ON THE CLASSIFICATION OF NON-NORMAL CUBIC HYPERSURFACES

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ABSTRACT. In this article we study the classification of non-normal cubic hypersurfaces over an algebraically closed field K of arbitrary characteristic. Let $X \subset \mathbb{P}_K^r$ be an irreducible non-normal cubic hypersurface which is not a cone. We prove that $r \leq 4$ and there are precisely five non-normal cubic equations (resp. six non-normal cubic equations) when char $K \neq 2, 3$ (resp. when char K = 2, 3), up to projective equivalence. Also we describe the normalization of X in detail.

1. INTRODUCTION

Throughout this paper we work over an algebraically closed field K of arbitrary characteristic. Let \mathbb{P}_K^r denote the projective r-space over K and $S = K[X_0, X_1, \ldots, X_r]$ the homogeneous coordinate ring of \mathbb{P}_K^r . Recall that the two projective subvarieties X and Y of \mathbb{P}_K^r are called *projectively equivalent* if there exists a projective transformation of \mathbb{P}_K^r which maps X to Y. This paper is devoted to give a complete classification all irreducible non-normal cubic hypersurfaces up to projective equivalence and to describe their normalization as specific as possible.

Let $X \subset \mathbb{P}^r$ be a singular irreducible cubic hypersurface defined by a homogeneous polynomial F of degree 3 in S. A classical result asserts that if char $K \neq 2, 3$, then a singular plane cubic curve is either a cusp curve defined by $X_0^2 X_2 + X_1^3$ or a nodal curve defined by $X_0^2 X_2 + X_1^3 + X_1^2 X_2$, up to projective equivalence. Recently S. B. Mulay [M] classified all the singular plane cubic equations for the remaining characteristics. When char K = 2, X is either a cusp curve defined by $X_0^2 X_2 + X_1^3$ or a nodal curve defined by $X_0^3 + X_1^3 + X_0 X_1 X_2$. Also when char K = 3, X is either a cusp curve defined by $X_0^2 X_2 + X_1^3 + X_0 X_1^2$ or a nodal curve defined by $X_0^2 X_2 + X_1^3 + X_1^2 X_2$ or a singular curve defined by $X_0^2 X_2 + X_1^3$. The third one is a strange curve in the sense that all tangent lines at the smooth points of X pass through a fixed point. Singular cubic surfaces in $\mathbb{P}^3_{\mathbb{C}}$ are classified by J. W.. Bruce and C. T. C. Wall [B-W] according to the type of their singularities. They proved that

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a non-normal cubic surface is either a cone over a singular plane cubic curve or defined by $X_0^2 X_2 + X_1^3 + X_1^2 X_3$ or by $X_0^2 X_2 + X_1^3 + X_0 X_1 X_3$.

Theorem 4.2 shows that there are exactly five (resp. six) irreducible nonnormal cubic equations if char $K \neq 2,3$ (resp. if char K = 2 or 3), up to projective equivalence. A list of all those cubic equations is provided in Theorem 3.1. Thus for any characteristic of K and for any $r \geq 2$, we complete the classification of non-normal cubic hypersurfaces in \mathbb{P}_{K}^{r} , from the viewpoint of the projective equivalence.

2. Preliminaries

(2.1) Let X be an n-dimensional irreducible projective variety and let L be an ample line bundle on X. The Hilbert function of (X, L) is defined by

$$\chi(X,L)(k) := \sum_{i=0}^{n} (-1)^{i} h^{i}(X,L^{\otimes k}).$$

For sufficiently large k it becomes a polynomial written as

$$\chi(X,L)(k) = \sum_{i=0}^{n} \chi_i(X,L) \binom{k+i-1}{i}$$

where the $\chi_i(X, L)$ are uniquely determined integers. The degree of (X, L) is equal to $\chi_n(X, L)$. Also the Δ -genus $\Delta(X, L)$ and the sectional genus g(X, L)of (X, L) respectively are defined by the formulas

 $\Delta(X,L) = n + \chi_n(X,L) - h^0(X,L)$

and

$$g(X, L) = 1 - \chi_{n-1}(X, L).$$

(2.2) Let $X \subset \mathbb{P}_K^r$ be an *n*-dimensional non-normal irreducible cubic hypersurface. We assume that X is not a cone. Let $\varphi : \widetilde{X} \to X$ be the normalization of X and let L denote the line bundle $\varphi^* \mathcal{O}_X(1)$ on \widetilde{X} .

Lemma 2.1. L is a very ample line bundle on \widetilde{X} and $h^0(\widetilde{X}, L) = r + 2$.

Proof. Note that $0 \leq \Delta(\tilde{X}, L) \leq \Delta(X, \mathcal{O}_X(1)) = 1$ (cf. [F, Theorem 4.2]). We will first show that $\Delta(\tilde{X}, L) = 0$. Suppose that $\Delta(\tilde{X}, L) = 1$. Then the image of the map defined by the complete linear series |L| is precisely $X \subset \mathbb{P}_K^r$. In particular, L is not very ample since X is non-normal. This implies that $g(\tilde{X}, L) = 0$ since if $g(\tilde{X}, L) \geq 1$, then L is very ample by [F, Theorem 3.5]. Then since \tilde{X} is normal and L is base point free, we have $\Delta(\tilde{X}, L) = 0$ by [F, Proposition 3.4] and [F-O-V, Corollary 1.5.10], which contradicts to $\Delta(\tilde{X}, L) = 1$. In conclusion, $\Delta(\tilde{X}, L) = 0$ and $h^0(\tilde{X}, L) = r + 2$. Now the very ampleness of L comes from [F, Theorem 3.5].

Lemma 2.2. Let $\widetilde{X} \subset \mathbb{P}_{K}^{r+1}$ be the linearly normal embedding defined by the complete linear series |L|. Then

- (1) X is a smooth rational normal scroll of degree 3. That is, either
 a. r = 2 and X = S(3) or
 b. r = 3 and X = S(1,2) or
 c. r = 4 and X = S(1,1,1).
- (2) There exists a closed point $p \in \mathbb{P}_{K}^{r+1}$ outside of \widetilde{X} such that $X = \pi_{p}(\widetilde{X})$ where π_{p} is the linear projection of \widetilde{X} from p. Moreover $\pi_{p} : \widetilde{X} \to X$ is the normalization of X.

Proof. (1) By Lemma 2.1, $\Delta(\tilde{X}, L) = 0$ and hence $\tilde{X} \subset \mathbb{P}_{K}^{r+1}$ is a variety of minimal degree and of degree 3 (cf. [F, Theorem 5.15]). Thus the wellknown classification result of varieties of minimal degree says that \tilde{X} is a rational normal scroll of degree 3. If \tilde{X} is not smooth, then it is a cone over a smooth rational normal scroll and so X is also a cone which contradicts to our assumption. Therefore \tilde{X} is smooth.

(2) The assertion comes immediately from the fact that $\varphi^* H^0(X, \mathcal{O}_X(1))$ is a codimension one subspace in $H^0(\widetilde{X}, L)$.

Lemma 2.3. The singular locus of X is an (r-2)-dimensional subspace.

Proof. Recall that the arithmetic depth of X, denoted by depth(X), is equal to r. Since X is the projected image of a rational normal scroll \widetilde{X} from a closed point $p \in \mathbb{P}_{K}^{r+1}$ outside of \widetilde{X} (Lemma 2.2), the singular locus of X is a linear space of dimension depth(X) - 2 = r - 2 by [B-S, Theorem 1.3]. \Box

(2.3) The following Lemma 2.4 which belongs to folklore will play a crucial role to classify non-normal cubic equations, up to projective equivalence. For the lack of suitable references we give a proof here.

Lemma 2.4. Let $V \subset H^0(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(2))$ be a two dimensional subspace.

- (1) If V has a base point, then there exist linearly independent linear forms $L_1, L_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $\{L_1^2, L_1L_2\}$ is a basis for V.
- (2) If char $K \neq 2$ and V is base point free, then there exist linearly independent linear forms $L_1, L_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that $\{L_1^2, L_2^2\}$ is a basis for V.
- (3) If char K = 2 and V is base point free, then there exist linearly independent linear forms $L_1, L_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that either $\{L_1^2, L_2^2\}$ or $\{L_1^2 + L_2^2, L_1L_2\}$ is a basis for V.

Proof. (1) Suppose that V has a base point. Then after an appropriate linear transformation we may assume that V is spanned by $M_1 = X_0(aX_0 + bX_1)$ and $M_2 = X_0(cX_0 + dX_1)$ for some $a, b, c, d \in K$. Note that $ad - bc \neq 0$. Then we have

$$V = \langle dM_1 - cM_2, aM_1 - dM_2 \rangle = \langle X_0^2, X_0 X_1 \rangle.$$

(2) Suppose that char $K \neq 2$ and V is base point free. Then we may assume that the quadrics $M_1 = X_0(X_0 + aX_1)$ and $M_2 = X_1(bX_0 + X_1)$, $a, b \in K$,

form a basis for V. Note that $ab \neq 1$ since V is base point free. If a = b = 0, then $V = \langle X_0^2, X_1^2 \rangle$. If a = 0 and $b \neq 0$, then

$$V = \langle M_1, \frac{b^2}{4}M_1 + M_2 \rangle = \langle X_0^2, (\frac{b}{2}X_0 + X_1)^2 \rangle.$$

The case where $a \neq 0$ and b = 0 can be dealt with by the same argument. If both a and b are nonzero, then there are non-zero constants α and β satisfying the quadratic equations $(a+\beta b)^2 - 4\beta = 0$ and $(\alpha a+b)^2 - 4\alpha = 0$, respectively. Therefore

$$V = \langle M_1 + \beta M_2, \alpha M_1 + M_2 \rangle = \langle (X_0 + \frac{a + \beta b}{2} X_1)^2, (\frac{\alpha a + b}{2} X_0 + X_1)^2 \rangle.$$

(3) Suppose that char K = 2 and V is base point free. As in the previous case, we may assume that $V = \langle X_0(X_0 + aX_1), X_1(bX_0 + X_1) \rangle$ for some $a, b \in K$ satisfying $ab \neq 1$. If a = b = 0, then $V = \langle X_0^2, X_1^2 \rangle$. If a = 0 and $b \neq 0$, then

$$V = \langle X_0^2, (bX_0 + X_1)X_1 \rangle = \langle b^2 X_0^2, (bX_0 + X_1)X_1 \rangle$$

Since $(bX_0 + X_1)^2 + X_1^2 = b^2 X_0^2$, we get the desired basis for V. The case where $a \neq 0$ and b = 0 can be dealt with by the same argument. Finally, when both a and b are nonzero we have

$$V = \langle b^2 X_0^2 + a b^2 X_0 X_1, (b X_0 + X_1) X_1 \rangle$$

= $\langle \{ (b X_0 + X_1) - X_1 \}^2 + a b \{ (b X_0 + X_1) - X_1 \} X_1, (b X_0 + X_1) X_1 \rangle$
= $\langle (b X_0 + X_1)^2 + (1 + a b) X_1^2, (b X_0 + X_1) X_1 \rangle$.
= $\langle (b X_0 + X_1)^2 + (s X_1)^2, (b X_0 + X_1) (s X_1) \rangle$.

where $s \in K$ satisfies $s^2 = 1 + ab$.

3. The non-normal cubic equations

This section is devoted to the classification of non-normal cubic hypersurfaces. In particular we prove

Theorem 3.1. Let $X \subset \mathbb{P}^r$ be an irreducible non-normal cubic hypersurface defined by a homogeneous polynomial F of degree 3 in S. We assume that X is not a cone.

- (a) Suppose that char $K \neq 2, 3$. Then there is a coordinate change of \mathbb{P}_{K}^{r} which transforms F to one of the following five cases: (a.1) r = 2 and $F_1 = X_0^2 X_2 + X_1^3$
 - (a.2) r = 2 and $F_2 = X_0^2 X_2 + X_1^3 + X_1^2 X_2$

 - (a.3) r = 3 and $F_3 = X_0^2 X_2 + X_1^3 + X_1^2 X_3$ (a.4) r = 3 and $F_4 = X_0^2 X_2 + X_1^3 + X_0 X_1 X_3$ (a.5) r = 4 and $F_5 = X_0^2 X_2 + X_1^3 + X_1^2 X_3 + X_0 X_1 X_4$.
- (b) Suppose that char K = 2. Then there is a coordinate change of \mathbb{P}_{K}^{r} which transforms F to one of the following six cases: (b.1) r = 2 and $F_1 = X_0^2 X_2 + X_1^3$

- $\begin{array}{ll} (b.2) \ r = 2 \ and \ F_2 = X_0^3 + X_1^3 + X_0 X_1 X_2 \\ (b.3) \ r = 3 \ and \ F_3 = X_0^2 X_2 + X_1^3 + X_1^2 X_3 \\ (b.4) \ r = 3 \ and \ F_3' = X_0^2 X_2 + X_1^3 + X_1^2 X_2 + X_0 X_1 X_3 \\ (b.5) \ r = 3 \ and \ F_4 = X_0^2 X_2 + X_1^3 + X_0 X_1 X_3 \\ (b.6) \ r = 4 \ and \ F_5 = X_0^2 X_2 + X_1^3 + X_1^2 X_3 + X_0 X_1 X_4. \end{array}$
- (c) Suppose that char K = 3. Then there is a coordinate change of \mathbb{P}_{K}^{r} which transforms F to one of the following six cases: (c.1) r = 2 and $F_{1} = X_{0}^{2}X_{2} + X_{1}^{3}$ (c.2) r = 2 and $F_{1} = X_{0}^{2}X_{2} + X_{1}^{3} + X_{0}X_{1}^{2}$ (c.3) r = 2 and $F_{2} = X_{0}^{2}X_{2} + X_{1}^{3} + X_{1}^{2}X_{2}$ (c.4) r = 3 and $F_{3} = X_{0}^{2}X_{2} + X_{1}^{3} + X_{1}^{2}X_{3}$ (c.5) r = 3 and $F_{4} = X_{0}^{2}X_{2} + X_{1}^{3} + X_{0}X_{1}X_{3}$ (c.6) r = 4 and $F_{5} = X_{0}^{2}X_{2} + X_{1}^{3} + X_{1}^{2}X_{3} + X_{0}X_{1}X_{4}$.

Proof. Let Y be the singular locus of X. By Lemma 2.3, Y is an (r-2)-dimensional linear subspace of \mathbb{P}^r_K . Thus we may assume that Y is defined by $X_0 = X_1 = 0$. The cubic equation F can be written as

$$F = f_3(X_0, X_1) + \sum_{i=2}^{r} f_{2,i}(X_0, X_1) X_i + \sum_{2 \le i \le j \le r} g_{1,i,j}(X_0, X_1) X_i X_j + g_3(X_2, \dots, X_r)$$

where $f_3(X_0, X_1)$ (resp. $f_{2,i}$ and $g_{1,i,j}$) is a homogeneous polynomial of degree 3 (resp. 2 and 1) in X_0 and X_1 and $g_3(X_2, \ldots, X_r)$ is a homogeneous polynomial of degree 3 in X_2, \ldots, X_r . The singularity conditions

$$F|_Y = 0$$
 and $\frac{\partial F}{\partial X_i}|_Y = 0$ for all $i = 0, 1, \dots, r$

of X enable us to show respectively that

$$g_3(X_2, \dots, X_r) = 0$$
 and $\sum_{2 \le i \le j \le r} g_{1,i,j}(X_0, X_1) X_i X_j = 0.$

Now, put $f_{2,i}(X_0, X_1) = a_i X_0^2 + b_i X_0 X_1 + c_i X_1^2$ for $2 \le i \le r$. Then we have

$$F = f_3(X_0, X_1) + (a_2X_2 + \ldots + a_rX_r)X_0^2 + (b_2X_2 + \ldots + b_rX_r)X_0X_1 + (c_2X_2 + \ldots + c_rX_r)X_1^2 = f_3(X_0, X_1) + H_1X_0^2 + H_2X_0X_1 + H_3X_1^2$$

where H_i , i = 1, 2, 3, are linear polynomials in X_2, \ldots, X_r . Let $f_3(X_0, X_1) = aX_0^3 + bX_0^2X_1 + cX_0X_1^2 + dX_1^3$ and let ℓ denote the dimension of the K-vector space $\langle H_1, H_2, H_3 \rangle$. Note that $1 \leq \ell \leq 3$ since X is irreducible.

<u>Case 1.</u> If $\ell = 1$, then there is a nonzero linear polynomial G in X_2, \ldots, X_r such that $H_1 = \alpha G$, $H_2 = \beta G$ and $H_3 = \gamma G$ for some $\alpha, \beta, \gamma \in K$, not all of them are zero. Letting $G = X_2$, we can reduce F to

$$F = f_3(X_0, X_1) + (\alpha X_0^2 + \beta X_0 X_1 + \gamma X_1^2) X_2.$$

Depending on the factorization of $\alpha X_0^2 + \beta X_0 X_1 + \gamma X_1^2$, F is transformed to

$$F = \begin{cases} f_3(X_0, X_1) + X_0^2 X_2 & \text{if } \beta^2 - 4\alpha\gamma = 0, \text{ and} \\ f_3(X_0, X_1) + X_0 X_1 X_2 & \text{if } \beta^2 - 4\alpha\gamma \neq 0. \end{cases}$$

Let $f_3(X_0, X_1) = aX_0^3 + bX_0^2X_1 + cX_0X_1^2 + dX_1^3$ where $a, b, c, d \in K$.

If $\beta^2 - 4\alpha\gamma = 0$, then d is nonzero because of the irreducibility of F. Let $s \in K$ be a nonzero constant satisfying $s^3 = d$. When char $K \neq 3$, the coordinate change

$$\begin{cases} Y_0 = X_0 \\ Y_1 = \frac{c}{3s^2} X_0 + sX_1 \\ Y_2 = (a - \frac{c^3}{27d^2}) X_0 + (b - \frac{c^2}{3a}) X_1 + X_2 \end{cases}$$

transforms F to $Y_0^2 Y_2 + Y_1^3$. When char K = 3 and c = 0, F is transformed to $Y_0^2 Y_2 + Y_1^3$ by the coordinate change

$$\begin{cases} Y_0 = X_0 \\ Y_1 = sX_1 \\ Y_2 = aX_0 + bX_1 + X_2. \end{cases}$$

Moreover, when char K = 3 and $c \neq 0$, the coordinate change

$$\begin{cases} Y_0 = \frac{c}{s^2} X_0 \\ Y_1 = s X_1 \\ Y_2 = \frac{s^4}{c^2} (a X_0 + b X_1 + X_2) \end{cases}$$

transforms F to $Y_0^2 Y_2 + Y_1^3 + Y_0 Y_1^2$. In the case where $\beta^2 - 4\alpha\gamma \neq 0$, both a and d are non-zero by the irreducibility of F. Let $i, s, t \in K$ be nonzero constants satisfying $i^2 = -1, s^3 = a$ and $t^3 = d$. If char $K \neq 2$, then the coordinate change

$$\begin{cases} Y_0 = i(sX_0 - tX_1) \\ Y_1 = sX_0 + tX_1 \\ Y_2 = \frac{1}{4st} \{ (b - 3s^2t)X_0 + (c - 3st^2)X_1 + X_2 \} \end{cases}$$

transforms F to $Y_0^2 Y_2 + Y_1^3 + Y_1^2 Y_2$. If char K = 2, then the coordinate change

$$\begin{cases} Y_0 = sX_0 \\ Y_1 = tX_1 \\ Y_2 = \frac{1}{st}(bX_0 + cX_1 + X_2) \end{cases}$$

transforms F to $Y_0^3 + Y_1^3 + Y_0Y_1Y_2$.

<u>Case 2.</u> If $\ell = 2$, then $\langle H_1, H_2, H_3 \rangle = \langle G_1, G_2 \rangle$ for two linearly independent linear polynomials G_1 and G_2 in X_2, \ldots, X_n . Letting $H_1 = \alpha_1 G_1 + \alpha_2 G_2$, $H_2 = \beta_1 G_1 + \beta_2 G_2$ and $H_3 = \gamma_1 G_1 + \gamma_2 G_2$, we have

$$F = f_3(X_0, X_1) + Q_1 G_1 + Q_2 G_2$$

where $Q_1 = \alpha_1 X_0^2 + \beta_1 X_0 X_1 + \gamma_1 X_1^2$ and $Q_2 = \alpha_2 X_0^2 + \beta_2 X_0 X_1 + \gamma_2 X_1^2$. Clearly Q_1 and Q_2 are linearly independent. Now consider the linear subsystem

$$V = \langle Q_1, Q_2 \rangle \subset H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$$

on \mathbb{P}^1 . By Lemma 2.4, F can be transformed to the following:

(i) Suppose that char $K \neq 2$. Then

$$F = \begin{cases} f_3(X_0, X_1) + X_0^2 X_2 + X_1^2 X_3 & \text{if } V \text{ is base point free, and} \\ f_3(X_0, X_1) + X_0^2 X_2 + X_0 X_1 X_3 & \text{if } V \text{ has a base point.} \end{cases}$$

(*ii*) Suppose that char K = 2. If V is base point free, then

$$F = \begin{cases} f_3(X_0, X_1) + X_0^2 X_2 + X_1^2 X_3 & \text{or} \\ f_3(X_0, X_1) + (X_0^2 + X_1^2) X_2 + X_0 X_1 X_3. \end{cases}$$

If V has a base point, then $F = f_3(X_0, X_1) + X_0^2 X_2 + X_0 X_1 X_3$. Put $f_3(X_0, X_1) = aX_0^3 + bX_0^2 X_1 + cX_0 X_1^2 + dX_1^3$ where $a, b, c, d \in K$. In the first case of (i) and (ii), F may be written as $Y_0^2 Y_2 + Y_1^3 + Y_1^2 Y_3$ where

$$\begin{cases} Y_0 = X_0 \\ Y_1 = X_1 \\ Y_2 = aX_0 + bX_1 + X_2 \\ Y_3 = cX_0 + (d-1)X_1 + X_3. \end{cases}$$

In the second case of (ii), we can transform F to $Y_0^2Y_2 + Y_1^3 + Y_1^2Y_2 + Y_0Y_1Y_3$ by the coordinate change

$$\begin{cases} Y_0 = X_0 \\ Y_1 = X_1 \\ Y_2 = aX_0 + (d-1)X_1 + X_2 \\ Y_3 = (b-d+1)X_0 + (c-a)X_1 + X_3). \end{cases}$$

When V has a base point, $d \neq 0$ because of the irreducibility of F. Thus we can transform F to $Y_0^2 Y_2 + Y_1^3 + Y_0 Y_1 Y_3$ by

$$\begin{cases} Y_0 = X_0 \\ Y_1 = sX_1 \\ Y_2 = aX_0 + bX_1 + X_2 \\ Y_3 = \frac{1}{s}(cX_1 + X_3) \end{cases}$$

where s is a constant satisfying $s^3 = d$.

<u>Case 3.</u> If $\ell = 3$, then F can be written as

$$F = f_3(X_0, X_1) + X_0^2 X_2 + X_0 X_1 X_3 + X_1^2 X_4.$$

Put $f_3(X_0, X_1) = aX_0^3 + bX_0^2X_1 + cX_0X_1^2 + dX_1^3$ where $a, b, c, d \in K$. We can transform F to $Y_0^2Y_2 + Y_1^3 + Y_1^2Y_3 + Y_0Y_1Y_4$ by the coordinate change

$$\begin{cases} Y_0 = X_0 \\ Y_1 = X_1 \\ Y_2 = aX_0 + X_2 \\ Y_3 = (d-1)X_1 + X_3 \\ Y_4 = bX_0 + cX_1 + X_4 \end{cases}$$

By Case $1 \sim 3$, we can transform F to F_i for some $1 \leq i \leq 5$, F'_1 or F'_3 . This finishes the classification of Theorem 3.1.

4. The Normalization

Our purpose in this section is to complete the classification of non-normal cubic hypersurfaces by showing that the cubic hypersurfaces listed in proof of Theorem 3.1 are not projectively equivalent to each other. To this aim, we investigate the normalization of them.

Let $X \subset \mathbb{P}^r_K$ be a non-normal irreducible cubic hypersurface which is not a cone. Let $\varphi : \widetilde{X} \to X$ be the normalization of X. According to Lemma 2.2, $\widetilde{X} \subset \mathbb{P}_K^{r+1}$ is either $S(3) \subset \mathbb{P}_K^3$ or $S(1,2) \subset \mathbb{P}_K^4$ or $S(1,1,1) \subset \mathbb{P}_K^5$, and $\varphi = \pi_p$ where p is a closed point in $\mathbb{P}_{K}^{r+1} \setminus \widetilde{X}$ and π_{p} is the linear projection of \widetilde{X} from p. A natural approach to understand $\pi_p: \widetilde{X} \to X$ would be the study of the secant cone $\operatorname{Sec}_p(X)$ and the secant locus $\Sigma_p(X)$ of X with respect to p. The secant cone is defined to be the union of all secant lines to X passing through p. That is,

$$\operatorname{Sec}_{p}(\widetilde{X}) = \bigcup_{\operatorname{length}(\mathcal{O}_{\widetilde{X} \cap \langle p, x \rangle}) > 1} \langle p, x \rangle$$

Also the secant locus is the scheme-theoretic intersection of \widetilde{X} and $\operatorname{Sec}_{p}(\widetilde{X})$ and so

$$\Sigma_p(X) = X \cap \operatorname{Sec}_p(X).$$

Note that the singular locus of X is exactly the image of the secant locus and so $\operatorname{Sec}_p(\widetilde{X}) = \langle p, \operatorname{Sing}(X) \rangle$ is the linear subspace of \mathbb{P}_K^{r+1} . This enables us to obtain the defining ideal of $\Sigma_p(\widetilde{X})$ from that of \widetilde{X} . Let $\psi: \Sigma_p(X) \to \operatorname{Sing}(X)$ be the restriction map of π_p to $\Sigma_p(X)$.

Proposition 4.1. Under the situation just stated, the followings hold:

- (a) When char $K \neq 2, 3, \Sigma_p(\widetilde{X})$ is equal to
 - (a.1) a double point if $F = X_0^2 X_2 + X_1^3$;
 - (a.2) a union of two distinct points if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_2;$ (a.3) a smooth plane conic if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3;$

(a.4) the union of two coplanar lines if $F = X_0^2 X_2 + X_1^3 + X_0 X_1 X_3$; (a.5) a smooth quadric surface if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3 + X_0 X_1 X_4$.

- (b) When char K = 2, $\Sigma_p(X)$ is equal to
 - (b.1) a double point if $F = X_0^2 X_2 + X_1^3$;
 - (b.2) a union of two distinct points if $F = X_0^3 + X_1^3 + X_0 X_1 X_2$;
 - (b.3) a smooth plane conic if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3$;

 - (b.4) a smooth plane conic if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_2 + X_0 X_1 X_3$; (b.5) the union of two coplanar lines if $F = X_0^2 X_2 + X_1^3 + X_0 X_1 X_3$;
 - (b.6) a smooth quadric surface if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3 + X_0 X_1 X_4$.

For (b.3), $\psi: \Sigma_p(X) \to Sing(X)$ is inseparable. For (b.4), $\psi: \Sigma_p(X) \to$ Sing(X) ramifies exactly at one point.

- (c) When char K = 3, $\Sigma_p(X)$ is equal to
 - (c.1) a double point if $F = X_0^2 X_2 + X_1^3$;
 - (c.2) a double point if $F = X_0^2 X_2 + X_1^3 + X_0 X_1^2$;
 - (c.3) a union of two distinct points if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_2;$
 - (c.4) a smooth plane conic if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3;$
 - (c.5) the union of two coplanar lines if $F = X_0^2 X_2 + X_1^3 + X_0 X_1 X_3$;
 - (c.6) a smooth quadric surface if $F = X_0^2 X_2 + X_1^3 + X_1^2 X_3 + X_0 X_1 X_4$.

Proof. Let $A_X = S/(F)$ resp. $A_{\tilde{X}}$ denote the coordinate ring of X resp. X. Let $x_i, i = 0, \ldots, r$, denote the image of the variable X_i in A_X . Then it follows that the normalization of A_X is $A_{\tilde{X}}$. Furthermore, $A_{\tilde{X}}$ is generated by a single element $\xi \in \mathbb{Q}(A_X)$, the quotient field of A_X , see Lemma 2.1. It is known that ξ is of degree one satisfying an integral equation of degree 2 over A_X , see [B-S, Theorems 1.3 and 6.9]. That means $A_{\tilde{X}} = A_X[\xi]$ with deg $\xi = 1$.

For each cubic equation F in Theorem 3.1, let $I_F \subset T = S[X_{r+1}]$ denote the defining ideal of $\widetilde{X} \subset \mathbb{P}_K^{r+1}$. Let ξ be the image of X_{r+1} under the natural surjective ring homomorphism $T \to A_{\tilde{X}}$. In the case of (b.2) define $\xi = x_0^2/x_1$. In all other cases put $\xi = (x_0 x_2)/x_1$.

Then the defining equation F of $X \subset \mathbb{P}^r_K$ provides an integral equation of ξ . By view of Theorem 3.1 we have to distinguish eight different cases. We obtain the quadratic equation as a defining equation of $\tilde{X} \subset \mathbb{P}_{K}^{r+1}$ in the following list:

1. $(a.1), (b.1), (c.1): X_3^2 + X_1X_2,$ 2. $(a.2), (c.3): X_3^2 + X_2^2 + X_1X_2,$ 3. $(a.3), (b.3), (c.4): X_4^{\overline{2}} + X_2X_3 + X_1X_2,$ 4. $(a.4), (b.5), (c.5): X_4^2 + X_3X_4 + X_1X_2,$ 5. $(a.5), (b.6), (c.6): X_5^2 + X_4X_5 + X_1X_2 + X_2X_3,$ 6. (b.2): $X_3^2 + X_0X_1 + X_2X_3$, 7. (b.4): $X_4^2 + X_3X_4 + X_2^2 + X_1X_2$, 8. (c.2): $X_3^2 + X_1X_3 + X_1X_2$,

In case of (b.2) the quadric $X_0^2 - X_1 X_3$ and in all the other cases the quadric $X_0X_2 - X_1X_{r+1}$ belongs to a minimal generating set of I_F . A second generating element of degree two is obtained by the integral equation of $x_{r+1} = \xi$, as listed above. Because X is either S(3), S(2,1), or S(1,1,1) the defining ideal I_F is generated by the 2×2 -minors of a certain 2×3 -matrix. Starting with the two elements as given and completing the matrices we obtain the following results: *Case a:* When char $K \neq 2$ and 3,

$$\begin{array}{l} (a.1) \ I_{F_1} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + X_3^2, X_1^2 + X_0 X_3 \rangle \\ (a.2) \ I_{F_2} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + X_2^2 + X_3^2, X_1^2 - X_2^2 + X_0 X_3 - X_3^2 \rangle \\ (a.3) \ I_{F_3} = \langle X_0 X_2 - X_1 X_4, X_4^2 + X_1 X_2 + X_2 X_3, X_1^2 + X_1 X_3 + X_0 X_4 \rangle \\ (a.4) \ I_{F_4} = \langle X_0 X_2 - X_1 X_4, X_1 X_2 + X_3 X_4 + X_4^2, X_1^2 + X_0 X_3 + X_0 X_4 \rangle \\ (a.5) \ I_{F_5} = \langle X_0 X_2 - X_1 X_5, X_1 X_2 + X_2 X_3 + X_4 X_5 + X_5^2, \\ X_1^2 + X_1 X_3 + X_0 X_4 + X_0 X_5 \rangle. \end{array}$$

Case b: When char K = 2,

$$\begin{array}{ll} (b.1) \ I_{F_1} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + X_3^2, X_1^2 + X_0 X_3 \rangle \\ (b.2) \ I_{F_2} = \langle X_0^2 - X_1 X_3, X_1^2 + X_0 X_2 + X_0 X_3, X_0 X_1 - X_2 X_3 - X_3^2 \rangle \\ (b.3) \ I_{F_3} = \langle X_0 X_2 - X_1 X_4, X_4^2 + X_1 X_2 + X_2 X_3, X_1^2 + X_1 X_3 + X_0 X_4 \rangle \\ (b.4) \ I_{F_3'} = \langle X_0 X_2 - X_1 X_4, X_1 X_2 + X_2^2 + X_3 X_4 + X_4^2, \\ & X_1^2 + X_2^2 + X_0 X_3 + X_0 X_4 + X_3 X_4 + X_4^2 \rangle \\ (b.5) \ I_{F_4} = \langle X_0 X_2 - X_1 X_4, X_1 X_2 + X_3 X_4 + X_4^2, X_1^2 + X_0 X_3 + X_0 X_4 \rangle \\ (b.6) \ I_{F_5} = \langle X_0 X_2 - X_1 X_5, X_1 X_2 + X_2 X_3 + X_4 X_5 + X_5^2, \\ & X_1^2 + X_1 X_3 + X_0 X_4 + X_0 X_4 + X_0 X_5 \rangle. \end{array}$$

Case c: When char K = 3,

$$\begin{array}{l} (c.1) \ I_{F_1} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + X_3^2, X_1^2 + X_0 X_3 \rangle \\ (c.2) \ I_{F_1'} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + + X_1 X_3 + X_3^2, X_0 X_1 + X_1^2 + X_0 X_3 \rangle \\ (c.3) \ I_{F_2} = \langle X_0 X_2 - X_1 X_3, X_1 X_2 + X_2^2 + X_3^2, X_1^2 - X_2^2 + X_0 X_3 - X_3^2 \rangle \\ (c.4) \ I_{F_3} = \langle X_0 X_2 - X_1 X_4, X_4^2 + X_1 X_2 + X_2 X_3, X_1^2 + X_1 X_3 + X_0 X_4 \rangle \\ (c.5) \ I_{F_4} = \langle X_0 X_2 - X_1 X_4, X_1 X_2 + X_3 X_4 + X_4^2, X_1^2 + X_0 X_3 + X_0 X_4 \rangle \\ (c.6) \ I_{F_5} = \langle X_0 X_2 - X_1 X_5, X_1 X_2 + X_2 X_3 + X_4 X_5 + X_5^2, \\ X_1^2 + X_1 X_3 + X_0 X_4 + X_0 X_5 \rangle \end{array}$$

In each case, the projection center p is

$$p = \begin{cases} [0, 0, 0, 1] & \text{for } I_{F_1}, I_{F'_1} \text{ and } I_{F_2}, \\ [0, 0, 0, 0, 1] & \text{for } I_{F_3}, I_{F'_3} \text{ and } I_{F_4}, \text{ and} \\ [0, 0, 0, 0, 0, 1] & \text{for } I_{F_5}. \end{cases}$$

Therefore $\operatorname{Sec}_p(\widetilde{X}) = \langle p, \operatorname{Sing}(X) \rangle$ is defined by $X_0 = X_1 = 0$. Since the defining ideal of $\Sigma_p(X)$ is equal to the sum $I_F + \langle X_0, X_1 \rangle$ of the two ideals I_F and $\langle X_0, X_1 \rangle$ in T, we have the following list:

When char
$$K \neq 2, 3, \begin{cases} I_{F_1} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3^2 \rangle, \\ I_{F_2} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2^2 + X_3^2 \rangle, \\ I_{F_3} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4^2 \rangle, \\ I_{F_4} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3 X_4 + X_4^2 \rangle, \text{ and} \\ I_{F_5} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4 X_5 + X_5^2 \rangle. \end{cases}$$

$$\text{When char } K = 2, \begin{cases} I_{F_1} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3^2 \rangle, \\ I_{F_2} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_3^2 \rangle, \\ I_{F_3} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4^2 \rangle, \\ I_{F'_3} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2^2 + X_3 X_4 + X_4^2 \rangle, \\ I_{F_4} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3 X_4 + X_4^2 \rangle, \text{ and} \\ I_{F_5} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4 X_5 + X_5^2 \rangle. \end{cases}$$

$$\text{When char } K = 3, \begin{cases} I_{F_1} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2^2 \rangle, \\ I_{F_1} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3^2 \rangle, \\ I_{F_2} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2^2 + X_3^2 \rangle, \\ I_{F_3} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4^2 \rangle, \\ I_{F_4} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3 X_4 + X_4^2 \rangle, \\ I_{F_4} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_3 X_4 + X_4^2 \rangle, \\ I_{F_5} + \langle X_0, X_1 \rangle = \langle X_0, X_1, X_2 X_3 + X_4 X_5 + X_5^2 \rangle. \end{cases}$$

Now all the statements about $\Sigma_p(X)$ and ψ come from this result.

Theorem 4.2. There are five (resp. six) irreducible non-normal cubic equations when char $K \neq 2, 3$ (resp. char K = 2 or 3), up to projective equivalence.

Proof. According to Theorem 3.1, X is defined by one of the cubic equations listed in the theorem. Thus it remains to show that they are not projectively equivalent. When char $K \neq 3$, Proposition 4.1 (1) and (2) complete the proof because of the uniqueness of the normalization. When char K = 3, it suffices to show that F_1 and F'_1 are not projectively equivalent. This comes from the fact that all the tangent lines of F_1 at smooth points pass through the fixed point [0, 1, 0] while the intersection of all tangent lines of F'_1 at smooth points is empty.

Remark 4.3. Suppose that char $K \neq 2, 3$. Then we can prove Theorem 4.2 by investigating the tangent cones at singular points of cubic hypersurfaces. Let $X \subset \mathbb{P}_K^r$ be a cubic hypersurface defined in Theorem 3.1.(*a*) and let *q* be a closed point in *X*. Recall that the tangent cone $C_q(X)$ to *X* at *q* is defined as follows:

Suppose that q = [1, 0, ..., 0] and let F^* be the lowest degree homogeneous part of the polynomial $F(1, X_1, ..., X_r)$. Then $C_q X$ is defined to be $\operatorname{Proj}(S/\langle F^* \rangle)$.

Since the singular locus of X is defined by $X_0 = X_1 = 0$, a singular point q of X is written as $q = [0, 0, a_2, \ldots, a_r]$.

When r = 2, q = [0, 0, 1]. If $F = F_1$, then

$$C_q(X) \cong \operatorname{Proj}(S/\langle X_0^2 \rangle)$$

is a double line and hence it is non-reduced. If $F = F_2$, then

$$C_q(X) \cong \operatorname{Proj}(S/\langle X_0^2 + X_1^2 \rangle)$$

is a union of two distinct lines and hence it is reduced. Therefore F_1 and F_2 are not projectively equivalent.

When r = 3, $q = [0, 0, a_2, a_3]$ for some $[a_2, a_3] \in \mathbb{P}^1_K$. If $F = F_3$, then

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2 X_0^2 + a_3 X_1^2 \rangle).$$

In particular, $C_q(X)$ is non-reduced if and only if $[a_2, a_3] \in \{[1, 0], [0, 1]\}$. If $F = F_4$, then

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2 X_0^2 + a_3 X_0 X_1 \rangle).$$

Therefore $C_q(X)$ is not reduced if and only if $[a_2, a_3] = [1, 0]$. This shows that F_3 and F_4 are not projectively equivalent.

When r = 4, $q = [0, 0, a_2, a_3, a_4]$ for some $[a_2, a_3, a_4] \in \mathbb{P}^2_K$. If $F = F_5$, then

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2 X_0^2 + a_3 X_1^2 + a_4 X_0 X_1 \rangle).$$

Thus $C_q(X)$ is not reduced if and only if $a_4^2 - 4a_2a_3 = 0$.

Remark 4.4. When char K = 2, the study of the tangent cone enables us to distinguish $Z(F_3)$ from $Z(F'_3)$ and $Z(F_4)$. Indeed if r = 3, then $q = [0, 0, a_2, a_3]$ for some $[a_2, a_3] \in \mathbb{P}^1_K$. For $F = F_3$,

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2 X_0^2 + a_3 X_1^2 \rangle)$$

while for $F = F'_3$,

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2(X_0^2 + X_1^2) + a_3X_0X_1 \rangle)$$

and for $F = F_4$,

$$C_q(X) \cong \operatorname{Proj}(S/\langle a_2 X_0^2 + a_3 X_0 X_1 \rangle)$$

Therefore $C_q(X)$ for $X = Z(F_3)$ is non-reduced for all $[a_2, a_3] \in \mathbb{P}^1_K$ while $C_q(X)$ for $X = Z(F'_3)$ and for $X = Z(F_4)$ is non-reduced if and only if $[a_2, a_3] = [1, 0]$.

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