ON ARITHMETIC PROPERTIES OF THE p-UNITS

SOOGIL SEO

In memory of Professor Robert Coleman

Abstract. We define certain deviation modules for the Hasse local-global norm theorem and Hilbert's theorem 90 for the p-units over the cyclotomic $\mathbb{Z}_p$-extension of a number field. We show that a necessary and sufficient condition for the two deviations are isomorphic over all sufficiently large intermediate fields in terms of an arithmetic property of the p-units which is called the generalized Gross conjecture. The proof is based on deep results of Kuz'min and Iwasawa on infinite class field theory. We will prove the Kuz'min's main result in a totally different way using the genus theory for p-ideal class group which depends on cohomology theory rather than the infinite class field theory used by Kuz'min and describe an explicit isomorphism.

1. Introduction

Let k be a number field and p an odd prime. For an extension field L of k, let $N_{L/k}$ denote the norm map from L to k and J_L the idele group of L. For a finite set T of primes of k, let $\mathcal{S}_k$ be a subgroup of T-units $U_k(T)$ of k.

We define the deviation module $\mathrm{HS}_{K/k}(\mathcal{S}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ of the Hasse local-global norm theorem and the deviation module $\mathrm{HT}_{K/k}((\mathcal{S}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ of the Hilbert's theorem 90 for the p-units over the cyclotomic $\mathbb{Z}_p$-extension of k.

Let $\mathrm{HS}_{K/k}((\mathcal{S}_K \otimes \mathbb{Z}_p)$ be a compact subgroup defined as

$$\mathrm{HS}_{K/k}(\mathcal{S}_K \otimes \mathbb{Z}_p) = \frac{(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{loc}}}{(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{glo}}}$$

where

$$(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{loc}} = \varprojlim_{k \subset L \subset K} ((\mathcal{S}_k \cap N_{L/k} J_L) \otimes \mathbb{Z}_p)$$

and

$$(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{glo}} = \varprojlim_{k \subset L \subset K} ((\mathcal{S}_k \cap N_{L/k} \mathcal{S}_L) \otimes \mathbb{Z}_p)$$

are the inverse limits with respect to the inclusion maps for all finite subextensions L of K over k. Similarly for any Galois extension K/k, we let

$$\mathrm{HT}_{K/k}((\mathcal{S}_K \otimes \mathbb{Z}_p) = \varinjlim_{k \subset L \subset K} H^1(G(L/k), \mathcal{S}_L \otimes \mathbb{Z}_p)$$

where the direct limit is taken for all finite cyclic extensions L of K over k.

In the following theorems let $k_\infty = k_\infty^{\mathrm{cyc}} = \bigcup_{n \geq 0} k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of k with k_n the unique subfield of k_∞ of degree p^n over k. For a field L, let $S_L = U_L(p)$ be the p-units of L. Let Γ denote the procyclic group $G(k_\infty/k)$ and for each $n \geq 0$, let $\Gamma_n = G(k_\infty/k_n)$ be the unique subgroup of Γ with index p^n.

Theorem 1.1. *For all sufficiently large n and $m-n$,*

$$\mathrm{tor}_{\mathbb{Z}_p}(\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p))\cong\mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m}\otimes\mathbb{Z}_p).$$

In proving Theorem 1.1, a crucial step is the following result Proposition 1.2 of Kuz'min. We prove it in a completely different way using only the cohomology theory without the infinite class field theory as was used in the original proof of Kuz'min. In special we do not use complicate topological properties of the idele group equipped with locally compact topological space. The only topological property used in our proof is the compactness of a finite $\mathbb{Z}$-module after tensoring with $\mathbb{Z}_p$. Let $S:=S_n=\{v|p\}$ be the set of primes of k_n dividing p. Let $Cl(k_n)$ be the ideal class group of k_n and let

$$Cl_n(S):=Cl(k_n)/\langle cl(v)\rangle_{v\in S}$$

be the S-ideal class group of k_n where $cl(v)\in Cl_n$ is the ideal class containing v.

We define the Tate module $T_p(k)$ of k following Kuz'min as the inverse limit of $Cl_n(S)$ with respect to the norm maps,

$$T_p(k):=\varprojlim_n(Cl_n(S)\otimes\mathbb{Z}_p).$$

For each generating class $(cl(\mathfrak{a}_n)\otimes 1)_n\in T_p(k)^\Gamma$ and $\alpha_n\in k_n^\times$ such that $(\gamma-1)\mathfrak{a}_n=\alpha_n\mathcal{O}_{k_n}(S)$, let

$$\psi_\infty: T_p(k)^\Gamma\to\mathrm{HS}_{k_\infty/k}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p)$$

be defined as

$$\psi_\infty((cl(\mathfrak{a}_n)\otimes 1))=N_{k_n/k}(\alpha_n)\otimes 1$$

which is well defined modulo $N_{k_n/k}(S_{k_n}(S)\otimes\mathbb{Z}_p)$ for all n.

Proposition 1.2. *ψ_∞ defines the isomorphism*

$$\psi_\infty: T_p(k)^\Gamma\overset{\cong}{\to}\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p).$$

In the original form of Proposition 1.2, the isomorphism hidden under infinite class field theory was not defined. Our proof provides us with an explicit description of the isomorphism ψ_∞.

We briefly introduce following Iwasawa (see §4 of [4]) and Kolster (see §1 of [6]) equivalent forms of the generalized Gross conjecture using the cohomologies over Γ;

i) $A_S(k_\infty)^\Gamma<\infty$,
ii) $H^1(\Gamma,A_S(k_\infty))=0$,
where $A_S(k_\infty)$ is the p-part of the S-ideal class group of k_∞^{cyc},
iii) $T_p(k)_\Gamma<\infty$,
iv) $T_p(k)^\Gamma<\infty$.

The equivalence of i), ii) is explained in §4 of [4] and the equivalence of i), iii) and iv) is explained in Theorem 1.14 of [6].

Finally we will show that the generalized Gross conjecture is equivalent to the claim that the deviation modules of Hasse's local-global norm theorem for the p-units and Hilbert's theorem 90 for the p-units over the cyclotomic $\mathbb{Z}_p$-extension of k are isomorphic for all sufficiently large intermediate fields.

Theorem 1.3. *The generalized Gross conjecture is true for all k_n with $n\geq 0$ if and only if for all sufficiently large n and $m-n$,*

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p)\cong\mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m}\otimes\mathbb{Z}_p).$$

2. Proofs of the main results

For a Galois extension K/k, let $\mathcal{S}_k$ and $\mathcal{S}_K$ be submodules of $k^\times$ and respectively $K^\times$. We define the deviation $\mathrm{HS}_{K/k}(\mathcal{S}_K)$ of Hasse's local-global norm theorem for $\mathcal{S}_\bullet$ over K/k to be

$$\mathrm{HS}_{K/k}(\mathcal{S}_K) = \frac{\mathcal{S}_{K/k}^{\mathrm{loc}}}{\mathcal{S}_{K/k}^{\mathrm{glo}}}$$

where

$$\mathcal{S}_{K/k}^{\mathrm{loc}} = \bigcap_{k \subset L \subset K} (\mathcal{S}_k \cap N_{L/k} J_L)$$

and

$$\mathcal{S}_{K/k}^{\mathrm{glo}} = \bigcap_{k \subset L \subset K} (\mathcal{S}_k \cap N_{L/k} \mathcal{S}_L)$$

for all finite subextensions L of K over k.

We will define another deviation of Hilbert's theorem 90. We first define for a finite extension. If K/k is a finite cyclic extension with generator σ, then we also define the deviation $\mathrm{HT}_{K/k}(\mathcal{S}_K)$ of Hilbert's theorem 90 for $\mathcal{S}_\bullet$ over K/k to be

$$\mathrm{HT}_{K/k}(\mathcal{S}_K) = \frac{(\mathcal{S}_K)_{N_{K/k}}}{(\sigma - 1)\mathcal{S}_K}$$

where $(\mathcal{S}_K)_{N_{K/k}}$ denotes the kernel of the norm map $N_{K/k} : \mathcal{S}_K \longrightarrow k^\times$. The last deviation is just one dimensional Tate-cohomology $H^1(G(K/k), \mathcal{S}_K)$ of $\mathcal{S}_\bullet$. For a Galois extension K/k, we let

$$\mathrm{HT}_{K/k}(\mathcal{S}_K) = \varinjlim_{k \subset L \subset K} \mathrm{HT}_{L/k}(\mathcal{S}_L) = \varinjlim_{k \subset L \subset K} H^1(G(L/k), \mathcal{S}_L)$$

where the direct limit is taken for all finite cyclic extension L of K over k.

For a field L, if $\mathcal{S}_L = L^\times$ is the full multiplicative group of L, then it is obvious that for any Galois extension K/k,

$$\mathrm{HS}_{K/k}(\mathcal{S}_K) = \mathrm{HT}_{K/k}(\mathcal{S}_K) = 1.$$

For a finite set T of primes of k, let $\mathcal{S}_k$ be a subgroup of T-units $U_k(T)$ of k. Then we also define a compact subgroup $\mathrm{HS}_{K/k}((\mathcal{S}_K \otimes \mathbb{Z}_p)$ to be

$$\mathrm{HS}_{K/k}(\mathcal{S}_K \otimes \mathbb{Z}_p) = \varprojlim_{k \subset L \subset K} \mathrm{HS}_{L/k}(\mathcal{S}_L \otimes \mathbb{Z}_p)$$

for all finite subextensions L of K over k. Then it follows from the compactness that

$$\mathrm{HS}_{K/k}(\mathcal{S}_K \otimes \mathbb{Z}_p) = \frac{(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{loc}}}{(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{glo}}}$$

where

$$(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{loc}} = \varprojlim_{k \subset L \subset K} ((\mathcal{S}_k \cap N_{L/k} J_L) \otimes \mathbb{Z}_p)$$

and

$$(\mathcal{S}_{K/k} \otimes \mathbb{Z}_p)^{\mathrm{glo}} = \varprojlim_{k \subset L \subset K} ((\mathcal{S}_k \cap N_{L/k} \mathcal{S}_L) \otimes \mathbb{Z}_p)$$

are the inverse limits with respect to the inclusion maps for all finite subextensions L of K over k. Thus we have

$$\varprojlim_{k\subset L\subset K}((\mathcal{S}_k\cap N_{L/k}J_L)\otimes\mathbb{Z}_p)=\bigcap_{k\subset L\subset K}(\mathcal{S}_k\otimes\mathbb{Z}_p)\cap N_{L/k}(J_L\otimes\mathbb{Z}_p)$$

$$\varprojlim_{k\subset L\subset K}((\mathcal{S}_k\cap N_{L/k}\mathcal{S}_L)\otimes\mathbb{Z}_p)=\bigcap_{k\subset L\subset K}(\mathcal{S}_k\otimes\mathbb{Z}_p)\cap N_{L/k}(\mathcal{S}_L\otimes\mathbb{Z}_p).$$

Similarly for any Galois extension K/k, we let

$$\begin{aligned}\mathrm{HT}_{K/k}((\mathcal{S}_K\otimes\mathbb{Z}_p) &= \varinjlim_{k\subset L\subset K}\mathrm{HT}_{K/k}(\mathcal{S}_{L/k}\otimes\mathbb{Z}_p)\\ &= \varinjlim_{k\subset L\subset K}H^1(G(L/k),\mathcal{S}_L\otimes\mathbb{Z}_p)\end{aligned}$$

where direct limit is taken for all finite cyclic extensions L of K over k.

As mentioned in the introduction we start with a deep result of Kuz'min on the Tate module of a number field k. He proves the isomorphism using infinite class field theory (see Proposition 1.1 of [10] and Proposition 7.5 of [7]). We will prove Proposition 2.1 in a different way where we do not use topological properties of locally compact idele groups in infinite class field theory as was needed in the Kuz'min's original proof.

For each generating class $(cl(\mathfrak{a}_n)\otimes 1)_n\in T_p(k)^\Gamma$ and $\alpha_n\in k_n^\times$ such that $(\gamma-1)\mathfrak{a}_n=\alpha_n\mathcal{O}_{k_n}(S)$, let

$$\psi_\infty: T_p(k)^\Gamma\to \mathrm{HS}_{k_\infty/k}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p)$$

be defined as

$$\psi_\infty((cl(\mathfrak{a}_n)\otimes 1))=N_{k_n/k}(\alpha_n)\otimes 1$$

which is well defined modulo $N_{k_n/k}(S_{k_n}(S)\otimes\mathbb{Z}_p)$ for all n.

Proposition 2.1. *For a field L, let $S_L=U_L(p)$ be the p-units of L. Then there exists an isomorphism*

$$\psi_\infty: T_p(k)^\Gamma\overset{\cong}{\to}\mathrm{HS}_{k_\infty/k}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p).$$

Proof. Let L/k be a finite Galois extension. For a finite set S of finite primes of k, let $I_k(S), P_k(S)$ and $U_k(S)$ be the group of S-ideals of k, the subgroup of principal S-ideals in $I_k(S)$ and the S-units of k respectively. Similarly we define $I_L(S), P_L(S)$ and $U_L(S)$ as the the group of S'-ideals of L, the subgroup of principal S'-ideals in $I_L(S)$ and the S'-units of L respectively where S' denotes the prime ideals of L dividing the prime ideals of S.

It follows from $P_L(S)=L^\times/U_k(S)$ and the S'-ideal class group $Cl_L(S)=I_L(S)/P_L(S)$ that there exist exact sequences of G-modules

$$1\longrightarrow P_L(S)^G\longrightarrow I_L(S)^G\longrightarrow Cl_L(S)^G\longrightarrow H^1(G,P_L(S))\longrightarrow 1, \tag{1}$$

$$1\longrightarrow U_k(S)\longrightarrow k^\times\longrightarrow P_L(S)^G\longrightarrow H^1(G,U_L(S))\longrightarrow 1, \tag{2}$$

$$1\longrightarrow H^1(G,P_L(S))\longrightarrow H^2(G,U_L(S))\longrightarrow H^2(G,L^\times). \tag{3}$$

From the equation (1) and $Cl_k(S) = I_k(S)/P_k(S)$, there exists the commutative diagram

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & P_k(S) & \longrightarrow & I_k(S) & \longrightarrow & Cl_k(S) & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \iota_{k,L}} & & & & \\ 1 & \longrightarrow & P_L(S)^G & \longrightarrow & I_L(S)^G & \longrightarrow & Cl_L(S)^G & \longrightarrow & H^1(G, P_L(S)) & \longrightarrow & 1. \end{array}$$

Applying the snake lemma to the above diagram, we obtain

$$\begin{aligned} (4)\quad 1 \longrightarrow \ker(\iota_{k,L}) \longrightarrow \quad & H^1(G, U_L(S)) \longrightarrow I_L(S)^G/I_k(S) \\ \longrightarrow \quad & \mathrm{coker}(\iota_{k,L}) \longrightarrow H^1(G, P_L(S)) \longrightarrow 1 \end{aligned}$$

where we used

$$P_L(S)^G/P_k(S) \cong H^1(G, U_L(S))$$

from the equation (2).

Moreover if $G(L/k) = \langle\gamma\rangle$ is cyclic, then from the equation (3), it follows that

$$H^1(G, P_L(S)) \cong H^{-1}(G, P_L(S)) \cong \frac{U_k(S) \cap N_{L/k}L^\times}{N_{L/k}U_L(S)}$$

where the isomorphism is the connecting homomorphism defined by

$$(\alpha) = \alpha\mathcal{O}_L(S) \mapsto N_{L/k}(\alpha)$$

for the ring $\mathcal{O}_L(S)$ of S-integers of L.

By letting $\mathrm{im}(I_L(S)^G)$ the image of $I_L(S)^G$ inside $Cl_L(S)^G$, we have the following isomorphism

$$\psi_L : \frac{Cl_L(S)^G}{\mathrm{im}(I_L(S)^G)} \cong \frac{U_k(S) \cap N_{L/k}L^\times}{N_{L/k}U_L(S)}$$

where the isomorphism is defined to be for each class $cl(\mathfrak{a}) \in Cl_L(S)^G$ of $\mathfrak{a} \in I_L(S)$ and $\alpha \in L^\times$ such that $(\gamma - 1)\mathfrak{a} = \alpha\mathcal{O}_L(S)$

$$\psi_L(cl(\mathfrak{a})) = N_{L/k}(\alpha)$$

which is well defined modulo $N_{L/k}U_L(S)$. Since all groups above are p-group, we have the following isomorphism of compact groups

$$\frac{Cl_L(S)^G \otimes \mathbb{Z}_p}{\mathrm{im}(I_L(S)^G \otimes \mathbb{Z}_p)} \cong \frac{(U_k(S) \cap N_{L/k}L^\times) \otimes \mathbb{Z}_p}{N_{L/k}(U_L(S)) \otimes \mathbb{Z}_p}.$$

It follows from the trivial isomorphisms

$$Cl_L(S)^G \otimes \mathbb{Z}_p \cong (Cl_L(S) \otimes \mathbb{Z}_p)^G, \ \ N_{L/k}(U_L(S)) \otimes \mathbb{Z}_p \cong N_{L/k}(U_L(S)) \otimes \mathbb{Z}_p)$$

that

$$\psi_L : \frac{(Cl_L(S) \otimes \mathbb{Z}_p)^G}{\mathrm{im}(I_L(S)^G \otimes \mathbb{Z}_p)} \cong \frac{(U_k(S) \cap N_{L/k}L^\times) \otimes \mathbb{Z}_p}{N_{L/k}(U_L(S) \otimes \mathbb{Z}_p)}.$$

For a subfield L' of L over k and for each class $cl(\mathfrak{a}) \in Cl_L(S)^G$ of $\mathfrak{a} \in I_L(S)$ such that $(\gamma - 1)\mathfrak{a} = \alpha\mathcal{O}_L(S)$, by applying the norm map $N_{L/L'}$, we have

$$(\gamma - 1)N_{L/L'}(\mathfrak{a}) = N_{L/L'}(\gamma - 1)\mathfrak{a} = N_{L/L'}(\alpha\mathcal{O}_L(S)) = N_{L/L'}(\alpha)\mathcal{O}_{L'}(S)$$

since the extensions $k \subset L' \subset L$ are abelian. Hence we have

$$\psi_{L'}(N_{L/L'}cl(\mathfrak{a})) = N_{L'/k}(N_{L/L'}(\alpha)) = N_{L/k}(\alpha) = \psi_L(cl(\mathfrak{a}))$$

which induces the following commutative diagram

$$\begin{array}{ccc} \Omega_L & \xrightarrow{\psi_L} & \Theta_L \\ \Big\downarrow N_{L/L'} & & \Big\downarrow id \\ \Omega_{L'} & \xrightarrow{\psi_{L'}} & \Theta_{L'} \end{array} \tag{5}$$

where $\Omega_\bullet$ and $\Theta_\bullet$ represent

$$\Omega_{L'} = \frac{(Cl_{L'}(S)\otimes \mathbb{Z}_p)^{G(L'/k)}}{\mathrm{im}(I_{L'}(S)^{G(L'/k)}\otimes \mathbb{Z}_p)}, \quad \Theta_{L'} = \frac{(U_k(S)\cap N_{L'/k}L'^{\times})\otimes \mathbb{Z}_p}{N_{L'/k}(U_{L'}(S)\otimes \mathbb{Z}_p)}.$$

Now let L vary all subfields k_n of the cyclotomic $\mathbb{Z}_p$-extension of k. For an inverse system of Galois modules $\{M_n\}_{n\geq 0}$ of $G_n = G(k_n/k)$, it is obvious that

$$\varprojlim H^0(G_n, M_n) = \varprojlim H^0(G_\infty, M_n) = H^0(G_\infty, \varprojlim M_n).$$

By taking the inverse limits in (5) with respect to the norm maps on the left and the inclusion maps on the right, ψ_{k_n} induces the isomorphism $\psi_\infty = \lim \psi_{k_n}$

$$\psi_\infty : \varprojlim \frac{(Cl_n(S)\otimes \mathbb{Z}_p)^{G_n}}{\mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \xrightarrow{\sim} \varprojlim \frac{(U_k(S)\cap N_{k_n/k}k_n^{\times})\otimes \mathbb{Z}_p}{N_{k_n/k}(U_n(S)\otimes \mathbb{Z}_p)}. \tag{6}$$

Since all the groups appearing above are compact, by taking the inverse limits which are exact over the following exact sequence

$$1 \longrightarrow \mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p) \longrightarrow (Cl_n(S)\otimes \mathbb{Z}_p)^{G_n} \longrightarrow \frac{(Cl_n(S)\otimes \mathbb{Z}_p)^{G_n}}{\mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \longrightarrow 1$$

it follows that

$$\varprojlim \frac{(Cl_n(S)\otimes \mathbb{Z}_p)^{G_n}}{\mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \cong \frac{(\varprojlim(Cl_n(S)\otimes \mathbb{Z}_p))^{\Gamma}}{\varprojlim\ \mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \tag{7}$$

and similarly that

$$\varprojlim \frac{(U_k(S)\cap N_{k_n/k}k_n^{\times})\otimes \mathbb{Z}_p}{N_{k_n/k}(U_n(S)\otimes \mathbb{Z}_p)} \cong \frac{\varprojlim(U_k(S)\cap N_{k_n/k}k_n^{\times})\otimes \mathbb{Z}_p}{\varprojlim N_{k_n/k}(U_n(S)\otimes \mathbb{Z}_p)}. \tag{8}$$

Note that each prime not in S is unramified over k_∞/k since S contains all primes lying over p. If $\mathfrak{a} \in I_n(S)^{G_n}$ then for each prime ideal $\mathfrak{P}$ lying over a prime $\mathfrak{p}$ dividing $\mathfrak{a}$, we have

$$\prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P} \mid \mathfrak{a}$$

since the Galois group acts transitively on $I_n(S)$ and hence

$$N_{k_n/k}(\prod_{\mathfrak{P}|\mathfrak{p}} \mathfrak{P}) = \mathfrak{p}^{\sum e(\mathfrak{P})f(\mathfrak{P})} = \mathfrak{p}^{p^n} \mid N_{k_n/k}\mathfrak{a}$$

where $e(\mathfrak{P})(=1)$ and $f(\mathfrak{P})$ represent the ramification index and respectively the residue class degree of $\mathfrak{P}$ over $\mathfrak{p}$. It follows from the definition of the norm map over the ideal class group that

$$\varprojlim\ \mathrm{im}(I_n(S)^{G_n} = 0 \tag{9}$$

where the inverse limits are taken with respect to the norm maps. It follows from (7) and (9) that

$$\varprojlim \frac{(Cl_n(S)\otimes \mathbb{Z}_p)^{G_n}}{\mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \cong (\varprojlim (Cl_n(S)\otimes \mathbb{Z}_p))^{\Gamma}. \tag{10}$$

Hence it follows from (6) and (10) that

$$\begin{aligned} T_p(k)^{\Gamma} = (\varprojlim (Cl_n(S)\otimes \mathbb{Z}_p))^{\Gamma} &\cong \varprojlim \frac{(Cl_n(S)\otimes \mathbb{Z}_p)^{G_n}}{\mathrm{im}(I_n(S)^{G_n}\otimes \mathbb{Z}_p)} \\ &\cong \varprojlim \frac{(U_k(S)\cap N_{k_n/k}k_n^{\times})\otimes \mathbb{Z}_p}{N_{k_n/k}(U_n(S)\otimes \mathbb{Z}_p)}. \end{aligned}$$

It follows from (8) that the last term above is isomorphic to

$$\frac{\varprojlim (U_k(S)\cap N_{k_n/k}k_n^{\times})\otimes \mathbb{Z}_p}{\varprojlim N_{k_n/k}(U_n(S)\otimes \mathbb{Z}_p)} = \mathrm{HS}_{k_\infty/k}(\mathcal{S}_{k_\infty}\otimes \mathbb{Z}_p).$$

This completes the proof of Proposition 2.1. □

Let $\gamma_{m,n} = \gamma^{l^n} (\mathrm{mod} \gamma^{l^m})$ denote the fixed generator of $G(k_m/k_n)$.

Theorem 2.2. *For a field L, let $S_L = U_L(p)$ be the p-units of L. Then for all sufficiently large n and $m-n$,*

$$\mathrm{tor}_{\mathbb{Z}_p}(\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty}\otimes \mathbb{Z}_p)) \cong \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m}\otimes \mathbb{Z}_p).$$

Proof. For each $n\geq 0$, let $\mathcal{O}_n$ and I_n be the ring of integers and the fractional ideals of k_n respectively. For $m\geq n$, let

$$j(n,m) : Cl_n(S) \longrightarrow Cl_m(S)$$

be the map induced from the natural map $i(n,m) : I_n \longrightarrow I_m$ defined as

$$i(n,m)(\mathfrak{a}) = \mathfrak{a}\mathcal{O}_m.$$

The following theorem is due to Kuz'min(see Theorem 3.1 of [7]).

Theorem 2.3. *For all sufficiently large n and $m-n$, the order of $\ker(j(n,m))$ is stabilized and*

$$\ker(j(n,m)) \cong \mathrm{tor}_{\mathbb{Z}_p}(T_p(k)^{\Gamma_n})$$

where $\ker(j(n,m))$ denotes the kernel of $j(n,m)$ and $\mathrm{tor}_{\mathbb{Z}_p}(T_p(k)^{\Gamma_n})$ the $\mathbb{Z}_p$-torsion part of $T_p(k)^{\Gamma_n}$.

It is also well known by Iwasawa from Theorem 12 of [3] that for all $m\geq n\geq 0$,

$$\ker(j(n,m)) \cong H^1(G(k_m/k_n), \mathcal{S}_{k_m}).$$

It follows from the flatness of $\mathbb{Z}_p$ over $\mathbb{Z}$ that

$$\begin{aligned} H^1(G(k_m/k_n), \mathcal{S}_{k_m}\otimes \mathbb{Z}_p) &\cong H^1(G(k_m/k_n), \mathcal{S}_{k_m})\otimes \mathbb{Z}_p \\ &\cong H^1(G(k_m/k_n), \mathcal{S}_{k_m}) \end{aligned}$$

where the last isomorphism follows from the fact that $H^1(G(k_m/k_n), \mathcal{S}_{k_m})$ is a p-group. From Proposition 2.1 we complete the proof of Theorem 2.2. □

We describe the deviations in term of the universal norm and the norm comparable properties. For a subgroup $\mathcal{H}_m$ of $k_m^\times$, let $\varprojlim_{m\geq n}(\mathcal{H}_m\otimes\mathbb{Z}_p)$ be the inverse limit of $\mathcal{H}_m\otimes\mathbb{Z}_p$ with respect to the norm maps. Let π denote the natural projection

$$\pi:\varprojlim_{m\geq n}(\mathcal{H}_m\otimes\mathbb{Z}_p)\to\mathcal{H}_n\otimes\mathbb{Z}_p.$$

The norm compatible subgroup $(\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm comp}$ and the universal norm subgroup $(\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm univ}$ of $\mathcal{H}_n\otimes\mathbb{Z}_p$ are defined as follows

$$(\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm univ}:=\bigcap_{m\geq n}N_{m,n}(\mathcal{H}_m\otimes\mathbb{Z}_p),\ \ (\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm comp}:=\pi(\varprojlim_{m\geq n}(\mathcal{H}_m\otimes\mathbb{Z}_p)).$$

Note that if $\mathcal{H}_m$ is a finite type over $\mathbb{Z}$, then the two modules are identical

$$(\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm comp}=(\mathcal{H}_n\otimes\mathbb{Z}_p)^{\rm univ}.$$

For a finite cyclic extension K/k and for a finite set S of primes of k, let $U_k(S)$ be the global S-units of k. Let S' be the set of primes of K lying over each prime $v\in S$,

$$S'=\{w|v\ ;\ v\in S\}.$$

We also let $U_K(S):=U_K(S')$ denote the global S'-units of K. Let

$$J_{K,S}:=\prod_{w\notin S'}U_v\times\prod_{w\in S'}k_v^\times$$

be the S-idele group of K where we identify $U_K(S)$ with a subgroup of $J_{K,S}$ via the diagonal imbedding $\delta_{K,S}:U_K(S)\longrightarrow J_{K,S}$. For $K=k_n$ and $S=\{v|p\}$, we write

$$U_n(p)=U_{k_n}(S),\ \ \delta_n=\delta_{k_n,S}.$$

We have the following exact sequence

$$1\longrightarrow\ker(\phi_n)\longrightarrow U_k(p)\xrightarrow{\phi_n}H^0(G_n,J_{k_n,S})$$

where $G_n=G(k_n/k)$ denotes the Galois group of k_n/k and

$$\phi_n(\alpha)=\delta_n(\alpha)\bmod N_nJ_{k_n,S}$$

the natural map induced from δ_n. Then Hasse's local-global norm theorem for $k^\times$ implies that

$$N_nU_n(p)\subset\ker(\phi_n)=U_k(p)\cap N_nk_n^\times.$$

By tensoring with $\mathbb{Z}_p$, the exact sequence induces

$$1\longrightarrow\ker(\overline{\phi}_n)=\ker(\phi_n)\otimes\mathbb{Z}_p\longrightarrow U_k(p)\otimes\mathbb{Z}_p\xrightarrow{\overline{\phi}_n}H^0(G_n,J_{k_n,S})$$

together with

$$N_n(U_n(p)\otimes\mathbb{Z}_p)\subset\ker(\phi_n)\otimes\mathbb{Z}_p=(U_k(p)\cap N_nk_n^\times)\otimes\mathbb{Z}_p.$$

Then

$$\ker(\overline{\phi}_\infty):=\bigcap_{n\geq0}\ker(\overline{\phi}_n)=\bigcap_{n\geq0}((U_k(p)\cap N_nk_n^\times)\otimes\mathbb{Z}_p).$$

Hence we are led to

$$(\mathcal{S}_{k_n}\otimes\mathbb{Z}_p)^{\rm loc}=\ker(\overline{\phi}_\infty),\ (\mathcal{S}_{k_n}\otimes\mathbb{Z}_p)^{\rm glo}=(U_n(p)\otimes\mathbb{Z}_p)^{\rm univ},$$

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty}\otimes\mathbb{Z}_p)=\frac{(\mathcal{S}_{k_n}\otimes\mathbb{Z}_p)^{\rm loc}}{(\mathcal{S}_{k_n}\otimes\mathbb{Z}_p)^{\rm glo}}=\frac{\ker(\overline{\phi}_\infty)}{(U_n(p)\otimes\mathbb{Z}_p)^{\rm univ}}.$$

The statement that the deviation modules of Hasse's local-global norm theorem for the p-units and Hilbert's theorem 90 for the p-units over the cyclotomic $\mathbb{Z}_p$-extension of k are isomorphic for all sufficiently large intermediate fields is now equivalent to the generalized Gross conjecture.

Theorem 2.4. *For a field L, let $S_L = U_L(p)$ be the p-units of L. The generalized Gross conjecture is true for all k_n with $n \geq 0$ if and only if for all sufficiently large n and $m-n$,*

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty} \otimes \mathbb{Z}_p) \cong \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m} \otimes \mathbb{Z}_p).$$

Proof. Firstly we claim that the generalized Gross conjecture is true for all k_n with $n \geq 0$ if and only if for all sufficiently large n and $m-n$,

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty} \otimes \mathbb{Z}_p) \cong \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m}).$$

Suppose now that the generalized Gross conjecture is true for all k_n with $n \geq 0$. Since $T_p(k)^{\Gamma_n} \cong T_p(k_n)^{\Gamma_n}$, Theorem 2.3 and the generalized Gross conjecture (iv) imply that for all sufficiently large n and $m-n$,

$$\ker(j(n,m)) \cong T_p(k_n)^{\Gamma_n}.$$

Hence Propositions 2.1 imply the isomorphism. The converse direction is obvious from Proposition 2.1 and Theorem 2.2 together with the fact that the generalized Gross conjecture descends over field extensions.

Secondly we show that the claim is equivalent to the theorem. It follows that

$$\begin{aligned} H^1(G(k_m/k_n), \mathcal{S}_{k_m} \otimes \mathbb{Z}_p) &\cong H^1(G(k_m/k_n), \mathcal{S}_{k_m}) \otimes \mathbb{Z}_p \\ &\cong H^1(G(k_m/k_n), \mathcal{S}_{k_m}). \end{aligned}$$

Hence it follows that

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty} \otimes \mathbb{Z}_p) = \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m}) = \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m} \otimes \mathbb{Z}_p).$$

This completes the proof of Theorem 2.4. □

Note that the generalized Gross conjecture is true for an abelian field k. Hence if k is abelian then for all sufficiently large n and $m-n$,

$$\mathrm{HS}_{k_\infty/k_n}(\mathcal{S}_{k_\infty} \otimes \mathbb{Z}_p) \cong \mathrm{HT}_{k_m/k_n}(\mathcal{S}_{k_m} \otimes \mathbb{Z}_p).$$

It could be also interesting to prove Theorem 2.4 using only the cohomology theory.

References

[1] L. Federer and B. Gross (with an appendix by W. Sinnott), Regulators and Iwasawa modules, Invent. Math. 62 (1981), 443-457.

[2] B. Gross, p-adic L-series at $s = 0$, J. Fac. Sci. Univ. Tokyo IA 28 (1981), 979-994.

[3] K. Iwasawa, On Z_l-extensions of algebraic number fields. Ann. of Math. 98 (1973), 246-326.

[4] K. Iwasawa, On cohomology groups of units for Z_p-extensions, Amer. J. Math. 105 (1983), 189-200.

[5] J.-F. Jaulent, L'arithmetique des l-extensions. Thèse, Université de Franche Comté, 1986.

[6] M. Kolster, An idelic approach to the wild kernel, Invent. Math. 103 (1991), 9-24.

[7] L. V. Kuz'min, The Tate module of algebraic number fields, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972), 267-327.

[8] L. V. Kuz'min, On formulas for the class number of real abelian fields, Izv. Ross. Akad. Nauk Ser. Mat. 60 (1996), 43-110.

[9] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields (2nd edition), Springer-Verlag 2008.

[10] S. Seo, On the Tate module of a number field, submitted.

Department of Mathematics, Yonsei University, 134 Sinchon-Dong, Seodaemun-Gu, Seoul 120-749, South Korea e-mail: sgseo@yonsei.ac.kr