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FREE GROUPS IN THE SECOND BOUNDED COHOMOLOGY

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Abstract. The second bounded cohomology of a free group of rank greater than 1 is infinite dimensional as a vector space over $\mathbb{R}$ [4]. For a group G and its n-th commutator subgroup $G^{(n)}$, the quotient $G/G^{(n)}$ is amenable and the homomorphism $\widehat{H}^2(G) \to \widehat{H}^2(G^{(n)})$ induced from the inclusion homomorphism $G^{(n)} \to G$ is injective. In this paper, we prove that if $G^{(n)}$ is free of rank greater than 1 for some finite ordinal n, then G is residually solvable and its second bounded cohomology is infinite dimensional. We prove its converse for a group generated by two elements. As for groups that are not residually solvable, we investigate the dimension of the second bounded cohomology of a perfect group. Also, some results on bounded cohomology of a connected CW complex X by applying a Quillen's plus construction X^+ to kill a perfect normal subgroup of $\pi_1 X$ are given.

1. Introduction

Bounded cohomology was first defined for discrete groups. It appeared in a version of a theorem of Hirsch and Thurston due to P. Trauber that the bounded cohomology of an amenable group is zero. Later, M. Gromov [5] defined the bounded cohomology of topological spaces and applied the theory of bounded cohomology to Riemannian geometry, thus demonstrating the importance of it.

In [6] N. Ivanov developed the R. Brooks's approach [2] to the theory of bounded cohomology from the view of relative homological algebra, and established the foundation of this theory.

In [2], [8], [4], [3], [9] and many other papers, the bounded cohomology groups of some important discrete groups are computed and many excellent examples are shown.

We review the definition of bounded cohomology.

For a discrete group G and a positive integer $n \geq 1$, let $C^n(G)$ be the space of all real valued functions $f\colon G^n \to \mathbb{R}$, where $G^n =$

[1]Keywords:The second bounded cohomology, free groups, residually solvable groups, perfect groups

[2]AMS classification: Primary: 55N35; Secondary: 20J06

$\underbrace{G \times G \times \cdots \times G}_{n}$. The boundary operator $\partial_n \colon C^n(G) \to C^{n+1}(G)$ for $n \geq 1$ is defined by the formula

$$\begin{aligned} \partial_n(f)(g_1, \cdots, g_{n+1}) &= f(g_2, \cdots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1} f(g_1, \cdots, g_n). \end{aligned} \tag{1.1}$$

Then it is easy to check $\partial_{n+1}\partial_n = 0$.

Definition 1.1. *The n-th cohomology of the complex*

$$0 \xrightarrow{\partial_{-1}=0} \mathbb{R} \xrightarrow{\partial_0=0} C(G) \xrightarrow{\partial_1} C^2(G) \xrightarrow{\partial_2} C^3(G) \xrightarrow{\partial_3} \cdots$$

is called the n-th cohomology of G with coefficients $\mathbb{R}$ and is denoted by $H^*(G)$.

Let $B^n(G)$ be the space of all bounded functions $f \colon G^n \to \mathbb{R}$, that is,

$$B^n(G) = \{ f \colon G^n \to \mathbb{R} \mid \|f\| < \infty \},$$

where $\|f\| = \sup\{ |f(g_1, \cdots, g_n)| \mid (g_1, \cdots, g_n) \in G^n \}$. It is easy to check that the sequence

$$0 \to \mathbb{R} \xrightarrow{d_0=0} B(G) \xrightarrow{d_1} B^2(G) \xrightarrow{d_2} B^3(G) \xrightarrow{d_3} \cdots \tag{1.2}$$

is a complex, where the boundary operator d_n is defined by the same formula for ∂_n as in (1.1).

Definition 1.2. *The n-th cohomology of the complex (1.2) is called the n-th* bounded cohomology *of G with trivial coefficients $\mathbb{R}$ and is denoted by $\widehat{H}^n(G)$.*

Similarly, we define bounded cohomology of a topological space X as follows. Recall that for every $n \geq 0$, the real n-dimensional singular cochain group $C^n(X)$ is defined as the set of all functions $f \colon S_n(X) \to \mathbb{R}$, where $S_n(X)$ is the set of n-dimensional singular simplices in X. As it is well known, the sequence

$$0 \to C^0(X) \xrightarrow{d_0} C^1(X) \xrightarrow{d_1} C^2(X) \xrightarrow{d_2} \cdots \tag{1.3}$$

is a complex, where d_* is defined by $d_n f(\sigma) = \sum_{i=0}^{n+1} (-1)^i f(\partial_i \sigma)$ and $\partial_i \sigma$ is the i-th face of the singular simplex σ.

Definition 1.3. *The cohomology of this complex (1.3) is called the singular cohomology group of X with coefficients $\mathbb{R}$ and denoted by $H^*(X)$.*

Let $B^n(X) \subset C^n(X)$ be the space of the real valued bounded functions $f\colon S_n(X) \to \mathbb{R}$, that is,

$$B^n(X) = \{f \in C^n(X) \mid \|f\| = \sup_{\sigma \in S_n(X)} |f(\sigma)| < \infty\}.$$

The space $B^*(X)$ is called the bounded cochain group and its elements the bounded cochains. It is easy to check that the sequence

$$0 \to B^0(X) \xrightarrow{d_0} B^1(X) \xrightarrow{d_1} B^2(X) \xrightarrow{d_2} \cdots \tag{1.4}$$

is a complex, where the boundary operator d_n is defined by the same formula as in (1.3).

Definition 1.4. *The cohomology of this complex (1.4) is called the* bounded cohomology *of X with trivial coefficients $\mathbb{R}$ and is denoted by $\widehat{H}^*(X)$.*

The first basic result on the theory of bounded cohomology is that the bounded cohomology of a simply connected space is zero as proved in [5] and [6].

As an important feature of this theory, N. Ivanov [6] proved the following.

Theorem 1.1. *Let X be a a topological space equipped with a universal covering. Then the bounded cohomology groups of X and of its fundamental group $\pi_1 X$ are (isometrically) isomorphic.*

Notice that Theorem 1.1 makes it possible to study this theory simultaneously from two view points: group theory and topology.

Let V denote either a discrete group or a space. Notice that, since $B^n(V)$ is a vector space over $\mathbb{R}$ for $n \geq 0$, $\widehat{H}^n(V)$ carries the structure of vector space over $\mathbb{R}$. Also, as we saw in the definition, there is a natural norm $\|\cdot\|$ on the space $B^n(V)$ and this natural norm turns it into a Banach space. Then there is a seminorm on $\widehat{H}^n(V)$ given by $\|[g]\| = inf\|f\|$, where $[g] \in \widehat{H}^n(V)$ and the infimum is taken over all bounded cochains $f \in \ker d_n$ lying in the bounded cohomology class corresponding to $[g]$. Thus this vector space $\widehat{H}^*(V)$ also carries the structure of seminormed space over $\mathbb{R}$.

From now on, we will understand **the dimension** of $\widehat{H}^n(G)$ (and of $\widehat{H}^*(X)$) as its dimension as a vector space over $\mathbb{R}$.

Amenable groups play a special role in the theory of bounded cohomology. Recall that a group G is called amenable if its action on the space of all bounded functions on G has a right invariant mean. For example, finite groups, abelian groups, solvable groups, subgroups and the homomorphic image of an amenable group are amenable. It is known that no group which contains a free group on two generators can be amenable.

In [5] and [6], the following is proved.

Theorem 1.2. *If a group G is amenable, then $\widehat{H}^n(G)$ is zero for every $n \geq 1$.*

From Theorem 1.1 and Theorem 1.2, the bounded cohomology of a space X is zero if its fundamental group is amenable.

We compute the bounded cohomology in dimensions 0 and 1. By Theorem 1.1, it is enough for us to compute them for a discrete group G. From the complex in (1.2), we have

$$\widehat{H}^0(G) = \ker(d_0) = \mathbb{R}.$$

Also

$$\begin{aligned}\ker(d_1) &= \{f \in B(G) \mid d_1(f) = 0\} \\ &= \{f \in B(G) \mid f(g_2) - f(g_1 g_2) + f(g_1) = 0 \quad \text{for } g_1, g_2 \in G\}.\end{aligned}$$

Thus $\ker(d_1)$ is the space of all bounded homomorphisms $G \to \mathbb{R}$. Since there are no bounded homomorphisms $G \to \mathbb{R}$, we have

$$\widehat{H}^1(G) = \ker(d_1) = 0.$$

Thus, for a discrete group G or a topological space X, *the second* bounded cohomology of G or X should be investigated first.

One of the notable differences between ordinary cohomology and bounded cohomology also follows from a free group F of rank greater than 1 as well as an amenable group. Recall that an amenable group does not contain a free group of rank 2 and its bounded cohomology is zero for every dimension. Also recall that $H^n(F)$ is zero for all $n \geq 2$. In particular, $H^2(F)$ is zero. On the other hand, we have the following.

Theorem 1.3. *Let F be a free group of rank greater than 1. Then $\widehat{H}^2(F)$ is infinite dimensional as a vector space over $\mathbb{R}$.*

R. Brooks [2] constructed infinitely many generators for $\widehat{H}^2(F)$. In [4] Grigorchuk proved Theorem 1.3 by constructing explicitly the infinitely many linearly independent generators based on pseudocharacters.

In [3] Fujiwara conjectured that the second bounded cohomology of a discrete group is either zero or infinite dimensional. However, S. Matsumoto and S. Morita [8] proved that

$$\widehat{H}^2(\mathrm{Homeo}_+(S^1)) \cong \widehat{H}^2\left(SL(2,\mathbb{R})\right) \cong \mathbb{R},$$

where $\mathrm{Homeo}_+(S^1)$ is the group of the orientation preserving homeomorphisms on S^1 (both $\mathrm{Homeo}_+(S^1)$ and $SL(2,\mathbb{R})$ are considered as discrete groups). Notice that the groups $\mathrm{Homeo}_+(S^1)$ and $SL(2,\mathbb{R})$ shown as counterexamples to Fujiwara's conjecture are perfect. Also notice that free groups are not perfect, and any commutator subgroup of a perfect group can not be free of rank greater than 1. Hence the relationship between the dimension of the second bounded cohomology of a group without nontrivial perfect normal subgroups and free groups of rank greater than 1 is interesting.

Now we describe the content of the paper. As the second bounded cohomology of a free group of rank greater than 1 is infinite dimensional, it seems natural to ask if there is some relationship between a free group and a group whose second second bounded cohomology is infinite dimensional. We see some relationship between them in Section 2 in terms of their commutator subgroups. We prove that if the n-th commutator subgroup $G^{(n)} \trianglelefteq G$ of a group G for some finite ordinal n is free of rank greater than 1, then G is residually solvable and $\widehat{H}^2(G)$ is infinite dimensional (Theorem 2.7). In particular, a group generated by two elements is our special interest (Theorem 2.8). In Section 3, we investigate the dimension of the second bounded cohomology of a finitely presented uniformly perfect group. Also, we see that the dimension of its second bounded cohomology is closely related to $H_2(G,\mathbb{Z})$, which is called the Schur multiplier of G (Theorem 3.6). In Section 4, we survey how Quillen's plus construction X^+ of a connected CW complex X for killing a perfect normal subgroup of $\pi_1 X$ is related to the bounded cohomology of X.

In the rest of this paper, G denotes a discrete group.

Acknowledgment This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2010-0001651)

2. Free groups in the second bounded cohomology

First, we consider $\mathrm{Homeo}_k(\mathbb{R}^n)$ the group of all homeomorphisms of $\mathbb{R}^n$ with compact support. It is known that this group contains a free group of rank 2, so that it is not amenable [4]. In [8] it is proved that $\widehat{H}^n(\mathrm{Homeo}_k(\mathbb{R}^n)) = 0$ for every $n \geq 1$. Thus there is a nonamenable group whose second bounded cohomology is zero and the converse of the Theorem 1.2 is not true. Also this example shows that, in general, even if a group contains a free group of rank greater than 1, its second bounded cohomology is not necessarily infinite dimensional.

In this section, we will investigate the relationship between free groups and the dimension of the second bounded cohomology of a residually solvable group.

Definition 2.1. *A group G is said to be residually solvable if for each $g \in G$ with $g \neq e$ there is a normal subgroup N of G such that $g \notin N$ and the quotient G/N is solvable.*

Residually solvable groups are explained in terms of their commutator subgroups.

Definition 2.2. *The derived series of a group G is the family of subgroups defined inductively,*

$$G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq G^{(n)} \supseteq G^{(n+1)} \supseteq \cdots ,$$

where $G^{(n)}$ is the commutator subgroup of $G^{(n-1)}$ for $n \geq 1$ so that

$$G^{(n)} = [G^{(n-1)}, G^{(n-1)}] = \left(G^{(n-1)}\right)'.$$

The transfinite derived series of G is an extension of its derived series to higher ordinals defined by the rules

$$G^{(\alpha)} = [G^{(\alpha-1)}, G^{(\alpha-1)}] \quad \text{and} \quad G^{(\lambda)} = \bigcap_{\beta<\lambda} G^{(\beta)},$$

where $\alpha \geq 1$ is a nonlimit ordinal and λ is a limit ordinal.

Theorem 2.1. *A group G is residually solvable if and only if $G^{(\omega)} = \cap_{n<\omega} G^{(n)}$ is trivial, where ω is the first limit ordinal.*

Proof. This is Theorem 3.3 in [11]. □

Corollary 2.2. *For a group G, either G or the quotient group $G/G^{(\omega)}$ is always residually solvable.*

Proof. If $G^{(\omega)}$ is trivial, then G is residually solvable by Theorem 2.1. Suppose $G^{(\omega)}$ is not trivial. Then

$$(G/G^{(\omega)})^{(n)} = G^{(n)}G^{(\omega)}/G^{(\omega)} = G^{(n)}/G^{(\omega)}.$$

Hence $(G/G^{(\omega)})^{(\omega)}$ is trivial, so that $G/G^{(\omega)}$ is residually solvable by Theorem 2.1. Thus, either G or the quotient group $G/G^{(\omega)}$ is always residually solvable. □

Remark 2.1. If G is a solvable group, then $G^{(n)}$ is trivial for some finite ordinal n. Hence $G^{(\omega)} = \cap_{n<\omega} G^{(n)}$ is trivial. Thus by Theorem 2.1 solvable groups are residually solvable. The free groups F of rank greater than 1 are residually solvable, but not solvable [11].

Notice that, since a solvable group is amenable, a free group is solvable if and only if its rank is 1 by Theorem 1.2 and Theorem 1.3.

Recall that a group presentation $G =< X \mid R >= F/K$ means that F is a free group on X and K is the normal closure of the set of defining relators R in F. Also the abelianization of G, denoted by G_{ab}, is that $G_{ab} = G/[G, G]$.

Let $S_{g,r}$ be an orientable compact surface that has genus $g \geq 1$ and r boundary components, where $r > 0$. Recall that its fundamental group has a presentation

$$\pi_1(S_{g,r}) =< a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_r \mid \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{r} c_j > .$$

Then, from Tietze transformations, it is known that $\pi_1(S_{g,r})$ is free of rank $2g + r - 1 \geq 2$ by omitting one of the generators and the single defining relator. Thus $\pi_1(S_{g,r})$ is residually solvable by Remark 2.1 and also $\widehat{H}^2(\pi_1(S_{g,r}))$ is infinite dimensional by Theorem 1.3. Also, by Theorem 1.1, $\widehat{H}^2(S_{g,r})$ is isomorphic to $\widehat{H}^2(\pi_1(S_{g,r}))$ and so infinite dimensional.

As in the ordinary cohomology, a five-term exact sequence for bounded cohomology holds.

Theorem 2.3. *Let $N \trianglelefteq G$ be a normal subgroup of G. Then there is an exact sequence*

$$0 \to \widehat{H}^2(G/N) \to \widehat{H}^2(G) \to \widehat{H}^2(N)^{G/N} \to \widehat{H}^3(G/N) \to \widehat{H}^3(G),$$

where $\widehat{H}^2(N)^{G/N}$ is the vector space of G/N-invariants of $\widehat{H}^2(N)$.

Proof. This is Theorem 12.4.2 in [9] □

Recall that, for $k \geq 2$, there is an exact sequence

$$1 \to SL(k, \mathbb{R}) \to GL(k, \mathbb{R}) \to GL(k, \mathbb{R})/SL(k, \mathbb{R}) \to 1.$$

As it is well known, since $SL(k, \mathbb{R})$ is a commutator subgroup of $GL(k, \mathbb{R})$, the quotient $GL(k, \mathbb{R})/SL(k, \mathbb{R})$ is abelian and so amenable. Hence by Theorem 1.2 and Theorem 2.3 there is an isomorphism

$$\widehat{H}^2(GL(k, \mathbb{R})) \cong \widehat{H}^2(SL(k, \mathbb{R})^{GL(k,\mathbb{R})/SL(k,\mathbb{R})}.$$

Similarly, recall that $PGL(k, \mathbb{R}) = GL(k, \mathbb{R})/Z(GL(k, \mathbb{R}))$ and also $PSL(k, \mathbb{R}) = SL(k, \mathbb{R})/Z(SL(k, \mathbb{R}))$, where $Z(\cdot)$ denotes the center of $(\cdot)$. Since the center is abelian and so amenable, there are isomorphisms

$$\widehat{H}^2(GL(k, \mathbb{R})) \cong \widehat{H}^2(PGL(k, \mathbb{R})) \text{ and } \widehat{H}^2(SL(k, \mathbb{R})) \cong \widehat{H}^2(PSL(k, \mathbb{R})).$$

Remark 2.2. Let $G = F/K$ for a free group F. Recall that, for every finite ordinal n, we have

$$G^{(n)} = F^{(n)}K/K = F^{(n)}/(F^{(n)} \cap K).$$

Similarly, as shown in [11], for the first limit ordinal ω we have

$$G^{(\omega)} = (\cap_{n<\omega} F^{(n)}K)/K.$$

Hence $G^{(n)}$ for every finite ordinal n and $G^{(\omega)}$ induce the following exact sequences respectively:

- $\mathrm{Exact}(G^{(n)}): \quad 1 \to K \to F^{(n)}K \to F^{(n)}K/K \to 1,$
- $\mathrm{Exact}(G^{(\omega)}): 1 \to K \to \cap_{n<\omega} F^{(n)}K \to (\cap_{n<\omega} F^{(n)}K)/K \to 1.$

Definition 2.3. *An exact sequence of groups* $1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$ *is said to be trivial if either* A *or* C *is trivial or the sequence splits.*

Proposition 2.4. *Let* $G = F/K$ *for a free group* F. *The exact sequence*

$$\mathit{Exact}(G^{(n)}): \quad 1 \to K \to F^{(n)}K \to F^{(n)}K/K \to 1$$

is trivial for some finite ordinal n *if and only if either* G *is solvable or the* n*-th commutator subgroup* $G^{(n)}$ *is free of rank greater than 1.*

Proof. If F is free of rank 1, then it is abelian. Hence its quotient F/K and so G is abelian. So G is solvable.

Now we assume that F is free of rank greater than 1. Suppose n is the smallest finite ordinal such that the exact sequence

$$\mathrm{Exact}(G^{(n)}): \quad 1 \to K \to F^{(n)}K \to F^{(n)}K/K \to 1$$

is trivial. Then by Definition 2.3 either K or $G^{(n)}$ is trivial, or the sequence $\mathrm{Exact}(G^{(n)})$ splits. If K is trivial, then $G = F$ and so G free of rank greater than 1. If $G^{(n)}$ is trivial, then G is solvable.

Now, let the sequence $\mathrm{Exact}(G^{(n)})$ split. Then $F^{(n)}K$ is a semidirect product of K and $F^{(n)}K/K$. So there is a subgroup Q of $F^{(n)}K$ such that $Q \cong F^{(n)}K/K$, $F^{(n)}K = QK$ and $K \cap Q$ is trivial. Since a subgroup of a free group is free and $Q \leq F^{(n)}K \leq F$, the group Q is free. Since $Q \cong F^{(n)}K/K = G^{(n)}$, the n-th commutator subgroup $G^{(n)}$ is free. In particular, if the rank of $G^{(n)}$ is 1, then $G^{(n)}$ is abelian and so $G^{(n+1)}$ is trivial. Thus G is solvable. Otherwise, $G^{(n)}$ is free of rank greater than 1.

Conversely, assume either G is solvable or $G^{(n)}$ is free of rank greater than 1. If G is solvable, then for some n the n-th commutator subgroup $G^{(n)}$ is trivial and so $F^{(n)}K/K$ is trivial. Hence $\mathrm{Exact}(G^{(n)})$ is trivial. If $G^{(n)}$ is free of rank greater than 1, then, as it is well known, the sequence $\mathrm{Exact}(G^{(n)})$ splits. □

Corollary 2.5. *A group* G *is residually solvable if and only if either* $\mathit{Exact}(G^{(n)})$ *for some finite ordinal* n *or* $\mathit{Exact}(G^{(\omega)})$ *is trivial.*

Proof. Let $G = F/K$ for a free group F. If the rank of F is 1, then G is abelian and so $G^{(1)}$ is trivial. Thus G is solvable (and so residually solvable) and the exact sequence $\mathrm{Exact}(G^{(1)})$ is trivial.

Let the rank of F be greater than 1. If K is trivial, then G is free of rank greater than 1. Then G is residually solvable and the exact sequence

$$\mathrm{Exact}(G^{(0)}) :\ 1 \to K(=1) \to F \xrightarrow{\cong} G \to 1$$

is trivial. Thus, in the rest of this proof, we assume the rank of F is greater than 1 and K is not trivial.

Suppose G is residually solvable and $\mathrm{Exact}(G^{(n)})$ is not trivial for every finite ordinal n. Then, since $G^{(\omega)}$ is trivial by Theorem 2.1, the exact sequence $\mathrm{Exact}(G^{(\omega)})$ is trivial.

Conversely, we first suppose a sequence $\mathrm{Exact}(G^{(n)})$ is trivial for some finite ordinal n. Then by Proposition 2.4, either G is solvable or the n-th commutator subgroup $G^{(n)}$ is free of rank greater than 1. If G is solvable, then it is residually solvable. If $G^{(n)}$ is free of rank greater than 1, it is residually solvable (Remark 2.1). So $(G^{(n)})^{(\omega)}$ is trivial by Theorem 2.1. Notice that

$$(G^{(n)})^{(\omega)} = G^{(n+\omega)} = G^{(\omega)}.$$

Hence $G^{(\omega)}$ is trivial and so G is residually solvable. Now, suppose the exact sequence

$$\mathrm{Exact}(G^{(\omega)}) :\ 1 \to K \to \cap_{n<\omega} F^{(n)}K \to G^{(\omega)} = (\cap_{n<\omega} F^{(n)}K)/K \to 1$$

is trivial. It is enough for us to prove that $G^{(\omega)}$ is trivial by Theorem 2.1. Since K is not trivial by assumption, the sequence $\mathrm{Exact}(G^{(\omega)})$ splits. So there is a subgroup $Q \le \cap_{n<\omega} F^{(n)}K$ such that $Q \cong G^{(\omega)}$, $QK = \cap_{n<\omega} F^{(n)}K$ and $Q \cap K$ is trivial. Notice that, since $Q \cap K$ is trivial and $F^{(n+1)} \le F^{(n)}$, this subgroup Q must lie in $F^{(n)}$ for every n. Thus $Q \subseteq \cap_{n<\omega} F^{(n)}$. Since F is free of rank greater than 1, it is residually solvable and so $\cap_{n<\omega} F^{(n)}$ is trivial by Theorem 2.1. Thus Q and so $G^{(\omega)}$ is trivial. □

Recall that the commutator subgroup $F^{(1)}$ of a free group F of rank greater than 1 is infinitely generated. Thus, in general, a subgroup of a finitely generated free group is not necessarily finitely generated.

Proposition 2.6. *Let $G = F/K$ for a free group F and K be nontrivial. If $\widehat{H}^2(G)$ is infinite dimensional, then K is infinitely generated.*

Proof. Notice that K is a nontrivial normal subgroup of a free group F. Suppose K is finitely generated. Then, as it is known [7], K either is trivial or has finite index in F. Since K is nontrivial by assumption,

$|F/K|$ is finite. Thus G is finite and so amenable. Then $\widehat{H}^2(G)$ must be zero by Theorem 1.2. Hence K must be infinitely generated. $\square$

Remark 2.3. We consider $F^{(n)}K$ in Proposition 2.4.

(1) Suppose $F^{(n)} \cap K$ is trivial. Then $F^{(n)}K$ is a semidirect product of K and $F^{(n)}$. Also,
$$G^{(n)} = F^{(n)}K/K = F^{(n)}/(F^{(n)} \cap K) = F^{(n)}$$
and so $G^{(n)}$ is free of rank greater than 1. In particular, since $F^{(n)}K/F^{(n)}$ is a subgroup of a solvable group $F/F^{(n)}$ and
$$F^{(n)}K/F^{(n)} = K/(F^{(n)} \cap K) = K,$$
the group K is also solvable. Hence K is a free solvable group and therefore infinite cyclic.

(2) Suppose $F^{(n)} \cap K$ is nontrivial and finitely generated. Then, as shown in Proposition 2.6, the subgroup $F^{(n)} \cap K$ is of finite index in $F^{(n)}$ and so $|G^{(n)}| = |F^{(n)}/(F^{(n)} \cap K)| < \infty$. Thus $G^{(n)}$ is finite and so amenable. Then $\widehat{H}^k(G^{(n)})$ is zero for every $k > 0$ and hence $\widehat{H}^2(G)$ is zero by Theorem 2.3.

Theorem 2.7. *Let $G = F/K$ for a free group F. If the n-th commutator subgroup $G^{(n)}$ is free of rank greater than 1, then there is an inclusion*
$$\widehat{H}^2(F) \subseteq \widehat{H}^2(G).$$
In particular, the group G is residually solvable and $\widehat{H}^2(G)$ is infinite dimensional.

Proof. Notice that, if the rank of F is 1, the free group F is abelian. So its quotient G is also abelian, and so it is solvable. Then, for any finite ordinal n, the n-th commutator subgroup $G^{(n)}$ of G can not be free of rank greater than 1. Hence the rank of F must be greater than 1.

Suppose $G^{(n)}$ is free of rank greater than 1. Then, as it is well known, its commutator subgroup $G^{(n+1)}$ is free of infinite rank. Also notice that the rank of $F^{(1)}$ is also infinite, because the rank of F is greater than 1. Recall that the rank of a free group is an invariant even in the case where it is an infinite cardinal. Hence the free groups $G^{(n+1)}$ and $F^{(1)}$ are isomorphic. Notice that $G^{(n+1)}$, $F^{(1)}$ and $F^{(n+1)}$ are the normal subgroups of G, F and $F^{(1)}$, respectively. Their quotients $G/G^{(n+1)}$, $F/F^{(1)}$ and $F^{(1)}/F^{(n+1)}$ are solvable and so amenable. Hence $\widehat{H}^k(G/G^{(n+1)})$, $\widehat{H}^k(F/F^{(1)})$ and $\widehat{H}^k(F^{(1)}/F^{(n+1)})$ are all zero for every $k \geq 1$ by Theorem 1.2. Also
$$G/G^{(n+1)} = (F/K)/(F^{(n+1)}K/K) = F/F^{(n+1)}K.$$

Then, by applying Theorem 2.3 to these pairs of groups $G^{(n+1)} \trianglelefteq G$, $F^{(1)} \trianglelefteq F$ and $F^{(n+1)} \trianglelefteq F^{(1)}$ repeatedly, we have

$$\begin{aligned}
\widehat{H}^2(G) &\cong \widehat{H}^2(G^{(n+1)})^{G/G^{(n+1)}} \\
&\cong \widehat{H}^2(F^{(1)})^{F/F^{(n+1)}K} \\
&\cong \widehat{H}^2(F^{(n+1)})^{(F^{(1)}/F^{(n+1)})\cdot(F/F^{(n+1)}K)} \\
&\cong \widehat{H}^2(F^{(n+1)})^{F/F^{(n+1)}K} \\
&\supseteq \widehat{H}^2(F^{(n+1)})^{F/F^{(n+1)}} \\
&\cong \widehat{H}^2(F),
\end{aligned}$$

where the inclusion $\supseteq$ in the second line from the last follows from the fact that $F/F^{(n+1)}K \subseteq F/F^{(n+1)}$. Thus we have

$$\widehat{H}^2(F) \subseteq \widehat{H}^2(G).$$

Since F is free of rank greater than 1, the dimension of $\widehat{H}^2(F)$ is infinite by Theorem 1.3. Hence $\widehat{H}^2(G)$ is also infinite dimensional. Finally, as we saw in Corollary 2.5, since $G^{(n)}$ is free of rank greater than 1 and so is residually solvable, we have $(G^{(n)})^{(\omega)} = G^{(\omega)}$ and so $G^{(\omega)}$ is trivial. Thus G is also residually solvable. $\square$

Now we consider finitely presented groups.

Definition 2.4. *A group G is called finitely presented if it has a presentation with a finite number of generators and of defining relators.*

Let G be a surface group, that is, the fundamental group of a closed orientable surface of genus g greater than 1. A presentation of G is given by

$$G = F/K =< a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] > .$$

It is known that the first commutator subgroup $G^{(1)}$ is free of rank greater than 1. Hence G is residually solvable and also its second bounded cohomology is infinite dimensional by Theorem 2.7, as it is also proved in [4].

Theorem 2.8. *Let G be generated by two elements. If $\widehat{H}^2(G)$ is not zero, then its first commutator subgroup $G^{(1)}$ is free of rank greater than 1. In particular, the group G is residually solvable and $\widehat{H}^2(G)$ is infinite dimensional.*

Proof. Since G is generated by two elements, every subgroup of G is either abelian or free of rank greater than 1 [7]. Hence $G^{(1)}$ is either abelian or free of rank greater than 1. If $G^{(1)}$ is abelian, then $G^{(2)}$ is trivial and so G is solvable. Thus G is amenable and so $\widehat{H}^2(G)$ is zero by Theorem 1.2. Hence $G^{(1)}$ must be free of rank greater than 1.

The second statement follows from Theorem 2.7. $\square$

We consider the group $SL(2,\mathbb{Z})$ and the modular group $PSL(2,\mathbb{Z})$. Recall that

$$PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/Z(SL(2,\mathbb{Z})).$$

Since the center $Z(SL(2,\mathbb{Z}))$ of $SL(2,\mathbb{Z})$ is abelian and so amenable, by Theorem 1.2 and Theorem 2.3, we have

$$\widehat{H}^2(SL(2,\mathbb{Z})) \cong \widehat{H}^2(PSL(2,\mathbb{Z})).$$

As it is well known, these groups are generated by two elements with presentations

$$SL(2,\mathbb{Z}) =< x, y \mid x^2y^{-3}, x^4 >, \text{ where } x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

and

$$PSL(2,\mathbb{Z}) =< a, b \,|\, a^2, b^3 >,$$

where a and b are the images of x and y respectively of the surjective homomorphism $SL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z})$. Also

$$SL(2,\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \quad \text{and} \quad PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

In [3] and [4] it is shown that $\widehat{H}^2(SL(2,\mathbb{Z}))$ and $\widehat{H}^2(PSL(2,\mathbb{Z}))$ are infinite dimensional. Hence their first commutator subgroups are free of rank greater than 1 (in fact 2) and also the groups $SL(2,\mathbb{Z})$ and $PSL(2,\mathbb{Z})$ are residually solvable by Theorem 2.7.

Question. Suppose that G is finitely presented residually solvable group. If $\widehat{H}^2(G)$ is infinite dimensional, then is some n-th commutator subgroup $G^{(n)}$ of G free of rank greater than 1 (or does it contain a free quotient of rank greater than 1)?

3. The second bounded cohomology of a uniformly perfect group

Notice that a (nontrivial) residually solvable group can not be equal to its commutator subgroup.

Definition 3.1. *A group is said to be perfect if it is equal to its commutator subgroup. A maximal perfect subgroup is a perfect subgroup which is not contained in any other larger perfect subgroup.*

Recall that if a perfect group G is finite, then this perfect group is amenable and so $\widehat{H}^n(G) = 0$ for all $n \geq 1$. Recall that $SL(2, \mathbb{R})$ is perfect and $\widehat{H}^2(SL(2, \mathbb{R})) = \mathbb{R}$. Thus $\widehat{H}^2(SL(2, \mathbb{R}))$ has dimension 1. On the other hand, since the free product of perfect groups is perfect, the group $SL(2, \mathbb{R}) * SL(2, \mathbb{R})$ is perfect. And $\widehat{H}^2(SL(2, \mathbb{R}) * SL(2, \mathbb{R}))$ is infinite dimensional. In general, for nontrivial groups G_1 and G_2 except for $G_1 = G_2 = \mathbb{Z}_2$, it is proved in [3] that $\widehat{H}^2(G_1 * G_2)$ is infinite dimensional.

A maximal perfect subgroup of a residually solvable group is trivial. In this section, contrary to residually solvable groups, we investigate the dimension of the second bounded cohomology of a perfect group, in particular, a uniformly perfect group.

First, we recall the Universal Coefficient Theorem [13]:

Theorem 3.1. *Let M be a trivial G-module. Then there is a split exact sequence*

$$0 \to Ext^1_{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), M) \to H^n(G, M) \to Hom_{\mathbb{Z}}(H_n(G, \mathbb{Z}), M) \to 0.$$

Corollary 3.2. *For $M = \mathbb{R}$, there is an isomorphism*

$$H^n(G) = H^n(G, \mathbb{R}) \cong Hom_{\mathbb{Z}}(H_n(G, \mathbb{Z}), \mathbb{R}).$$

Proof. Since $\mathbb{R}$ is divisible, we have $Ext^1_{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), \mathbb{R}) = 0$. Hence the isomorphism follows from Theorem 3.1. □

Definition 3.2. *A group G is said to be uniformly perfect if there is a positive integer N such that every element of G can be presented as a product of at most N commutators.*

Uniformly perfect groups are perfect. It is clear that every finite perfect group is uniformly perfect. The alternating groups A_n for $n \geq 5$ are uniformly perfect. Also it is proved in [14] that the infinite alternating group $A_\infty = \lim_{\rightarrow} A_n$ is uniformly perfect, in fact, every element of A_∞ is a commutator. For a field $\mathbf{F}$, every element of $SL(n, \mathbf{F})$, for every n, is a product of at most 2 commutators. In particular, $SL(2, \mathbb{R})$ is uniformly perfect.

However, as we saw in Section 2, the group $SL(2, \mathbb{Z})$ is residually solvable. So it is not perfect.

In [8], the following theorem is proved:

Theorem 3.3. *If G is uniformly perfect, then homomorphism $\widehat{H}^2(G) \to H^2(G)$ is injective.*

Remark 3.1. Though a free product of perfect groups is perfect, a free product of uniformly perfect groups is not necessarily uniformly perfect.

For example, consider the uniformly perfect group $G = SL(2,\mathbb{R})$. The canonical embedding $G \hookrightarrow G * G$ induces homomorphisms

$$\widehat{H}^n(G * G) \xrightarrow{\varphi} \widehat{H}^n(G) \text{ and } H^n(G * G) \xrightarrow{\psi} H^n(G).$$

Then there is a commutative diagram

$$\begin{array}{ccc} \widehat{H}^2(G * G) & \xrightarrow{\varphi} & \widehat{H}^2(G) \oplus \widehat{H}^2(G) \\ \rho\big\downarrow & & \nu\big\downarrow \\ H^2(G * G) & \xrightarrow{\psi} & H^2(G) \oplus H^2(G). \end{array}$$

Since $G = SL(2,\mathbb{R})$ is uniformly perfect, the homomorphism ν is injective. It is known as in [8] that $H^2(SL(2,\mathbb{R})) = \mathbb{R}$. As it is well known, the homomorphism ψ is an isomorphism by Mayer-Vietoris sequence and so $H^2(G * G) = \mathbb{R} \oplus \mathbb{R}$. Recall that the second bounded cohomology of free product, except for $\mathbb{Z}_2 * \mathbb{Z}_2$, is infinite dimensional. Thus $\widehat{H}^2(SL(2,\mathbb{R}) * SL(2,\mathbb{R}))$ is infinite dimensional, and so the map ρ can not be injective. By Theorem 3.3, the group $SL(2,\mathbb{R}) * SL(2,\mathbb{R})$ is not uniformly perfect.

Remark 3.2. As it is well known, the group $H_2(G,\mathbb{Z})$ is called the Schur multiplier of G. It is an important invariant of a group that has applications in many areas, especially, for the central extension of a perfect group which we will see later. If $G = F/K$ with F free, it is a theorem of Hopf (see [12]) that $H_2(G,\mathbb{Z}) \cong (K \cap [F,F])/[F,K]$.

We first see how the dimension of the second bounded cohomology of a finitely presented uniformly perfect group is related to the group $H_2(G,\mathbb{Z})$.

Proposition 3.4. *Let G be finitely presented with m generators and n relators. Also, let r be the rank of G_{ab}. Then $H_2(G,\mathbb{Z})$ can be generated by $n - m + r$ elements.*

We refer the reader to [12] for a proof.

Corollary 3.5. *Let G be finitely presented with m generators and n relators, and r the rank of G_{ab}. Then $H^2(G) \cong \mathbb{R}^k$, where $0 \leq k \leq n - m + r$.*

Proof. Since G is finitely presented, by Proposition 3.4 the abelian group $H_2(G,\mathbb{Z})$ can be generated by $n - m + r$ elements. Hence we can write

$$H_2(G,\mathbb{Z}) \cong \mathbb{Z}^k \oplus T,$$

where T is its torsion subgroup of and its rank k is that $0 \leq k \leq n - m + r$. Then

$$\begin{aligned} H^2(G) &= Hom_{\mathbb{Z}}(H_2(G, \mathbb{Z}), \mathbb{R}) \\ &\cong Hom_{\mathbb{Z}}(\mathbb{Z}^k \oplus T, \mathbb{R}) \\ &\cong \mathbb{R}^k, \end{aligned}$$

where the first isomorphism follows from Corollary 3.2. Hence $H^2(G) \cong \mathbb{R}^k$ for $0 \leq k \leq n - m + r$. □

Definition 3.3. *A finite presentation is called balanced if it has the same number of generators and defining relators.*

Theorem 3.6. *Let G be uniformly perfect. If G is finitely presented with m generators and n relators, then the dimension of $\widehat{H}^2(G)$ is at most $n - m$. In particular, if G has a balanced finite presentation, then $\widehat{H}^2(G) = 0$.*

Proof. Since G is uniformly perfect, it is perfect and so its abelianization $G/[G, G]$ is trivial. Then from Corollary 3.5, there is an isomorphism $H^2(G) \cong \mathbb{R}^k$ for $0 \leq k \leq n - m$. Also by Theorem 3.3, there is an injective homomorphism

$$\widehat{H}^2(G) \hookrightarrow H^2(G) \cong \mathbb{R}^k.$$

Hence the dimension of $\widehat{H}^2(G)$ is less than or equal to $n - m$.

If a uniformly perfect group G has a balanced finite presentation, then $n = m$ by definition and so $\widehat{H}^2(G) = 0$. □

It is known in [1] that a perfect group G with a balanced finite presentation is the fundamental group of a homology 4-sphere, that is, $G = \pi_1 M$ for a smooth n-dimensional manifold M with $H_*(M, \mathbb{Z}) \cong H_*(S^4, \mathbb{Z})$. Notice that $H^1(G) = 0$ and also $H^2(G) = 0$ from the Hurewicz homomorphism. Since a uniformly perfect group is perfect, a uniformly perfect group G with a balanced finite presentation is also the fundamental group of a homology 4-sphere M. In this case, by Theorem 3.6 we have $\widehat{H}^2(M) = \widehat{H}^2(G) = 0$.

We consider another property of a perfect group. Recall that a central extension of a group G is an exact sequence

$$0 \to C \to E \to G \to 1,$$

where C lies in the center of E. This central extension is said to be universal if for any central extension

$$0 \to C' \to E' \to G \to 1$$

of G, there is a unique homomorphism $E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{=} & & \downarrow \\ 0 & \longrightarrow & C' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

Not every group has a universal central extension in general, but we have the following:

Theorem 3.7. *A group G is perfect if and only if G has a universal central extension*

$$0 \to C \to E \to G \to 0,$$

where E is perfect and $H_2(E, \mathbb{Z}) = 0$. Furthermore, $C \cong H_2(G, \mathbb{Z})$.

The proof can be found in [13].

Proposition 3.8. *Let $0 \to H_2(G, \mathbb{Z}) \to E \to G \to 1$ be a universal central extension of a perfect group G. Then there is an isomorphism of vector spaces*

$$\widehat{H}^2(G) \cong \widehat{H}^2(E).$$

Proof. Let $C = H_2(G, \mathbb{Z})$, so that $G = E/C$. Since C is abelian, it is amenable and so $\widehat{H}^n(C)$ is zero for every $n \geq 1$. Hence it follows from Theorem 2.3. □

As shown in [6], since C is an amenable normal subgroup of E, $\widehat{H}^n(E/C)$ and $\widehat{H}^n(E)$ are isomorphic for every $n \geq 1$.

Recall that the inclusion homomorphism $B^*(G) \hookrightarrow C^*(G)$ induces a homomorphism $\widehat{H}^*(G) \xrightarrow{\Phi} H^*(G)$. It is easy to see that this induced homomorphism is in general neither injective nor surjective. For example, if G is free of rank 1, then it is not surjective. Also if G is free of rank greater than 1, then it can not be injective.

Definition 3.4. *The kernel of $\widehat{H}^*(G) \xrightarrow{\Phi} H^*(G)$ is called the singular part of $\widehat{H}^*(G)$ and denoted by $\widehat{H}^*_s(G)$. Also the image of Φ is called the bounded part of $H^*(G)$ and denoted by $H^*_b(G)$.*

The singular part $\widehat{H}^2_s(G)$ plays an important role for the dimension of $\widehat{H}^2(G)$. There is another description of the singular part of $\widehat{H}^2(G)$.

Definition 3.5. *An element $f \in C(G)$ is called a pseudocharacter if $\partial^1 f \in B^2(G)$ and also $f(g^n) = nf(g)$ for $n \in \mathbb{Z}$. The space of pseudocharacters of G is denoted by $PX(G)$. We also denote by $X(G)$ the space of characters of G, that is, the space of homomorphisms $G \to \mathbb{R}$.*

Notice that

$$X(G) = H^1(G) = Hom_{\mathbb{Z}}(H_1(G,\mathbb{Z}),\mathbb{R}).$$

Theorem 3.9. *There are isomorphisms of vector spaces*

$$i)\ \widehat{H}^*(G) \cong \widehat{H}^*_s(G) \oplus H^*_b(G)$$
$$ii)\ \widehat{H}^2_s(G) \cong PX(G)/X(G).$$

Proof. The first is Corollary 1.15 and the second is Theorem 3.5 in [4]. □

Proposition 3.10. *Let G be finitely presented. Suppose $\widehat{H}^2(G)$ is infinite dimensional. Then $PX(G)/X(G)$ is infinite dimensional.*

Proof. Let G have a finite presentation $< X \mid \mathcal{R} >$, where $|X|$ and $|\mathcal{R}|$ are finite. If $|X| = 1$, the group G abelian and so amenable. Then $\widehat{H}^2(G)$ is zero by Theorem 1.2. Hence we have $1 < |X| < \infty$. Then, by Corollary 3.5, the dimension of $H^2(G)$ as a vector space over $\mathbb{R}$ is finite. Then, since $\widehat{H}^2(G)$ is infinite dimensional, the kernel of the homomorphism $\widehat{H}^2(G) \to H^2(G)$ is infinite dimensional. Hence, by Definition 3.4 and Theorem 3.9, the space $PX(G)/X(G)$ is infinite dimensional. □

Recall that the commutator subgroup of $SL(2,\mathbb{Z})$ is free of rank 2 and of index 12. So the abelianization of $SL(2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{12}$ the cyclic group of order 12, so that its rank is zero. Also recall that $SL(2,\mathbb{Z})$ has a balanced finite presentation with 2 generators and 2 defining relators. Then by Corollary 3.5, we have $H^2(SL(2,\mathbb{Z})) = 0$. Thus the homomorphism $\widehat{H}^2(SL(2,\mathbb{Z})) \to H^2(SL(2,\mathbb{Z}))$ is zero. Hence from Theorem 3.9, we have

$$\begin{aligned}\widehat{H}^2(SL(2,\mathbb{Z})) &= \widehat{H}^2_s(SL(2,\mathbb{Z}))\\ &= PX(SL(2,\mathbb{Z}))/X(SL(2,\mathbb{Z})) = PX(SL(2,\mathbb{Z})),\end{aligned}$$

where the last equality follows from the following fact

$$\begin{aligned}X(SL(2,\mathbb{Z})) &= H^1(SL(2,\mathbb{Z}))\\ &= Hom_{\mathbb{Z}}(H_1(SL(2,\mathbb{Z}),\mathbb{Z}),\ \mathbb{R})\\ &= Hom_{\mathbb{Z}}(SL(2,\mathbb{Z})_{ab},\ \mathbb{R})\\ &= Hom_{\mathbb{Z}}(\mathbb{Z}_{12},\ \mathbb{R}) = 0.\end{aligned}$$

Notice that $PX(SL(2,\mathbb{Z}))$ is infinite dimensional.

Corollary 3.11. *Let G be generated by two elements. If there is at least one pseudocharacter that is not a character, then $\widehat{H}^2(G)$ is infinite dimensional. In particular, if G is also finitely related, then $PX(G)/X(G)$ is infinite dimensional.*

Proof. Let $\alpha \in PX(G) \setminus X(G)$. Then, since $PX(G)/X(G)$ is not zero, $\widehat{H}^2(G)$ is not zero either by Theorem 3.9. Hence $\widehat{H}^2(G)$ is infinite dimensional by Theorem 2.8.

Now suppose G has finitely many defining relators. Then G is finitely presented and so by Proposition 3.10 the vector space $PX(G)/X(G)$ is infinite dimensional. □

From Corollary 3.11, if G is a finitely presented group with two generators, then the existence of only one pseudocharacter that is not a character guarantees that there are infinitely many linearly independent pseudocharacters providing $\widehat{H}^2(G)$ with infinite dimension.

Proposition 3.12. *If G is uniformly perfect, then $PX(G) = 0$.*

Proof. Since G is perfect, it is clear that

$$X(G) = H^1(G) = Hom_{\mathbb{Z}}(H_1(G, \mathbb{Z}), \mathbb{R}) = 0.$$

Since G is uniformly perfect, the homomorphism $\widehat{H}^2(G) \xrightarrow{\Phi} H^2(G)$ is injective by Theorem 3.3. Hence,

$$0 = \ker \Phi = \widehat{H}^2_s(G) \cong PX(G)/X(G) = PX(G).$$

□

Remark 3.3. Let $0 \to C \to E \to G \to 1$ be a universal central extension of a perfect group G.

(1) Since G and E are perfect, by Corollary 3.2 we have $H^1(G) = H^1(E) = 0$. Also, since $H_2(E, \mathbb{Z}) = 0$ by Theorem 3.7, we have $H^2(E) = 0$.
(2) Since C is an amenable normal subgroup, as shown in [6] there is an isomorphism

$$\widehat{H}^*(E) \cong \widehat{H}^*(E/C) = \widehat{H}^*(G).$$

(3) Since $0 = H^1(E) = X(E)$ and $H^2(E) = 0$, we have

$$\widehat{H}^2(G) = \widehat{H}^2(E/C) \cong \widehat{H}^2(E) = \widehat{H}^2_s(E) \cong PX(E).$$

Thus each pseudocharacter of E determines a unique bounded 2-cocycle of G.

Proposition 3.13. *Let*

$$0 \to C \xrightarrow{i} E \xrightarrow{p} G \to 1$$

be a universal central extension of a uniformly perfect group G, where $C = H_2(G, \mathbb{Z})$. Then there is an injective homomorphism

$$PX(E) \to Hom_{\mathbb{Z}}(H_2(G, \mathbb{Z}), \mathbb{R}).$$

Furthermore, suppose G is finitely presented. Then the dimension of $\widehat{H}^2(G)$ is no more than the rank of $H_2(G, \mathbb{Z})$.

Proof. Recall that there is a five-term exact sequence of the ordinary cohomology

$$0 \to H^1(G) \to H^1(E) \to H^1(C)^{E/C} \to H^2(G) \to H^2(E).$$

Since C is in the center of E, the action of E/C on $H^*(C)$ is trivial, so that $H^1(C)^{E/C} = H^1(C)$. Also, since $H^1(E) = H^2(E) = 0$ by Remark 3.3.(1), there is an isomorphism $H^1(C) \xrightarrow{\cong} H^2(G)$. Also by Remark 3.3.(2) there are isomorphisms

$$\widehat{H}^2(G) \cong \widehat{H}^2(E) \cong PX(E).$$

Consider the following diagram

$$\begin{array}{ccc}
\widehat{H}^2(G) & \xrightarrow[\cong]{p_*} & \widehat{H}^2(E) = PX(E) \\
{\scriptstyle\Phi}\big\downarrow & & \\
H^2(G) & \xleftarrow[\cong]{d_*} & H^1(C) = Hom_{\mathbb{Z}}(H_2(G, \mathbb{Z}), \mathbb{R}).
\end{array}$$

Since G is uniformly perfect, the homomorphism Φ is injective by Theorem 3.3. Then from the composition $d_*^{-1} \circ \Phi \circ p_*^{-1}$ there is an injective homomorphism

$$PX(E) \xrightarrow{\rho_*} Hom_{\mathbb{Z}}(H_2(G, \mathbb{Z}), \mathbb{R}).$$

Suppose G is finitely presented. By Corollary 3.4, we have $H^2(G) \cong \mathbb{R}^k$, where k is the rank of $H_2(G, \mathbb{Z})$ and $0 \le k < \infty$. Thus the dimension of $H^2(G)$ is k. So, by the injectivity of Φ, the dimension of $\widehat{H}^2(G)$ is at most k. Hence the dimension of $\widehat{H}^2(G)$ is no more than the rank of $H_2(G, \mathbb{Z})$. □

Corollary 3.14. *Let G be a finitely presented uniformly perfect group. If $\widehat{H}^2(G) \neq 0$, then $H_2(G, \mathbb{Z})$ is not a torsion group.*

Proof. If $\widehat{H}^2(G) \neq 0$, then the dimension of $\widehat{H}^2(G)$ is at least 1. Also by Proposition 3.13, the rank of $H_2(G,\mathbb{Z})$ is no less than the dimension of $\widehat{H}^2(G)$. Hence $H_2(G,\mathbb{Z})$ has rank at least 1 and so it can not be a torsion group. □

Let $0 \to C \to E \to G \to 0$ be a universal central extension of a uniformly perfect group G, where $C = H_2(G,\mathbb{Z})$. Since $\widehat{H}^2(C) = 0$, we have

$$PX(C) = X(C) = H^1(C) = Hom_{\mathbb{Z}}(H_2(G,\mathbb{Z}), \mathbb{R}).$$

Then from Proposition 3.13, there is an injective homomorphism

$$\rho_*\colon PX(E) \to PX(C).$$

Let

$$PX(E,C) = \{f \in PX(E) \mid f(x) = 0 \text{ for all } x \in C\}.$$

Then $PX(E,C) = \ker(\rho_*) = 0$. Thus every pseudocharacter vanishing on C vanishes on E.

4. An application of Quillen's plus construction

For a connected CW complex X, there is a construction, which is known as the Quillen's plus construction, for killing a perfect normal subgroup of $\pi_1 X$, while *preserving the ordinary homology:*

Theorem 4.1. *Let X be a connected CW complex and N be a perfect normal subgroup of $\pi_1(X)$. Then there is a new CW complex X^+ by attaching only 2-cells and 3-cells to X, so that the pair (X^+, X) satisfies the following conditions:*

1. *The map $\pi_1 X \to \pi_1 X^+$ induced by the inclusion $X \to X^+$ is the quotient map $\pi_1 X \to \pi_1 X/N$.*
2. *The pair (X^+, X) is homologically acyclic.*
3. *The space X^+ is unique up to homotopy.*

For a proof we refer the reader to [13].

Definition 4.1. *For a connected CW complex X and a perfect normal subgroup N of $\pi_1 X$, we call a Quillen's plus construction X^+ such that $\pi_1(X^+) = (\pi_1 X)/N$ the Quillen's space for X relative to $N \trianglelefteq \pi_1 X$.*

To see how Quillen's plus construction behaves in bounded cohomology, we briefly review the relative bounded cohomology.

In [10], the relative bounded cohomology of any continuous map $Y \xrightarrow{f} X$ of spaces X and Y, not necessarily $Y \subset X$, is defined as the cohomology of mapping cones of $B^*(X) \xrightarrow{f_*} B^*(Y)$, that is, as the cohomology of bounded cochain complex of $\{B^n(X) \bigoplus B^{n-1}(Y), d_n\}$, where the boundary operator d_n is defined by the equation $d_n(\alpha,\beta) = (\delta_n(\alpha), -f_*(\alpha)-\partial_{n-1}(\beta))$ from boundary operators δ_* and ∂_* of $B^*(X)$ and $B^*(Y)$ respectively. It is denoted by $\widehat{H}^*(Y \xrightarrow{f} X)$. Then for any continuous map $Y \xrightarrow{f} X$, there is a long exact sequence

$$\to \widehat{H}^{n-1}(Y) \to \widehat{H}^n(Y \xrightarrow{f} X) \to \widehat{H}^n(X) \xrightarrow{f_*} \widehat{H}^n(Y) \to .$$

In particular, for a pair (X,Y) of spaces with $Y \subset X$, by considering it as an inclusion $Y \hookrightarrow X$ the group $\widehat{H}^*(Y \hookrightarrow X)$ is isomorphic to the usual sense of the relative bounded cohomology $\widehat{H}^*(X,Y)$.

Similarly, for any group homomorphism $A \xrightarrow{\varphi} G$, the relative bounded cohomology of $A \xrightarrow{\varphi} G$ is defined as the cohomology of the complex of mapping cones $\{B^n(G) \oplus B^{n-1}(A), d_n\}$, and denoted by $\widehat{H}^*(A \xrightarrow{\varphi} G)$. In this manner, for any continuous map of semilocally simply connected spaces $Y \xrightarrow{f} X$ it is proved that

$$\widehat{H}^*(Y \xrightarrow{f} X) \cong \widehat{H}^*(\pi_1 Y \xrightarrow{f_\sharp} \pi_1 X).$$

Also there is a commutative diagram of the long exact sequence

$$\begin{array}{ccccccc}
\to \widehat{H}^{n-1}(Y) & \longrightarrow & \widehat{H}^n(Y \xrightarrow{f} X) & \longrightarrow & \widehat{H}^n(X) & \longrightarrow & \widehat{H}^n(Y) \to \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\to \widehat{H}^{n-1}(\pi_1 Y) & \longrightarrow & \widehat{H}^n(\pi_1 Y \xrightarrow{f_\sharp} \pi_1 X) & \longrightarrow & \widehat{H}^n(\pi_1 X) & \longrightarrow & \widehat{H}^n(\pi_1 Y) \to .
\end{array} \tag{4.1}$$

Remark 4.1. *The Excision Axiom does not hold in the theory of bounded cohomology.* For example, let $X = T$ be a torus and $Y \cong D^2 \subset X$, and $Z = p$ be a point in Y. Then $X - p$ is a punctured torus with $\pi_1(X-p) = \mathbb{Z} * \mathbb{Z}$ and $Y - p$ deformation retracts to a circle S^1. Recall that $\widehat{H}^1(\cdot) = 0$. Then we have the long exact sequence

$$0 \to \widehat{H}^2(T, D^2) \to \widehat{H}^2(T) \to \widehat{H}^2(D^2) \to \widehat{H}^3(T, D^2) \to \cdots .$$

Since $\pi_1(T)$ is amenable, the group $\widehat{H}^2(T) = 0$ and hence $\widehat{H}^2(T, D^2) = \widehat{H}^2(X,Y) = 0$. However, notice that there is an exact sequence

$$0 \to \widehat{H}^2(T-p, D^2-p) \to \widehat{H}^2(T-p) \to \widehat{H}^2(D^2-p) \to \widehat{H}^3(T-p, D^2-p) \to .$$

Also $\pi_1(D^2 - p) = \pi_1(S^1) = \mathbb{Z}$, so that $\pi_1(D^2 - p)$ is amenable. Then $\widehat{H}^2(D^2 - p) = 0$ and

$$\widehat{H}^2(T - p, D^2 - p) \cong \widehat{H}^2(T - p) = \widehat{H}^2(\mathbb{Z} * \mathbb{Z}).$$

Hence $\widehat{H}^2(T - p, D^2 - p)$ is infinite dimensional by Theorem 1.3.

Remark 4.2. Let X be a countable connected CW complex.

(1) Suppose $\pi_1 X$ is perfect. Then Quillen's space X^+ for X relative to $\pi_1 X \trianglelefteq \pi_1 X$ is a simply-connected CW complex.
(2) Suppose that $\pi_1 X$ has a perfect commutator subgroup, that is, $(\pi_1 X)^{(1)} = [\pi_1 X, \pi_1 X]$ is perfect. Then the Quillen's space X^+ for X relative to $(\pi_1 X)^{(1)} \trianglelefteq \pi_1 X$ gives
$$\pi_1(X^+) = \frac{\pi_1 X}{[\pi_1 X, \pi_1 X]} = (\pi_1 X)_{ab}.$$
Thus $\pi_1(X^+)$ is abelian.

In both cases, $H_n(X, \mathbb{Z})$ and $H_n(X^+, \mathbb{Z})$ are isomorphic by Theorem 4.1. Hence, by Corollary 3.2, the groups $H^n(X; \mathbb{R})$ and $H^n(X^+; \mathbb{R})$ are also isomorphic, and also $H^n(X^+, X; \mathbb{R}) = 0$ for all $n \geq 1$.

On the other hand, in both cases $\pi_1(X^+)$ are amenable. Hence $\widehat{H}^n(X^+) = 0$ for every $n \geq 1$. However, $\widehat{H}^n(X)$ may not be zero. Instead $\widehat{H}^n(X)$ and $\widehat{H}^{n+1}(X^+, X) = \widehat{H}^{n+1}(X \hookrightarrow X^+)$ are isomorphic.

Thus for the theory of bounded cohomology the condition (2) in Theorem 4.1 does not hold in general.

Remark 4.3. Let X be a countable connected CW complex and N be a perfect normal subgroup of $\pi_1 X$. Recall that there is a normal covering $Y \to X$ such that $\pi_1 Y = N$, and it induces a homomorphism $\pi_1 Y \to \pi_1 X$. Then the fundamental group of Quillen's space X^+ for X relative to $N \trianglelefteq \pi_1 X$ is that $\pi_1(X^+) = \pi_1 X / \pi_1 Y = \pi_1 X / N$. Thus we have an exact sequence
$$0 \to N \to \pi_1 X \to \pi_1 X^+ \to 0$$
and it induces a commutative diagram of the five-term exact sequences
$$\begin{array}{ccccccccc}
\widehat{H}^2(\pi_1 X/N) & \longrightarrow & \widehat{H}^2(\pi_1 X) & \longrightarrow & \widehat{H}^2(N)^{\pi_1 X/N} & \longrightarrow & \widehat{H}^3(\pi_1 X/N) & \longrightarrow & \widehat{H}^3(\pi_1 X) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\widehat{H}^2(X^+) & \longrightarrow & \widehat{H}^2(X) & \longrightarrow & \widehat{H}^2(Y)^{G/N} & \longrightarrow & \widehat{H}^3(X^+) & \longrightarrow & \widehat{H}^3(X),
\end{array}$$
where the first arrows in both rows are injective. Notice that the dimension of $\widehat{H}^2(X)$ is greater than or equal to the dimension of $\widehat{H}^2(X^+)$.

Remark 4.4. Let A be a closed subcomplex of a CW complex X. Notice that the cone CA on A is contractible. As it is well known, the map $X \cup CA \to X/A$ is a homotopy equivalence. Hence we have
$$X/A = (X \cup CA)/CA \simeq X \cup CA.$$
In fact, it is a homotopy equivalence of pairs
$$(X/A, *) \simeq (X \cup CA, CA) \simeq (X \cup CA, v),$$
where v is the vertex of the cone.

Hence there are the isomorphisms on the ordinary cohomology

$$\widetilde{H}^*(X/A,*) \cong H^*(X \cup CA, CA) \cong H^*(X,A),$$

where $\widetilde{H}^*(\cdot)$ denotes the reduced ordinary cohomology and the second isomorphism follows directly from the *Excision Axiom* by excising the vertex v of the cone CA.

However, since the Excision Axiom does not hold in the theory of bounded cohomology as shown in Remark 4.1, we will see that the groups $\widehat{H}^*(X/A)$ and $\widehat{H}^*(X,A)$ are not isomorphic in general.

Proposition 4.2. *Let X be a connected CW complex and X^+ the Quillen's space for X relative to $N \trianglelefteq \pi_1 X$. Then there is an isomorphism*

$$\widehat{H}^*(X^+/X) \cong \widehat{H}^*(X^+ \cup CX).$$

Proof. Notice that (X^+, X) is a CW pair with X as a subcomplex of X^+. Also $X^+/X = (X^+ \cup CX)/CX$. Since CX is contractible, the quotient map $X^+ \cup CX \to (X^+ \cup CX)/CX$ is a homotopy equivalence. Hence $\widehat{H}^*(X^+/X) \cong \widehat{H}^*(X^+ \cup CX)$. □

Theorem 4.3. *Let X be a connected CW complex and X^+ the Quillen's space for X relative to $N \trianglelefteq \pi_1 X$. Then $\pi_1(X^+/X)$ is trivial. Furthermore, $\widehat{H}^n(X^+/X) = 0$ for all $n \geq 1$.*

Proof. Recall that the spaces X^+/X and $X^+ \cup CX$ are homotopy equivalent. Hence it is enough for us to prove that $\pi_1(X^+ \cup CX)$ is trivial. Since $X^+ \cap CX = X$, we have the following squares such that the first square of inclusion maps of spaces induces the second commutative square of groups

$$\begin{array}{ccc} X & \xrightarrow{i} & X^+ \\ \downarrow & & \downarrow \\ CX & \longrightarrow & (X^+ \cup CX) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1 X & \xrightarrow{i_*} & \pi_1(X^+) \\ \downarrow & & \downarrow \\ \pi_1(CX) & \longrightarrow & \pi_1(X^+ \cup CX). \end{array}$$

Notice that the second square is an amalgamation diagram, where all fundamental groups are computed at a fixed vertex $x \in X$. Then $\pi_1(X^+ \cup CX) = \pi_1(X^+) *_{\pi_1 X} \pi_1(CX)$. Since CX is contractible, the group $\pi_1(CX)$ is trivial. Hence $\pi_1(X^+ \cup CX)$ is the quotient of $\pi_1(X^+)$ by the smallest normal subgroup of $\pi_1(X^+)$ containing the image $i_*\pi_1(X)$. By Quillen's plus construction, the homomorphism i_* is surjective

$$\pi_1 X \xrightarrow{i_*} \pi_1 X^+ = \pi_1 X/N.$$

Thus $i_*\pi_1 X = \pi_1(X^+)$. Then it shows that

$$\begin{aligned}\pi_1(X^+/X) &= \pi_1(X^+ \cup CX) = \pi_1(X^+) *_{\pi_1 X} \pi_1(CX)\\ &= (\pi_1 X^+)/(i_*\pi_1 X) = \pi_1(X^+)/\pi_1(X^+) = 1.\end{aligned}$$

Thus $\pi_1(X^+/X) = 1$ and so the quotient space X^+/X is a simply connected CW complex. Hence $\widehat{H}^n(X^+/X) = 0$ for all $n \geq 1$. $\square$

Remark 4.5. If X is a connected CW complex and X^+ is the Quillen's space for X relative to $N \trianglelefteq \pi_1 X$, the quotient space X^+/X is a bouquet of, possibly infinitely many, 2-cells and 3-cells. Then from Theorem 4.3 the fundamental group of a bouquet of, possibly infinitely many, 2-cells and 3-cells is trivial.

Proposition 4.4. *Let X be a connected CW complex and X^+ the Quillen's space for X relative to $N \trianglelefteq \pi_1 X$. Then $\widehat{H}^2(X^+, X) = 0$. In particular, if $\pi_1 X$ itself is perfect, then there is an isomorphism*

$$\widehat{H}^n(X) \cong \widehat{H}^{n+1}(X^+, X).$$

Proof. Recall that $\widehat{H}^1(\cdot) = 0$. The inclusion map $X \xrightarrow{i} X^+$ induces a surjective homomorphism $\pi_1 X \xrightarrow{p} \pi_1(X^+) = (\pi_1 X)/N$. Let $G = \pi_1 X$. As in the diagram (4.1), there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to \widehat{H}^2(X \xrightarrow{i} X^+) & \longrightarrow & \widehat{H}^2(X^+) & \xrightarrow{i_*} & \widehat{H}^2(X) & \longrightarrow & \widehat{H}^3(X \xrightarrow{i} X^+) \to \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ 0 \to \widehat{H}^2(G \xrightarrow{p} G/N) & \longrightarrow & \widehat{H}^2(G/N) & \xrightarrow{p_*} & \widehat{H}^2(G) & \longrightarrow & \widehat{H}^3(G \xrightarrow{p} G/N) \to . \end{array}$$

From Theorem 2.3, the homomorphism p_* is injective. So the homomorphism i_* is also injective. Hence $\widehat{H}^2(X^+, X) = \widehat{H}^2(X \hookrightarrow X^+) = 0$.

Suppose $\pi_1 X$ is perfect. Then X^+ is simply connected and so $\widehat{H}^n(X^+)$ is zero for every $n \geq 1$. Hence there is an isomorphism

$$\widehat{H}^n(X) \cong \widehat{H}^{n+1}(X^+, X).$$

$\square$

Let X^+ be the Quillen's space for a connected CW complex X relative to $N \trianglelefteq \pi_1 X$ for a perfect group N. Notice that by Remark 4.4 we have $H^n(X^+/X) = H^n(X^+, X) = 0$ for every $n \geq 1$. Also $\widehat{H}^n(X^+/X) = 0$ for all $n \geq 1$ by Theorem 4.3. But $\widehat{H}^n(X^+, X)$ is not zero for all $n \geq 1$ in general.

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