |\arg z| < \frac{3}{2}\pi$ . We apply this to  $W_{\frac{k}{2}, s-\frac{1}{2}}(2\pi m y)$  with  $L = k - 1$ . This makes the integral associated to the error terms converge, i.e.

$$\begin{aligned} & \int_1^{\infty} e^{-\pi m y} |2\pi m y|^{\frac{k}{2}} \cdot C |2\pi m y|^{-L-1} e^{-\pi y R^g[X]} y^{\frac{k}{2} - \frac{3}{2}} dy \\ & = C |2\pi m|^{\frac{k}{2} - L - 1} \sqrt{\pi(m + R^g[X])}^{2L - 2k + 3} \int_{\pi(m + R^g[X])}^{\infty} e^{-y} y^{k - L - \frac{5}{2}} dy. \end{aligned}$$

The latter integral is uniformly bounded since  $k - L - \frac{5}{2} \notin \mathbb{Z}$  and  $2L - 2k + 3 > 0$ , so it converges to zero as  $m + R^g[X] \rightarrow 0$ . Hence we may use this asymptotic formula and get

$$\begin{aligned} & \int_1^{\infty} \overline{W_{\frac{k}{2}, s-\frac{1}{2}}(2\pi m y)} e^{-\pi y R^g[X]} y^{\frac{k}{2} - \frac{3}{2}} dy \\ & \sim \sum_{n=0}^{k-1} \left\{ 1 + \frac{\{(\bar{s} - \frac{1}{2})^2 - (\frac{k}{2} - \frac{1}{2})^2\} \cdots \{(\bar{s} - \frac{1}{2})^2 - (\frac{k}{2} - n + \frac{1}{2})^2\}}{n! (2\pi m)^n} \right\} (2\pi m)^{\frac{k}{2}} \\ & \quad \times \sqrt{\pi(m + R^g[X])}^{2n - 2k + 3} \Gamma\left(k - n - \frac{3}{2}\right), \end{aligned}$$

which completes the proof.

**4.2. Fourier expansion.** In this subsection we compute the Fourier expansion of  $F_{\alpha}^{reg}(g(\mu, \nu); f)$ . Let  $\Lambda$  be one of the  $\Lambda_i$ 's above and  $J$  the corresponding  $J_i$ . Write  $V \in \Lambda^*$  as  $V = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_4 \end{pmatrix}$ , and for  $X = \begin{pmatrix} m & r \\ r & p \end{pmatrix} \in \mathcal{V}$  put  $X - V = \begin{pmatrix} m_1 & r_2 \\ r_2 & p_4 \end{pmatrix}$ . Take a set of coset representatives  $\mathcal{R}$  for  $(\frac{1}{0} \frac{\mathbb{Z}}{1}) \backslash SL_2(\mathbb{Z}) / \Gamma(N)$  such that  $\rho = \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix} \in \mathcal{R}$  has  $\rho_3 \neq 0$ . Let  $C_{\alpha} = \frac{(k-1)! N i^{k-\alpha}}{(\mathbf{N}(I)^2 D)^{1/2} \pi^{k/2} 2^{k-2}}$ . For a given  $f \in WMF_{k, -\lambda}^+(\Gamma(N))$  we have  $m \in \mathbb{Z}$  such that  $f|_k\gamma^{-1} = O(e^{2\pi m y})$  for all  $\gamma \in (\frac{1}{0} \frac{\mathbb{Z}}{1}) \backslash SL_2(\mathbb{Z}) / \Gamma(N)$ . Let  $\hat{m}$  be  $(2mP^2)^{1/2}$  if  $m \geq 0$  and 0 otherwise. Since  $F_{\alpha}^{reg}(g, V, \Lambda; f) = (-1)^{\alpha} F_{-\alpha}^{reg}(\bar{g}, \bar{V}, \bar{\Lambda}; f)$  for  $\alpha \leq 0$ , it is enough to

compute the Fourier expansion in the case  $\alpha \geq 0$ . To state the result, we define

$$\begin{aligned}
p'(\nu, s, \rho) &= \nu^{2-2\bar{s}} \pi^{-\frac{k}{2}+1-\bar{s}} q(k, s) \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} (P^{-1}t)^{1-2\bar{s}} e(-2P^{-1}t\rho_4v_4), \\
\tau(\rho) &= \sum_{v' \in \bar{J}^{-1}\mathcal{D}_K^{-1}/(-\rho_3)J} \sum_{v'' \in J/(-\rho_3)J} e(-2\rho_3^{-1}(\rho_4\mathbf{N}(v'') + \rho_1\mathbf{N}(v')) + \mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}\{-2\bar{v}'v''\rho_3^{-1}\}), \\
p(\nu, w) &= \frac{(k-1)!P^{2w-1}Ni}{2^{k-1}(\mathbf{N}(I)^2D)^{1/2}} \sum_{\rho \in \mathcal{R}} \rho_3^{-1} \tau(\rho) e(2\rho_4\rho_2\mathbf{N}(v_2)) \\
&\quad \times (\overline{\tilde{a}_\rho(0, s)} p'(\nu, s+w, \rho) + \overline{a_\rho(0, s)} p'(\nu, 1-s+w, \rho)),
\end{aligned}$$

where  $q(k, s) = \sum_{\substack{2\beta+\gamma=k-1 \\ \beta, \gamma \geq 0}} \frac{(2i)^\gamma}{\beta!\gamma!} \Gamma\left(\bar{s} + \frac{k-1+\gamma}{2}\right)$ .

**THEOREM 4.2.** *Let  $\alpha \geq 0$ . Let  $f \in WMF_{k, -\lambda}^+(\Gamma(N))$  with a Fourier expansion of the form (1). Let  $V = \begin{pmatrix} v_1 & v_2 \\ v_2 & v_4 \end{pmatrix} \in \Lambda^*$ . Then for  $\nu > \hat{m}$  the lifted form  $F_\alpha^{reg}(g(\mu, \nu); f)$  has the Fourier expansion given by the constant term of the Laurent expansion at  $w = 0$  of the analytic continuation of*

$$\begin{aligned}
&C_\alpha \sum_{\rho \in \mathcal{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1 \pmod{MP}}}^\infty t^{-1} \sum_{r \in \bar{J}^{-1}\mathcal{D}_K^{-1}-0} |r|^{-k} (r/|r|)^\alpha \overline{a_\rho(\mathbf{N}\mathbf{N}(r), s)} \\
&\quad \times \tau(\rho, r, v_2) q_\alpha(2\pi P^{-1}t\nu|r|, \lambda) e(2P^{-1}t \mathrm{tr}(\bar{r}\mu) - \rho_4v_4) + \delta_{\alpha, 0} p(\nu, w) \\
&\quad + (k-1)! \left[ \left( \frac{k-1+\alpha}{2} \right)! (4\pi)^{(k-1-\alpha)/2} \right]^{-1} \delta_{v_1(M), 0} \nu \langle \phi_{\alpha, (k-1-\alpha)/2}(z, 0, v_2), f \rangle^{reg},
\end{aligned}$$

where  $\tau(\rho, r, v_2)$  and  $q_\alpha(2\pi P^{-1}t\nu|r|, \lambda)$  are the functions defined in (5) and (6).

In [10, Theorem 4.1] the Fourier expansion of  $F_\alpha^{reg}(g(\mu, \nu); f)$  has been derived in the case when  $f$  is a Maass cusp form. Compared to this, our result coincide with [10, Theorem 4.1] in the first two parts and contains extra terms whose main terms are  $\nu^{2-2\bar{s}}$  and  $\nu^{2\bar{s}}$ .

**PROOF.** We proceed as in the proof of Theorem 4.1 in [10] together with the convergence analysis. Let us first consider the  $\alpha \geq 0$  case. Note that  $\langle \theta_{k, \alpha}, f \rangle^{reg} = \langle \theta_{k, \alpha} |_{k\rho^{-1}}, f |_{k\rho^{-1}} \rangle^{reg}$ . This is because  $\sum_{\gamma \in \Gamma(N) \backslash SL_2(\mathbb{Z})} \theta_{k, \alpha} |_{k\gamma} f |_{k\gamma} y^k$  is  $SL_2(\mathbb{Z})$ -invariant. Using the Rankin-Selberg method we unfold the integral. First  $\phi_{\alpha, \beta}(z, 0, v) \in G\{\Gamma(N), \alpha+2\beta+1, -\beta(\alpha+\beta)\}$  for  $v \in \bar{J}^{-1}\mathcal{D}_K^{-1}$ , when we consider also the behaviors under Maass operators, where  $DN(I)|N$ . We compute the lifting using Proposition 4.6 of [10] to obtain that  $\langle \theta_{k, \alpha}, f \rangle^{reg}$

is equal to

$$\begin{aligned} & \int_{\Gamma(N)\backslash\mathfrak{H}}^{reg} \left[ \sum_{\beta,\gamma} \frac{(k-1)!i^\gamma\nu^{\gamma+1}}{(\alpha+\beta)! \gamma! 2^\beta (2\pi)^\beta P} \times \left\{ \delta_{v_1(M),0} \delta_{\gamma,0} \phi_{\alpha,\beta}(z,0,v_2) \right. \right. \\ & + \sum_{\rho \in \mathcal{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} (P^{-1}t)^\gamma \sum_{\sigma \in \Gamma_\infty \backslash \Gamma(N)} \left[ (\phi_{\alpha,\beta}(z, P^{-1}t\mu\rho_4, v_2 - P^{-1}t\mu\rho_3)|_{\alpha+2\beta+1}\rho^{-1} \right. \\ & \left. \left. \times y^{-\gamma} e(iP^{-2}t^2\nu^2y^{-1} - P^{-1}t(\text{Tr}_{\mathbb{R}}^{\mathbb{C}}(v_2\bar{\mu}\rho_4) + 2\rho_4v_4))|_{k\sigma})(z) \right] \right\} \overline{f|_{k\rho^{-1}}y^k} \frac{dx dy}{y^2}. \end{aligned}$$

The constant term  $\delta_{v_1(M),0} \delta_{\gamma,0} \phi_{\alpha,\beta}(z,0,v_2)$  in the expansion of  $\theta_{k,\alpha}$  is a modular form for  $\Gamma(N)$  of weight  $k = \alpha + 2\beta + 1$  if this is nontrivial, i.e.,  $\gamma = 0$ , so as above we have

$$\langle \delta_{v_1(M),0} \phi_{\alpha,(k-1-\alpha)/2}(z,0,v_2), f \rangle^{reg} = \langle \delta_{v_1(M),0} \phi_{\alpha,(k-1-\alpha)/2}(z,0,v_2)|_{k\rho^{-1}}, f|_{k\rho^{-1}} \rangle^{reg},$$

which gives the constant term in the expansion of  $F_\alpha^{reg}(g(\mu,\nu); f)$ . The strategy is to single out this constant term, and the idea for the convergence of it is that  $\phi_{\alpha,\beta}$  contains only Fourier coefficients with positive indices, so that only decreasing Whittaker functions and exponentials survive after integration with respect to  $x$  (up to the constant Fourier coefficients, which are dealt with using the extra  $y^{-w}$ ).

First we consider the constant term part. If we let  $v_1 \equiv 0 \pmod{N}$ , then

$$\langle \phi_{\alpha,\frac{k-1-\alpha}{2}}(z,0,v_2), f \rangle^{reg} = \int_{SL_2(\mathbb{Z})\backslash\mathfrak{H}}^{reg} \sum_{\gamma \in \Gamma(N)\backslash SL_2(\mathbb{Z})} (\phi_{\alpha,\frac{k-1-\alpha}{2}}|_{k\gamma})(z,0,v_2) \overline{f|_{k\gamma}(z)} y^k \frac{dx dy}{y^2}.$$

It suffices to consider the integration over the subregion satisfying the condition  $y > 1$  for convergence check. So we consider the integral taken over that region. Next we separate the integral into two parts

$$\begin{aligned} \langle \phi_{\alpha,\frac{k-1-\alpha}{2}}(z,0,v_2), f \rangle^{reg} &= \int_1^\infty \int_0^1 \sum_{\substack{\gamma \in \Gamma(N)\backslash SL_2(\mathbb{Z}) \\ c=0}} (\phi_{\alpha,\frac{k-1-\alpha}{2}}|_{k\gamma})(z,0,v_2) \overline{f|_{k\gamma}(z)} y^{k-w} \frac{dx dy}{y^2} \\ &+ \int_1^\infty \int_0^1 \sum_{\substack{\gamma \in \Gamma(N)\backslash SL_2(\mathbb{Z}) \\ c \neq 0}} \{ (-1)^{\alpha+1} c^{k-2} \sum_{v' \in \bar{J}^{-1} \mathcal{D}_K^{-1}/cJ} \tau'(\gamma,0,v_2,v') \\ &\times \phi_{\alpha,\frac{k-1-\alpha}{2}}(c^2z, cbv_2, -c^{-1}(av_2 - v')) \} \overline{f|_{k\gamma}(z)} y^{k-w} \frac{dx dy}{y^2}, \end{aligned}$$

where we have used the expression of  $\phi_{\alpha,\beta}|_{k\gamma}$  in [10, Proposition 4.7] in the second part where

$$\tau'(\gamma, u, v, v') = (\mathbf{N}(I)^2 D)^{-1/2} \sum_{v'' \in J/cJ} e(2c^{-1}(c\mathbf{N}(v'') + d\mathbf{N}(v')) - \text{Tr}_{\mathbb{R}}^{\mathbb{C}}(\bar{v}'(bv - du - 2v''c^{-1}))).$$

Note that in the first part, each  $(\phi_{\alpha,\frac{k-1-\alpha}{2}}|_{k\gamma})(z,0,v_2)$  are just translations by  $m \in \mathbb{Z}/N\mathbb{Z}$ . So the first part is equal to

$$\int_1^\infty \int_0^N \phi_{\alpha,\frac{k-1-\alpha}{2}}(z,0,v_2) \overline{f(z)} y^{k-w} \frac{dx dy}{y^2}.$$

Then the term concerning  $W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\mathbf{N}(r-v_2)y)$  is of exponential decay and we can make the exponent of  $y$  less than  $-1$  for  $Re(w)$  sufficiently large, hence we conclude that the whole integral can be convergent when we make use of the regularization technique.

Next we consider the second part. First input the Fourier expansion of  $\phi_{\alpha, \frac{k-1-\alpha}{2}}$  and the Fourier expansion of  $f|_k\gamma(z)$  in the second part, and integrate it with respect to  $x$ . Then it suffices to show that the summation of each integration for each  $r$  yields an absolutely convergent series using Lebesgue's dominated convergence theorem. Here the important summation is the sum over  $r \in J$ , because this is an infinite sum. But the growth conditions  $a_{\gamma^{-1}}(n, s) = O(e^{C\sqrt{n}})$  for some  $C > 0$  and  $W_{\nu, \mu}(y) \sim e^{-y/2}y^\nu$  as  $|y| \rightarrow \infty$  guarantee the convergence corresponding to  $W_{\frac{k}{2}, s-\frac{1}{2}}$ . It yields the convergence of the constant term of lifting.

Next we consider the remaining 'non-constant term' part to compute. Note that  $((f|_k\rho^{-1})|_k\gamma)(z) = (f|_k\rho^{-1})(z)$  for all  $\gamma \in \Gamma(N)$ , because  $\Gamma(N) \triangleleft SL_2(\mathbb{Z})$ . Then we can replace the integral over a fundamental domain of  $SL_2(\mathbb{Z})$  by an integral over a fundamental domain of  $N\mathbb{Z}$ , since the argument appearing in [2, Theorem 7.1] holds equally well here:

$$\int_0^\infty \int_0^N \left[ \sum_{\beta, \gamma} \frac{(k-1)!i^\gamma\nu^{\gamma+1}}{(\alpha+\beta)!\gamma!2^\beta(2\pi)^\beta P} \sum_{\rho \in \mathcal{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} (P^{-1}t)^\gamma (\phi_{\alpha, \beta}(z, P^{-1}t\mu\rho_4, v_2 - P^{-1}t \\ \times \mu\rho_3)|_{\alpha+2\beta+1}\rho^{-1} \cdot y^{-\gamma} e(iP^{-2}t^2\nu^2y^{-1} - P^{-1}t(\text{Tr}_{\mathbb{R}}^{\mathbb{C}}(v_2\bar{\mu}\rho_4) + 2\rho_4v_4))) \right] \overline{f|_k\rho^{-1}} y^{k-w} \frac{dx dy}{y^2}.$$

Then we use Proposition 4.7 of [10] to expand  $\phi_{\alpha, \beta}(z, P^{-1}t\mu\rho_4, v_2 - P^{-1}t\mu\rho_3)|_{\alpha+2\beta+1}\rho^{-1}$ . Now we are not considering the issue of convergence and we consider this later. When we compute the above integral term by term, then the essential part we have to consider is

$$\int_0^\infty \int_0^N (\rho_3^2 y)^{-\beta} L_\beta^{(\alpha)}(4\pi y \mathbf{N}(\rho_3 r - \rho_4 v_2 + v')) e(2\mathbf{N}(\rho_3 r - \rho_4 v_2 + v')z) y^{-\gamma} e(iP^{-2}t^2\nu^2y^{-1}) \\ \times \overline{f|_k\rho^{-1}} y^{k-w} \frac{dx dy}{y^2}.$$

In the case  $\rho_3 r - \rho_4 v_2 + v' \neq 0$  this is equal to

$$N\rho_3^{-2\beta} \overline{a_\rho(N\mathbf{N}(\rho_3 r - \rho_4 v_2 + v'), s)} (4\pi\mathbf{N}(\rho_3 r - \rho_4 v_2 + v'))^{-k/2} \\ \times \int_0^\infty y^{-\beta} L_\beta^{(\alpha)}(4\pi\mathbf{N}(\rho_3 r - \rho_4 v_2 + v')y) e^{-2\pi\mathbf{N}(\rho_3 r - \rho_4 v_2 + v')y} \\ \times y^{-\gamma} e^{-\pi P^{-2}t^2\nu^2y^{-1}} y^{-k/2} \overline{W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\mathbf{N}(\rho_3 r - \rho_4 v_2 + v')y)} y^{k-2-w} dy.$$

Using Lemma 4.8 of [10] and the Fourier expansion of  $f|_k\rho^{-1}$  we can evaluate the integral. Since there are exponentially decaying terms inside the integral, this converges regardless

of  $w$ , so we can evaluate at  $w = 0$  to get

$$\begin{aligned}
& 2N\rho_3^{-2\beta} \overline{a_\rho(N\mathbf{N}(\rho_3r - \rho_4v_2 + v'), s)} (4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v'))^{-k/2} \\
& \times \sum_{m=0}^{\beta} \sum_{e=0}^{\beta+\gamma-m} (-1)^{\beta+\gamma+e} \binom{\beta+\alpha}{\beta-m} \binom{\beta+\gamma-m}{e} (k/2-s)_{\gamma+\beta-m-e} (m!)^{-1} \\
& \times (4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v'))^{m+(e+1)/2} (\pi P^{-2}t^2\nu^2)^{m+(\alpha-\gamma+e)/2} \\
& \times \overline{K_{2s-1-e}(4\pi P^{-1}t\nu(\mathbf{N}(\rho_3r - \rho_4v_2 + v')))},
\end{aligned}$$

where  $K$  is the  $K$ -Bessel function. In the case  $\rho_3r - \rho_4v_2 + v' = 0$  the integral is convergent for sufficiently large  $Re(w)$  and the result is

$$\begin{aligned}
& N\rho_3^{-2\beta} \binom{\alpha+\beta}{\beta} \left\{ \overline{\tilde{a}_\rho(0, s)} (\pi P^{-2}t^2\nu^2)^{-\beta-\gamma+\frac{k}{2}-\bar{s}-w} \Gamma\left(w + \bar{s} - \frac{k}{2} + \gamma + \beta\right) \right. \\
& \left. + \overline{a_\rho(0, s)} (\pi P^{-2}t^2\nu^2)^{-\beta-\gamma+\frac{k}{2}+\bar{s}-1-w} \Gamma\left(w + 1 - \bar{s} - \frac{k}{2} + \beta + \gamma\right) \right\}.
\end{aligned}$$

To justify this splitting of the integral, we need to justify the convergence for the change of order of integration and two infinite summations, i.e., sums over  $t$  and  $r$ . If  $\rho_3r - \rho_4v_2 + v' = 0$ , the sum over  $t$  converges for sufficiently large  $Re(w)$ . When  $\rho_3r - \rho_4v_2 + v' \neq 0$ , we have to study the convergence of

$$\begin{aligned}
& \sum_{\substack{t=1 \\ t\rho_3 = -Pv_1(MP)}} \sum_{r \in J} (P^{-1}t)^\gamma \tau'(\rho^{-1}, P^{-1}t\mu\rho_4, v_2 - P^{-1}t\mu\rho_3, v') e(-P^{-1}t(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(v_2\bar{\mu}\rho_4))) \\
& \times \int_0^\infty N y^{-\gamma} e(iP^{-2}t^2\nu^2 y^{-1}) y^{-\beta} L_\beta^{(\alpha)}(4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v')y) (\rho_3^{-1}(\rho_3r - \rho_4v_2 + v'))^\alpha \\
& \times e(2\mathbf{N}(\rho_3r - \rho_4v_2 + v')z + \mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}\{(2r\rho_3 - \rho_4v_2 + v')(\overline{\rho_2v_2 + P^{-1}t\mu})\}) \\
& \times \overline{a_\rho(N\mathbf{N}(\rho_3r - \rho_4v_2 + v'), s)} |4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v')y|^{-\frac{k}{2}} \\
& \times \overline{W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v')y)} y^{k-2-w} dy.
\end{aligned}$$

Then we see that summing first with taking absolute values and integrating gives a finite value, when we consider exponentially decaying terms  $W_{\frac{k}{2}, s-\frac{1}{2}}(4\pi\mathbf{N}(\rho_3r - \rho_4v_2 + v')y)$  and  $e(iP^{-2}t^2\nu^2 y^{-1})$ , in spite of the subexponentially growing term  $\overline{a_\rho(N\mathbf{N}(\rho_3r - \rho_4v_2 + v'), s)}$ . Therefore by Lebesgue's dominated convergence theorem we can carry out the previous term-by-term integration. The part corresponding to  $\rho_3r - \rho_4v_2 + v' = 0$  is simplified by



using  $(r - \rho_3^{-1}\rho_4v_2 + \rho_3^{-1}v')^\alpha = 0$  for  $\alpha > 0$  into

$$\begin{aligned} & \delta_{\alpha,0} \frac{(k-1)!P^{2w-1}N}{2^{k-1}(\mathbf{N}(I)^2D)^{1/2}} i \left\{ \nu^{2-2\bar{s}-2w} \pi^{-\frac{k}{2}+1-\bar{s}-w} q(k, s, w) \right. \\ & \times \sum_{\rho \in \mathcal{R}} \rho_3^{-1} \overline{\tilde{a}_\rho(0, s)} \tau(\rho) e(2\rho_4\rho_2\mathbf{N}(v_2)) \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} (P^{-1}t)^{1-2\bar{s}-2w} e(-2P^{-1}t\rho_4v_4) \\ & + \nu^{2\bar{s}-2w} \pi^{-\frac{k}{2}+\bar{s}-w} q(k, 1-s, w) \sum_{\rho \in \mathcal{R}} \rho_3^{-1} \overline{a_\rho(0, s)} \tau(\rho) e(2\rho_4\rho_2\mathbf{N}(v_2)) \\ & \left. \times \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} (P^{-1}t)^{2\bar{s}-1-2w} e(-2P^{-1}t\rho_4v_4) \right\} = \delta_{\alpha,0} p(\nu, w). \end{aligned}$$

Finally, the last part corresponding to  $\rho_3r - \rho_4v_2 + v' \neq 0$  can be computed as in [10]. Note that rearranging the sum does not make a problem because it does not change the order between summations over  $t$  and  $r$ , which are infinite sums. Finally we arrive at the following, completing the proof in the  $\alpha \geq 0$  case: let

$$(5) \quad \begin{aligned} \tau(\rho, r, v_2) &= \rho_3^{-1} e(-2\rho_3^{-1}(\rho_1\mathbf{N}(r) + \mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(v_2\bar{r}) + \rho_4\mathbf{N}(v_2))) \\ &\quad \times \sum_{j \in J/\rho_3J} e(-2\rho_3^{-1}(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(\bar{j}(r + \rho_4v_2)) + \rho_4\mathbf{N}(j))), \end{aligned}$$

$$(6) \quad \begin{aligned} q_{\alpha, \beta, \gamma}(y, \lambda) &:= \sum_{m=0}^{\beta} \sum_{e=0}^{\beta+\gamma-m} (-1)^{\gamma+e} \binom{\beta+\gamma-m}{e} [(\beta-m)!(\alpha+m)!m!\gamma!]^{-1} 2^\gamma \\ &\quad \times \binom{k}{2-s}_{\beta+\gamma-m-e} y^{2m+e+\alpha+1} K_{2s-1-e}(2y), \end{aligned}$$

where  $(a)_n = a(a-1)\cdots(a-n+1)$  with  $(a)_0 = 1$ . Let  $q_\alpha(y, \lambda) = \sum_{\beta, \gamma} q_{\alpha, \beta, \gamma}(y, \lambda)$ , where the sum is over nonnegative integers  $\beta$  and  $\gamma$  such that  $2\beta + \gamma = k - \alpha - 1$  and  $\alpha + \beta \geq 0$ . Then we get the following simplified expression

$$\begin{aligned} & C_\alpha \sum_{\rho \in \mathcal{R}} \sum_{\substack{t=1 \\ t\rho_3 \equiv -Pv_1(MP)}} t^{-1} \sum_{r \in \bar{J}^{-1}\mathcal{D}_K^{-1}-0} \mathbf{N}(r)^{-k/2} (r/|r|)^\alpha \\ & \times \overline{a_\rho(N\mathbf{N}(r), s)} \tau(\rho, r, v_2) e(2P^{-1}t\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(\bar{r}\mu) - \rho_4v_4) q_\alpha(2\pi P^{-1}t\nu|r|, \lambda). \end{aligned}$$

Last equality follows from the fact that  $v_2 \in \bar{J}^{-1}\mathcal{D}_K^{-1}$ . Finally, since the relationship between  $\alpha \geq 0$  and  $\alpha \leq 0$  is simply a restatement of Theorem 3.2 (2), this finishes the proof of Theorem 4.2.  $\square$

We can simplify Theorem 4.2 by mixing in characters as in [10, Theorem 4.2]. Applying the *Doi-Naganuma lifting with character* defined in the section 3 of [10], we have the following simplification: For a character  $\chi \pmod{N}$ , let

$$WMF_{k, -\lambda, \chi}^+(\Gamma_0(N)) := \{f \in WMF_{k, -\lambda}^+(\Gamma(N)) \mid f|_k\sigma = \chi(d)f \text{ for all } \sigma \in \Gamma_0(N)\}.$$

We define

$$\begin{aligned}\widehat{p}(\nu, w) &= (k-1)!N\#\{\rho \in \mathcal{R} | (\rho_3, N) = 1\} \\ &\quad \times (\overline{\widetilde{a}_S(0, s)}r_{k, \mathbf{DN}(I)^2}(s+w) + \overline{a_S(0, s)}r_{k, \mathbf{DN}(I)^2}(1-s+w)), \\ r_{k, \mathbf{DN}(I)^2}(s) &= \nu^{2-2\bar{s}} \sum_{\beta, \gamma} \beta!^{-1} \gamma!^{-1} i^{\gamma-1} 2^{-2\beta} (\mathbf{DN}(I)^2)^{2\beta+\frac{1}{2}} \\ &\quad \times \Gamma\left(\bar{s} + \frac{k-1+\gamma}{2}\right) \sum_{t=1}^{\infty} \chi\chi_{-D}(t)t^{1-2\bar{s}}.\end{aligned}$$

**THEOREM 4.3.** *For  $P = 1$  and  $M = N$  the Doi-Naganuma lifting with character of  $f$  in  $WMF_{k, -\lambda, \chi}^+(\Gamma_0(N))$  is given by the constant term of the Laurent expansion at  $w = 0$  of the analytic continuation of*

$$\begin{aligned}F_{\alpha}^{reg}(g(\mu, \nu); f) &= C'_{\alpha} \sum_{t=1}^{\infty} t^{-1} \chi\chi_{-D}(t) \sum_{r \in J-0} |r|^{-k} (r/|r|)^{\alpha b(\mathbf{NN}(r)/\mathbf{DN}(I)^2)} \\ &\quad \times q_{|\alpha|}(2\pi\nu t|r|/[\mathbf{DN}(I)^2]^{1/2}, \lambda) e(2\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(r\partial^{-1}\mathbf{N}(I)^{-1}t\bar{\mu})) + \delta_{\alpha, 0}\widehat{p}(\nu, w).\end{aligned}$$

Here  $\partial \in \mathfrak{H}$  is the purely imaginary generator of  $\mathcal{D}_K$ ,  $\chi\chi_{-D}$  is a (not necessarily primitive) character (mod  $N$ ),  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and  $C'_{\alpha} = C_{|\alpha|}[\mathbf{DN}(I)^2/N]^{k/2} Ni^{2-\alpha} \#\{\rho \in \mathcal{R} | (\rho_3, N) = 1\}$ .

Using the argument in the proof of Theorem 4.2 and [10, Theorem 4.2], we can prove Theorem 4.3 so we omit the proof.

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## REFERENCES

1. T. ASAI, On the Doi-Naganuma lifting associated with imaginary quadratic fields, *Nagoya Math. J.* Vol. 71 (1978), 149-167.
2. R. BORCHERDS, Automorphic forms with singularities on Grassmannians, *Invent. Math.* 132 (1998), 491-562.
3. K. BRINGMANN AND K. ONO, Dyson's ranks and Maass forms, *Ann. of Math.* (2) 171 (2010), 419-449.
4. K. BRINGMANN AND D. PENNISTON, Arithmetic properties of non-harmonic weak Maass forms, *Proc. Amer. Math. Soc.* Vol. 137 (2009), 825-833.
5. J.H. BRUINIER, Borchers products on  $O(2, l)$  and Chern classes of Heegner divisors, *Lecture Notes in Math.* 1780 (2002).
6. J.H. BRUINIER AND J. FUNKE, Traces of CM values of modular functions, *J. Reine Angew. Math.* 594 (2006), 1-33.

7. J.H. BRUINIER AND K. ONO, Heegner divisors, L-functions, and Maass forms, *Ann. of Math. (2)* Vol. 172 (2010), 2135–2181.
8. B.A. CIPRA, On the Niwa-Shintani theta-kernel lifting of modular forms, *Nagoya Math. J.* Vol. 91 (1983), 49–117.
9. K. DOI AND H. NAGANUMA, On the functional equation of certain Dirichlet series, *Invent. Math.* Vol. 9 (1969), 1–14.
10. S. FRIEDBERG, On Maass Wave Forms and the Imaginary Quadratic Doi-Naganuma Lifting, *Math. Ann.* 263 (1983), 483–508.
11. S. FRIEDBERG, On the imaginary quadratic Doi-Naganuma lifting of modular forms of arbitrary level, *Nagoya Math. J.* Vol. 92, (1983), 1–20.
12. S. FRIEDBERG, Differential operators and theta series, *Trans. Amer. Math. Soc.* Vol. 287 (1985), 569–589.
13. J. HARVEY AND G. MOORE, Algebras, BPS states, and strings, *Nuclear Phys. B* 463, no.2-3 (1996), 315–368.
14. E. HECKE, *Mathematische Werke*, Vandenhoeck & Ruprecht, Gottingen, 1970, no. 14.
15. S. HUSSEINI AND M. KNOPP, Eichler cohomology and automorphic forms, *Illinois J. Math.* Vol. 15 (1971), 565–577.
16. S. KATOK AND P. SARNAK, Heegner points, cycles, and Maass forms, *Israel J. Math.* 84 (1984), 193–227.
17. S. KUDLA, Derivatives of Eisenstein series and arithmetic geometry, *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), 173–183, Higher Ed. Press, Beijing, 2002.
18. S. KUDLA AND J. MILLSON, Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables, *Inst. Hautes Études Sci. Publ. Math. No. 71* (1990), 121–172.
19. S. KUDLA, M. RAPOPORT AND T. YANG, Derivatives of Eisenstein series and Faltings heights, *Compos. Math.* 140 (2004), 887–951.
20. H. NAGANUMA, On the coincidence of two Dirichlet series associated with cusp forms of Hecke’s “Neben”-type and Hilbert modular forms over a real quadratic field, *J. Math. Soc. Japan* Vol. 25 (1967), 547–555.
21. T. SHINTANI, On construction of holomorphic cusp forms of half integral weight, *Nagoya Math. J.* Vol. 58 (1975), 83–126.
22. E.T. WHITTAKER, G.N. WATSON, *A course of modern analysis* (4ed. CUP).
23. D. ZAGIER, Traces of singular moduli, in: *Motives, Polylogarithms and Hodge Theory, Part I*, F. Bogomolov and L. Katzarkov, eds., *International Press*, Somerville 2002.
24. D. ZAGIER, Ramanujan’s mock theta functions and their applications [d’après Zwegers and Bringmann-Ono], *Séminaire Bourbaki*, 60<sup>ème</sup> année, 2006–2007, no. 986.
25. S. ZWEGERS, Mock theta functions, Ph. D. Thesis, U. Utrecht, 2002.

DEPARTMENT OF MATHEMATICAL SCIENCES AND ASARC  
 KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY  
 DAEJEON 305-701  
 REPUBLIC OF KOREA  
*E-mail address:* seokho.jin1@gmail.com

SCHOOL OF MATHEMATICS  
 KOREA INSTITUTE FOR ADVANCED STUDY

HOEGIRO 85, DONGDAEMUN-GU, SEOUL 130-722  
REPUBLIC OF KOREA  
*E-mail address:* subong@kias.re.kr