

THE COMPLETE DETERMINATION OF WIDE RICAUD-DEGERT TYPE WHICH IS NOT 5 MODULO 8 WITH CLASS NUMBER ONE

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let d be a square free integer and $h(d)$ be the class number of $\mathbb{Q}(\sqrt{d})$. Let $d = n^2 + r$ be a square free integer such that $r|4n$ and $-n < r \leq n$. In this case, we call d a Richaud-Degert type. If $|r| \neq 1, 4$, then it is called a wide-Richaud-Degert type. And if $|r| = 1$ or 4 , then it is called a narrow-Richaud-Degert type.

There had been many conjectures about the upper bound of Richaud-Degert type d with $h(d) = 1$. Yokoi [9] conjectured that $h(n^2 + 4) > 1$ if $n > 17$. Chowla [3] conjectured that $h(4n^2 + 1) > 1$ if $n > 13$. Biró [1] [2] proved above two conjectures. Also Mollin[5] conjectured that $h(n^2 - 4) = 1$ if $n > 21$. Mollin's conjecture was solved by Byeon, Kim and the author [13]. This determines all real quadratic fields of narrow-Richaud-Degert types with class number 1.

In this paper, we prove special case of Mollin-William's conjecture. From this, we show that there are exactly 14 wide R-D types $d (\not\equiv 5 \pmod{8})$ with $h(d) = 1$.

Mollin and William's Conjecture *Let $d = n^2 \pm 2$ be a squarefree integer. Then $h(d) > 1$ if $n > 20$.*

Theorem 1.1. *Let $d (\not\equiv 5 \pmod{8})$ be wide-Richaud-Degert type. Then $h(d) = 1$ if and only if*

$$d = 3, 6, 7, 11, 14, 23, 33, 38, 47, 62, 83, 167, 227, 398.$$

2. COMPUTATION OF THE SPECIAL VALUES OF ZETA FUNCTIONS ASSOCIATED WITH $\mathbb{Q}(\sqrt{n^2 - 2})$

Let $d = n^2 - 2$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{d})$ and $O(K)$ a ring of integers of K . Then $\epsilon = n^2 - 1 + n\sqrt{n^2 - 2}$ is a fundamental unit of K , and $\{1, \omega\}$ is an integral basis for $O(K)$, where $\omega = \sqrt{n^2 - 2}$. For an integral ideal \mathbf{a} , let $N(\mathbf{a})$ be the number of the cosets of $O(K)/\mathbf{a}$, and for an element α of K , let $N_K(\alpha) = \alpha \cdot \bar{\alpha}$. We note that the norm of ϵ , $N_K(\epsilon) = 1$. Let $I(K)$ be the set of nonzero fractional ideals of K .

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And let K^+ be the set of totally positive elements in K and $i(K^+)$ be the set of principal fractional ideals generated by the elements in K^+ . Let χ be an odd primitive character with conductor q . Then by the fact of $N_K(\epsilon) = 1$ and $N_K(\omega) < 0$, we have the following proposition.

Proposition 2.1. *If $h(d) = 1$, then*

$$I(K) = (q) \cdot i(K^+) \cup (q\omega) \cdot i(K^+).$$

Proof: See the page 242-243 in [4] □

Thus if $h(d) = 1$, then we have

$$\begin{aligned} (1) \quad \zeta_K(s, \chi) &:= \sum_{\substack{\mathbf{a} \in I(K) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} \\ &= \sum_{\substack{\mathbf{a} \in (q) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} + \sum_{\substack{\mathbf{a} \in (q\omega) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}. \end{aligned}$$

By defining

$R(\mathbf{b}) := \{a+b\epsilon \mid a, b \in \mathbb{Q} \text{ with } 0 < a \leq 1, 0 \leq b < 1 \text{ and } \mathbf{b} \cdot (a+b\epsilon) \subset O(K)\}$ for an integral ideal \mathbf{b} in K , we have the following proposition.

Proposition 2.2. *An integral ideal \mathbf{a} of K is in $\mathbf{b} \cdot i(K^+) := \{\mathbf{b} \cdot \mathbf{c} \mid \mathbf{c} \in i(K^+)\}$ if and only if*

$$\mathbf{a} = \mathbf{b} \cdot (a + b\epsilon + n_1 + n_2\epsilon)$$

for $a + b\epsilon \in R(\mathbf{b})$ and nonnegative integers n_1, n_2 .

Proof: See Lemma 2.2 in [13] □

In the following lemma, we find the complete set of (x, y) for which $x + y\epsilon \in R((q))$.

Lemma 2.3.

$$\begin{aligned} &\{(x, y) \mid x + y\epsilon \in R((q))\} \\ &= \{(x, y) \mid x = -\frac{r_{C,D}(n)}{q} + \frac{D + qj}{nq} + \sigma_1(j) \text{ and } y = \frac{D + qj}{nq} \\ &\quad \text{for } j = 0, 1, 2, \dots, (n-1) \text{ and } 0 \leq C, D \leq q-1\}, \end{aligned}$$

where

$$\sigma_1(j) = \begin{cases} 1 & \text{if } 0 \leq j \leq \left[\frac{n r_{C,D}(n) - D}{q} \right], \\ 0 & \text{if } \left[\frac{n r_{C,D}(n) - D}{q} \right] + 1 \leq j \leq n-1 \end{cases}$$

and

$$r_{C,D}(n) = nD - C - q \left[\frac{nD - C}{q} \right].$$

Proof: Since $\{C+D\omega \text{ for } 0 \leq C, D \leq q-1\}$ represents every elements in $O(K)/qO(K)$, we have

$$\begin{aligned} & \{(x, y) | x + y\epsilon \in R((q))\} \\ &= \{(x, y) | x + y\epsilon \in q^{-1}O(K) \text{ and } 0 < x \leq 1, 0 \leq y < 1\} \\ &= \{(x, y) | q(x + y\epsilon) = C + D\omega + q(i + j\omega) \text{ for } 0 \leq C, D \leq q-1 \\ &\quad \text{and } 0 < x \leq 1, 0 \leq y < 1\}. \end{aligned}$$

And the equation $\omega = \frac{\epsilon}{n} - \frac{n^2-1}{n}$ implies that

$$q(x + y\epsilon) = C + qi + (D + qj)\left(\frac{\epsilon}{n} - \frac{n^2-1}{n}\right).$$

Hence

$$y = \frac{D + qj}{2nq} \text{ for } j = 0, 1, 2, \dots, (n-1)$$

and

$$\begin{aligned} x &= \frac{C}{q} - \frac{(D + qj)(n^2 - 1)}{nq} + \left[1 + \frac{(D + qj)(n^2 - 1)}{nq} - \frac{C}{q} \right] \\ &\quad \text{for } j = 0, 1, 2, \dots, (n-1). \end{aligned}$$

And the equation

$$\frac{(D + qj)(n^2 - 1)}{nq} - \frac{C}{q} = \left[\frac{nD - C}{q} \right] + \frac{r_{C,D}(n)}{q} - \frac{D + qj}{qn} + nj$$

implies

$$\begin{aligned} & \frac{C}{q} - \frac{(D + qj)(n^2 - 1)}{nq} + \left[1 + \frac{(D + qj)(n^2 - 1)}{nq} - \frac{C}{q} \right] \\ &= 1 - \frac{r_{C,D}(n)}{q} + \frac{D + qj}{qn} + \left[\frac{r_{C,D}(n)}{q} - \frac{D + qj}{qn} \right], \end{aligned}$$

where

$$r_{C,D}(n) = nD - C - q \left[\frac{nD - C}{q} \right].$$

By the fact of

$$\left[\frac{r_{C,D}(n)}{q} - \frac{D + qj}{qn} \right] + 1 = \begin{cases} 1 & \text{if } 0 \leq j \leq \left[\frac{nr_{C,D}(n)-D}{q} \right], \\ 0 & \text{if } 1 + \left[\frac{nr_{C,D}(n)-D}{q} \right] \leq j \leq n-1, \end{cases}$$

we complete the proof. \square

We also find the set of (x, y) with $x + y\epsilon \in R((q\omega))$, in the following lemma.

Lemma 2.4.

$$\begin{aligned} & \{(x, y) | x + y\epsilon \in R((q\omega))\} \\ = & \{(x, y) | x = 1 + \frac{D}{q} - \frac{(n^2 - 1)(C + iq)}{qn(n^2 - 2)} + \sigma_2(i) \text{ and } y = \frac{C + iq}{qn(n^2 - 2)} \\ & \text{for } i = 0, 1, 2, \dots, n(n^2 - 2) - 1 \text{ and } 0 \leq C, D \leq q - 1\}, \end{aligned}$$

where

$$\sigma_2(i) = k \Leftrightarrow l(k) \leq i < l(k+1)$$

for $k = -1, 0, 1, \dots, (n^2 - 2)$ and

$$l(k) = \left\lceil \frac{n(n^2 - 2)(qk + D)}{q(n^2 - 1)} - \frac{C}{q} \right\rceil.$$

Proof: The set $\{C + D\omega \text{ for } 0 \leq C, D \leq q - 1\}$ represents every elements in $O(K)/qO(K)$. So

$$\begin{aligned} & \{(x, y) | x + y\epsilon \in R((q\omega))\} \\ = & \{(x, y) | x + y\epsilon \in (q\omega)^{-1}O(K) \text{ and } 0 < x \leq 1, 0 \leq y < 1\} \\ = & \{(x, y) | q\omega(x + y\epsilon) = C + D\omega + q(i + j\omega) \text{ for } 0 \leq C, D \leq q - 1 \\ & \text{and } 0 < x \leq 1, 0 \leq y < 1\}. \end{aligned}$$

From $\bar{\omega} = -\frac{\epsilon}{n} + \frac{n^2 - 1}{n}$, we deduce that

$$\begin{aligned} & q\omega(x + y\epsilon) = (C + qi) + \omega(D + qj) \\ \iff & -q(n^2 - 2)(x + y\epsilon) = -(n^2 - 2)(D + qj) + \frac{n^2 - 1}{n}(C + iq) - \frac{C + iq}{n}\epsilon. \end{aligned}$$

So

$$y = \frac{C + iq}{qn(n^2 - 1)} \text{ for } i = 0, 1, 2, \dots, n(n^2 - 2) - 1$$

and

$$\begin{aligned} x = 1 + \frac{D}{q} - \frac{(n^2 - 1)(C + iq)}{qn(n^2 - 2)} + \left[\frac{(n^2 - 1)(C + iq)}{qn(n^2 - 2)} - \frac{D}{q} \right] \\ \text{for } i = 0, 1, 2, \dots, n(n^2 - 2) - 1 \end{aligned}$$

By defining

$$\sigma_2(i) := \left[\frac{(n^2 - 1)(C + iq)}{qn(n^2 - 2)} - \frac{D}{q} \right],$$

we have that $\sigma_2(i) = k$ for $k = -1, 0, 1, \dots, (n^2 - 2)$ if and only if

$$\left\lceil \frac{n(n^2 - 2)(qk + D)}{q(n^2 - 1)} - \frac{C}{q} \right\rceil \leq i < \left\lceil \frac{n(n^2 - 2)(q(k+1) + D)}{q(n^2 - 1)} - \frac{C}{q} \right\rceil.$$

□

By Lemma 2.3 and 2.4, we deduce the proposition.

Proposition 2.5. *If $h(n^2 - 2) = 1$, then*

$$\begin{aligned} \zeta_K(\chi, 0) &= \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (n^2 - 2)D^2) \left[\sum_{j=0}^{n-1} S\left(-\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j), \frac{D+qj}{nq}\right) \right. \\ &\quad \left. - \sum_{i=0}^{(n^2-2)n-1} S\left(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\right) \right] \end{aligned}$$

where $S(x, y) = B_1(x)B_1(y) + \frac{1}{4}(\epsilon + \bar{\epsilon})(B_2(x) + B_2(y))$, for the first and second Bernoulli polynomials B_1, B_2 .

Proof: By proposition 2.2, we have

$$\begin{aligned} (2) \quad & \sum_{\substack{\mathbf{a} \in (q) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} = \sum_{x+y\epsilon \in R((q))} \sum_{n_1, n_2=0}^{\infty} \frac{\chi(N((q) \cdot (x+y\epsilon + n_1 + n_2\epsilon)))}{N((q) \cdot (x+y\epsilon + n_1 + n_2\epsilon))^s} \\ &= \sum_{x+y\epsilon \in R((q))} \chi(N_K(q(x+y\epsilon))) \sum_{n_1, n_2=0}^{\infty} N_K(q(x+y\epsilon + n_1 + n_2\epsilon))^{-s} \end{aligned}$$

and

$$\begin{aligned} (3) \quad & \sum_{\substack{\mathbf{a} \in (q\omega) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} = \sum_{x+y\epsilon \in R((q\omega))} \sum_{n_1, n_2=0}^{\infty} \frac{\chi(N((q\omega) \cdot (x+y\epsilon + n_1 + n_2\epsilon)))}{N((q\omega) \cdot (x+y\epsilon + n_1 + n_2\epsilon))^s} \\ &= \sum_{x+y\epsilon \in R((q\omega))} \chi(-N_K(q\omega(x+y\epsilon))) \sum_{n_1, n_2=0}^{\infty} \left(-N_K(q\omega(x+y\epsilon + n_1 + n_2\epsilon)) \right)^{-s} \end{aligned}$$

And we recall Shintani's result in [7] [8]:

$$(4) \quad \sum_{n_1, n_2=0}^{\infty} N_K(x+y\epsilon + n_1 + n_2\epsilon)^{-s}|_{s=0} = S(x, y).$$

We note that for $x+y\epsilon \in R((q))$,

$$\begin{aligned} (5) \quad & N_K(q(x+y\epsilon)) \\ &= N_K(C + D\omega + q(i + j\omega)) \equiv N_K(C + D\omega) \pmod{q} = C^2 - (n^2 - 2)D^2, \end{aligned}$$

and for $x + y\epsilon$ is in $R((q\omega))$,

$$(6) \quad \begin{aligned} N_K(q\omega(x + y\epsilon)) \\ = N_K(C + D\omega + q(i + j\omega)) \equiv N_K(C + D\omega) \pmod{q} = C^2 - (n^2 - 2)D^2. \end{aligned}$$

From Lemma 2.3, 2.4 and the equations (1)-(6), we deduce the Proposition, immediately. \square

We observe that $-1 \leq \sigma_2(i) \leq (n^2 - 2)$ for $0 \leq i \leq n(n^2 - 2) - 1$ and $\sigma_2(i) = k \Leftrightarrow l(k) \leq i < l(k+1)$. So we have

$$(7) \quad \begin{aligned} \sum_{i=0}^{n(n^2-2)-1} \sigma_2(i) &= \sum_{i=0}^{l(0)-1} (-1) + \sum_{k=0}^{n^2-3} k \sum_{i=l(k)}^{l(k+1)-1} 1 + (n^2 - 2) \sum_{i=l(n^2-2)}^{n(n^2-2)-1} 1 \\ \sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)^2 &= \sum_{i=0}^{l(0)-1} 1 + \sum_{k=0}^{n^2-3} k^2 \sum_{i=l(k)}^{l(k+1)-1} 1 + (n^2 - 2)^2 \sum_{i=l(n^2-2)}^{n(n^2-2)-1} 1 \\ \sum_{i=0}^{n(n^2-2)-1} \sigma_2(i)i &= \sum_{i=0}^{l(0)-1} (-i) + \sum_{k=0}^{n^2-3} k \sum_{i=l(k)}^{l(k+1)-1} i + (n^2 - 2) \sum_{i=l(n^2-2)}^{n(n^2-2)-1} i \end{aligned}$$

where $l(k)$ is explained within the following lemma.

Lemma 2.6. Let $r_{C,D}(n) = nD - C - q\left[\frac{nD-C}{q}\right]$, $s_{C,D}(n) = r_{C,D}(n)n - D - q\left[\frac{r_{C,D}(n)n-D}{q}\right]$, $u_{C,D}(n) = \left[\frac{nD-C}{q}\right]$, $v_{C,D}(n) = \left[\frac{r_{C,D}(n)n-D}{q}\right]$ and $w_{C,D}(n) = \left\lceil \frac{s_{C,D}(n)n-r_{C,D}(n)}{q} \right\rceil$. Assume $n > q$, then

$$l(k) = \begin{cases} u_{C,D}(n) + nk - k_1 + 1, & \text{if } k_1 < w_{C,D}(n) \text{ and } k_2 \leq v_{C,D}(n) \\ u_{C,D}(n) + nk - k_1, & \text{if } k_1 < w_{C,D}(n) \text{ and } k_2 > v_{C,D}(n) \\ u_{C,D}(n) + nk - k_1 + 1, & \text{if } k_1 \geq w_{C,D}(n) \text{ and } k_2 < v_{C,D}(n) \\ u_{C,D}(n) + nk - k_1 + 1, & \text{if } k_1 < w_{C,D}(n) \text{ and } k_2 \geq v_{C,D}(n) \end{cases},$$

$$\text{where } l(k) = \left\lceil \frac{n(n^2-2)(qk+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil.$$

Proof: Note that

$$(8) \quad l(k) = \left\lceil \frac{n(n^2-2)(qk+D)}{q(n^2-1)} - \frac{C}{q} \right\rceil = nk + \left\lceil \frac{nD-C}{q} - \frac{k}{n-\frac{1}{n}} - \frac{D}{q(n-\frac{1}{n})} \right\rceil.$$

And if we express k by $k_1n + k_2$, for $k_2 = 0, 1, \dots, n - 1$, then

$$(9) \quad \left\lceil \frac{nD - C}{q} - \frac{k}{n - \frac{1}{n}} - \frac{D}{q(n - \frac{1}{n})} \right\rceil = \left\lceil \frac{nD - C}{q} \right\rceil + \left\lceil \frac{r_{C,D}(n)}{q} - \frac{\frac{D}{q} + \frac{k_1}{n} + k_2}{n - \frac{1}{n}} \right\rceil - k_1.$$

And we observe that

$$(10) \quad \begin{aligned} & \left\lceil \frac{r_{C,D}(n)}{q} - \frac{\frac{D}{q} + \frac{k_1}{n} + k_2}{n - \frac{1}{n}} \right\rceil \\ &= \left\lceil \frac{\frac{r_{C,D}(n)n - D}{q} - k_2}{n - \frac{1}{n}} + \frac{\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} - k_1}{n(n - \frac{1}{n})} + \frac{s_{C,D}(n)n - r_{C,D}(n) - q\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}}{qn(n - \frac{1}{n})} \right\rceil. \end{aligned}$$

Firstly, we consider the case of

$$(11) \quad k_1 < \left\lceil \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rceil \text{ and } k_2 \leq \left[\frac{r_{C,D}(n)n - D}{q} \right].$$

For $n > q$ we have $\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} < n - 1$ and $\frac{r_{C,D}(n)n - D}{q} < n - 1$, since $0 \leq r_{C,D}(n), s_{C,D}(n)$ and $D < q - 1$. So

$$(12) \quad \left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right] \leq n - 2 \text{ and } \left[\frac{r_{C,D}(n)n - D}{q} \right] \leq n - 2.$$

If $s_{C,D}(n)n - r_{C,D}(n) - q\left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}\right] = 0$, then $\left\lceil \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rceil = \left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right]$. Thus from (11), (12), we have

$$1 \leq \left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right] - k_1 \leq n - 2$$

and

$$0 \leq \left[\frac{r_{C,D}(n)n - D}{q} \right] - k_2 \leq n - 2.$$

Hence

$$\frac{1}{n(n - \frac{1}{n})} \leq \frac{\left[\frac{r_{C,D}(n)n - D}{q} \right] - k_2}{n - \frac{1}{n}} + \frac{\left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right] - k_1}{n(n - \frac{1}{n})} \leq \frac{n^2 - n - 2}{n^2 - 1} < 1.$$

So

$$(13) \quad \left\lceil \frac{\left[\frac{r_{C,D}(n)n - D}{q} \right] - k_2}{n - \frac{1}{n}} + \frac{\left[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right] - k_1}{n(n - \frac{1}{n})} \right\rceil = 1.$$

And if $s_{C,D}(n)n - r_{C,D}(n) - q[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] \neq 0$, then $\lceil \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \rceil = [\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] + 1$. From (11), (12), we have

$$0 \leq [\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] - k_1 \leq n - 2$$

and

$$0 \leq [\frac{r_{C,D}(n)n - D}{q}] - k_2 \leq n - 2$$

and

$$1 \leq s_{C,D}(n)n - r_{C,D}(n) - q[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] \leq q - 1.$$

Thus

$$\begin{aligned} 0 &< \frac{1}{qn(n - \frac{1}{n})} \\ &\leq \frac{[\frac{r_{C,D}(n)n - D}{q}] - k_2}{n - \frac{1}{n}} + \frac{[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] - k_1}{n(n - \frac{1}{n})} + \frac{s_{C,D}(n)n - r_{C,D}(n) - q[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}]}{qn(n - \frac{1}{n})} \\ &\leq \frac{n^2 - n - 2}{n^2 - 1} + \frac{q - 1}{qn(n - \frac{1}{n})} < 1 \end{aligned}$$

So

$$(14) \quad \left\lceil \frac{[\frac{r_{C,D}(n)n - D}{q}] - k_2}{n - \frac{1}{n}} + \frac{[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}] - k_1}{n(n - \frac{1}{n})} + \frac{s_{C,D}(n)n - r_{C,D}(n) - q[\frac{s_{C,D}(n)n - r_{C,D}(n)}{q}]}{qn(n - \frac{1}{n})} \right\rceil = 1$$

From (8)-(10) and (13)-(14), we have

$$l(k) = [\frac{nD - C}{q}] + nk - k_1 + 1$$

where

$$k_1 < \left\lceil \frac{s_{C,D}(n)n - r_{C,D}(n)}{q} \right\rceil \text{ and } k_2 \leq \left[\frac{r_{C,D}(n)n - D}{q} \right].$$

In this way, we can prove the other cases. \square

Above expression of $l(k)$ help the computation of

$$\sum_{i=0}^{(n^2-2)n-1} S(1 - \frac{(C + iq)(n^2 - 1)}{qn(n^2 - 2)} + \frac{D}{q} + \sigma_2(i), \frac{C + iq}{qn(n^2 - 2)}).$$

Proposition 2.7. *If $n > q$ then*

$$\begin{aligned}
 (i) \quad & \sum_{j=0}^{n-1} S\left(-\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j), \frac{D+qj}{nq}\right) \\
 = & -\frac{1}{2} + \frac{n}{4} + \frac{D^2n}{2q^2} + \frac{r_{C,D}(n)}{q} - \frac{nr_{C,D}(n)^2}{2q^2} - \frac{v_{C,D}(n)}{2} + \frac{r_{C,D}(n)v_{C,D}(n)}{q} \\
 & - \frac{nD}{2q} + \frac{ns_{C,D}(n)^2}{2q^2} - \frac{ns_{C,D}(n)}{2q} \\
 (ii) \quad & \sum_{i=0}^{(n^2-2)n-1} S\left(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\right) \\
 = & \frac{1}{2} - \frac{n}{4} - \frac{nr_{C,D}(n)C}{q^2} + \frac{nr_{C,D}(n)}{2q} - \frac{3nr_{C,D}(n)^2}{2q^2} + \frac{u_{C,D}(n)}{2} + \frac{Cv_{C,D}(n)}{q} \\
 & + \frac{r_{C,D}(n)v_{C,D}(n)}{q} + \frac{w_{C,D}(n)}{2} - \frac{D}{q} + \frac{3nD^2}{2q^2} - \frac{2u_{C,D}(n)D}{q} + \frac{ns_{C,D}(n)^2}{2q^2} \\
 & - \frac{u_{C,D}(n)s_{C,D}(n)}{q} - \frac{w_{C,D}(n)s_{C,D}(n)}{q} + \frac{ns_{C,D}(n)D}{q^2}
 \end{aligned}$$

where $r_{C,D}(n)$, $s_{C,D}(n)$, $u_{A,B}(n)$, $v_{A,B}(n)$, $w_{C,D}(n)$, are in Lemma 2.6.

Proof:

(i) By computer work, we obtain the following equation

$$\begin{aligned}
 & \sum_{j=0}^{n-1} S\left(-\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j), \frac{D+qj}{nq}\right) \\
 = & \frac{nD^2}{2q^2} - \frac{n^2r_{C,D}(n)D}{q^2} + \frac{n^3r_{C,D}(n)^2}{2q^2} + \frac{nv_{C,D}(n)D}{q} - \frac{n^2r_{C,D}(n)v_{C,D}(n)}{q} + \frac{nv_{C,D}(n)^2}{2} \\
 & - \frac{n^2r_{C,D}(n)}{2q} + \frac{nv_{C,D}(n)}{2} - \frac{1}{2} + \frac{n}{4} + \frac{D^2n}{2q^2} + \frac{r_{C,D}(n)}{q} - \frac{nr_{C,D}(n)^2}{2q^2} - \frac{v_{C,D}(n)}{2} \\
 & + \frac{r_{C,D}(n)v_{C,D}(n)}{q}
 \end{aligned}$$

From the equation

$$s_{C,D}(n) = nr_{C,D}(n) - D - q\left[\frac{nr_{C,D}(n) - D}{q}\right] = nr_{C,D}(n) - D - qv_{C,D}(n)$$

we have

$$\begin{aligned} & \frac{nD^2}{2q^2} - \frac{n^2r_{C,D}(n)D}{q^2} + \frac{n^3r_{C,D}(n)^2}{2q^2} + \frac{nv_{C,D}(n)D}{q} - \frac{n^2r_{C,D}(n)v_{C,D}(n)}{q} + \frac{nv_{C,D}(n)^2}{2} \\ &= \frac{n(nr_{C,D}(n) - D - qv_{C,D}(n))^2}{2q^2} = \frac{ns_{C,D}(n)^2}{2q^2} \end{aligned}$$

and

$$-\frac{nr_{C,D}(n)^2}{2q} + \frac{nv_{C,D}(n)}{2} = \frac{-n(nr_{C,D}(n) - D - qv_{C,D}(n))}{2q} - \frac{nD}{2q} = -\frac{ns_{C,D}(n)}{2q} - \frac{nD}{2q}.$$

By above equations, we can obtain (i), directly.

(ii) Using the equation (7) and Lemma 2.6, we have

$$\begin{aligned} & \sum_{i=0}^{(n^2-2)n-1} S\left(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\right) \\ &= \left(\frac{C^2}{nq^2} + \frac{2Cr_{C,D}(n)}{nq^2} + \frac{r_{C,D}(n)^2}{nq^2} + \frac{2Cu_{C,D}(n)}{nq} + \frac{2r_{C,D}(n)u_{C,D}(n)}{nq}\right) \\ &\quad + \left(-\frac{C}{nq} - \frac{r_{C,D}(n)}{nq} - \frac{u_{C,D}(n)}{n}\right) + \left(\frac{Cw_{C,D}(n)}{nq} + \frac{r_{C,D}(n)w_{C,D}(n)}{nq} + \frac{u_{C,D}(n)w_{C,D}(n)}{n}\right) \\ &\quad + \left(-\frac{Cu_{C,D}(n)}{nq} - \frac{r_{C,D}(n)u_{C,D}(n)}{nq}\right) + \left(-\frac{nr_{C,D}(n)w_{C,D}(n)}{q} + \frac{Dw_{C,D}(n)}{q}\right. \\ &\quad \left.+ v_{C,D}(n)w_{C,D}(n)\right) + \left(\frac{n^3r_{C,D}(n)^2}{2q^2} - \frac{n^2r_{C,D}(n)v_{C,D}(n)}{q} + \frac{nv_{C,D}(n)^2}{2} + \frac{nD^2}{2q^2}\right) \\ &\quad + \left(-\frac{nr_{C,D}(n)u_{C,D}(n)}{q} + \frac{Du_{C,D}(n)}{q} + u_{C,D}(n)v_{C,D}(n)\right) + \frac{u_{C,D}(n)}{2} + \frac{cv_{C,D}(n)}{q} \\ &\quad + \frac{r_{C,D}(n)v_{C,D}(n)}{q} + \frac{w_{C,D}(n)}{2} - \frac{2u_{C,D}(n)D}{q} + \frac{1}{2} - \frac{n}{4}. \end{aligned}$$

We note that $r_{C,D}(n) = nD - C - qu_{C,D}(n)$ implies that

$$\begin{aligned} (15) \quad & \frac{C^2}{nq^2} + \frac{2Cr_{C,D}(n)}{nq^2} + \frac{r_{C,D}(n)^2}{nq^2} + \frac{2Cu_{C,D}(n)}{nq} + \frac{2r_{C,D}(n)u_{C,D}(n)}{nq} \\ &= \frac{(C+r_{C,D}(n)+qu_{C,D}(n))^2}{nq^2} - \frac{u_{C,D}(n)^2}{n} = \frac{nD^2}{q^2} - \frac{u_{C,D}(n)^2}{n} \end{aligned}$$

and

$$(16) \quad -\frac{C}{nq} - \frac{r_{C,D}(n)}{nq} - \frac{u_{C,D}(n)}{n} = -\frac{C+r_{C,D}(n)+qu_{C,D}(n)}{nq} = -\frac{D}{q}$$

and

$$(17) \quad \begin{aligned} & \frac{Cw_{C,D}(n)}{nq} + \frac{r_{C,D}(n)w_{C,D}(n)}{nq} + \frac{u_{C,D}(n)w_{C,D}(n)}{n} \\ &= \frac{(C + r_{C,D}(n) + qu_{C,D}(n))w_{C,D}(n)}{nq} = \frac{Dw_{C,D}(n)}{q} \end{aligned}$$

and

$$(18) \quad \begin{aligned} & -\frac{Cu_{C,D}(n)}{nq} - \frac{r_{C,D}(n)u_{C,D}(n)}{nq} \\ &= -\frac{(C + r_{C,D}(n) + u_{C,D}(n)q)u_{C,D}(n)}{nq} + \frac{u_{C,D}(n)^2}{n} = -\frac{Du_{C,D}(n)}{q} + \frac{u_{C,D}(n)^2}{n} \end{aligned}$$

Also the equation $s_{C,D}(n) = r_{C,D}(n)n - D - qv_{C,D}(n)$ implies that

$$(19) \quad \begin{aligned} & \left(-\frac{nr_{C,D}(n)w_{C,D}(n)}{q} + \frac{Dw_{C,D}(n)}{q} + v_{C,D}(n)w_{C,D}(n) \right) \\ &= -\frac{w_{C,D}(n)(r_{C,D}(n)n - D - qv_{C,D}(n))}{q} = -\frac{w_{C,D}(n)s_{C,D}(n)}{q} \end{aligned}$$

and

$$(20) \quad \begin{aligned} & \left(\frac{n^3r_{C,D}(n)^2}{2q^2} - \frac{n^2r_{C,D}(n)v_{C,D}(n)}{q} + \frac{nv_{C,D}(n)^2}{2} + \frac{nD^2}{2q^2} \right) \\ &= \frac{n(nr_{C,D}(n) - D - qv_{C,D}(n))^2}{2q^2} - \frac{nDv_{C,D}(n)}{q} + \frac{n^2r_{C,D}(n)D}{q^2} \\ &= \frac{ns_{C,D}(n)^2}{2q^2} + \frac{nD(r_{C,D}(n)n - D - qv_{C,D}(n))}{q^2} + \frac{nD^2}{q^2} \\ &= \frac{ns_{C,D}(n)^2}{2q^2} + \frac{nDs_{C,D}(n)}{q^2} + \frac{nD^2}{q^2} \end{aligned}$$

and

$$(21) \quad \begin{aligned} & \left(-\frac{nr_{C,D}(n)u_{C,D}(n)}{q} + \frac{Du_{C,D}(n)}{q} + u_{C,D}(n)v_{C,D}(n) \right) \\ &= -\frac{u_{C,D}(n)(r_{C,D}(n)n - D - qv_{C,D}(n))}{q} = -\frac{w_{C,D}(n)s_{C,D}(n)}{q} \end{aligned}$$

From (15)-(21), we can deduce (ii). \square

Finally, we have the following Theorem.

Theorem 2.8. If $h(n^2 - 2) = 1$ and $n > q$, then

$$\begin{aligned} & \zeta_K(0, \chi) \\ &= \frac{1}{2q^2} \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (n^2 - 2)D^2) \cdot \left[n \left(q^2 - qs_{C,D}(n) + 2r_{C,D}(n)^2 + 2D^2 - 2qD \right. \right. \\ &\quad \left. \left. - 2qr_{C,D}(n) \right) + w_{C,D}(n) \left(-q^2 + 2qs_{C,D}(n) \right) - 2q^2 - 4Dr_{C,D}(n) + qC \right. \\ &\quad \left. + qs_{C,D}(n) - 2s_{C,D}(n)r_{C,D}(n) + 3qr_{C,D}(n) + 3qD - 2CD \right], \end{aligned}$$

where $r_{C,D}(n)$, $s_{C,D}(n)$, $u_{A,B}(n)$, $v_{A,B}(n)$, $w_{C,D}(n)$, are in Lemma 2.6.

Proof: By Propositon 2.5 and 2.7,

$$\begin{aligned} & \zeta_K(0, \chi) \\ &= \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (n^2 - 2)D^2) \cdot \left[\sum_{j=0}^{n-1} S\left(-\frac{r_{C,D}(n)}{q} + \frac{D+qj}{nq} + \sigma_1(j), \frac{D+qj}{nq}\right) \right. \\ &\quad \left. - \sum_{i=0}^{n(n^2-2)-1} S\left(1 - \frac{(C+iq)(n^2-1)}{qn(n^2-2)} + \frac{D}{q} + \sigma_2(i), \frac{C+iq}{qn(n^2-2)}\right) \right] \\ &= \frac{1}{2q^2} \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (n^2 - 2)D^2) \cdot \left[\left(-2nD^2 + 4qu_{C,D}(n)D \right) + \left(-qnD \right. \right. \\ &\quad \left. \left. - q^2u_{C,D}(n) \right) + \left(2qu_{C,D}(n)s_{C,D}(n) - 2nDs_{C,D}(n) \right) + \left(-q^2v_{C,D}(n) \right. \right. \\ &\quad \left. \left. - qnr_{C,D}(n) \right) + \left(2Cnr_{C,D}(n) - 2qCv_{C,D}(n) \right) - 2q^2 + nq^2 + 2qr_{C,D}(n) \right. \\ &\quad \left. - qns_{C,D}(n) - q^2w_{C,D}(n) + 2qD + 2qw_{C,D}(n)s_{C,D}(n) + 2nr_{C,D}(n)^2 \right] \end{aligned}$$

The equation $nD - C - qu_{C,D}(n) = r_{C,D}(n)$ implies that

$$\begin{aligned} (22) \quad & -2nD^2 + 4qu_{C,D}(n)D = 4D(qu_{C,D}(n) - nD) + 2nD^2 \\ & = -4Dr_{C,D}(n) - 4DC + 2nD^2 \end{aligned}$$

and

$$\begin{aligned} (23) \quad & -qnD - q^2u_{C,D}(n) = q(nD - qu_{C,D}(n)) - 2qnD \\ & = qr_{C,D}(n) + qC - 2qnD \end{aligned}$$

and

$$\begin{aligned} (24) \quad & 2qu_{C,D}(n)s_{C,D}(n) - 2nDs_{C,D}(n) \\ & = -2s_{C,D}(n)r_{C,D}(n) - 2s_{C,D}(n)C \end{aligned}$$

Also the equation $r_{C,D}(n)n - D - qv_{C,D}(n) = s_{C,D}(n)$ implies that

$$(25) \quad \begin{aligned} -q^2 v_{C,D}(n) - qnr_{C,D}(n) &= -q(qv_{C,D}(n) - nr_{C,D}(n)) - 2qnr_{C,D}(n) \\ &= qD + qs_{C,D}(n) - 2qnr_{C,D}(n) \end{aligned}$$

and

$$(26) \quad -2qnv_{C,D}(n) + 2Cnr_{C,D}(n) = 2CD + 2Cs_{C,D}(n)$$

By (22)-(26), we can prove Theorem. \square

Corollary 2.9. *If $h(n^2 - 2) = 1$, $n > q$ and $n = qk + r$ for $0 \leq r < q$ then $\zeta_K(0, \chi) = \frac{1}{2q^2}(B_\chi(r)k + A_\chi(r))$, where*

$$\begin{aligned} A_\chi(r) &= \frac{1}{2q^2} \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (r^2 - 2)D^2) \cdot \left[r \left(q^2 - qs_{C,D}(r) + 2r_{C,D}(r)^2 + 2D^2 - 2qD \right. \right. \\ &\quad \left. \left. - 2qr_{C,D}(r) \right) + w_{C,D}(r) \left(-q^2 + 2qs_{C,D}(r) \right) - 2q^2 - 4Dr_{C,D}(r) + qC \right. \\ &\quad \left. - 2s_{C,D}(r)r_{C,D}(r) + qs_{C,D}(r) + 3qr_{C,D}(r) + 3qD - 2CD \right], \\ B_\chi(r) &= \frac{1}{2q^2} \sum_{0 \leq C, D \leq q-1} \chi(C^2 - (r^2 - 2)D^2) \cdot \left[q^3 - 2q^2s_{C,D}(r) + 2r_{C,D}(r)^2q \right. \\ &\quad \left. + 2D^2q - 2Dq^2 - 2q^2r_{C,D}(r) + 2qs_{C,D}(r)^2 \right]. \end{aligned}$$

Proof: Since χ has a conductor q , $\chi(C^2 - (n^2 - 2)D^2) = \chi(C^2 - (r^2 - 2)D^2)$ for $n = qk + r$. We note that $s_{C,D}(qk + r) = s_{C,D}(r)$, $r_{C,D}(qk + r) = r_{C,D}(r)$ and $w_{C,D}(qk + r) = s_{C,D}(r)k + w_{C,D}(r)$. This prove the corollary. \square

3. COMPUTATION OF SPECIAL VALUES OF ZETA FUNCTION ASSOCIATED WITH $\mathbb{Q}(\sqrt{n^2 + 2})$

Let $d = n^2 + 2$ be a positive square free integer, $K = \mathbb{Q}(\sqrt{d})$. Then the fundamental unit ϵ of K is $n^2 + 1 + n\sqrt{n^2 + 2}$. And let $\alpha = n + \sqrt{n^2 + 2}$. We note that $N_K(\epsilon) = 1$ and $N_K(\alpha) = -2 < 0$. So if $h(d) = 1$ then

$$I(K) = (q) \cdot i(K^+) \cup (q\alpha) \cdot i(K^+),$$

where $I(K)$ and $i(K^+)$ are defined in the previous section. So we have

$$\zeta_K(s, \chi) = \sum_{\substack{\mathbf{a} \in (q) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s} + \sum_{\substack{\mathbf{a} \in (q\alpha) \cdot i(K^+) \\ \text{integral}}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}.$$

We define

$$R(\mathbf{b}) := \{a+b\epsilon \mid a, b \in \mathbb{Q} \text{ with } 0 < a \leq 1, 0 \leq b < 1 \text{ and } \mathbf{b} \cdot (a+b\epsilon) \subset O(K)\}$$

for an integral ideal \mathbf{b} in K . Then in the following lemma, we find the complete set of (x, y) such that $x + y\epsilon \in R((q))$ or $x + y\epsilon \in R((q\alpha))$.

Lemma 3.1.

$$(i) \quad \{(x, y) \mid x + y\epsilon \in R((q))\} = \{(x, y) \mid x = \frac{C}{q} - \frac{D + qj}{qn} + \delta_1(j), y = \frac{D + qj}{qn} \text{ for } j = 0, 1, 2, \dots, n-1 \text{ and } 0 \leq C, D \leq q-1\},$$

$$\text{where } \delta_1(j) = \begin{cases} 1, & \text{if } \lceil \frac{Cn-D}{q} \rceil \leq j \leq n-1 \\ 0, & \text{if } 0 \leq j \leq \lceil \frac{Cn-D}{q} \rceil - 1 \end{cases}.$$

$$(ii) \quad \{(x, y) \mid x + y\epsilon \in R((q\alpha))\} = \{(x, y) \mid x = -\frac{C + qi}{2nq} + \frac{t_{C,D}(n)}{q} + \delta_2(i), y = \frac{C + qi}{2qn} \text{ for } i = 0, 1, 2, \dots, 2n-1 \text{ and } 0 \leq C, D \leq q-1\},$$

$$\text{where } \delta_2(i) = \begin{cases} 0, & \text{if } 0 \leq i \leq \lceil \frac{2nt_{C,D}(n)-C}{q} \rceil - 1 \\ 1, & \text{if } \lceil \frac{2nt_{C,D}(n)-C}{q} \rceil \leq i \leq 2n-1 \end{cases} \text{ and } t_{C,D}(n) = D - Cn - q[\frac{D-Cn}{q}].$$

Proof: The set $\{C + D\alpha \mid 0 \leq C, D \leq q-1\}$ is the complete representatives of the coset O_K/qO_K , since $\{1, \alpha\}$ is a basis for O_K . So for $x + y\epsilon$ in $R((q))$ we have

$$q(x + y\epsilon) = (C + D\alpha) + q(i + j\alpha)$$

and for $x + y\epsilon$ in $R((q\alpha))$ we also have

$$q\alpha(x + y\epsilon) = (C + D\alpha) + q(i + j\alpha),$$

for $0 \leq C, D \leq q-1$ and integers i, j . By the equation $\alpha = \frac{\epsilon-1}{n}$ and similar computations in Lemma 2.3 and 2.4, we can prove the Lemma. \square

Proposition 3.2. *If $h(n^2 + 2) = 1$, then*

$$\begin{aligned}\zeta_K(\chi, 0) &= \sum_{0 \leq C, D \leq q-1} \chi(C^2 - 2D^2 + 2nCD) \left[\sum_{j=0}^{n-1} S\left(\frac{C}{q} - \frac{D+qj}{qn} + \delta_1(j), \frac{D+qj}{qn}\right) \right. \\ &\quad \left. - \sum_{i=0}^{2n-1} S\left(\frac{t_{C,D}(n)}{q} - \frac{C+iq}{2qn} + \delta_2(i), \frac{C+iq}{2qn}\right) \right],\end{aligned}$$

where δ_1 , δ_2 and $t_{C,D}$ are defined in Lemma 3.1, and $S(x,y)$ is in proposition 2.5.

Proof: As Proposition 2.5, we have

$$\begin{aligned}(27) \quad \zeta_K(0, \chi) &= \sum_{x+y\epsilon \in R((q))} \chi(N_K(q(x+y\epsilon))) S(x, y) + \sum_{x+y\epsilon \in R((\alpha q))} \chi(N_K(q\alpha(x+y\epsilon))) S(x, y).\end{aligned}$$

And for $x+y\epsilon \in R((q))$,

$$q(x+y\epsilon) = C + D\alpha + q(i+j\alpha).$$

Thus

$$(28) \quad N_K(q(x+y\epsilon)) \equiv N_K(C + D\alpha) (\text{mod } q) = C^2 - 2D^2 + 2nCD.$$

And if $x+y\epsilon$ is in $R((q\alpha))$, then

$$q\alpha(x+y\epsilon) = C + D\alpha + q(i+j\alpha),$$

for $0 \leq C, D \leq q-1$ and integers i, j . So

$$(29) \quad N_K(q\alpha(x+y\epsilon)) \equiv N_K(C + D\alpha) (\text{ mod } q) = C^2 - 2D^2 + 2nCD$$

From (27)-(29), we can obtain above Propositon, immediately. \square

Finally, we obtain the following Theorem.

Theorem 3.3. *If $h(n^2 + 2) = 1$, then*

$$\begin{aligned}\zeta_K(\chi, 0) &= \frac{1}{12q^2} \sum_{0 \leq C, D \leq q-1} \chi(C^2 - 2D^2 + 2nCD) (6q^2 A_{C,D}(n) + 2q^2 n + 3C^2 n \\ &\quad - 12qA_{C,D}(n)C + 6D^2 n - 6qDn - 6nt_{C,D}(n)^2 - 6qnt_{C,D}(n) - 3ne_{C,D}(n) \\ &\quad + 3qne_{C,D}(n) - 6q^2 B_{C,D}(n) + 12qB_{C,D}(n)t_{C,D}(n) + 3qCn),\end{aligned}$$

where $A_{C,D}(n) = \lceil \frac{nC-D}{q} \rceil$, $B_{C,D}(n) = \lceil \frac{2nt_{C,D}(n)-C}{q} \rceil$, $t_{C,D}(n) = D - nC + qA_{C,D}(n)$ and $e_{C,D}(n) = C - 2nt_{C,D}(n) + qB_{C,D}(n)$

Proof: The equation $D - nC + qA_{C,D}(n) = t_{C,D}(n)$ implies that

$$(30) \quad \begin{aligned} & \frac{A_{C,D}(n)^2}{2} + \frac{D^2n}{2q^2} - \frac{CDn^2}{q^2} + \frac{C^2n^3}{2q^2} + \frac{A_{C,D}(n)Dn}{q} - \frac{A_{C,D}(n)Cn^2}{q} \\ &= \frac{n(D - nC + qA_{C,D}(n))^2}{2q^2} = \frac{nt_{C,D}(n)^2}{2q^2} \end{aligned}$$

and

$$(31) \quad \frac{A_{C,D}(n)n}{2} + \frac{Dn}{2q} - \frac{Cn^2}{2q} = \frac{n(D - nC + qA_{C,D}(n))}{2q} = \frac{nt_{C,D}(n)}{2q}.$$

By (30), (31), we have

$$(32) \quad \begin{aligned} & \sum_{i=0}^{n-1} S\left(\frac{C}{q} - \frac{D+qj}{nq} + \delta_1(j), \frac{D+qj}{qn}\right) \\ &= \left(\frac{A_{C,D}(n)^2}{2} + \frac{D^2n}{2q^2} - \frac{CDn^2}{q^2} + \frac{C^2n^3}{2q^2} + \frac{A_{C,D}(n)Dn}{q} - \frac{A_{C,D}(n)Cn^2}{q} \right) \\ &\quad - \left(\frac{A_{C,D}(n)n}{2} + \frac{Dn}{2q} - \frac{Cn^2}{2q} \right) + \frac{A_{C,D}(n)}{2} + \frac{n}{12} + \frac{C^2n}{2q^2} - \frac{A_{C,D}(n)C}{q} + \frac{D^2n}{2q^2} - \frac{Dn}{2q} \\ &= \frac{nt_{C,D}(n)^2}{2q^2} - \frac{nt_{C,D}(n)}{2q} + \frac{A_{C,D}(n)}{2} + \frac{n}{12} + \frac{C^2n}{2q^2} - \frac{A_{C,D}(n)C}{q} + \frac{D^2n}{2q^2} - \frac{Dn}{2q}. \end{aligned}$$

Also $C - 2nt_{C,D}(n) + qB_{C,D}(n) = e_{C,D}(n)$ implies that

$$(33) \quad \begin{aligned} & \frac{B_{C,D}(n)^2n}{4} + \frac{C^2n}{4q^2} + \frac{B_{C,D}(n)Cn}{2q} - \frac{cn^2t_{C,D}(n)}{q^2} - \frac{B_{C,D}(n)n^2t_{C,D}(n)}{q} + \frac{n^3t_{C,D}(n)^2}{q^2} \\ &= \frac{n(C - 2nt_{C,D}(n) + qB_{C,D}(n))^2}{4q^2} = \frac{ne_{C,D}(n)^2}{4q^2} \end{aligned}$$

and

$$(34) \quad \frac{B_{C,D}(n)n}{4} + \frac{Cn}{4q} - \frac{n^2t_{C,D}(n)}{2q} = \frac{n(C - 2nt_{C,D}(n) + qB_{C,D}(n))}{4q} = \frac{ne_{C,D}(n)}{4q}.$$

From (33) and (34), we deduce that

$$\begin{aligned}
 (35) \quad & \sum_{i=0}^{2n-1} S\left(\frac{t_{C,D}(n)}{q} - \frac{C+qi}{2nq} + \delta_2(i), \frac{C+iq}{2nq}\right) \\
 & = \left(\frac{B_{C,D}(n)^2 n}{4} + \frac{C^2 n}{4q^2} + \frac{B_{C,D}(n)Cn}{2q} - \frac{cn^2 t_{C,D}(n)}{q^2} - \frac{B_{C,D}(n)n^2 t_{C,D}(n)}{q} + \frac{n^3 t_{C,D}(n)^2}{q^2} \right) \\
 & \quad - \left(\frac{B_{C,D}(n)n}{4} + \frac{Cn}{4q} - \frac{n^2 t_{C,D}(n)}{2q} \right) + \frac{B_{C,D}(n)}{2} - \frac{n}{12} - \frac{B_{C,D}(n)t_{C,D}(n)}{q} + \frac{nt_{C,D}(n)^2}{q^2} \\
 & = \frac{C^2 n}{4q^2} - \frac{Cn}{4q} = \frac{ne_{C,D}(n)^2}{4q^2} - \frac{ne_{C,D}(n)}{4q} + \frac{B_{C,D}(n)}{2} - \frac{n}{12} - \frac{B_{C,D}(n)t_{C,D}(n)}{q} + \frac{nt_{C,D}(n)^2}{q^2}.
 \end{aligned}$$

By combining Proposition 3.2, (32) and (35), we can obtain the theorem. \square

Corollary 3.4. *If $h(n^2+2)=1$ and $n=qk+r$ for $r=0, 1, \dots, q-1$, then*

$$\zeta_K(\chi, 0) = \frac{1}{12q^2}(F_\chi(r)k + E_\chi(r))$$

where

$$\begin{aligned}
 E_\chi(r) = & \sum_{0 \leq C,D \leq q-1} \chi(C^2 - 2D^2 + 2rCD)(6q^2 A_{C,D}(r) + 2q^2 r + 3C^2 r \\
 & - 12qA_{C,D}(r)C + 6D^2 r - 6qDr - 6nt_{C,D}(r)^2 - 6qrt_{C,D}(r) - 3ne_{C,D}(r) \\
 & + 3qne_{C,D}(r) - 6q^2 B_{C,D}(r) + 12qB_{C,D}(r)t_{C,D}(r) + 3qCr),
 \end{aligned}$$

and

$$\begin{aligned}
 F_\chi(r) = & \sum_{0 \leq C,D \leq q-1} \chi(C^2 - 2D^2 + 2rCD)(-9C^2 q + 6D^2 q - 3e_{C,D}(n)^2 q \\
 & + 9Cq^2 - 6Dq^2 + 3e_{C,D}(n)q^2 + 2q^3 - 18q^2 t_{C,D}(n) + 18qt_{C,D}(n)^2)
 \end{aligned}$$

and $A_{C,D}$, $B_{C,D}$, $t_{C,D}$ and $e_{C,D}$ are defined in Theorem 3.3.

Proof: We note that $A_{C,D}(qk+r) = Ck + A_{C,D}(r)$, $B_{C,D}(qk+r) = 2t_{C,D}(r)k + B_{C,D}(r)$, $t_{C,D}(qk+r) = t_{C,D}(r)$ and $e_{C,D}(qk+r) = e_{C,D}(r)$. Since the character χ has the conductor q , above equations induce above Corollary. \square

4. PROOF OF THEOREM

Let $d = n^2 + 2$ be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$. Let q be an positive integer with $(q, d) = 1$, χ an odd primitive character with conductor q and L_χ a field over \mathbb{Q} generated by the values of $\chi(a)$ for $a = 1, 2, \dots, q$. And we define $m_\chi := \sum_{a=1}^q a\chi(a)$. Then from the same argument in Section 2 of [1] and Corollary 2.9 and 3.5, we have if $n = qk + r$ and $h(n^2 - 2) = 1$ with $n > q$, then

$$B_\chi(r)k + A_\chi(r) \equiv 0 \pmod{I},$$

and if $h(n^2 + 2) = 1$, then

$$F_\chi(r)k + E_\chi(r) \equiv 0 \pmod{I},$$

for a prime ideal I of L_χ dividing the principal ideal (m_χ) . If the integers q and p satisfy the Condition(*) in [10], then for r such that $B_\chi(r) \notin I$ [resp. $F_\chi(r) \notin I$], there exists a unique $T_{A,B}^\chi(r)$ [resp. $T_{E,F}^\chi(r)$] $\in \{0, 1, 2, \dots, p-1\}$ such that

$$\begin{aligned} -q \frac{A_\chi(r)}{B_\chi(r)} + r + I &= T_{A,B}^\chi(r) + pZ \\ -q \frac{E_\chi(r)}{F_\chi(r)} + r + I &= T_{E,F}^\chi(r) + pZ. \end{aligned}$$

So we have

$$(36) \quad n \equiv T_{A,B}^\chi(r) \pmod{p} \text{ for } n = qk + r \text{ with } h(n^2 - 2) = 1 \text{ and } n > q$$

$$(37) \quad n \equiv T_{E,F}^\chi(r) \pmod{p} \text{ for } n = qk + r \text{ with } h(n^2 + 2) = 1.$$

We will denote by $q \rightarrow p$, if q and p satisfy Condition(*) in [10]. From the Section 4 in [1], we have

$$175 \rightarrow 61, \quad 61 \rightarrow 1861, \quad 175 \rightarrow 1861$$

Now, we find another p and q satisfying the Condition(*) in [10]. Consider the function $f_{25} : (\mathbb{Z}/25\mathbb{Z})^* \rightarrow \mathbb{Z}/20\mathbb{Z}$ for which $2^{f_{25}(a)} \equiv a \pmod{25}$ and the function $g_7 : (\mathbb{Z}/7\mathbb{Z})^* \rightarrow \mathbb{Z}/6\mathbb{Z}$ for which $3^{g_7(a)} \equiv a \pmod{7}$. Above two functions are well defined, since $(\mathbb{Z}/25\mathbb{Z})^*$ [resp. $(\mathbb{Z}/7\mathbb{Z})^*$] is a cyclic group generated by 2 [resp. 3]. Define

$$\chi_4 : (\mathbb{Z}/175\mathbb{Z})^* \rightarrow \mathbb{C}$$

by $\chi_4(a) = \zeta_{30}^{6f_{25}(a_{25})} \cdot \zeta_{30}^{25g_7(a_7)}$, where $a \equiv a_{25} \pmod{25}$, $a \equiv a_7 \pmod{7}$ and ζ_{30} is a primitive 30-th root of unity. Then χ_4 is an odd primitive character with a conductor 175. And $I_4 = (601, \zeta_{30} - 450)$ is the prime

ideal in L_{χ_4} lying over rational prime 601 of degree 1, for which $m_{\chi_4} \equiv 0 \pmod{I_4}$. From this, we have

$$(38) \quad 175 \rightarrow 601.$$

And we define the functions $T_{A,B}^{\chi_i}(r)$ as follows:

$$-175 \frac{A_{\chi_1}(r)}{B_{\chi_1(r)}} + r + I_1 = T_{A,B}^{\chi_1}(r) + 61\mathbb{Z}$$

$$-61 \frac{A_{\chi_2}(r)}{B_{\chi_2(r)}} + r + I_2 = T_{A,B}^{\chi_2}(r) + 1861\mathbb{Z}$$

$$-175 \frac{A_{\chi_3}(r)}{B_{\chi_3(r)}} + r + I_3 = T_{A,B}^{\chi_3}(r) + 1861\mathbb{Z}$$

$$-175 \frac{A_{\chi_4}(r)}{B_{\chi_4(r)}} + r + I_4 = T_{A,B}^{\chi_4}(r) + 601\mathbb{Z}$$

where the characters χ_i and ideals I_i are defined in Example 1, Example 3 and Example 4 of Section 4 in [1], respectively for $i = 1, 2, 3$.

For a residue a_{175} modulo 175 with $B_{\chi_1}(a_{175}) \notin I_1$ [resp. $B_{\chi_3}(a_{175}) \notin I_3$], we define b_{61} [resp. d_{1861}] by residues modulo 61 [resp. 1861] for which

$$b_{61} = T_{A,B}^{\chi_1}(a_{175})$$

$$d_{1861} = T_{A,B}^{\chi_3}(a_{175}).$$

And for a residue b_{61} modulo 61 with $B_{\chi_2}(b_{61}) \notin I_2$, we define c_{1861} by a residue modulo 1861 such that

$$c_{1861} = T_{A,B}^{\chi_2}(b_{61}).$$

Let $U_m = \{a \in \mathbb{Z} | (\frac{a^2-2}{p}) = -1, \text{ for any prime } p \text{ dividing } m\}$. By computer work we, can check that if $a_{175} \in U_{175}$ then $B_{\chi_1}(a_{175}) \notin I_1$ and $B_{\chi_3}(a_{175}) \notin I_3$ and if $b_{61} = T_{A,B}^{\chi_1}(a_{175})$ for $a_{175} \in U_{175}$ then $B_{\chi_2}(b_{61}) \notin I_2$. Hence it is possible to calculate $T_{A,B}^{\chi_1}(a_{175})$, $T_{A,B}^{\chi_3}(a_{175})$ and $T_{A,B}^{\chi_2}(b_{61})$, for $a_{175} \in U_{175}$. From the computer work, we obtain the following table:

$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}	$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}
± 7	± 7	± 7	± 7	± 8	± 8	± 8	± 8
± 13	± 13	± 13	± 13	± 15	± 38	± 1266	± 1060
± 20	± 20	± 20	± 20	± 22	± 23	± 595	± 1022
± 27	± 34	± 851	± 389	± 28	± 59	± 1859	± 962
± 35	± 51	± 1851	± 288	± 42	± 43	± 329	± 392
± 43	± 40	± 1821	± 1353	± 48	± 34	± 851	± 306
± 50	± 16	± 1075	± 193	± 55	± 35	± 1272	± 566
± 57	± 32	± 845	± 1559	± 62	± 6	± 301	± 1647
± 63	± 43	± 329	± 399	± 70	± 58	± 1858	± 49
± 77	± 4	± 4	± 1760	± 78	± 9	± 1690	± 561
± 83	± 26	± 589	± 427	± 85	± 49	± 1501	± 1072

Also for r with $F_{\chi_i}(r) \notin I_i$, the functions $T_{E,F}^{\chi_i}(r)$ are defined as follows:

$$\begin{aligned} -175 \frac{E_{\chi_1}(r)}{F_{\chi_1}(r)} + r + I_1 &= T_{E,F}^{\chi_1}(r) + 61\mathbb{Z} \\ -61 \frac{E_{\chi_2}(r)}{F_{\chi_2}(r)} + r + I_2 &= T_{E,F}^{\chi_2}(r) + 1861\mathbb{Z} \\ -175 \frac{E_{\chi_3}(r)}{F_{\chi_3}(r)} + r + I_3 &= T_{E,F}^{\chi_3}(r) + 1861\mathbb{Z} \\ -175 \frac{E_{\chi_4}(r)}{F_{\chi_4}(r)} + r + I_4 &= T_{E,F}^{\chi_4}(r) + 601\mathbb{Z} \end{aligned}$$

Then for a residue e_{175} modulo 175 with $F_{\chi_1}(e_{175}) \notin I_1$ [resp. $F_{\chi_3}(e_{175}) \notin I_3$], we define f_{61} [resp. h_{1861}] by residues modulo 61 [resp. 1861] for which

$$\begin{aligned} f_{61} &= T_{E,F}^{\chi_1}(e_{175}) \\ h_{1861} &= T_{E,F}^{\chi_3}(e_{175}). \end{aligned}$$

And for a residue f_{61} modulo 61 with $F_{\chi_2}(f_{61}) \notin I_2$, we define g_{1861} by a residue modulo 1861 such that

$$g_{1861} = T_{E,F}^{\chi_2}(f_{61}).$$

Let $V_m = \{a \in \mathbb{Z} | (\frac{a^2+2}{p}) = -1, \text{ for any prime } p \text{ dividing } m\}$. Then by computer work we also can check that if $e_{175} \in V_{175}$ then $F_{\chi_1}(e_{175}) \notin I_1$ and $F_{\chi_3}(e_{175}) \notin I_3$ and if $f_{61} = T_{E,F}^{\chi_1}(e_{175})$ for $e_{175} \in V_{175}$ then $F_{\chi_2}(f_{61}) \notin I_2$. So we can calculate $T_{E,F}^{\chi_1}(e_{175})$, $T_{E,F}^{\chi_3}(e_{175})$ and $T_{E,F}^{\chi_2}(f_{61})$, for $e_{175} \in V_{175}$. So we obtain the following table:

$e_{175} \in V_{175}$	f_{61}	g_{1861}	h_{1861}	$e_{175} \in V_{175}$	f_{61}	g_{1861}	h_{1861}
± 1	± 1	± 1	± 1	± 5	± 5	± 5	± 5
± 6	± 6	± 6	± 6	± 9	± 9	± 9	± 9
± 15	± 15	± 15	± 15	± 16	± 50	± 491	± 935
± 19	± 19	± 244	± 1534	± 20	± 44	± 403	± 943
± 26	± 41	± 1235	± 1243	± 29	± 19	± 244	± 1567
± 30	± 24	± 610	± 363	± 34	± 12	± 32	± 1589
± 36	± 28	± 458	± 200	± 40	± 30	± 1654	± 578
± 41	± 21	± 804	± 1762	± 44	± 30	± 1654	± 186
± 50	± 47	± 1124	± 213	± 51	± 45	± 728	± 181
± 54	± 54	± 778	± 1097	± 55	± 39	± 240	± 40
± 61	± 51	± 753	± 858	± 64	± 25	± 155	± 817
± 65	± 4	± 4	± 1691	± 69	± 26	± 1280	± 784
± 71	± 27	± 190	± 339	± 75	± 60	± 1860	± 70
± 76	± 57	± 1857	± 651	± 79	± 42	± 1617	± 1056
± 85	± 44	± 403	± 1623	± 86	± 8	± 1448	± 1048

To prove our theorem, we need the following class number 1 criteria.

Lemma 4.1. [12]

- (i) $h((2k+1)^2 - 2) = 1 \rightarrow 4k^2 + 4k - 1 - 4t^2$ are primes for $0 \leq t \leq n$
- (ii) $h(4k^2 - 2) = 1 \rightarrow 2k^2 - 1 - 2t^2$ are primes for $0 \leq t \leq n-1$
- (iii) $h((2k+1)^2 + 2) = 1 \rightarrow (2k+1)^2 + 2 - 4t^2$ are primes for $0 \leq t \leq n$
- (v) $h(4k^2 - 2) = 1 \rightarrow 4k^2 + 2 - (2t-1)^2$ are primes for $1 \leq t \leq n$

In the following Proposition, we find the upper bound of n with $h(n^2 - 2) = 1$.

Proposition 4.2. Let $n^2 - 2$ be a square free integer. Then $h(n^2 - 2) > 1$, for $n > 1244$.

Proof: If $n \notin U_{175}$ and $n = 2i + 1$, then there exists an integer $t_0 \in \mathbb{Z}$ such that $(2i+1)^2 - 2 - (2t_0)^2 \equiv 0 \pmod{5}$ or $(2i+1)^2 - 2 - (2t_0)^2 \equiv 0 \pmod{7}$. Similarly, for $n = 2j \notin U_{175}$, there is an integer $s_0 \in \mathbb{Z}$ such that $2j^2 - 1 - 2s_0^2 \equiv 0 \pmod{5}$ or $2j^2 - 1 - 2s_0^2 \equiv 0 \pmod{7}$, since $4j^2 - 2 - 4s_0^2 \equiv 2j^2 - 1 - 2s_0^2 \pmod{5}$ and $4k^2 - 2 - 4s_0^2 \equiv 2k^2 - 1 - 2s_0^2 \pmod{7}$. For example, if we take $t_0 = 5$ then $(2i+1)^2 - 2 - (2t_0)^2 = 7(28l^2 + 36l - 3)$ for $i = 7l + 4$. From lemma 4.1-(i), we have $h((2i+1)^2 - 2) > 1$ for $i > 5$ and $i = 7l + 4$. And if we take $s_0 = 4$, then we have $2j^2 - 1 - 2s_0^2 = 5(10l^2 + 12l - 3)$ for $j = 5l + 3$. So $h((2j)^2 - 2) > 1$ for $j > 3$ and $j = 5l + 3$ by Lemma 4.1-(ii). By applying this method to other cases, we can find the upper bound of i [resp. j] with $h((2i+1)^2 - 2) > 1$ [resp. $h((2j)^2 - 2) > 1$], for i with

$2i+1 \notin U_{175}$ [resp. $2j \notin U_{175}$]. The upper bounds do not exceed 15. So we have

$$(39) \quad h(n^2 - 2) > 1, \text{ for } n > 15 \text{ with } n \notin U_{175}.$$

Suppose $n \equiv a_{175} \pmod{175}$ for $a_{175} \in U_{175}$ and $a_{175} \neq \pm 7, \pm 8, \pm 13, \pm 20$ and $h(n^2 - 2) = 1$, $n > 175$, then from the table, we have $c_{1861} \neq d_{1861}$. This is a contradiction to (36). So

(40)

$$h(n^2 - 2) > 1$$

for $n \not\equiv \pm 7, \pm 8 \pm 13 \pm 20 \pmod{175}$ and $n \in U_{175}$ with $n > 175$.

By computer work, we have $T_{A,B}^{\chi_4}(\pm 20) = \pm 20$. That is $n \equiv \pm 20 \pmod{601}$ for $n \equiv \pm 20 \pmod{175}$ with $h(n^2 - 2) = 1$ and $n > 175$. If we take $t_0 = 20$, then $(2i+1)^2 - 2 - (2t_0)^2$ is a multiple of 601 for $2i+1 \equiv \pm 20 \pmod{601}$. And if we also take $s_0 = 621$ then $2j^2 - 1 - 2s_0^2$ is a multiple of 601 for $2j \equiv \pm 20 \pmod{601}$. Thus by Lemma 4.1-(i),(ii), we have

$$(41) \quad h(n^2 - 1) > 1, \text{ for } n \equiv \pm 20 \pmod{175} \text{ with } n > 1244.$$

If $n = \pm 7$ or ± 8 or $\pm 13 \pmod{175}$ then $n \in U_{175}$ but if we assume $n > 175$ and $h(n^2 - 2) = 1$, then $n = \pm 7$ or ± 8 or $\pm 13 \pmod{61}$. So $n \notin U_{61}$. We can find t_0 [resp. s_0] making $(2i+1)^2 - 2 - (2t_0)^2$ [resp. $2j^2 - 1 - 2s_0^2$] a multiples of 61 for $2i+1 \equiv \pm 7 \text{ or } \pm 8 \text{ or } \pm 13 \pmod{61}$ [resp. $2j \equiv \pm 7 \text{ or } \pm 8 \text{ or } \pm 13 \pmod{61}$] like above cases. They give the upper bound of n such that $h(n^2 - 2) > 1$, for $n = \pm 7$ or ± 8 or $\pm 13 \pmod{175}$. The upper bound does not exceed 1244. So we have

$$(42) \quad h(n^2 - 2) > 1, \text{ for } n \equiv \pm 7, \pm 8, \pm 13 \pmod{175} \text{ with } n > 1244.$$

By (39)-(42), we can complete the proof. \square

Now, we find the upper bound of n with $h(n^2 + 2) = 1$

Proposition 4.3. *Let $n^2 + 2$ be a square free integer. Then $h(n^2 + 2) > 1$, for $n > 1596$.*

Proof: For $n \notin V_{175}$, it is possible to find t_0 [resp. s_0] such that $(2i+1)^2 + 2 - (2t_0)^2$ [resp. $(2j)^2 + 2 - (2s_0 - 1)^2$] are the multiple of 5 or 7 with $n = 2i+1$ [resp. $n = 2j$]. They give the upper bounds of i [resp. j] with $h((2i+1)^2 + 2) = 2$ [resp. $h((2j)^2 + 2) = 2$]. From this, we have

$$(43) \quad h(n^2 + 2) > 1, \text{ for } n > 15 \text{ and } n \notin V_{175}.$$

If $n \equiv e_{175} \pmod{175}$ for $e_{175} \in U_{175}$ and $e_{175} \neq \pm 1, \pm 5, \pm 6, \pm 9, \pm 15$ and $h(n^2 + 2) = 1$, then from the table, we have $g_{1861} \neq h_{1861}$. This is a contradiction to (37). So

$$(44) \quad h(n^2 + 2) > 1, \text{ for } n \not\equiv \pm 1, \pm 5, \pm 6, \pm 9, \pm 15 \pmod{175} \text{ and } n \in V_{175}.$$

If $n \equiv \pm 6 \pmod{175}$ with $h(n^2 + 2) = 1$, then $n \equiv \pm 6 \pmod{1861}$. Suppose $n \equiv \pm 6 \pmod{1861}$ and $n = 2i+1$ then $i = 1861l+933$ or $1861l+927$. Take $t_0 = 133$, then $(2i+1)^2 - 2 - (2t_0)^2$ is $1861(7444l^2 + 7468l + 1835)$ or $1861(7444l^2 + 7420l + 1811)$. Suppose $n \equiv \pm 6 \pmod{1861}$ and $n = 2j$ then $j = 1861l + 3$ or $1861l + 1858$. If we take $s_0 = 798$ then $(2j)^2 + 2 - (2s_0 - 1)^2 = 1861(7444l^2 + 24l - 1367)$ or $1861(7444l^2 + 14864l + 6053)$. Thus by Lemma 4.1-(iii),(v), we have

$$(45) \quad h(n^2 + 2) > 1, \text{ for } n \equiv \pm 6 \pmod{175} \text{ with } n > 1596.$$

If $n = \pm 1, \pm 5, \pm 9 \pmod{175}$ then $n \in V_{175}$ but by assumption of $h(n^2 + 2) = 1$, we have $n = \pm 1, \pm 5, \pm 9 \pmod{61}$. And $n \notin V_{61}$. So we can find t_0 [resp. s_0] such that $(2i+1)^2 + 2 - (2t_0)^2$ [resp. $(2j)^2 + 2 - (2s_0 - 1)^2$] are the multiples of 61 for $2i+1 \equiv \pm 1, \pm 5, \pm 9 \pmod{61}$ [resp. $2j \equiv \pm 1, \pm 5, \pm 9 \pmod{61}$]. They give the upper bound of n such that $h(n^2 + 2) = 1$ and $n = \pm 1, \pm 5, \pm 9 \pmod{175}$ as above case. From this, we have

$$(46) \quad h(n^2 + 2) > 1, \text{ for } n \equiv \pm 1, \pm 5, \pm 9 \pmod{175} \text{ with } n > 114.$$

By computer work, we have $T_{A,B}^{X_4}(\pm 15) = \pm 15$. So if $n \equiv \pm 15 \pmod{175}$ with $h(n^2 + 2) = 1$, then $n \equiv \pm 15 \pmod{601}$. If we take $t_0 = 105$ [resp. $s_0 = 196$] then $(2i+1)^2 + 2 - (2t_0)^2$ [resp. $(2j)^2 + 2 - (2s_0 - 1)^2$] is the multiple of 601 for any $2i+1 \equiv \pm 15 \pmod{601}$ [resp. $2j \equiv \pm 15 \pmod{601}$]. Thus by Lemma 4.1-(iii),(v), we have

$$(47) \quad h(n^2 + 2) > 1, \text{ for } n \equiv \pm 15 \pmod{175} \text{ with } n > 392.$$

By (43)-(47), we complete the proof. \square

Proof of Mollin and William's Conjecture: By Proposition 4.2 4.3, we have that

$$h(n^2 \pm 2) > 1 \text{ for } n > 1596.$$

And in [15], Mollin and William prove that if $d = n^2 \pm 2$ is a square free integer with $n < 5000$ then

$$h(d) = 1 \text{ if and only if } d = 3, 6, 7, 11, 14, 38, 47, 62, 83, 167, 227, 398.$$

By combining above two results, we prove the conjecture.

Now, we observe the following proposition.

Proposition 4.4. [11] Let d be Richaud-Degert type.

- I. $d = n^2 + r \equiv 2, 3 \pmod{4}$
 - (i) $|r| \neq 1, 4, h(d) > 1$ except $r = \pm 2$
 - (ii) $|r| = 1, h(d) > 1$ except $d = 2, 3$
- II. $d = n^2 + r \equiv 1 \pmod{8}$
 - (i) $|r| \neq 1, 4, h(d) > 1$ except $d = 33$
 - (ii) $|r| = 1, h(d) > 1$ except $d = 17$

Since Mollin and William's conjecture is true, Theorem 1.1 is a direct consequence of Proposition 4.4.

5. APPENDIX

In this section, we will provide the MATHEMATICA program to evaluate the values $T_{A,B}^{\chi_i}(r)$ and $T_{E,F}^{\chi_i}(r)$ in section 4.
(The function $f[x_, y_]$ computes the logarithm of x with base 2 modulo y . And $g[x_, y_]$ computes the logarithm of x with base 3 modulo y .)

```

f[x_, y_] := ( j = 0; m = Mod[x, y];
    If [Mod[x, y] == 0, Return[0]];
    While[ Mod[m, y] >1, m = Mod[m*2, y]; j = j + 1];
    Return[y - 1 - j]);
g[x_, y_] := (j = 0; m = Mod[x, y];
    If [Mod[x, y] == 0, Return[0]];
    While[ Mod[m, y] >1, m = Mod[m*3, y]; j = j + 1];
    Return[y - 1 - j]);
g7[x_] := g[x, 7];
f25[x_] := (j = 0; m = Mod[x, 25];
    If[ Mod[m, 5] == 0, Return[0]];
    While[Mod[m, 25] >1, m = Mod[m*2, 25]; j = j + 1];
    Return[20 - j]);
f61[x_] := f[x, 61];
(The function  $iv[x_, y]$  computes the multiplicative inverse of  $x$  modulo  $y$ .)
iv[x_, y_] := (
    i = 1;
    While[Mod[ i*x, y] >1, i++];
    Return[i] );
(The functions  $chi[a_]$  computes  $\chi_i(a)$  modulo  $I_i$ , for  $i = 1, 2, 3, 4$ )
ch1[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];

```

```

Return[Mod[PowerMod[8, f25[Mod[a, 25]], 61]*  

PowerMod[47, g7[Mod[a, 7]], 61], 61]]);  

ch2[a_] := (If[Mod[a, 61] == 0, Return[0]];  

Return[PowerMod[1833, f61[Mod[a, 61]], 1861]]);  

ch3[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];  

Return[Mod[PowerMod[380, f25[Mod[a, 25]], 1861]*  

PowerMod[1406, g7[Mod[a, 7]], 1861], 1861]]);  

ch4[a_] :=  

(If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];  

Return[Mod[PowerMod[432, f25[Mod[a, 25]], 601]*  

PowerMod[577, g7[Mod[a, 7]], 601], 601]]));

```

(The followings are needed to compute $A_{\chi_i}(r)$ and $B_{\chi_i}(r)$ modulo I_i .)

```

u[q_, n_, c_, d_] := Floor[(n d - c)/q]  

r[q_, n_, c_, d_] := n d - c - q Floor[(n d - c)/q]  

v[q_, n_, c_, d_] := Floor[(r[q, n, c, d]n - d)/q]  

s[q_, n_, c_, d_] := r[q, n, c, d]n - d - q Floor[(r[q, n, c, d]n - d)/q]  

w[q_, n_, c_, d_] := -Floor[(r[q, n, c, d] - n s[q, n, c, d])/q]

A[q_, n_, c_, d_] := -2 q^2 + n q^2 + 3 q r[q, n, c, d] - q n s[q, n, c,  

d] - q^2 w[q, n, c, d] + 3q d + 2 q w[q, n, c, d] s[q, n, c, d]  

+ 2n r[q, n, c, d]^2 - 2 s[q, n, c, d]r[q, n, c, d] - 4 d r[q, n, c, d]  

- 2 c d + 2 n d^2 + q c - 2 q n d + q s[q, n, c, d] - 2 q n r[q, n, c, d]
B[q_, n_, c_, d_] := q^3 - 2 q^2 s[q, n, c, d] + 2 r[q, n, c, d]^2 q  

+ 2 d^2 q - 2 q^2 d - 2 q^2 r[q, n, c, d] + 2 q s[q, n, c, d]^2

```

```

SB1[n_, c_, d_] := Mod[ch1[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 61]  

SB2[n_, c_, d_] := Mod[ch2[c^2 - (n^2 - 2)d^2]*B[61, n, c, d], 1861]  

SB3[n_, c_, d_] := Mod[ch3[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 1861]  

SB4[n_, c_, d_] := Mod[ch4[c^2 - (n^2 - 2)d^2]*B[175, n, c, d], 601]  

SA1[n_, c_, d_] := Mod[ch1[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 61]  

SA2[n_, c_, d_] := Mod[ch2[c^2 - (n^2 - 2)d^2]*A[61, n, c, d], 1861]  

SA3[n_, c_, d_] := Mod[ch3[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 1861]  

SA4[n_, c_, d_] := Mod[ch4[c^2 - (n^2 - 2)d^2]*A[175, n, c, d], 601]

```

(The functions RAi[a_] and RBi[a_] computes $A_{\chi_i}(a)$ and $B_{\chi_i}(a)$ modulo I_i respectively, where q is the conductor for the character χ_i for $i = 1, 2, 3, 4$.)

```

RB1[a_] :=
Mod[Sum[Mod[Sum[SB1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RA1[a_] :=
Mod[Sum[Mod[Sum[SA1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RB2[a_] :=

```

```

Mod[Sum[Mod[Sum[SB2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RA2[a_] :=
Mod[Sum[Mod[Sum[SA2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RB3[a_] :=
Mod[Sum[Mod[Sum[SB3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RA3[a_] :=
Mod[Sum[Mod[Sum[SA3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RB4[a_] :=
Mod[Sum[Mod[Sum[SB4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
RA4[a_] :=
Mod[Sum[Mod[Sum[SA4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]

```

(The functions TABi[a_] compute $T_{A,B}^{\chi_i}(a)$, for i=1,2,3,4.)

```

TAB1[a_] := Mod[-RA1[a]*175*iv[RB1[a], 61] + a, 61]
TAB2[a_] := Mod[-RA2[a]*61*iv[RB2[a], 1861] + a, 1861]
TAB3[a_] := Mod[-RA3[a]*175*iv[RB1[a], 1861] + a, 1861]
TAB4[a_] := Mod[-RA4[a]*175*iv[RB4[a], 601] + a, 601]

```

(The followings are needed to compute $E_{\chi_i}(r)$ and $F_{\chi_i}(r)$ modulo I_i .)

```

A[c_, d_, n_, q_] := -Floor[(d - c n)/q]
t[c_, d_, n_, q_] := d - c n - q Floor[(d - n c)/q]
B[c_, d_, n_, q_] := -Floor[(c - 2 n t[c, d, n, q])/q]
e[c_, d_, n_, q_] := c - 2n t[c, d, n, q] + q B[c, d, n, q]

E[c_, d_, n_, q_] := 3 c^2 n + 6 d^2 n - 3 e[c, d, n, q]^2 n
-12 A[c, d, n, q] c q + 3 c n q - 6 d n q + 3 e[c, d, n, q] n q
+6 A[c, d, n, q] q^2 - 6 B[c, d, n, q] q^2 + 2 n q^2
+12 B[c, d, n, q] q t[c, d, n, q] - 6 n q t[c, d, n, q]
- 6 n t[c, d, n, q]^2
F[c_, d_, n_, q_] := -9 c^2 q + 6 d^2 q - 3 e[c, d, n, q]^2 q + 9 c q^2
- 6 d q^2 + 3 e[c, d, n, q] q^2 + 2 q^3 - 18 q^2 t[c, d, n, q]
+ 18 q t[c, d, n, q]^2

SF1[n_, c_, d_] := Mod[ch1[c^2-2 d^2+2 n c d]*F[175, n, c, d], 61]
SF2[n_, c_, d_] := Mod[ch2[c^2-2 d^2+2 n c d]*F[61, n, c, d], 1861]
SF3[n_, c_, d_] := Mod[ch3[c^2-2 d^2+2 n c d]*F[175, n, c, d], 1861]
SF4[n_, c_, d_] := Mod[ch4[c^2-2 d^2+2 n c d]*F[175, n, c, d], 601]
SE1[n_, c_, d_] := Mod[ch1[c^2-2 d^2+2 n c d]*E[175, n, c, d], 61]
SE2[n_, c_, d_] := Mod[ch2[c^2-2 d^2+2 n c d]*E[61, n, c, d], 1861]
SE3[n_, c_, d_] := Mod[ch3[c^2-2 d^2+2 n c d]*E[175, n, c, d], 1861]
SE4[n_, c_, d_] := Mod[ch4[c^2-2 d^2+2 n c d]*E[175, n, c, d], 601]

```

(The functions REi[a_] and RFi[a_] computes $E_{\chi_i}(a)$ and $F_{\chi_i}(a)$ modulo I_i respectively, where q is the conductor for the character χ_i for $i = 1, 2, 3, 4$.)

```

RF1[a_] :=
  Mod[Sum[Mod[Sum[SF1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RE1[a_] :=
  Mod[Sum[Mod[Sum[SE1[a, c, d], {c, 0, 174}], 61], {d, 0, 174}], 61]
RF2[a_] :=
  Mod[Sum[Mod[Sum[SF2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RE2[a_] :=
  Mod[Sum[Mod[Sum[SE2[a, c, d], {c, 0, 60}], 1861], {d, 0, 60}], 1861]
RF3[a_] :=
  Mod[Sum[Mod[Sum[SF3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RE3[a_] :=
  Mod[Sum[Mod[Sum[SE3[a, c, d], {c, 0, 174}], 1861], {d, 0, 174}], 1861]
RF4[a_] :=
  Mod[Sum[Mod[Sum[SF4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]
RE4[a_] :=
  Mod[Sum[Mod[Sum[SE4[a, c, d], {c, 0, 174}], 601], {d, 0, 174}], 601]

(The functions TEFi[a_] compute  $T_{E,F}^{\chi_i}(a)$ , for  $i=1,2,3,4$ .)

TEF1[a_] := Mod[-RE1[a]*175*iv[RF1[a], 61] + a, 61]
TEF2[a_] := Mod[-RE2[a]*61*iv[RF2[a], 1861] + a, 1861]
TEF3[a_] := Mod[-RE3[a]*175*iv[RF3[a], 1861] + a, 1861]
TEF4[a_] := Mod[-RE4[a]*175*iv[RF4[a], 601] + a, 601]
```

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