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Generation of ray class fields by elliptic units

Ho Yun Jung, Ja Kyung Koo and Dong Hwa Shin

Abstract

We show that certain special value of a Siegel function generates the ray class field over the Hilbert class field for an imaginary quadratic field, from which we settle the Schertz's conjecture.

1. *Introduction*

For any pair $(r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we define a *Siegel function* $g_{(r_1,r_2)}(\tau)$ on the complex upper half plane $\mathfrak{H}$ by a product of a Klein form and the square of Dedekind eta function, that is,

$$g_{(r_1,r_2)}(\tau) = \mathfrak{k}_{(r_1,r_2)}(\tau)\eta^2(\tau)$$

with

$$\eta(\tau) = \sqrt{2\pi} e^{\frac{2\pi i}{8}} q_\tau^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q_\tau^n).$$

Its Fourier expansion is given by

$$g_{(r_1,r_2)}(\tau) = -q_\tau^{\frac{1}{2}\mathbf{B}_2(r_1)} e^{\pi i r_2(r_1-1)}(1 - q_z) \prod_{n=1}^{\infty}(1 - q_\tau^n q_z)(1 - q_\tau^n q_z^{-1}), \tag{1.1}$$

where $\mathbf{B}_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial, $q_\tau = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. Then it is a modular unit in the sense of [**7**].

Let $\mathfrak{a}$ be a fractional ideal of an imaginary quadratic field not containing 1 with oriented basis ω_1 and ω_2, that is, $\mathfrak{a} = [\omega_1, \omega_2] = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\frac{\omega_1}{\omega_2} \in \mathfrak{H}$. Writing $1 = r_1\omega_1 + r_2\omega_2$ for some $(r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we define

$$g(1,\ [\omega_1, \omega_2]) = g_{(r_1,r_2)}\left(\frac{\omega_1}{\omega_2}\right),$$

which depends on the choice of ω_1 and ω_2. When a product of these values becomes a unit, we call it an *elliptic unit*([**10**]). By taking 12-th power the above value depends only on $\mathfrak{a}$ itself. So we write $g^{12}(1,\ \mathfrak{a})$ instead of $g^{12}(1, [\omega_1, \omega_2])$.

For later use of Shimura's reciprocity law we need a criterion to determine the levels of Siegel functions in a modular function field and transformation formulas. Let $N \geq 2$ be an integer and let $\mathcal{F}_N$ be the modular function field of level N defined over the N-th cyclotomic field $\mathbb{Q}_N$. We say that a family of integers $\{m(r)\}_{r=(r_1,r_2)\in\frac{1}{N}\mathbb{Z}^2\setminus\mathbb{Z}^2}$ with $m(r) = 0$ except finitely many r

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satisfies the *quadratic relation* modulo N if

$$\sum_r m(r)(Nr_1)^2 \equiv \sum_r m(r)(Nr_2)^2 \equiv 0 \pmod{\gcd(2,N)\cdot N}$$
$$\sum_r m(r)(Nr_1)(Nr_2) \equiv 0 \pmod{N}.$$

Proposition 1.1. *Let $\{m(r)\}_{r\in\frac{1}{N}\mathbb{Z}^2\setminus\mathbb{Z}^2}$ be a family of integers such that $m(r)=0$ except finitely many r. Then a product of Siegel functions*

$$\prod_r g_r^{m(r)}(\tau)$$

belongs to $\mathcal{F}_N$, if $\{m(r)\}_r$ satisfies the quadratic relation modulo N and 12 divides $\gcd(12,N)\cdot\sum_r m(r)$.

Proof. See [**7**] Chapter 3. □

As its immediate corollary we have

Corollary 1.2. *For $(r_1,r_2)\in\frac{1}{N}\mathbb{Z}^2\setminus\mathbb{Z}^2$ the function $g_{(r_1,r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau)$ belongs to $\mathcal{F}_N$.*

Proposition 1.3. *An element $\alpha\in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}\cong \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ acts on the function $g_{(r_1,r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau)$ by the rule*

$$\big(g_{(r_1,r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau)\big)^\alpha = g_{(r_1,r_2)\alpha}^{\frac{12N}{\gcd(6,N)}}(\tau).$$

In particular,

$$g_{(-r_1,-r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau) = g_{(r_1,r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau) = g_{(\langle r_1\rangle,\langle r_2\rangle)}^{\frac{12N}{\gcd(6,N)}}(\tau),$$

where $\langle X\rangle$ is the fractional part of $X\in\mathbb{R}$ such that $0\le\langle X\rangle<1$.

Proof. See [**7**]. □

Let $\mathfrak{f}$ be a nontrivial integral ideal of an imaginary quadratic field K. And, let

$$\sigma : \mathrm{Cl}(\mathfrak{f}) \longrightarrow \mathrm{Gal}(K(\mathfrak{f})/K)$$

be the Artin map between the ray class group $\mathrm{Cl}(\mathfrak{f}) = I(\mathfrak{f})/P_1(\mathfrak{f})$ and the Galois group of the ray class field $K(\mathfrak{f})$ modulo $\mathfrak{f}$ over K. For a character $\chi\neq 1$ on $\mathrm{Cl}(\mathfrak{f})$ we consider the sum

$$A_\mathfrak{f}(\chi) = \frac{1}{12N(\mathfrak{f})}\sum_{C\in \mathrm{Cl}(\mathfrak{f})} \overline{\chi}(C)\log\left|g^{12N(\mathfrak{f})}(1,\ \mathfrak{f})^{\sigma(C)}\right|, \tag{1.2}$$

where $N(\mathfrak{f})$ is the smallest positive integer in $\mathfrak{f}$ and $\overline{\chi}$ is the character obtained by taking complex conjugation on χ. When $\mathfrak{f}_\chi$ is the conductor of χ, $A_{\mathfrak{f}_\chi}(\chi)$ appears as a factor in the value of the L-function $L(s,\chi)$ of K at $s=1$([**9**]). Moreover we can factor out the Euler factor in (1.2), namely

$$A_\mathfrak{f}(\chi) = \frac{w(\mathfrak{f}_\chi)}{w(\mathfrak{f})}\Big(\prod_{\mathfrak{p}\mid\mathfrak{f}\mathfrak{f}_\chi^{-1}}\big(1-\overline{\chi}(\mathfrak{p})\big)\Big)A_{\mathfrak{f}_\chi}(\chi), \tag{1.3}$$

where $w(\mathfrak{f})$ and $w(\mathfrak{f}_\chi)$ denote the number of roots of unity in K congruent to 1 modulo $\mathfrak{f}$ and $\mathfrak{f}_\chi$, respectively([**11**]). Assuming that $\mathfrak{f}$ is the exact conductor of the extension $K(\mathfrak{f})/K$, Schertz gave in [**11**] a criterion for the value $g^{12N(\mathfrak{f})n}(1,\ \mathfrak{f})$ with $n=1,2,\cdots$ being a generator of $K(\mathfrak{f})$ over the Hilbert class field $K(1)$. This criterion seems to cover many cases, however, it is too theoretic in practical use so that we can hardly check the condition of applying the criterion. He also found some conditions on $\mathfrak{f}$ for which there exists a character of $\mathrm{Cl}(\mathfrak{f})$ such that the Euler factor in (1.3) does not vanish. Then $A_\mathfrak{f}(\chi)$ is not zero because $A_{\mathfrak{f}_\chi}(\chi)$ never vanishes. By slight modification of $A_\mathfrak{f}(\chi)$ and Galois theory he proved that $K(1)\big(g^{12N(\mathfrak{f})n}(1,\ \mathfrak{f})\big)$ is all of $K(\mathfrak{f})$.

Besides, he conjectured that his theorem holds without any condition on $\mathfrak{f}$. To establish Schertz's conjecture we observe the expansion formula (1.1) instead. Without a link with L-series we shall show that for an integer $N\geq 2$ there is a universal generator of the ray class field $K(N)$ modulo $N\mathcal{O}_K$ over $K(1)$ for all imaginary quadratic fields $K\neq\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$. Since the coefficients of a Siegel function is quite small, its values are dominated by its leading term in general. Owing to this fact we can show that the conjugates of our candiate for a primitive generator are all distinct. Precisely, we shall compare their absolute values.

Here we observe that it is not necessary to assume that the exact conductor of the extension $K(N)/K(1)$ is $N\mathcal{O}_K$.

2. *Ray class fields of imaginary quadratic fields*

Let $K\neq\mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field with the ring of integers $\mathcal{O}_K=\mathbb{Z}[\theta]$ with $\theta\in\mathfrak{H}$ and discriminant $d_K(\leq -7)$. Let $N\geq 2$ be an integer. By the main theorem of complex multiplication we have $K(1)=K(j(\theta))$ and $K(N)=K\mathcal{F}_N(\theta)$([**12**] or [**8**]). Letting $\mathrm{irr}(\theta,\ \mathbb{Q})=X^2+BX+C$ we consider a group

$$W_{N,\theta}=\left\{\begin{pmatrix}t-Bs & -Cs\\ s & t\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\mid t,s\in\mathbb{Z}/N\mathbb{Z}\right\}.$$

Then by the Shimura's reciprocity law([**12**] or [**4**]) we have a surjection

$$\begin{aligned}W_{N,\theta}&\longrightarrow \mathrm{Gal}(K(N)/K(1))\\ \alpha&\longmapsto \overline{\alpha}=(h(\theta)\mapsto h^\alpha(\theta))\text{ where }h\in\mathcal{F}_N\text{ is defined and finite at }\theta\end{aligned}\tag{2.1}$$

with kernel $\{\pm 1_2\}$([**4**] or [**2**]).

Throughout this paper we let $N\geq 4$ and $\zeta_N=e^{\frac{2\pi i}{N}}$. We shall prove that the value $g^{\frac{12Nn}{\gcd(6,N)}}(1,\ N\mathcal{O}_K)$ for $n=1,\ 2,\ \cdots$ generates $K(N)$ over the Hilbert class field $K(1)$. If we put $A=|q_\theta|^{\frac{1}{N}}=|e^{2\pi i\theta}|^{\frac{1}{N}}$, then $A<1$ and

$$A^k\leq A^{\frac{N}{2}}=|e^{\pi i\theta}|=e^{-\frac{\sqrt{-d_K}\pi}{2}}\leq e^{-\frac{\sqrt{7}\pi}{2}}\quad\text{for all }k\geq\frac{N}{2}.\tag{2.2}$$

Lemma 2.1. *We have the following inequalities:*

(i) $\left|\frac{1-\zeta_N}{1-\zeta_N^c}\right|<\frac{1}{\sqrt{2}}$ *for* $2\leq c\leq\frac{N}{2}$.
(ii) $\frac{1}{1-A^k}<1+A^{k-\frac{N}{2}}$ *for all* $k\geq\frac{N}{2}$.
(iii) $A^{\frac{N}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{a}{N}))}\left|\frac{1-\zeta_N}{1-A^a}\right|<0.9$ *for* $1\leq a\leq\frac{N}{2}$.

Proof. (i) It suffices to show the inequality

$$2|1-\zeta_N|^2\leq|1-\zeta_N^c|^2.$$

Now that

$$2|1-\zeta_N|^2 = 4-4\cos\frac{2\pi}{N}$$
$$|1-\zeta_N^c|^2 = 2-2\cos\frac{2c\pi}{N} \geq 2-2\cos\frac{4\pi}{N},$$

it is enough to prove

$$4-4\cos\frac{2\pi}{N} \leq 2-2\cos\frac{4\pi}{N}.$$

Observe that

$$2-2\cos\frac{4\pi}{N}-\left(4-4\cos\frac{2\pi}{N}\right) = 2\left(-1-\cos\frac{4\pi}{N}+2\cos\frac{2\pi}{N}\right)$$
$$= 2\left(-2\cos^2\frac{2\pi}{N}+2\cos\frac{2\pi}{N}\right) = 4\cos\frac{2\pi}{N}\left(-\cos\frac{2\pi}{N}+1\right) \geq 0$$

because $N \geq 4$.
(ii) The inequality is equivalent to $A^{\frac{N}{2}} + A^k < 1$. But this is obvious from (2.2).
(iii) By the fact $A < 1$ and the shape of the graph $Y = \mathbf{B}_2(X)$ on the interval $0 \leq X \leq \frac{1}{2}$, we get

$$A^{\frac{N}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{a}{N}))} \leq A^{\frac{N}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{1}{N}))} = A^{\frac{1}{2}(1-\frac{1}{N})} \leq A^{\frac{3}{8}}.$$

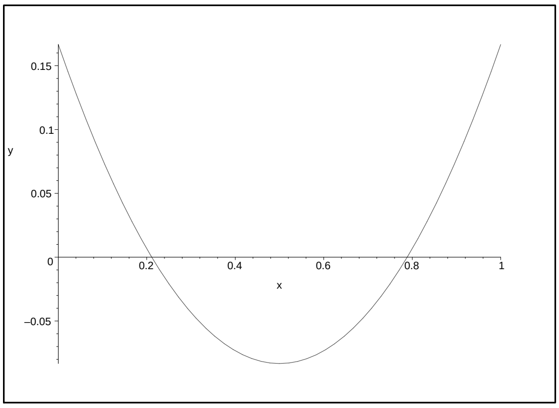

FIGURE 1. *The graph of* $Y = \mathbf{B}_2(X)$

Hence it follows that

$$A^{\frac{N}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{a}{N}))}\left|\frac{1-\zeta_N}{1-A^a}\right| \leq A^{\frac{3}{8}}\frac{|1-\zeta_N|}{1-A},$$

from which it suffices to prove

$$A^{\frac{3}{8}}\frac{|1-\zeta_N|}{1-A} < 0.9.$$

Since $|1-\zeta_N| = 2\sin\frac{\pi}{N}$ and

$$\frac{A^{\frac{3}{8}}}{1-A} \leq \frac{e^{-\frac{3\sqrt{7}\pi}{8N}}}{1-e^{-\frac{\sqrt{7}\pi}{N}}} \quad \text{by (2.2)},$$

we achieve

$$A^{\frac{3}{8}}\frac{|1-\zeta_N|}{1-A} \leq \frac{2e^{-\frac{3\sqrt{7}\pi}{8N}}\sin\frac{\pi}{N}}{1-e^{-\frac{\sqrt{7}\pi}{N}}}.$$

And, it is routine to check the function

$$\frac{2e^{-\frac{3\sqrt{7}}{8}X}\sin X}{1-e^{-\sqrt{7}X}} \quad \text{for } 0 < X \leq \frac{\pi}{4}$$

is less than 0.9. See the Figure 2(we used MAPLE 8 for the graph):

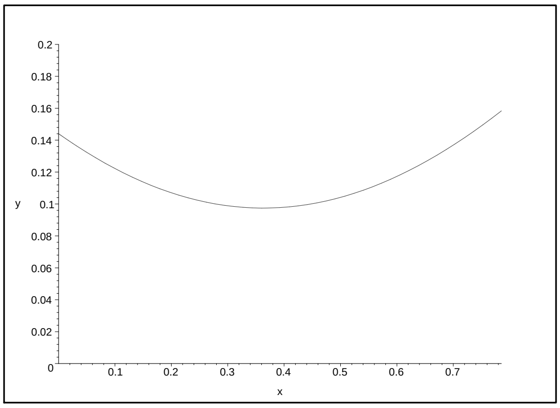

FIGURE 2. *The graph of* $Y = 0.9 - \frac{2e^{-\frac{3\sqrt{7}}{8}X}\sin X}{1-e^{-\sqrt{7}X}}$ *for* $0 < X \leq \frac{\pi}{4}$

□

Furthermore, we have the inequality

$$1 + X < e^X \quad \text{for } X > 0. \tag{2.3}$$

LEMMA 2.2. *We have the inequality*

$$\left|g_{(0,\frac{1}{N})}(\theta)\right| < \left|g_{(\frac{a}{N},\frac{b}{N})}(\theta)\right|$$

for $a,\ b \in \mathbb{Z}$ *such that* $a \not\equiv 0 \pmod N$.

Proof. We may assume that $1 \leq a \leq \frac{N}{2}$ by Proposition 1.3. Observe that

$$\begin{aligned}
\left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(\frac{a}{N},\frac{b}{N})}(\theta)}\right| &= \left|\frac{q_\theta^{\frac{1}{2}\mathbf{B}_2(0)}(1-\zeta_N)\prod_{n=1}^\infty(1-q_\theta^n\zeta_N)(1-q_\theta^n\zeta_N^{-1})}{q_\theta^{\frac{1}{2}\mathbf{B}_2(\frac{a}{N})}(1-q_\theta^{\frac{a}{N}}\zeta_N^b)\prod_{n=1}^\infty(1-q_\theta^{n+\frac{a}{N}}\zeta_N^b)(1-q_\theta^{n-\frac{a}{N}}\zeta_N^{-b})}\right| \text{ by (1.1)}\\
&\leq A^{\frac{N}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{a}{N}))}\left|\frac{1-\zeta_N}{1-A^a}\right|\frac{\prod_{n=1}^\infty(1+A^{Nn})^2}{\prod_{n=1}^\infty(1-A^{Nn+a})(1-A^{Nn-a})}\\
&< 0.9\prod_{n=1}^\infty(1+A^{Nn})^2(1+A^{Nn+a-\frac{N}{2}})(1+A^{Nn-a-\frac{N}{2}}) \quad \text{by Lemma 2.1(iii) and (ii)}\\
&< 0.9\prod_{n=1}^\infty(1+A^{Nn})^2(1+A^{Nn-\frac{N}{2}})^2 \quad \text{by Lemma 2.1(iii) and } 1 \leq a \leq \frac{N}{2}\\
&< 0.9\prod_{n=1}^\infty e^{2A^{Nn}+2A^{Nn-\frac{N}{2}}} = 0.9\, e^{\frac{2A^{\frac{N}{2}}}{1-A^{\frac{N}{2}}}} \quad \text{by (2.3)}\\
&\leq 0.9\, e^{\frac{2e^{-\frac{\sqrt{7}\pi}{2}}}{1-e^{-\frac{\sqrt{7}\pi}{2}}}} < 1 \quad \text{by (2.2)}.
\end{aligned}$$

This yields the lemma. □

Lemma 2.3. *We also have*

$$\left|g_{(0,\frac{1}{N})}(\theta)\right| < \left|g_{(0,\frac{b}{N})}(\theta)\right|$$

for $b \in \mathbb{Z}$ *with* $b \not\equiv 0, \ \pm 1 \pmod N$.

Proof. The proof is almost the same as that of the previous lemma. We may assume that $2 \le b \le \frac{N}{2}$. Put $A = |q_\theta|^{\frac{1}{N}} = |e^{2\pi i\theta}|^{\frac{1}{N}}$, and note that

$$\begin{aligned}
\left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(0,\frac{b}{N})}(\theta)}\right| &= \left|\frac{q_\theta^{\frac{1}{2}\mathbf{B}_2(0)}(1-\zeta_N)\prod_{n=1}^{\infty}(1-q_\theta^n\zeta_N)(1-q_\theta^n\zeta_N^{-1})}{q_\theta^{\frac{1}{2}\mathbf{B}_2(0)}(1-\zeta_N^b)\prod_{n=1}^{\infty}(1-q_\theta^n\zeta_N^b)(1-q_\theta^n\zeta_N^{-b})}\right| \quad \text{by (1.1)} \\
&\le \left|\frac{1-\zeta_N}{1-\zeta_N^b}\right| \frac{\prod_{n=1}^{\infty}(1+A^{Nn})^2}{\prod_{n=1}^{\infty}(1-A^{Nn})^2} \\
&< \frac{1}{\sqrt{2}}\prod_{n=1}^{\infty}(1+A^{Nn})^2(1+A^{Nn-\frac{N}{2}})^2 \quad \text{by Lemma 2.1(i) and (ii)} \\
&< \frac{1}{\sqrt{2}}\prod_{n=1}^{\infty}e^{2A^{Nn}+2A^{Nn-\frac{N}{2}}} \quad \text{by (2.3)} \\
&= \frac{1}{\sqrt{2}}e^{2\left((1-A^{\frac{N}{2}})^{-1}-1\right)} \le \frac{1}{\sqrt{2}}e^{2\left((1-e^{-\frac{\sqrt{7}\pi}{2}})^{-1}-1\right)} < 1 \quad \text{by (2.2).}
\end{aligned}$$

□

Theorem 2.4. *Let* $N \ge 4$. *For an imaginary quadratic field* $K \ne \mathbb{Q}(\sqrt{-1}), \ \mathbb{Q}(\sqrt{-3})$ *the value*

$$g^{\frac{12Nn}{\gcd(6,N)}}(1, \ N\mathcal{O}_K) = g_{(0,\frac{1}{N})}^{\frac{12Nn}{\gcd(6,N)}}(\theta) \qquad \text{for} \quad n = 1, \ 2, \ \cdots$$

generates $K(N)$ *over* $K(1)$. *It is a real algebraic integer. So its minimal polynomial over* $K(1)$ *has coefficients in* $\mathbb{Z}[j(\theta)]$. *In particular, if* N *has at least two prime factors, then it is a unit.*

Proof. For simplicity, we put $m = \frac{12Nn}{\gcd(6,N)}$. Let $\alpha = \begin{pmatrix} t-Bs & -Cs \\ s & t\end{pmatrix}$ be an element of $W_{N,\theta}$ such that $\big(g_{(0,\frac{1}{N})}^m(\theta)\big)^{\overline{\alpha}} = g_{(0,\frac{1}{N})}^m(\theta)$. Then by (2.1) and Proposition 1.3

$$\big(g_{(0,\frac{1}{N})}^m(\theta)\big)^{\overline{\alpha}} = \big(g_{(0,\frac{1}{N})}^m(\tau)\big)^{\alpha}(\theta) = g_{(0,\frac{1}{N})\alpha}^m(\theta) = g_{(\frac{s}{N},\frac{t}{N})}^m(\theta) = g_{(0,\frac{1}{N})}^m(\theta).$$

By Lemma 2.2 we obtain $s \equiv 0 \pmod N$, which yields

$$g_{(0,\frac{t}{N})}^m(\theta) = g_{(0,\frac{1}{N})}^m(\theta). \tag{2.4}$$

Now by Lemma 2.3 we get $t \equiv \pm 1 \pmod N$. Thus $\alpha = \pm 1_2 \in W_{N,\theta}$, which claims that $K(1)\big(g_{(0,\frac{1}{N})}^m(\theta)\big) = K(N)$.

The functions $\big\{g_{(r_1,r_2)}^m(\tau)\big\}$, indexed by $(r_1,r_2) \in \frac{1}{N}\mathbb{Z}^2$ with (r_1,r_2) primitive modulo $\mathbb{Z}^2$, form a Fricke family([**7**] Chapter 2 §1). Applying [**7**] Chapter 2 Proposition 1.4 we derive

$$\overline{g^m(1, \ N\mathcal{O}_K)} = g^m(\overline{1}, \ \overline{N\mathcal{O}_K}) = g^m(1, \ N\mathcal{O}_K),$$

where $-$ means complex conjugation, which implies that $g^m(1, \ N\mathcal{O}_K)$ is a real algebraic number as an evaluation of the modular function $g_{(0,\frac{1}{N})}^m(\tau)$ at $\tau = \theta$. Since $j(\theta)$ is real and $[K:\mathbb{Q}]=2$, we establish $[K\big(j(\theta),\ g^m_{(0,\frac{1}{N})}(\theta)\big):K\big(j(\theta)\big)]=[\mathbb{Q}\big(j(\theta),\ g^m_{(0,\frac{1}{N})}(\theta)\big):\mathbb{Q}\big(j(\theta)\big)]$. Furthermore, we see from [**6**] §3 that the function $g^m_{(0,\frac{1}{N})}(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$. Hence the value $g^m_{(0,\frac{1}{N})}(\theta)$ is a real algebraic integer and its minimal polynomial over $K(1)=K(j(\theta))$ has coefficient in $\mathbb{Z}[j(\theta)]$. In particular, if N has at least two prime factors, the function $1/g^m_{(0,\frac{1}{N})}(\tau)$ is also integral over $\mathbb{Z}[j(\tau)]$. Therefore, $g^m_{(0,\frac{1}{N})}(\theta)$ is a unit. $\square$

REMARKS.

(i) Theorem 2.4 holds even in the cases $N=2$ and 3. It suffices to prove that $|g_{(0,\frac{1}{2})}|\neq|g_{(\frac{1}{2},\frac{b}{2})}|$ for $b=0,\ 1$, and $|g_{(0,\frac{1}{3})}|\neq|g_{(\frac{1}{3},\frac{b'}{3})}|$ for $b'=0,\ 1,\ 2$, which is easier than the above procedures.

(ii) It is also conjectured in [**11**] that the generators given by Theorem 2.4 are even generators over quadratic imaginary base field. On the other hand, in the forthcoming paper [**5**] we showed that the singular value $g^{\frac{12N}{\gcd(6,N)}}_{(0,\frac{1}{N})}(\theta)$ indeed generates the ray class field $K(N)$ over the base field K by further analyzing the action of Galois group $\mathrm{Gal}(K(1)/K)$.

(iii) The exponent of $g^{\frac{12N}{\gcd(6,N)}}_{(0,\frac{1}{N})}(\theta)$ could be quite high for numerical purposes. So one usually takes suitable products of Siegel functions with lower exponents(see [**1**]).

(iv) Note that in order for the singular value $g^{\frac{12N}{\gcd(6,N)}}_{(0,\frac{1}{N})}(\theta)$ to be a unit it suffices N to have more than one prime ideal factor in K.

EXAMPLE 1. Let $K=\mathbb{Q}(\sqrt{-19})$ and $N=4$. Then $h_K=1$([**3**]), and so $K(1)=K$. Taking $\theta=\frac{-1+\sqrt{-19}}{2}$ we have $\mathrm{irr}(\theta,\ \mathbb{Q})=X^2+X+5$ and

$$W_{4,\theta}/\{\pm 1_2\}=\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&3\\1&0\end{pmatrix},\begin{pmatrix}2&3\\1&1\end{pmatrix},\begin{pmatrix}3&3\\1&2\end{pmatrix},\begin{pmatrix}0&3\\1&3\end{pmatrix},\begin{pmatrix}3&2\\2&1\end{pmatrix}\right\}.$$

Hence the minimal polynomial of $g^{24}_{(0,\frac{1}{4})}(\theta)$ is

$$\begin{aligned}\mathrm{irr}(g^{24}_{(0,\frac{1}{4})}(\theta),K)&=\big(x-g^{24}_{(0,\frac{1}{4})}(\theta)\big)\big(x-g^{24}_{(\frac{1}{4},0)}(\theta)\big)\big(x-g^{24}_{(\frac{1}{4},\frac{1}{4})}(\theta)\big)\\&\quad\big(x-g^{24}_{(\frac{1}{4},\frac{2}{4})}(\theta)\big)\big(x-g^{24}_{(\frac{1}{4},\frac{3}{4})}(\theta)\big)\big(x-g^{24}_{(\frac{2}{4},\frac{1}{4})}(\theta)\big)\\&=x^6-885360x^5-84804816x^4-2089917952x^3\\&\quad-9283226880x^2-784074436608x+4096.\end{aligned}$$

EXAMPLE 2. Let $K=\mathbb{Q}(\sqrt{-43})$ and $N=6(=2\cdot 3)$. Then $h_K=1$([**3**]) so that $K(1)=K$. And, setting $\theta=\frac{-1+\sqrt{-43}}{2}$ we have $\mathrm{irr}(\theta,\ \mathbb{Q})=X^2+X+11$ and

$$\begin{aligned}W_{6,\theta}/\{\pm 1_2\}=\Big\{&\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&1\\1&0\end{pmatrix},\begin{pmatrix}2&1\\1&1\end{pmatrix},\begin{pmatrix}3&1\\1&2\end{pmatrix},\begin{pmatrix}4&1\\1&3\end{pmatrix},\begin{pmatrix}5&1\\1&4\end{pmatrix},\\&\begin{pmatrix}0&1\\1&5\end{pmatrix},\begin{pmatrix}3&2\\2&1\end{pmatrix},\begin{pmatrix}5&2\\2&3\end{pmatrix},\begin{pmatrix}1&2\\2&5\end{pmatrix},\begin{pmatrix}4&3\\3&1\end{pmatrix},\begin{pmatrix}5&3\\3&2\end{pmatrix}\Big\}.\end{aligned}$$

Hence the minimal polynomial of $g^{12}_{(0,\frac{1}{6})}(\theta)$ is

$$\begin{aligned}\mathrm{irr}(g^{12}_{(0,\frac{1}{6})}(\theta),K) &= \left(x-g^{12}_{(0,\frac{1}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{1}{6},0)}(\theta)\right)\left(x-g^{12}_{(\frac{1}{6},\frac{1}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{1}{6},\frac{2}{6})}(\theta)\right)\\ &\quad\left(x-g^{12}_{(\frac{1}{6},\frac{3}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{1}{6},\frac{4}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{1}{6},\frac{5}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{2}{6},\frac{1}{6})}(\theta)\right)\\ &\quad\left(x-g^{12}_{(\frac{2}{6},\frac{3}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{2}{6},\frac{5}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{3}{6},\frac{1}{6})}(\theta)\right)\left(x-g^{12}_{(\frac{3}{6},\frac{2}{6})}(\theta)\right)\\ &= x^{12}-60x^{11}++884737794x^{10}-30965791100x^{9}\\ &\quad+153944392815x^{8}+782759084947677000x^{7}-4267079045220x^{6}\\ &\quad+28203637156200x^{5}-12634001239185x^{4}+1496984221300x^{3}\\ &\quad-63700846206x^{2}+884736660x+1,\end{aligned}$$

which concludes that $g^{12}_{(0,\frac{1}{6})}(\theta)$ is a unit.

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Ho Yun Jung, Ja Kyung Koo and
Dong Hwa Shin
Department of Mathematical Sciences
KAIST
Daejeon 373-1
Korea

DOSAL@kaist.ac.kr
jkkoo@math.kaist.ac.kr
shakur01@kaist.ac.kr