GEOMETRIC PROPERTIES OF PROJECTIVE MANIFOLDS OF SMALL DEGREE

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Abstract. We study geometric structures of smooth projective varieties of small degree in birational geometric viewpoints. First, using the positivity of double point divisors, we classify non-degenerate smooth projective varieties $X \subset \mathbb{P}^r$ of degree $d \leq r+2$, and consequently, we show that every such X is simply connected and rationally connected except in a few cases. This is a generalization of P. Ionescu's work ([Io4]). We also show the finite generation of Cox rings of smooth projective varieties $X \subset \mathbb{P}^r$ of degree $d \leq r$ with a counter-example for $d = r+1$. On the other hand, we prove that a non-degenerate non-uniruled smooth projective variety $X \subset \mathbb{P}^r$ of dimension n and degree $d \leq n(r-n)+2$ is Calabi-Yau.

Introduction

Every n-dimensional smooth projective variety can be embedded in $\mathbb{P}^{2n+1}$ with arbitrarily large degree. In contrast, there is a restriction to admit an embedding with small degree. We expect that small degree varieties have special geometric properties which in turn allow us to classify them. First, we consider the following problem, which is a natural generalization of L. Ein's classification ([E]) of surfaces in $\mathbb{P}^r$ of degree $d \leq r+3$.

Problem A. *Classify non-degenerate smooth projective varieties $X \subset \mathbb{P}^r$ of dimension n and degree $d \leq r+n+1$.*

Many classical results on classifications of lower dimensional projective varieties were achieved by systematic studies of adjunction mappings. However, those methods cannot be directly generalized to higher dimensional cases. By recent developments of higher dimensional algebraic geometry, we can study geometric properties (e.g., positivity of anticanonical divisors, rationally connectedness, vanishing of higher cohomology groups, etc.) of small degree varieties before detailed classifications. Then, we obtain natural descriptions of small degree varieties in terms of important classes of varieties (e.g., Fano, Calabi-Yau, ruled varieties, etc.) from birational geometry. It is an interesting problem to determine embedded structures of projective varieties via their intrinsic properties or vice versa.

In this paper, we adopt the approach described in the above to extend P. Ionescu's work ([Io4]) into the case $d \leq r+2$. Ionescu proved that if $d \leq r$, then X is simply connected. He deduced it from a classification result: such X is either a Fano variety with $b_2(X) = 1$ or a rational variety (there are 6 infinite series

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and 14 sporadic examples). In particular, this gave an answer to a question of F. Russo and F. Zak which asks if X is regular (i.e., $h^1(\mathcal{O}_X) = 0$) when $d \le r$. Our main result is the following theorem, which is a consequence of Theorem C and Theorem 6.1.

Theorem B. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree $d \le r+2$. Assume that X is neither a curve nor an elliptic scroll. Then, X is always simply connected, and it is rationally connected unless it is a K3 surface in $\mathbb{P}^4$ of degree 6 or a hypersurface of degree $d = r+1$ or $r+2$.*

In contrast with Ionescu's approach in [Io4], we directly study the positivity of the anticanonical divisors, which immediately imply some geometric properties, especially, simply connectedness, rationally connectedness, and finite generation of Cox ring. It seems difficult to derive our main results from Ionescu's classification ([Io2, Theorem I]) of smooth projective varieties $X \subset \mathbb{P}^r$ of degree $d \le 2e+2$ or similar results based on adjunction theory.

For a while, we consider the case $d \le r+1$ in detail.

Theorem C. *Let $X \subset \mathbb{P}^r$ be an n-dimensional non-degenerate smooth projective variety of degree $d \le r+1$. Then, one of the following holds:*

(*a*) *X is a weak Fano variety.* (*If $d \le r$, then X is a Fano variety.*)
(*b*) *X is a Roth variety or a rational scroll.*
(*c*) *$d = r+1, r = n+1 \ge 3$ and X is a Calabi-Yau hypersurface.*
(*d*) *$d = r+1$ and X is an elliptic normal scroll or an elliptic normal curve.*

In particular, X is simply connected if and only if it is from (*a*), (*b*), *or* (*c*), *and X is rationally connected if and only if it is from* (*a*) *or* (*b*).

We will study adjunction mappings of varieties from the case (a) to obtain a more detailed classification, which is Theorem E. For definitions and basic properties of scrolls and Roth varieties, see Sections 1 and 2 in which we classify those varieties of degree $d \le r+1$. We refer to [KMM] for rationally connected varieties. Note that every rationally connected variety is simply connected.

Remark. (1) The fundamental group of an elliptic scroll or an elliptic curve is $\mathbb{Z}\oplus\mathbb{Z}$.
(2) Every case of Theorem C really occurs. The existence of cases (b) and (c) is clear. See Remark 1.6 for case (d) and see Examples 5.8 and 5.9 for case (a).
(3) Every variety from (b), (c), or (d) is projectively normal (i.e., the natural map $H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(\mathcal{O}_X(k))$ is surjective for every $k \ge 1$) unless it is a non-linearly normal rational scroll (see Remark 1.6 and Proposition 2.3).
(4) When X is a curve, $d = r+1$ is equivalent to that X is of almost minimal degree. Thus, Theorem C can be seen as a higher dimensional generalization of the classification of curves of almost minimal degree (Theorem 1.2) in a topological viewpoint because non-simply connected varieties appear only when $d \ge r+1$. See also Theorem 1.1 for the scroll case.
(5) Theorem C classifies n-dimensional polarized pairs (X, H) with $\Delta(X, H) \le n$. The Δ-*genus* for a n-dimensional polarized pair (X, H), where H is a very ample divisor on a smooth projective variety X of dimension n, is defined by

$$\Delta(X, H) := n + H^n - h^0(\mathcal{O}_X(H)).$$

For basic properties, we refer to [F3].

We will prove Theorem C in Subsection 3.2 using A. Noma's work on the double point divisor from inner projection ([N]), which we review in Subsection 3.1. Theorem C is an improvement of [N, Corollary 7.4], in which Noma showed that if $d \le r$ and X is neither a Roth variety nor a scroll, then $-K_X$ is ample. The Ionescu's work ([Io4]) relies on the study of the adjunction mapping given by $|K_X+(n-1)H|$, where H is a hyperplane section. It is related to the ramification divisor ($\sim K_X+(n+1)H$) which is obtained by the ramified locus of an outer projection $\pi_{\Lambda_1}\colon X \to \mathbb{P}^n$ centered at a general $(r-n-1)$-dimensional linear subspace Λ_1. On the other hand, the non-isomorphic locus of an outer projection $\pi_{\Lambda_2}\colon X \to \mathbb{P}^{n+1}$ centered at a general $(r-n-2)$-dimensional linear subspace Λ_2 defines the double point divisor from outer projection linearly equivalent to

$$D_{out} := -K_X + (d-n-2)H.$$

Note that D_{out} is base point free (see [BM, Technical appendix 4]), and hence, X is a Fano variety provided that $d \le n+1$. The double point divisor from inner projection can be defined similarly, and it is linearly equivalent to

$$D_{inn} := -K_X + (d-r-1)H.$$

Noma proved that D_{inn} is semiample if it is defined ([N, Theorem 4]). Our classification is accomplished by showing that D_{inn} is big when $d \le r+1$ except some special cases which exactly correspond to the other cases in Theorem C. We emphasize that a bound of the sectional genus (Lemma 3.6) from the embedded structure plays a crucial role in proving the bigness of $-K_X$, which is an intrinsic property.

Rationally connectedness and simply connectedness of small degree varieties mean that they are simple in geometric and topological ways. It turns out that small degree varieties are also simple in *algebraic way* by the following.

Corollary D. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree d. If $d \le r$, then the Cox ring of X is finitely generated.*

Corollary D will be proved in Section 4, where the definition of Cox ring is given. Every variety from (a) or (b) in Theorem C has finitely generated Cox ring, and every hypersurface of dimension $n \ge 3$ has also finitely generated Cox ring. However, there is a quartic surface whose Cox ring is not finitely generated (see Example 4.1). The Cox rings of non-regular varieties (i.e., $h^1(\mathcal{O}_X) \ne 0$) cannot be defined.

We note that simply connectedness, rationally connectedness, and finite generation of Cox rings of weak Fano varieties, scrolls, and hypersurfaces are well understood. For this purpose, in Sections 2 and 4, we will study those properties of Roth varieties in detail.

In Section 5, we will study adjunction mappings. In particular, we obtain the following classification of the weak Fano case in Theorem C, which will be shown in Subsection 5.2.

Theorem E. *Let $X \subset \mathbb{P}^r$ be an n-dimensional non-degenerate smooth projective variety of degree $d \le r+1$, and let H be a general hyperplane section. Assume that $n \ge 2$ and X is a weak Fano variety but not a rational scroll. Then, one of the following holds:*

(a) *X is prime Fano, i.e., $-K_X = \ell H$ for some $\ell > 0$ and $\operatorname{Pic}(X) = \mathbb{Z}[H]$.*
(b) *X is a del Pezzo variety, i.e., $-K_X = (n-1)H$.*

(c) *X is a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ or $\mathbb{P}^4$, or a quadric hypersurface $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subset \mathbb{P}^3$.*
(d) *$|K_X + (n-1)H|$ induces a hyperquadric fibration over $\mathbb{P}^1$.*
(e) *$|K_X + (n-1)H|$ induces a linear fibration over $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*

In particular, if X is not a Fano variety but a weak Fano variety, then it is a rational variety.

Del Pezzo varieties are completely classified by Fujita ([F1] and [F2]; see Theorem 5.2 for complete list). We will also discuss the classification of prime Fano varieties of degree $d \leq r+1$ in Subsection 5.4.

Remark. (1) Every case of Theorem E really occurs (see Subsection 5.3).
(2) Not every prime Fano variety is rational. E.g., smooth cubic threefolds in $\mathbb{P}^4$, which are also del Pezzo varieties, are not rational.
(3) Every Roth variety of degree $d \leq r+1$ that is not a rational scroll has a hyperquadric fibration over $\mathbb{P}^1$ which is induced by $|K_X + (n-1)H|$ (see Remark 2.5). The natural fibrations of scrolls are induced by $|K_X + nH|$.

The main ingredients of the proof of Theorem E are adjunction theory and methods in [Io4], more precisely, Proposition 5 and Lemma 6 in that paper. On the contrary to [Io4], we may assume that X is a weak Fano variety, and in particular, X is simply connected. Then, we can simplify arguments by excluding many unnecessary cases in the first place.

Our methods can be applied to the case $d = r+2$. Section 6 will be devoted to classify non-degenerate smooth projective varieties $X \subset \mathbb{P}^r$ of degree $d \leq r+2$ (see Theorems 6.1 and 6.2). In our proofs of Theorem C and Theorem 6.1, we sometimes use ad-hoc arguments, which force us to be faced with a difficulty as the degree increases. To solve Problem A, we need to develop more systematic methods.

On the other hand, in view of A. Buium's classification ([Bui]) of non-ruled surfaces in $\mathbb{P}^r$ of degree $d \leq 2r+1$, it is natural to consider the following problem.

Problem F. *Classify non-degenerate smooth projective varieties $X \subset \mathbb{P}^r$ of dimension n and degree $d \leq nr+1$ which are not uniruled.*

In [Bui], Buium pointed out that every non-ruled surface of degree $d \leq 2r+1$ is regular. We expect that there is an organized way to prove such results in higher dimensional case without classification.

Remark. When X is a curve, $d \leq nr+1$ is equivalent to that X is of almost minimal degree, and hence, Theorem 1.2 gives an answer to Problem F.

In this paper, we prove the following, which will be used for classifying projective varieties of degree $d = r+2$ (see Proof of Theorem 6.1).

Theorem G. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of dimension n, codimension e, and degree d. If $d \leq ne+2$, then either X is a uniruled variety or $d = ne+2$ and X is a Calabi-Yau variety.*

This is a higher dimensional generalization of a classical theorem in surface case (see e.g., [Bui, Lemmas 1.3 and 1.6]), and it seems to be well-known to specialists. However, for lack of suitable references, we give a brief proof in Section 7. It will be done by classifying Castelnuovo varieties (see Definition 7.1) of minimal degree combining with Zak's result ([Z, Corollary 1.6]), which says that if $d \leq ne+1$, then X is uniruled.

Conventions. Throughout the paper, we work over the complex number field $\mathbb{C}$, and we use the following terminologies.

(1) A smooth projective variety X is called resp. *Fano* or *weak Fano* if the anti-canonical divisor $-K_X$ is resp. ample or nef and big.
(2) A smooth projective variety X is called *Calabi-Yau* if $\mathcal{O}_X(K_X) = \mathcal{O}_X$ and $h^i(\mathcal{O}_X) = 0$ for $0 < i < \dim X$.
(3) A variety X is called *uniruled* if for any general point x on X, there is a rational curve passing through x, and it is called *rationally connected* if for any general points x_1 and x_2 on X, there is a rational curve connecting x_1 and x_2.

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1. Scrolls

In this section, we collect basic facts on scrolls. A vector bundle E on a smooth projective curve C is called *very ample* if the tautological line bundle $\mathcal{O}_{\mathbb{P}_C(E)}(1)$ is very ample on $\mathbb{P}_C(E)$. For a very ample vector bundle E of rank n on C, we have an embedding $\mathbb{P}_C(E) \subset \mathbb{P}^r$ given by a linear subsystem of $|\mathcal{O}_{\mathbb{P}_C(E)}(1)|$. We call $\mathbb{P}_C(E)$ a *scroll over* C. Note that

$$h^0(\mathbb{P}_C(E), \mathcal{O}_{\mathbb{P}_C(E)}(1)) = h^0(C, E) \text{ and } d = \deg_{\mathbb{P}^r}(\mathbb{P}_C(E)) = \deg_C(\det E).$$

A scroll is called *rational* (resp. *elliptic*) if it is defined over a rational (resp. an elliptic) curve. A scroll is simply connected (and rationally connected) if and only if it is a rational scroll. Throughout the paper, we assume that $n \geq 2$ (and hence, $r \geq 3$) for any scroll.

The main result of this section classifies scrolls of small degree, which is an easy consequence of [IT2].

Theorem 1.1. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree $d \leq r+1$. If X is a scroll over a curve C, then either X is a rational scroll or $d = r+1$ and X is an elliptic normal scroll.*

Proof. It follows from Propositions 1.3 and 1.4, which we will prove in the remaining of this section. ☐

First, we recall the classification of curves of almost minimal degree. We include the whole proof to motivate our proof of Proposition 1.4.

Theorem 1.2. *Let $C \subset \mathbb{P}^r$ be a non-degenerate smooth projective curve of degree d. If $d \leq r+1$, then either C is a rational curve or $d = r+1$ and C is an elliptic normal curve.*

Proof. Denote by H the hyperplane section. If H is special (i.e., $h^1(\mathcal{O}_C(H)) \neq 0$), then by Clifford Inequality, we have

$$r+1 \leq h^0(\mathcal{O}_C(H)) \leq \frac{d}{2} + 1 \leq \frac{r+1}{2} + 1.$$

Thus, we get $r \leq 1$, which is a contradiction. Hence, H is non-special. By Riemann-Roch Formula, we have

$$r + 1 \leq h^0(\mathcal{O}_C(H)) = d - g + 1 \leq r + 1 - g + 1,$$

where g is the genus of the curve C. Then, $g \leq 1$, and the equality holds exactly when $d = r + 1 = h^0(\mathcal{O}_C(H))$. □

Recall the lower bound of the degree of scrolls over a curve of positive genus.

Proposition 1.3 (Corollary 3 of [Io4])**.** *Let C be a smooth projective curve of positive genus, and let E be a very ample vector bundle on C, which defines a linearly normal scroll $\mathbb{P}_C(E) \subset \mathbb{P}^r$ over C of degree d. Then, we have $d \geq r + 1$.*

We classify scrolls of minimal degree over a curve of positive genus.

Proposition 1.4. *Let C be a smooth projective curve of positive genus, and let E be a very ample vector bundle on C, which defines a linearly normal scroll $\mathbb{P}_C(E) \subset \mathbb{P}^r$ over C of degree d. If $d = r + 1$, then C is an elliptic curve.*

Proof. Suppose that the genus $g = g(C) \geq 2$. Recall the main theorem of [IT2]:

$$h^0(E) + n - 2 \leq h^0(\det E). \tag{1.1}$$

According to [IT2, Corollary 4], we divide into two cases. Firstly, if $\det E$ is special (i.e., $h^1(C, \det E) \neq 0$), then by Clifford Inequality, we have

$$r + 1 = h^0(E) \leq h^0(\det E) - n + 2 \leq \frac{d}{2} - n + 3 = \frac{r+1}{2} - n + 3.$$

Thus, we obtain $r + 2n \leq 5$, which is a contradiction. Secondly, if $\det E$ is non-special, then by Riemann-Roch Formula, we have

$$r + 1 = h^0(E) \leq h^0(\det E) - n + 2 = d - g - n + 3 = r - g - n + 4.$$

Thus, we obtain $g + n \leq 3$, which is also a contradiction. Hence, $g = 1$. □

The following lemma will be used in Section 5. Note that there is an n-dimensional scroll in $\mathbb{P}^{2n-1}$ (e.g., Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$).

Lemma 1.5. *Let $X \subset \mathbb{P}^r$ be an n-dimensional scroll over a curve of genus g. Then, $r \geq 2n - 1$ for $g = 0$, and $r \geq 2n$ for $g \geq 1$.*

Proof. By Barth-Larsen Theorem ([L1, Corollaries 3.2.2 and 3.2.3]), the lemma follows. □

Finally, we recall the following well-known facts on elliptic scrolls.

Remark 1.6. (1) (Proposition 5.2 of [Io3]) There exist n-dimensional elliptic normal scrolls in $\mathbb{P}^{2n+k}$ of degree $2n + k + 1$ for all $k \geq 0$.
(2) (Theorem 5.1A of [But]) Elliptic normal scrolls and elliptic normal curves are projectively normal.

2. Roth varieties

In this section, we collect basic facts on Roth varieties. Let $E = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n+1})$ be a globally generated vector bundle of rank $n+1$ on $\mathbb{P}^1$ with all $a_i \geq 0$. Denote $S_{a_1,\ldots,a_{n+1}} := \mathbb{P}_{\mathbb{P}^1}(E)$, and let $\pi_1 \colon S_{a_1,\ldots,a_{n+1}} \to \mathbb{P}^1$ be the natural projection. Let F be a fiber of π_1, and H be a base point free divisor on $S_{a_1,\ldots,a_{n+1}}$ with $\mathcal{O}_{S_{a_1,\ldots,a_{n+1}}}(H) = \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(E)}(1)$.

Definition 2.1 (Definition 3.1 and Theorem 3.7 of [Il])**.** Let $n \geq 2$ be an integer. Consider the birational morphism $\pi_2 \colon S_{0,0,a_1,\ldots,a_{n-1}} \to \overline{S} \subset \mathbb{P}^r$ given by the complete linear system $|H|$ for $a_i \geq 1$. Then, the singular locus of $\overline{S}$ is a line $L \subset \mathbb{P}^r$. For every integer $b \geq 1$, take a smooth variety $\widetilde{X} \in |bH + F|$ such that $\pi_2|_{\widetilde{X}} \colon \widetilde{X} \to X := \pi_2(\widetilde{X})$ is an isomorphism, and $L \subset X \subset \mathbb{P}^r$. Then, X is called a *Roth variety*.

We have an embedding $X \subset \overline{S} \subset \mathbb{P}^r$. By [Il, Proposition 3.5 and Theorem 3.14 (4)], $X \subset \mathbb{P}^r$ is non-degenerate and projectively normal, i.e., every linearly normal Roth variety is projectively normal. For more detail, we refer to [Il, Section 3].

Note that $H+F$ is very ample, because $\mathcal{O}_{S_{0,0,a_1,\ldots,a_{n-1}}}(H+F)$ is the tautological line bundle of $S_{1,1,a_1+1,\ldots,a_{n-1}+1}$. Thus, $bH+F$ is very ample for every integer $b \geq 1$.

Proposition 2.2. *Every Roth variety is simply connected.*

Proof. By Lefschetz Hyperplane Theorem (see e.g., [L1, Theorem 3.1.21]), the assertion follows. □

Recall that by [Il, Theorem 3.7], we have

$$\sum_{i=1}^{n-1} a_i = r - n \text{ and } d = b(r-n) + 1$$

for a linearly normal Roth variety $X \subset \mathbb{P}^r$ of dimension n and degree d. Note that a Roth variety with $b = 1$ is a rational scroll $S_{1,a_1,\ldots,a_{n-1}}$ ([Il, Theorem 3.14 (1)]). We can completely classify linearly normal Roth varieties $X \subset \mathbb{P}^r$ with $\deg_{\mathbb{P}^r}(X) \leq r+1$.

Proposition 2.3. *Let $X \subset \mathbb{P}^r$ be a non-degenerate Roth variety of dimension n, codimension e, and degree d. If $d \leq r+1$, then one of the following holds:*

(1) *$b = 1$ and X is a (not necessarily linearly normal) rational scroll.*
(2) *$b = 2$, $d = r = 2n-1$, and X is projectively normal.*
(3) *$b = 2$, $d = r+1 = 2n+1$, and X is projectively normal.*

Proof. Assume that $X \subset \mathbb{P}^r$ is linearly normal. First, we show that $e \geq n-1$. Since the cases $n = 2$ and 3 are trivial, we may assume that $n \geq 4$. If $e \leq n-2$, then by Barth-Larsen Theorem ([L1, Corollary 3.2.3]), $\operatorname{Pic}(X) \simeq \mathbb{Z}$, but by Lefschetz Theorem for Picard Group ([L1, Example 3.1.25]), $\operatorname{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$, and hence, we get a contradiction. Thus, $e \geq n-1$. Recall that $d = be+1 \leq n+e+1$. We obtain $e \leq \frac{n}{b-1}$. It follows that $b \leq 2$.

The case $b = 1$ is trivial by [Il, Theorem 3.14 (1)]. Now, put $b = 2$. We have $d = 2e+1 \leq n+e+1$, so we obtain $e \leq n$. Thus, we have two cases $e = n-1$ (then, $d = r$), and $e = n$ (then, $d = r+1$). By considering the Picard group and linearly normality in small codimension, we see that every linearly normal Roth variety $X \subset \mathbb{P}^r$ with $b = 2$ and $d = r$ cannot have isomorphic projection. Thus, every Roth variety $X \subset \mathbb{P}^r$ with $b = 2$ and $d \leq r+1$ is linearly normal, and hence, it is projectively normal. $\square$

Now, we investigate the rationally connectedness of Roth varieties.

Proposition 2.4. *Let $X \subset \mathbb{P}^r$ be a linearly normal Roth variety of $\dim(X) = n$ and $\deg(X) = b(r-n)+1$. Then, X is rationally connected if and only if $b \leq n$.*

Proof. Let S be a rational scroll with the projection $\pi_1 \colon S \to \mathbb{P}^1$, and let $X \subset S$ be a Roth variety. Then, there is a surjective morphism $\pi_1|_X \colon X \to \mathbb{P}^1$. Let F_X be a general fiber of $\pi_1|_X$, i.e., it is a restriction of a general fiber F of π_1 to X. We note that the natural map $H^0(\mathcal{O}_S(F)) \to H^0(\mathcal{O}_X(F_X))$ is an isomorphism. By Bertini Theorem, F_X is a smooth hypersurface in $F \simeq \mathbb{P}^n$. Since $X \in |bH + F|$, we have $\deg_{\mathbb{P}^n}(F_X) = (bH+F).H^{n-1}.F = b$. If $b \leq n$, then F_X is rationally connected. Thus, by [GHS, Corollary 1.3], X is rationally connected. Conversely, if $b \geq n+1$, then by [Il, Theorem 3.14 (5)], X is a Castelnuovo variety (see Definition 7.1). Thus, $h^0(\mathcal{O}_X(K_X)) \neq 0$, and hence, X cannot be rationally connected. $\square$

Remark 2.5. In the proof, we saw that every Roth variety $X \subset \mathbb{P}^r$ has a fibration $\pi_1|_X \colon X \to \mathbb{P}^1$ such that the fiber F_X is a hypersurface of degree b via $\mathcal{O}_X(1)$. In particular, every Roth variety $X \subset \mathbb{P}^r$ of degree $d \leq r+1$ is a rational scroll or a hyperquadric fibration over $\mathbb{P}^1$.

Corollary 2.6. *Every Roth variety $X \subset \mathbb{P}^r$ of degree $d \leq r+1$ is rational.*

3. Positivity of double point divisors and weak Fano varieties

In this section, we prove Theorem C after reviewing the Noma's work on double point divisors ([N]).

3.1. **Double point divisors from inner projection.** In this subsection, we summarize the Noma's work ([N]), which will be useful for our proof of Theorem C. Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of dimension n, codimension e, and degree d. Throughout the section, we assume that $n \geq 2$ and $e \geq 2$. First, we need the following definition.

Definition 3.1. Let $x_1, \ldots, x_m$ $(1 \leq m \leq e-1)$ be general points on X, and define

$$E_{x_1,\ldots,x_m} := \overline{\{z \in X \setminus \{x_1,\ldots,x_m\} \mid \dim \langle x_1,\ldots,x_m,z\rangle \cap X = 1\}}.$$

We say that X *satisfies the property* (E_m) if $\dim E_{x_1,\ldots,x_m} = n-1$ for general points $x_1, \ldots, x_m$.

Theorem 3.2 (Theorem 3 in [N])**.** *If $X \subset \mathbb{P}^r$ satisfies the property (E_{e-1}), then X is either a scroll over a smooth projective curve or the Veronese surface in $\mathbb{P}^5$.*

Now, we assume that $X \subset \mathbb{P}^r$ does not satisfy property (E_{e-1}). Let $x_1, \ldots, x_{e-1}$ be general points of X, and $\Lambda := \langle x_1, \ldots, x_{e-1}\rangle$ be their linear span. Note that $\Lambda \cap X = \{x_1, \ldots, x_{e-1}\}$ by the general position lemma (see e.g., [N, Lemma 1.1]). Consider the inner projection from the center Λ and the blow-up $\sigma \colon \widetilde{X} \to X$ at $x_1, \ldots, x_{e-1}$. We have the following diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\sigma"] \arrow[dr, "\widetilde{\pi}"'] & X \arrow[r, hook] \arrow[d, dashed, "\pi"] & \mathbb{P}^{n+e} \arrow[d, dashed, "\pi_\Lambda"] \\
 & \overline{X}_\Lambda \arrow[r, hook] & \mathbb{P}^{n+1}.
\end{tikzcd}$$

Note that $\deg(\overline{X}_\Lambda) = d - e + 1$. Since X does not satisfy property (E_{e-1}) by the assumption, the birational morphism $\widetilde{\pi}$ has no exceptional divisor. Therefore, there is an effective divisor $D(\widetilde{\pi})$ on $\widetilde{X}$ by the birational double point formula ([L2, Lemma 10.2.8]) such that

$$\mathcal{O}_{\widetilde{X}}(D(\widetilde{\pi})) \simeq \widetilde{\pi}^*(\omega^\circ_{\overline{X}}) \otimes \omega_{\widetilde{X}}{}^{-1}.$$

Let $D_{inn}(\pi) := \overline{\sigma(D(\widetilde{\pi})|_{\widetilde{X}\setminus E_1\cup\cdots\cup E_{e-1}}}$ be a divisor on X, where $E_1,\ldots,E_{e-1}$ are exceptional divisors of σ. Then, the effective divisor $D_{inn}(\pi)$, called the *double point divisor from inner projection*, is linearly equivalent to

$$D_{inn} = -K_X + (d - r - 1)H.$$

By varying the centers of projections, Noma proved in [N, Theorem 1] that the base locus of $|D_{inn}|$ lies in the set of non-birational centers of simple inner projections, i.e.,

$$\mathrm{Bs}(|D_{inn}|) \subset \mathcal{C}(X) := \{u \in X \mid l(X \cap \langle u, x\rangle) \geq 3 \text{ for general } x \in X\}.$$

Theorem 3.3 (Theorem 2 of [N]). *If $\dim \mathcal{C}(X) \geq 1$, then X is a Roth variety.*

To sum up, we have the following.

Theorem 3.4 (Theorem 4 of [N]). *Suppose that X is not a scroll over a smooth projective curve, the Veronese surface in $\mathbb{P}^5$, or a Roth variety. Then, the double point divisor D_{inn} from inner projection is semiample.*

3.2. **Proof of Theorem C.** First, recall some notations. Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of dimension n, codimension e, and degree $d \leq r+1$. We denote by H a hyperplane section.

By the classification of curves of almost minimal degree (Theorem 1.2), there is nothing to prove when X is a curve. Thus, assume that $n \geq 2$. When $X \subset \mathbb{P}^r$ is a hypersurface, X is simply connected by Barth-Larsen Theorem ([L1, Corollary 3.2.2]). Moreover, by the adjunction theorem, X is a Fano variety if $d \leq r$, and X is a Calabi-Yau variety if $d = r+1$. Note that a smooth hypersurface $X \subset \mathbb{P}^r$ is rationally connected if and only if $\deg(X) \leq r$, i.e., X is a Fano variety.

We further assume that $e \geq 2$. The case that $X \subset \mathbb{P}^r$ is a Roth variety or a scroll over a curve is already treated in Sections 1 and 2. Thus, suppose that X is neither a Roth variety nor a scroll over a curve. Note that the Veronese surface in $\mathbb{P}^5$ is clearly a Fano variety. By Theorem 3.4, we only have to consider the case that $\mathcal{C}(X)$ is finite and the double point divisor $D_{inn} = -K_X + (d-r-1)H$ is semiample (thus, nef). If $d \leq r$, then we may write

$$-K_X = D_{inn} + (r + 1 - d)H,$$

and hence, $-K_X$ is ample, i.e., X is a Fano variety (see also [N, Corollary 7.6]).

Now, consider the case $d = r+1$. Then, $D_{inn} = -K_X$ is a non-trivial semiample divisor. We will show that $-K_X$ is big, and hence, X is a weak Fano variety. Note that every weak Fano variety is rationally connected and simply connected (see e.g., [HM, Corollary 1.4]). If $X \subset \mathbb{P}^r$ is not linearly normal, then it is obtained by an isomorphic projection from $X \subset \mathbb{P}^{r+1}$ with degree $d = r+1$, and hence, it is a Fano variety. Thus, we may assume that $X \subset \mathbb{P}^r$ is linearly normal. We divide into three cases: (1) $e \leq n-2$, (2) $e = n-1$, and (3) $e \geq n$.

If $e \leq n-2$, then by Barth-Larsen Theorem ([L1, Corollary 3.2.3]), $-K_X = \ell H$ for some integer $\ell > 0$, and hence, $-K_X$ is ample.

Suppose that $e = n-1$. By Barth-Larsen Theorem ([L1, Corollary 3.2.2]), X is simply connected. Note that $d = 2n = 2e+2$, and $n \geq 3$ since $e \geq 2$. By the classification results by Ionescu (see tables in Introductions of [Io1] and [Io3]), case f) of Theorem I in [Io2] does not occur in our case. Thus, by [Io2, Theorem I], X is a Fano variety, a scroll, a linear fibration over a rational surface, or a hyperquadric fibration because X is simply connected and $-K_X$ is non-trivial. Recall that we already exclude scrolls. First, we show that X cannot have a hyperquadric fibration. Suppose that X has a hyperquadric fibration. Let F be a hyperquadric fiber of X, and let $H_F := H|_F$. Then, we have (cf., [IT1, p.217])

$$4n = 2nH_F^{n-1} = c_{n-1}(N_{X|\mathbb{P}^{2n-1}}|_F) = c_{n-1}(N_{F|\mathbb{P}^{2n-1}}) = 2(2n-1),$$

which is a contradiction. Now, we show that if X has a linear fibration over a rational surface B, then it is a weak Fano variety. Since $X \subset \mathbb{P}^r$ is linearly normal and $d = 2n$, by [IT1, Proposition 4], $B \simeq \mathbb{P}^2$ and $d = \frac{n(n+1)}{2}$. Thus, $n = 3$. The only possible X is a Bordiga threefold by the classification (see tables in Introduction of [Io1]). In Example 5.9, we will give a detailed description of the Bordiga threefold. Note that X is neither a scroll (over a curve) nor a Roth variety, and hence, $-K_X$ is semiample. By Riemann-Roch Formula, we obtain $(-K_X)^3 = 6$. Thus, X is a weak Fano variety. In particular, we have shown the following.

Lemma 3.5. *Let $X \subset \mathbb{P}^{2n-1}$ be an n-dimensional non-degenerate smooth projective variety of degree $2n$. Then, it does not admit hyperquadric fibration, and if it has a linear fibration over a surface, then it is the Bordiga threefold.*

Now, we consider the last case $e \geq n$ ($d = r+1 = n+e+1$) and X is not a scroll. Let C be a curve section ($C \subset \mathbb{P}^{e+1}$), and let g be the sectional genus, i.e., the genus of C. Denote $H_C := H|_C$. The following bound of the sectional genus plays a crucial role (cf. Lemma 4 of [Io4]).

Lemma 3.6. *If $e \geq n$ and $r+1 \geq d$, then $n \geq g$ and $d \geq 2g+1$.*

Proof. If H_C is special, then by Clifford Inequality, we have

$$e+2 \leq h^0(\mathcal{O}_C(H_C)) \leq \frac{d}{2}+1 \leq \frac{n+e+1}{2}+1.$$

Then, we get $2e+4 \leq n+e+1+2$, and hence, $e \leq n-1$, which is a contradiction. Thus, H_C is non-special. By Riemann-Roch Formula, we have

$$e+2 \leq h^0(\mathcal{O}_C(H_C)) = d+1-g \leq (n+e+1)+1-g.$$

Then, we get $g \leq n$ and $g+e+1 \leq d$. Since $g \leq n \leq e$, we obtain $2g+1 \leq g+e+1 \leq d$. □

Remark 3.7. We obtain the same inequality $n \geq g$ by applying Castelnuovo's Bound on sectional genus.

If $g = 0$, then by Fujita's classification (see e.g., [Io4, Theorem A]), X is a Fano variety or a rational scroll. Thus, we may assume that $g \geq 1$. Then, we can prove the following.

Lemma 3.8. *If $n \geq g \geq 1$ and $d \geq 2g+1$, then $d + \frac{n}{n-1}\{(n-1)d-2g+2\} - nd > 0$.*

Proof. Since $3n > 2g+1$, we have

$$(2g+1)(n-1) = 2gn + n - 2g - 1 > 2gn - 2n = (2g-2)n.$$

Thus, $n-1 > \frac{2g-2}{2g+1}n$, and hence, there exists a rational number $\epsilon > 0$ such that

$$\frac{2g-2}{2g+1}n + \epsilon = n-1.$$

Note that

$$\frac{n-1}{n} + \frac{(n-1)d-2g+2}{d} = \frac{2g-2}{2g+1} + \frac{\epsilon}{n} + (n-1) + \frac{-2g+2}{d}.$$

Since $d \geq 2g+1$, we get $\frac{2g-2}{2g+1} \geq \frac{2g-2}{d}$. Then, we have

$$\frac{n-1}{n} + \frac{(n-1)d-2g+2}{d} = (n-1) + \frac{2g-2}{2g+1} + \frac{-2g+2}{d} + \frac{\epsilon}{n} > n-1.$$

By multiplying $\frac{nd}{n-1}$, we obtain $d + \frac{n}{n-1}\{(n-1)d-2g+2\} > nd$. □

Let $D := H$ and $E := \frac{1}{n-1}K_X + H = \frac{1}{n-1}\{K_X + (n-1)H\}$. By [Io1, Theorem 1.4], if $K_X + (n-1)H$ is not base point free, then X is a Fano variety or a scroll. Thus, we may assume that $K_X + (n-1)H$ is base point free (thus, nef), and hence, E is a nef $\mathbb{Q}$-divisor. Note that

$$2g-2 = \{K_X + (n-1)H\}.H^{n-1} = K_X.H^{n-1} + (n-1)d,$$

and hence, we get

$$K_X.H^{n-1} = -(n-1)d + 2g - 2.$$

Now, we have

$$(D-nE).D^{n-1} = \left(H - \frac{n}{n-1}K_X - nH\right).H^{n-1} = d + \frac{n}{n-1}\{(n-1)d-2g+2\} - nd.$$

By Lemma 3.8, we obtain $D^n > nE.D^{n-1}$. Thus, the divisor

$$D - E = H - \left(\frac{1}{n-1}K_X + H\right) = -\frac{1}{n-1}K_X$$

is big by [L1, Theorem 2.2.15]. We complete the proof of Theorem C.

4. Cox rings

First, we recall the definition of Cox ring. Let X be a regular smooth projective variety (i.e., $h^1(\mathcal{O}_X) = 0$). Then, $\operatorname{Pic}(X)$ is finitely generated so that we can choose generators $L_1, \ldots, L_m$ of $\operatorname{Pic}(X)$. The *Cox ring* of X with respect to $L_1, \ldots, L_m$ is defined by

$$\operatorname{Cox}(X) := \bigoplus_{(a_1,\ldots,a_m)\in\mathbb{Z}^m} H^0(L_1^{\otimes a_1} \otimes \cdots \otimes L_m^{\otimes a_m}).$$

Note that the finite generation of $\operatorname{Cox}(X)$ is independent of the choice of generators of $\operatorname{Pic}(X)$ (see [HK, Remark in p.341]), and $\operatorname{Cox}(X)$ is finitely generated if and only if X is a Mori dream space ([HK, Proposition 2.9]). Typical examples of Mori dream spaces are toric varieties and regular varieties with Picard number one. For further detail on Cox rings and Mori dream spaces, we refer to [HK].

Example 4.1. Let $X \subset \mathbb{P}^r$ be a smooth hypersurface. By Lefschetz Theorem for Picard Group ([L1, Example 3.1.25]), if $r \geq 4$, then X has finitely generated Cox ring. If $r = 2$, then X has finitely generated Cox ring if and only if X is a rational curve. When $r = 3$, determining finite generation of Cox ring is a delicate problem even in the case of quartic surfaces. Although very general smooth quartic surfaces have finitely generated Cox rings, there is a smooth quartic surface whose automorphism group is infinite (see e.g., [Og, Theorem 1.2]) so that its Cox ring is not finitely generated by [AHL, Theorems 2.7 and 2.11].

Now, we prove Corollary D.

Proof of Corollary D. Since $d \leq r$, by Theorem C, X is a Fano variety, a Roth variety, or a rational scroll. By [BCHM, Corollary 1.3.2], the Cox ring of a Fano variety is finitely generated. Moreover, every rational scroll is a toric variety, so the Cox ring is a polynomial ring (see e.g., [HK, Corollary 2.10]). Thus, we only have to consider the case that X is a Roth variety.

Every Roth variety X is supported on a very ample divisor of a rational scroll $S = \mathbb{P}_{\mathbb{P}^1}(E)$ for some globally generated vector bundle E on $\mathbb{P}^1$. Recall that $d = b(r-n)+1$, and $b = 1$ or 2 when $d \leq r$. If $b = 1$, then X is a rational scroll, and hence, $\mathrm{Cox}(X)$ is finitely generated. Assume that $b = 2$. If $n = 2$, then $r = 3$ by Proposition 2.3, i.e., X is a cubic surface in $\mathbb{P}^3$. Since X is a Fano surface, its Cox ring is finitely generated. The assertion for $n \geq 3$ follows from Proposition 4.2. □

Proposition 4.2. *The Cox ring of a Roth variety of dimension $n \geq 3$ is finitely generated.*

Proof. Let $X \subset \mathbb{P}^r$ be a Roth variety of dimension $n \geq 3$ and degree $d = b(r-n)+1$. We only have to consider the case that $b \geq 2$. Recall that X is a divisor of a rational scroll $S = S_{0,0,a_1,\ldots,a_{n-1}}$ with all $a_i \geq 1$, and let $\pi_1 \colon S \to \mathbb{P}^1$ be the natural projection with a general fiber F. Recall that we have a birational morphism $\pi_2 \colon S \to \overline{S} \subset \mathbb{P}^r$ given by the complete linear system $|H|$, where $\mathcal{O}_{S_{0,0,a_1,\ldots,a_{n-1}}}(H) = \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(E)}(1)$. The singular locus of $\overline{S}$ is a line $L \subset \mathbb{P}^r$, which is contained in X. There are effective divisors $L_1 \in |H - a_1F|, \ldots, L_{n-1} \in |H - a_{n-1}F|$ (we can arrange them to be $(\mathbb{C}^*)^{n+1}$-invariant divisors by the maximal torus action on S) such that

$$\pi_2^{-1}(L) = L_1 \cap \cdots \cap L_{n-1} \simeq L \times \mathbb{P}^1.$$

Note that $L_1 \cap \cdots \cap L_{n-1} \cap X = L$.

By Lefschetz Theorem for Picard Group, the map $\mathrm{Pic}(S) \to \mathrm{Pic}(X)$ is an isomorphism. In particular, the Picard number of X is two. For all i, we have an isomorphism

$$H^0(\mathcal{O}_S(H - a_iF)) \to H^0(\mathcal{O}_X(H_X - a_iF_X)),$$

where $F_X = F|_X$ is the restriction, from the exact sequence

$$0 \to \mathcal{O}_S(-bH - F) \to \mathcal{O}_S \to \mathcal{O}_X \to 0.$$

Let $D_1 := L_1|_X, \ldots, D_{n-1} := L_{n-1}|_X$. We denote by $D_{out} := -K_X + (d-n-2)H_X$ the double point divisor from outer projection, where $H_X = H|_X$ is the restriction. By [Il, Proposition 3.8], $D_{out}.L = 0$. Since D_{out} is base point free, we can choose an effective divisor $D_n \in |D_{out}|$ such that $D_n \cap L = \emptyset$. Thus, we obtain

$$D_1 \cap \cdots \cap D_{n-1} \cap D_n = (L_1 \cap \cdots \cap L_{n-1} \cap X) \cap D_n = L \cap D_n = \emptyset.$$

On the other hand, let $D_1' := F_1|_X, D_2' := F_2|_X$, where F_1 and F_2 are distinct fibers of π_1. Then, we have $D_1' \cap D_2' = \emptyset$. Divisor classes of $D_1, \ldots, D_{n-1}$ are in outside of the nef cone of X, and the divisor class of D_n is a ray generator of the nef cone of X. The other ray generator of the nef cone of X is the divisor class of F_X which is also a ray generator of the effective cone of X. Thus, we have

$$\mathrm{Cone}(D_1, \cdots, D_n) \cap \mathrm{Cone}(D_1', D_2') = \{0\}$$

By [It, Theorem 1.3], $\mathrm{Cox}(X)$ is finitely generated. $\square$

Remark 4.3. We first show that a Roth surface $X \subset \mathbb{P}^r$ with $\deg(X) = r+1$ and $b = 2$ has a finitely generated Cox ring. By Proposition 2.3, $r = 4$, and by Corollary 2.6, X is a rational surface. We have $-K_X = H_X - F_X$, where H_X is the hyperplane section and F_X is a general hyperquadric fiber. Thus, $(-K_X)^2 = 1$. By Riemann-Roch Theorem, $-K_X$ is big, and hence, the Cox ring of X is finitely generated by [TVAV, Theorem 1]. However, we do not know whether Cox rings of all Roth surfaces are finitely generated.

5. Adjunction mappings

In this section, we study adjunction mappings of weak Fano varieties of small degree, and then, we prove Theorem E. The end product of the minimal model program (see [KM] for basics and [BCHM] for recent progress) should be either a Mori fiber space or a minimal model. Recall Theorem C: if $X \subset \mathbb{P}^r$ is an n-dimensional smooth projective variety of degree $d \leq r+1$, then it is (a) a weak Fano variety, (b) a Roth variety or a rational scroll, (c) a Calabi-Yau hypersurface, or (d) an elliptic scroll or an elliptic curve. Note that smooth elliptic curves and Calabi-Yau hypersurfaces are minimal models. It is natural to study a contraction appeared in the minimal model program for the other cases in Theorem C. The fibration of a scroll can be regarded as a contraction of a K_X-negative extremal ray, and hence, every scroll is a Mori fiber space. Similarly, the adjunction mapping (if it is defined) also gives a K_X-negative contraction of Roth varieties and weak Fano varieties. It turns out that the base space and the fibers of the adjunction mapping are very simple in our case. More precisely, we will prove that the base space is $\mathbb{P}^1$, $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and the fiber is either a hyperquadric or a linear subspace.

5.1. **Basics of adjunction mappings.** We recall basic notions in adjunction theory (for further detail, see [BS]). Throughout the section, we denote by H a general hyperplane section of $X \subset \mathbb{P}^r$. By [Io1, Theorem 1.4], if $K_X + (n-1)H$ is not base point free, then X is a prime Fano variety or a scroll (over a curve) when $n \geq 3$. In the case $n = 2$, we should add a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$ or $\mathbb{P}^4$ and a quadric hypersurface $Q^2 \in \mathbb{P}^3$. Now, assume that $K_X + (n-1)H$ is base point free. Then, we can define a surjective morphism $\varphi\colon X \to B$ given by $|K_X + (n-1)H|$, which is called an *adjunction mapping*.

Proposition 5.1 (Proposition 1.11 of [Io1])**.** *Let $X \subset \mathbb{P}^r$ be a smooth projective variety of dimension n. Assume that $K_X + (n-1)H$ is base point free so that we have the adjunction mapping $\varphi\colon X \to B$. Then, one of the following holds:*

1. $\dim B = 0$*, or equivalently,* $-K_X = (n-1)H$.
2. $n \geq 2$ *and* φ *gives a hyperquadric fibration over a smooth curve* B.
3. $n \geq 3$ *and* φ *gives a linear fibration over a smooth surface* B.
4. $\dim B = n$.

For reader's convenience, we give the complete list of del Pezzo varieties (i.e., $-K_X = (n-1)H$), which was classified by Fujita ([F1] and [F2]; see also [Io4, Theorem B] and [IP, Section 12.1]).

Theorem 5.2 (Fujita)**.** *Let $X \subsetneq \mathbb{P}^r$ be a non-degenerate linearly normal smooth del Pezzo variety of dimension $n \geq 2$ and degree d. Then, $d = r - n + 2$ and X is one of the following:*

(1) *a cubic hypersurface of dimension $n \geq 3$;*
(2) *a complete intersection of type $(2, 2)$;*
(3) *the Plücker embedding of the Grassmannian $Gr(2, 5)$ or its linear section;*
(4) *the Veronese threefold $v_2(\mathbb{P}^3)$;*
(5) *a del Pezzo surface embedded by the anticanonical divisor $-K_X$;*
(6) *the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ or its hyperplane section;*
(7) *the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;*
(8) *$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ embedded by the tautological line bundle.*

A del Pezzo surface whose anticanonical divisor is very ample is either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at r points in general position for $r \leq 6$.

Remark 5.3. In the above list, varieties from (1)∼(3) are prime Fano, and varieties from (5)∼(8) have the Picard number $\rho \geq 2$ except for the third Veronese surface $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ from (5).

5.2. **Proof of Theorem E.** To prove Theorem E, we need two lemmas which concern resp. hyperquadric fibrations and linear fibrations (cf. resp. Lemma 7 and Proposition 5 of [Io4]).

Lemma 5.4. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree $d \leq r + 1$. Assume that the adjunction mapping $\varphi\colon X \to C$ induces a hyperquadric fibration over a curve C. Then, $C \simeq \mathbb{P}^1$.*

Proof. Note that X is not an elliptic scroll. By Theorem C, we have $q = h^1(\mathcal{O}_X) = 0$, which coincides with the genus of C by [Io4, Lemma 6]. □

Lemma 5.4 also follows from [FG, Corollary 3.3].

Lemma 5.5. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth projective variety of degree $d \leq r+1$. Assume that the adjunction mapping $\varphi\colon X \to S$ induces a linear fibration over a surface S. Then, $S \simeq \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. We will closely follow [Io4, Proof of Proposition 5]. Note that the case $d \leq r$ is treated in [Io4, Proposition 5]. We may assume that $d = r + 1$. By Barth-Larsen Theorem ([L1, Corollary 3.2.3]), we can further assume that $e \geq n - 1$. For the case $e = n - 1$, we already verified the assertion in Lemma 3.5. Thus, assume that $e \geq n$. Since X is not an elliptic scroll, we have $q = h^1(\mathcal{O}_X) = 0$.

Let $S' = X \cap H_1 \cap \cdots \cap H_{n-2}$ be a smooth surface, where each H_i is a generic hyperplane section. By the adjunction formula and Lemma 3.6, we have

$$2g - 2 = H_{S'}^2 + H_{S'}.K_{S'} = d + H_{S'}.K_{S'} \geq 2g + 1 + H_{S'}.K_{S'},$$

and hence, $H_{S'}.K_{S'} < 0$. In particular, $h^0(\mathcal{O}_{S'}(K_{S'})) = h^2(\mathcal{O}_{S'}) = 0$. Now, observe that $n \geq 3$. We have the following exact sequence

$$0 \to \mathcal{O}_X(K_X + (n-2)H) \to \mathcal{O}_X(K_X + (n-1)H) \to \mathcal{O}_H(K_H + (n-2)H_H) \to 0.$$

Since $K_X + (n-1)H$ is not big, $h^0(\mathcal{O}_X(K_X + (n-2)H)) = 0$. Furthermore, by Kodaira Vanishing, $h^1(\mathcal{O}_X(K_X + (n-2)H)) = 0$. Let g be the sectional genus of $X \subset \mathbb{P}^r$. Then, we have

$$h^0(\mathcal{O}_X(K_X + (n-1)H)) = h^0(\mathcal{O}_H(K_H + (n-2)H_H)) = \cdots = h^0(\mathcal{O}_{S'}(K_{S'} + H_{S'})),$$

and by [Io1, Lemma 1.1], $h^0(\mathcal{O}_{S'}(K_{S'}+H_{S'})) = g$. Thus, we get $\varphi\colon X \to S \subset \mathbb{P}^{g-1}$. Let H_S be a generic hyperplane section of $S \subset \mathbb{P}^{g-1}$, and let $Y := \varphi^{-1}(H_S)$. Note that Y is a scroll of dimension $n-1$ and degree $d_Y = (K_X+(n-1)H).H^{n-1} = 2g-2$ over the curve H_S. Let m be the dimension of the smallest linear subspace of $\mathbb{P}^r$ containing Y, i.e., $Y \subset \mathbb{P}^m$ is non-degenerate. By Lemma 1.5, $m \geq 2(n-1)-1 = 2n-3$. By Lemma 3.6, $m \geq 2n-3 \geq 2g-3 = d_Y - 1$. Thus, by Theorem 1.1, the genus g' of H_Y is 0 or 1. Suppose that $g' = 1$. By Lemma 1.5, we must have $m \geq 2(n-1)$. It follows that $m \geq 2n-2 \geq 2g-2 = d_Y$, which is a contradiction to Theorem 1.1. Thus, $g' = 0$. By Fujita's classification ([Io4, Theorem A]), $S \simeq \mathbb{P}^2$ (in this case, we have $g = \Delta(X,H) = 3$ or $g = 6$) or S is a scroll over $\mathbb{P}^1$, i.e., Hirzebruch surface.

It suffices to show that if S is a Hirzebruch surface, then $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Suppose that $S \simeq F_a := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$ be a Hirzebruch surface for some integer $a \geq 0$. Then, $H_S = C + bF$, where C is a section with $C^2 = -a$ and F is a general fiber of the projection $F_a \to \mathbb{P}^1$, such that $b > a$. Let $Y_0 := \varphi^{-1}(C)$ and $Y_1 := \varphi^{-1}(F)$ be $(n-1)$-dimensional rational scrolls. For each i, let m_i be the dimension of the smallest linear subspace of $\mathbb{P}^r$ containing Y_i, and let d_i be the degree of $Y_i \subset \mathbb{P}^{m_i}$. Then, by Lemma 1.5, $m_i \geq 2(n-1)-1 = 2n-3$. Thus, $d_i \geq m_i-(n-1)+1 \geq n-1$. It follows that

$$2g-2 = d_Y = d_0 + bd_1 \geq d_0 + d_1 \geq 2(n-1),$$

and hence, we get $g \geq n$. By Lemma 3.6, we obtain $g = n$. Thus, we must have $b = 1$, and hence, $a = 0$, i.e., $S = \mathbb{P}^1 \times \mathbb{P}^1$. In this case, we have $n = g = h^0(\mathcal{O}_S(H_S)) = 4$. (By the same argument, we can show that if $S \simeq \mathbb{P}^2$ and $g = 6$, then $n = 6$.) □

Now, we consider the following special case.

Lemma 5.6. *Let $X \subset \mathbb{P}^{2n-1}$ be a non-degenerate smooth projective variety of dimension $n \geq 3$ and degree $d \leq 2n$, and let H be a general hyperplane section. If $-K_X = (n-2)H$, then X is a prime Fano variety.*

Proof. By Ionescu's classification (see tables in Introductions of [Io1] and [Io3]), X is a complete intersection provided that $n \leq 4$. Thus, assume $n \geq 5$. We claim that the Picard number $\rho(X)$ of X is 1. By [W1, Theorem A], if $n \geq 7$, then $\rho(X) = 1$. Consider the case $n = 6$. By [W2, Theorem B], either $\rho(X) = 1$ or $X \simeq \mathbb{P}^3 \times \mathbb{P}^3$. However, $\mathbb{P}^3 \times \mathbb{P}^3$ cannot have an embedding in $\mathbb{P}^{11}$ with degree $d \leq 12$ so that $\rho(X) = 1$ for $n = 6$. Suppose that $n = 5$. A general hyperplane section Y of X is a Fano fourfold with $-K_Y = 2H_Y$ and $H_Y^4 = 10$. Clearly, a general member of $|H_Y|$ is irreducible and smooth. Thanks to Wiśniewski's classification ([W2]; see also [IP, Secion 12.7]), we see that there is no such Fano fourfold with Picard number $\rho(Y) \geq 2$. By Lefschetz Theorem for Picard Group, we obtain $\rho(X) = \rho(Y) = 1$. We have shown the claim.

Since the Picard group of a Fano variety is torsion-free ([IP, Proposition 2.1.2]), $\mathrm{Pic}(X)$ is generated by an ample divisor L. Let $H = mL$ for some integer $m \geq 1$. Then, we have $d = H^n = m^n L^n \leq 2n$. Since $n \geq 5$, we must have $m = 1$. Thus, X is prime Fano. □

We are ready to prove Theorem E.

Proof of Theorem E. Let $e = r - n$ be the codimension. We divide into three cases. Firstly, assume that $e \leq n-2$. By Barth-Larsen Theorem ([L1, Corollary 3.2.3]), X is a prime Fano variety. Secondly, assume that $e = n-1$ (then, $d \leq 2e+2$). By Ionescu's classification (see tables in Introductions of [Io1] and [Io3]), we can easily verify the assertion for $n \leq 4$. For $n \geq 5$, by [Io2, Theorem I], the assertion follows from Lemmas 5.4, 5.5, and 5.6. Finally, suppose that $e \geq n$ (then, $d \leq 2e+1$). We denote by g the sectional genus. By Lemma 3.6, $g \leq n$. If $g \leq 1$, then the assertion follows from Fujita's classification (see [Io1, Proposition 2.4] and [Io4, Theorems A and B]). If $g = 2$, then X has a hyperquadric fibration over $\mathbb{P}^1$ by [Io1, Corollary 3.3] for $n \geq 3$ and Castelnuovo's Theorem (see e.g., [Io1, Proposition 3.1]) for $n = 2$. If $g = 3$ (resp. $g = 4$), then the assertion follows from [Io1, Theorem 4.2] (resp. [BS, Theorem 11.6.3] together with Lemmas 5.4 and 5.5). It remains the case $5 \leq g \leq n$. If $K_X + (n-1)H$ is not base point free, then X is a prime Fano variety by [Io1, Theorem 1.4]. Now, suppose that we can define the adjunction mapping $\varphi\colon X \to B$. By [Io2, Theorem I], we only have to consider the cases $\dim B = 1$ or 2, and then, the remaining part is an immediate consequence of Lemmas 5.4 and 5.5. □

Remark 5.7. We can analyze in detail the cases (d) and (e) in Theorem E. If X is from (d), then it is a divisor of a rational scroll ([Io4, Lemma 6]). If X is from (e) and the base is $\mathbb{P}^2$, then $g = \Delta(X,H) = 3$ (e.g., Bordiga threefold in Example 5.9) or $n = g = 6$ (e.g., Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^4$). If X is from (e) and the base is $\mathbb{P}^1 \times \mathbb{P}^1$, then $n = g = \Delta(X,H) = 4$, and by [LN, Theorem 6.3], there is only one case (see Example 5.10).

5.3. **Examples.** We give examples of weak Fano varieties having fibrations coming from adjunction mappings.

Example 5.8. (1) Let $F_2 := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ be a Hirzebruch surface with the fibration $f\colon F_2 \to \mathbb{P}^1$. Then, F_2 is not a Fano surface but a weak Fano surface. For the embedding $X := \varphi_{|H|}(F_2) \subset \mathbb{P}^{11}$ given by a very ample divisor $H := 2C + 5F$, where F is a fiber of f and C is a section of f with $C^2 = -2$, we have $H^2 = 12$, i.e., $\deg(X) = 12$. Note that $K_X + H \sim F$ and $F.H = 2$, and hence, the adjunction mapping $\varphi_{|K_X+H|}$ gives a hyperquadric fibration over $\mathbb{P}^1$.

(2) Let $\pi\colon \widetilde{F}_2 \to F_2$ be the blow-up at a point not in C, and let E be the exceptional divisor. Then, $\widetilde{F}_2$ is also not a Fano surface but a weak Fano surface. Furthermore, the very ample divisor $H' := \pi^*(H) - E$ gives an embedding $X' := \varphi_{|H'|}(\widetilde{F}_2) \subset \mathbb{P}^{10}$ with $\deg(X') = 11$. Note that $K_{X'} + H' \sim \pi^*F$ and $\pi^*F.H' = 2$, and hence, the adjunction mapping $\varphi_{|K_{X'}+H'|}$ gives a hyperquadric fibration over $\mathbb{P}^1$ with a singular fiber.

Example 5.9. There exists a stable vector bundle E of rank 2 on $\mathbb{P}^2$ such that it is given by an extension

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to \mathcal{I}_Y(4) \to 0,$$

where Y is a closed subscheme of $\mathbb{P}^2$ consisting of 10 distinct points, $c_1(E) = 4, c_2(E) = 10$, and $E|_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(2)$ when L is a generic line (see [Io1, Proposition 7.5] and [Ot]). Then, for $X := \mathbb{P}_{\mathbb{P}^2}(E) \subset \mathbb{P}^5$, we have $\deg(X) = 6$, and the adjunction mapping induces a linear fibration over $\mathbb{P}^2$. This X is called the *Bordiga threefold*. Now, we show that X is a weak Fano variety but not a Fano variety. Recall that X is a weak Fano variety with $(-K_X)^3 = 6$. By the classification of Mori and Mukai ([MM]; see also [IP, Section 12.3]), there is no rational Fano threefold with $(-K)^3 = 6$, and hence, X is not a Fano variety.

More examples having fibrations over $\mathbb{P}^1$ or $\mathbb{P}^2$ can be found in [Io4].

Example 5.10. Let $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and $E := \mathcal{O}_Q(1,1)^{\oplus 3}$. Consider the embedding $X := \mathbb{P}_Q(E) \subset \mathbb{P}^{11}$ given by the complete linear system $|\mathcal{O}_{\mathbb{P}_Q(E)}(1)|$. Then, $\deg(X) = 12$. Moreover, X is a Fano variety, and the adjunction mapping induces a linear fibration over a quadric hypersurface in $\mathbb{P}^3$.

5.4. **Prime Fano varieties of small degree.** Finally, we further investigate the prime Fano case in Theorem E. Let $X \subset \mathbb{P}^r$ be a non-degenerate prime Fano variety of dimension n, codimension e, and degree $d \leq r+1$. If $e \leq \frac{n-1}{2}$, then Hartshorne conjectured that X must be a complete intersection ([Ht]). If $e \geq n$, then by [Io2, Theorem I], X is a del Pezzo variety. It only remains the case $e+1 \leq n \leq 2e$. We may write $-K_X = \ell H$ for some integer $\ell > 0$, where H is the ample generator of $\mathrm{Pic}(X)$. If $\ell \geq n-1$, then $(X, \mathcal{O}_X(H)) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, X is a hyperquadric in $\mathbb{P}^{n+1}$, or X is a del Pezzo variety (see e.g., [IP, Corollary 2.1.14]). Thus, we may assume that $\ell \leq n-2$.

Proposition 5.11. *Let $X \subset \mathbb{P}^r$ be a non-degenerate prime Fano variety of dimension n, codimension e, and degree d with $-K_X = \ell H$. If $e+1 \leq n \leq 2e$, $d \leq r+1$, and $\ell \leq n-2$, then $\ell = n-2$.*

Proof. By [Io4, Theorem A, Theorem B, and Proposition 10], we only have to consider the case $d = r+1$. By the adjunction formula, we have

$$2g - 2 = (n - \ell - 1)d, \tag{5.1}$$

where g is the sectional genus of X. Recall Castelnuovo's Bound:

$$g \leq \frac{m(m-1)}{2}e + m\epsilon, \tag{5.2}$$

where $m = \lfloor \frac{d-1}{e} \rfloor$ and $\epsilon = d - me - 1$. Suppose that $n = 2e$ ($d = 3e+1$). Then, $m = 3$ and $\epsilon = 0$, and hence, by (5.2), we get $g \leq 3e$. By (5.1), we have $(n - \ell - 1)(3e+1) \leq 6e - 2$, so we obtain $n - \ell - 1 \leq 1$, i.e., $\ell \geq n-2$. Now, suppose that $n \leq 2e-1$ ($d = n+e+1$). Then, $m = 2$ and $\epsilon = n - e$, and hence, by (5.2), we get $g \leq 2n - e$. By (5.1), we have $(n-\ell-1)(n+e+1) \leq 4n - 2e - 2$. If $n - \ell - 1 \geq 2$, then $2e + 2 \leq n$, which is a contradiction. Thus, $\ell \geq n-2$. ☐

For reader's convenience, we give the complete list of Mukai varieties (i.e., Fano varieties with $-K_X = (n-2)H$ and $\mathrm{Pic}(X) = \mathbb{Z}[H]$), which are completely classified by Mukai ([M]; see also [IP, Theorem 5.2.3]), of degree $d \leq r+1$.

Theorem 5.12 (Mukai)**.** *Let $X \subset \mathbb{P}^r$ be a non-degenerate Mukai variety of dimension n and degree d. If $d \leq r+1$, then X satisfies one of the following:*

(1) *a complete intersection of type $(2,3)$ and $n \geq 3$ or of type $(2,2,2)$ and $n \geq 4$;*
(2) *$n = 5, 6, d = 10$, and the intersection $C \cap Q \subset \mathbb{P}^{10}$, where $C \subset \mathbb{P}^{10}$ is the cone over the Plücker embedding of the Grassmannian $Gr(2,5) \subset \mathbb{P}^9$ and $Q \subset \mathbb{P}^{10}$ is a quadric hypersurface, or its hyperplane section;*
(3) *$6 \leq n \leq 10, d = 12$, and the spinor variety $X \subset \mathbb{P}^{15}$ or its linear sections;*
(4) *$n = 7, 8, d = 14$, and the Plücker embedding of the Grassmannian $Gr(2,6) \subset \mathbb{P}^{14}$ or its hyperplane section.*

6. Smooth projective varieties of degree at most $r+2$

In this section, we prove the following theorem, which immediately implies Theorem B.

Theorem 6.1. *Let $X \subset \mathbb{P}^r$ be an n-dimensional non-degenerate smooth projective variety of degree $d \le r+2$. Then, one of the following holds:*

(1) *X is rationally connected. (If $e \ge n+1 \ge 4$, then $-K_X$ is big.)*
(2) *X is a Roth variety or a rational scroll.*
(3) *$r = n+1$ or $r = 4, n = 2$ and X is a Calabi-Yau variety.*
(4) *$r = n+1$, $d = r+2$ and X is a hypersurface of general type.*
(5) *X is a curve of genus $g \le 2$ or an elliptic scroll.*

In particular, X is simply connected if and only if it is from (1), (2), (3), or (4) and it is not a plane quintic curve, and X is rationally connected if and only if it is from (1) or (2).

Proof. By Theorem C, we may assume that X is linearly normal and $d = r+2$. If X is a curve of genus g or a scroll over a curve of genus g, then by the same arguments in Section 1, we get $g \le 2$. Since the equality of (1.1) does not hold when $g = 2$ ([IT2]), the assertion follows. Let $e := r - n$ be the codimension of $X \subset \mathbb{P}^r$, and let H be a general hyperplane section. If $e = 1$, then there is nothing to prove. Now, assume that $e \ge 2$ and X is neither a scroll nor a Roth variety. By Theorem 3.4, $D_{inn} = -K_X + H$ is semiample. If $e \le n-2$, then by Barth-Larsen Theorem, X must be a Fano variety or a Calabi-Yau variety. If $n = e = 2$ ($d = 6$), then we can verify the assertion by the classification result (see [Io1, table in Introduction]). We note that there exists a K3 surface $S \subset \mathbb{P}^4$ of degree 6. Assume that $e \ge n-1$. Since $d = n+e+2 \le en+1$ unless $e = n = 2$, it follows that X is uniruled by Theorem G. Moreover, if $n \ge 3$ and $e \le n-2$, then X is uniruled, and hence, it cannot be Calabi-Yau. In particular, every variety from (3) is simply connected. Moreover, by Lefschetz Hyperplane Theorem ([L1, Theorem 3.1.21]) and Proposition 2.2, varieties from (2) or (4) are also simply connected. Moreover, by Proposition 2.4, Roth varieties $X \subset \mathbb{P}^r$ of degree $d \le r+2$ are rationally connected.

Now, suppose that X is not from (2), (3), (4), or (5). We prove that X is rationally connected, and hence, it is simply connected. If $e \le n-2$, then we already shown that X is Fano, and hence, it is rationally connected. If $e = n-1$, then X is simply connected by Barth-Larsen Theorem ([L1, Corollary 3.2.2]). In this case, $d = 2n+1$, and $\Delta(X, H) = n+1$. The case $n = 3$ can be directly checked by the classification result (see [Io1, table in Introduction]). If $n \ge 4$, then $d > \frac{3}{2}\Delta(X, H) + 1$. By [Fu, Corollary 4.5], we only have to show that if X has a hyperquadric fibration over a curve C or it has a linear fibration over a surface S with $h^0(\mathcal{O}_S(K_S)) = 0$, then X is rationally connected. In the first case, C is rational by [Io4, Lemma 6]. In the second case, S is also rational by [IT1, Proposition 4]. Thus, in both cases, X is rational.

We assume that $e \ge n$. First, we show that $h^1(\mathcal{O}_X) = 0$. We claim that $-K_X+H$ is big except when a general surface section of X is a K3 surface, which is simply connected. In the exceptional case, X is simply connected by Lefschetz Hyperplane Theorem. Let g be the sectional genus of $X \subset \mathbb{P}^r$. By Castelnuovo's Bound for sectional genus, we have $g \le n+2$ for $n = e$, and $g \le n+1$ for $n \le e-1$. In the first case, we have $d = 2e+2$. If $g = e+2$, then the curve section of X is a Castelnuovo curve, and hence, by [Hr, p.67] or [Il, Proposition 3.13], the surface section of X is also a Castelnuovo surface (see Definition 7.1). By Lemma 7.2, this surface is K3. Thus, we can assume that $g \leq n+1$ so that $2g-2 \leq 2n < n+e+2 = d$. On the other hand, let $D := nH$ and $E := K_X + (n-1)H$. By [Io1, Theorem 1.4], we may assume that E is nef. We have

$$nE.D^{n-1} = n^n(K_X + (n-1)H).H^{n-1} = n^n(2g-2) < n^n d = D^n.$$

By [L1, Theorem 2.2.15], $D - E = -K_X + H$ is big.

We proceed the induction on n to show that $h^1(\mathcal{O}_X) = 0$ provided that X is not a scroll over a curve. The case $n = 2$ can be done by case-by-case analysis as follows. We already showed the assertion when $n = e = 2$. When $e \geq 3$, we have $g \leq 3$. Then, the assertion follows from [Io4, Theorem A] for $g = 0$, [Io1, Proposition 2.6] for $g = 1$, [Io1, Proposition 3.1] for $g = 2$, and [Io1, Theorem 4.1] for $g = 3$. Suppose that $n \geq 3$. By Bertini Theorem, $H \subset \mathbb{P}^{r-1}$ is a non-degenerate smooth projective variety of degree $d = (r-1)+3$. Note that H should not be a scroll over a curve. We have the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0. \tag{6.1}$$

Note that $-K_X + H$ is nef and big and $H = K_X + (-K_X + H)$, and hence, by Kawamata-Viehweg Vanishing, $h^1(\mathcal{O}_X(H)) = 0$. From (6.1) with twisting $\mathcal{O}_X(H)$, $h^1(\mathcal{O}_X) = 0$ if and only if the natural map $H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_H(1))$ is surjective. Suppose that $H^0(\mathcal{O}_X(1)) \to H^0(\mathcal{O}_H(1))$ is not surjective. Since $X \subset \mathbb{P}^r$ is linearly normal, $H \subset \mathbb{P}^{r-1}$ must not be linearly normal. Then, $H \subset \mathbb{P}^{r-1}$ can be obtained by an isomorphic projection from $H \subset \mathbb{P}^r$ with degree of H is $r+2$. By induction hypothesis, H is regular. By Kodaira Vanishing, $h^1(\mathcal{O}_X(-H)) = h^2(\mathcal{O}_X(-H)) = 0$. Thus, from (6.1), we have $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_H) = 0$.

We still assume that X is not from (2), (3), (4), or (5), and $e \geq n$. Then, we have $d = n+e+2 \leq 2e+2$. By [Io2, Theorem I], we only have to consider the case that the adjunction mappings of X induce hyperquadric fibration over a curve C or linear fibrations over a ruled surface S. In the first case, by [Io4, Lemma 6], C is a rational curve. In the second case, $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_S) = 0$, and hence, S is a rational surface. Thus, in both cases, X is rational.

It remains to show that $-K_X$ is big when $e \geq n+1 \geq 4$. Recall that g is the sectional genus of X. We proved that $n+1 \geq g$, which implies that $d \geq 2g+1$. As in Lemma 3.8, it follows that $d + \frac{n}{n-1}\{(n-1)d - 2g + 2\} - nd > 0$, and hence, we can conclude that $-K_X$ is big by [L1, Theorem 2.2.15] except when $n = 3, e = 4$ and $g = 4$ $(d = 9)$. By the classification of degree nine varieties (see [FL, Proposition 3.1]), we can easily check that $-K_X$ is also big for the exceptional case. $\square$

By the same arguments in Section 5, we can prove the analogous statement to Theorem E. We leave it to the interested reader.

Theorem 6.2. *Let $X \subset \mathbb{P}^r$ be an n-dimensional non-degenerate smooth projective variety of degree $d = r+2$, and let H be a general hyperplane section. Assume that $n \geq 5$ and X is from the case (1) in Theorem 6.1. Then, one of the following holds:*

(1) *X is prime Fano, i.e., $-K_X = \ell H$ for some $\ell > 0$ and $\mathrm{Pic}(X) = \mathbb{Z}[H]$.*
(2) *$|K_X + (n-1)H|$ induces a hyperquadric fibration over $\mathbb{P}^1$.*
(3) *$|K_X + (n-1)H|$ induces a linear fibration over a smooth del Pezzo surface.*

In particular, if X is not a prime Fano, then it is a rational variety.

Finally, we give examples of varieties from (1) in Theorem 6.1. There are some exceptional cases for $n = 2, 4$ and 5, which can be completely classified.

Example 6.3. (1) Let $\pi\colon S \to \mathbb{P}^2$ be the blow-up at 9 points in general position, and let $E_1, \dots, E_9$ be the exceptional divisors. Then, the very ample divisor $H := \pi^*(4L) - E_1 - \cdots - E_9$ gives an embedding $S \subset \mathbb{P}^5$ with $\deg(S) = 7$. where L is a line in $\mathbb{P}^2$. Note that $-K_S$ is not big. Moreover, the Cox ring of S is not finitely generated because there are infinitely many (-1)-curves.
(2) Let $X \subset \mathbb{P}^5$ be a *Palatini threefold* (see e.g., [Ot]). Then, $\deg(X) = 7$ and X has a linear fibration over a cubic surface in $\mathbb{P}^3$.
(3) There is a threefold $X \subset \mathbb{P}^5$ of degree 7 such that $|K_X + H|$ induces a hypercubic fibration over $\mathbb{P}^1$ (see [Io1, table in Introduction]).

7. Smooth projective varieties of degree at most $ne + 2$

In this section, we prove Theorem G. For this purpose, we need to study Castelnuovo varieties.

Definition 7.1 (p.44 of [Hr])**.** Let $X \subset \mathbb{P}^r$ be a smooth projective variety of dimension n, codimension e, and degree d. Then, X is called a *Castelnuovo variety* if $d \geq ne + 2$ and

$$p_g(X) = h^0(\mathcal{O}_X(K_X)) = \binom{m}{n+1}e + \binom{m}{n}\epsilon,$$

where $m = \lfloor \frac{d-1}{e} \rfloor$ and $\epsilon = d - me - 1$.

See [Hr] for more detail on (possibly singular) Castelnuovo varieties.

Lemma 7.2. *Let $X \subset \mathbb{P}^r$ be a non-degenerate smooth Castelnuovo variety of dimension n, codimension e, and degree d. If $d = ne + 2$, then X is a Calabi-Yau variety.*

Proof. Note that every Castelnuovo variety is arithmetically Cohen-Macaulay (see [Hr, p.66]), and in particular, $h^i(\mathcal{O}_X) = 0$ for $0 < i < n$. According to Harris' classification (see [Hr, p.65]), one of the following holds: (1) X is supported on a rational normal scroll S, (2) X is a complete intersection of type $(2, n+1)$ (since $d = 2(n+1)$), or (3) X is a divisor of a cone over the Veronese surface in $\mathbb{P}^5$. Suppose that we are in Case (1). Recall that $X \in |-K_S|$ when $d = ne + 2$ (see [Hr, p.56]), and thus, $\mathcal{O}_X(K_X) = \mathcal{O}_X$ by the adjunction formula, i.e., X is Calabi-Yau. If we are in Case (2), then by the adjunction formula, X is also a Calabi-Yau variety. Since $d = 4n + 2$ is not divisible by 4 (see [Hr, p.64]), Case (3) cannot occur. □

Proof of Theorem G. In [Z, Proof of Corollary 1.6], using the Castelnuovo's bound on the sectional genus and the concavity theorem (see e.g., [Z, Theorem 1.1] or [L1, Example 1.6.4]), Zak showed that $K_X.H^{n-1} < 0$ (resp. $K_X.H^{n-1} \leq 0$) provided that $d \leq ne + 1$ (resp. $d \leq ne + 2$). It only remains the case $H^n = d = ne + 2$ and $K_X.H^{n-1} = 0$. Let g be the sectional genus of $X \subset \mathbb{P}^r$. Then, we have

$$2g - 2 = (K_X + (n-1)H).H^{n-1} = (n-1)H^n = (n-1)(ne+2) = 2\binom{m}{2}e + 2\binom{m}{1}\epsilon - 2,$$

where $m = n$ and $\epsilon = 1$, and hence, the generic curve section C is a Castelnuovo curve. By [Hr, p.67] or [Il, Proposition 3.13], X is a Castelnuovo variety, and hence, the assertion follows from Lemma 7.2. □

Remark 7.3. There is a non-degenerate smooth projective surface $S \subset \mathbb{P}^r$ of codimension e and degree $d = 2e + 3$ such that S is neither Castelnuovo nor uniruled (see [Bui, Proposition 2.1 and Corollary 3.2]).

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