# Solvable subgroups of automorphisms of a compact Kähler manifold

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**Abstract**

Let $G$ be a solvable subgroup of the automorphism group $\mathrm{Aut}(X)$ of a compact Kähler manifold $X$ of complex dimension $n$, and let $N(G)$ be the normal subgroup of $G$ consisting of elements with null entropy. Let us denote by $G^*$ the image of $G$ under the natural map from $\mathrm{Aut}(X)$ to $GL(V, \mathbf{R})$, where $V$ is the Dolbeault cohomology group $H^{1,1}(X, \mathbf{R})$. Assume that the Zariski closure of $G^*$ in $GL(V_{\mathbf{C}})$ is connected. In this paper we show that $G/N(G)$ is a free abelian group of rank $r(G) \leq n-1$ and that the rank estimate is optimal. This gives a proof of the conjecture of Tits type. Our approach also gives some non-obvious implications on the structure of solvable subgroups of automorphisms of a compact Kähler manifold that are analogous to abelian subgroups of automorphisms. Moreover, if the rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$ and the identity component of $\mathrm{Aut}(X)$ is trivial, then it will be shown by using a theorem of Lieberman that $N(G)$ is a finite set.

## 1 Introduction and Main Results

Let $X$ be a compact Kähler manifold of complex dimension $n$, and let us denote by $\mathrm{Aut}(X)$ the biholomorphism (or automorphism) group of $X$. In this paper we study the structure of solvable subgroups of $\mathrm{Aut}(X)$. One motivation for this study comes from the paper [4] of Dinh and Sibony that deals with only abelian subgroups of automorphisms of a compact Kähler manifold. It will be worth pointing out the fact that the class of solvable subgroups of automorphisms of a compact Kähler manifold is not exhausted by abelian subgroups.

In order to describe our results in more detail, we first need to set up some notation and terminology. Let $f$ be an automorphism of $X$. The spectral radius

$$\rho(f) = \rho(f^*|_{H^2(X,\mathbf{C})})$$

of the action of $f$ on the cohomology ring $H^2(X,\mathbf{C})$ is defined to be the maximum of the absolute values of eigenvalues on the $\mathbf{C}$-linear extension of $f^*|_{H^2(X,\mathbf{R})}$. It is easy to show from the definition that $\rho(f^{\pm 1})$ is always less than or equal to $\rho(f^{\mp})^{n-1}$ (e.g., see [6]). We call $f$ *of null entropy* (resp. *of positive entropy*) if the spectral radius $\rho(f)$ is equal to 1 (resp. $> 1$). It is known by the results of Gromov and Yomdin in [5] and [11] that

$$\rho(f^*|_{H^2(X,\mathbf{C})}) = \rho(f^*|_{H^{1,1}(X,\mathbf{C})}).$$

We say that a subgroup $G$ of automorphisms is *of null entropy* (resp. *of positive entropy*) if all non-trivial elements of $G$ are of null entropy (resp. of positive entropy).

Now let $G^*$ denote the image of a subgroup $G$ of $\mathrm{Aut}(X)$ under the natural map from $\mathrm{Aut}(X)$ to $GL(V,\mathbf{R})$, where $V$ is the Dolbeault cohomology group $H^{1,1}(X,\mathbf{R})$. Then clearly $G^*$ is a subgroup of $GL(V,\mathbf{R})$. We call a group *virtually solvable* if it has a solvable subgroup of finite index. Let $V_{\mathbf{C}}$ be the complexification of $V$ so that $V_{\mathbf{C}}$ is now a finite dimensional vector space over $\mathbf{C}$. Then $G^*$ can be regarded as a subgroup of $GL(V_{\mathbf{C}})$ in a natural way, and a solvable subgroup $G^*$ of $GL(V_{\mathbf{C}})$ is called *connected* if its Zariski closure $\bar{G}^*$ in $GL(V_{\mathbf{C}})$ is connected. One important point to note here is that $G^*$ itself may not be connected, in general. From now on, for the sake of simplicity, we will say that $G$ is *connected* if $G^*$ is connected.

It is easy to see that given a virtually solvable subgroup $G^*$ of $GL(V_{\mathbf{C}})$, one can find a connected solvable finite-index subgroup $G_1$ of $G^*$. Moreover, any subgroup of a solvable group $G^*$ and any quotient group of $G^*$ are also solvable, and the closure of $G^*$ is solvable as well (see Section 2 of [7]).

For a subgroup of $GL(V_{\mathbf{C}})$, we then recall that Tits proved the following well-known alternative theorem in [10].

**Theorem 1.1** (Tits)**.** *Let $G^*$ be a subgroup of $GL(V_{\mathbf{C}})$. Then either $G^*$ is virtually solvable or it contains a non-commutative free group $\mathbf{Z} * \mathbf{Z}$.*

On the other hand, in the paper [4] Dinh and Sibony proved that if $G$ is an abelian subgroup of $\mathrm{Aut}(X)$ and $N(G)$ is the set consisting of elements with null entropy, then $N(G)$ is a normal subgroup of $G$ and $G/N(G)$ is a free abelian group of rank $\leq n-1$. So, in view of the above Tits' alternative and Dinh-Sibony's theorem, it is natural to ask the following interesting conjecture which is usually called a *conjecture of Tits type*.

**Conjecture 1.2.** *Let $X$ be a compact Kähler manifold of complex dimension $n$, and let $G$ be a connected solvable subgroup of the automorphism group* $\mathrm{Aut}(X)$. *Let*

$$N(G) = \{f \in G \,|\, f \text{ is of null entropy}\}.$$

*Then $G/N(G)$ is a free abelian group of rank $\leq n-1$.*

It is easy to see that the rank estimate is optimal from the case $X = E^n$, where $E$ is an elliptic curve. In fact, the conjecture has been proved by Keum, Oguiso, and Zhang in the paper [7] except for the rank estimate (see also [2] or Section 2). In the same paper [7], they also exhibited various examples such as complex tori, hyperkähler manifolds and minimal threefolds for which the full conjecture of Tits type holds. After that, a proof of the conjecture in full generality has appeared in the paper [12] of Zhang.

The aim of this paper is to give another proof of the conjecture of Tits type whose original idea traces back to the preprint [8] and which is more applicable to other related problems, and to show some more non-trivial results concerning the structure of solvable subgroups of automorphisms of a compact Kähler manifold. We remark that results of this paper are some applications of the excellent paper [4] of Dinh and Sibony and a theorem of Birkhoff-Perron-Frobenius (or a theorem of Lie-Kolchin type in [7]). One argument of Zhang originated from the paper of Dinh and Sibony also plays an important role in Section 3 (cf. Theorem 3.5).

To be precise, our first main result is

**Theorem 1.3.** *Let $X$ be a compact Kähler manifold of complex dimension $n$, and let $G$ be a connected solvable subgroup of the automorphism group* $\mathrm{Aut}(X)$. *Let*

$$N(G) = \{f \in G \,|\, f \text{ is of null entropy}\}.$$

*Then the following properties hold:*

(a) *$G/N(G)$ is a free abelian group of rank $r(G) \leq n-1$. Furthermore, the rank estimate is optimal.*

(b) *Let $h_k$ be the real dimension of the cohomology group $H^{k,k}(X, \mathbf{R})$. If $r(G) = n-1$, then $h_k$ satisfies*

$$h_k \geq \binom{n-1}{k}, \quad 1 \leq k \leq n-1. \tag{1.1}$$

*In addition, if $k$ divides $n-1$, then the lower bound of (1.1) can be improved by one, i.e., we have*

$$h_k \geq \binom{n-1}{k} + 1.$$

(c) *Let $\bar{\mathcal{K}}$ denote the closure of the Kähler cone $\mathcal{K}$. Then there exist $(r(G)+1)$ many non-zero classes $c_1, \ldots, c_{r(G)+1}$ in $\bar{\mathcal{K}}$ such that*

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{r(G)+1} \neq 0.$$

Clearly this theorem generalizes results of Dinh and Sibony for abelian subgroups of automorphisms with positive entropy to solvable subgroups. The proofs of Theorem 1.3 (a) and Theorem 1.3 (b)–(c) will be given in Theorem 3.5 and Proposition 3.6 of Section 3, respectively.

In their paper [4], Dinh and Sibony also proved that if $G$ is abelian and the rank $r(G)$ is equal to $n-1$, then $N(G)$ is finite ([4], Proposition 4.7). Let $\mathrm{Aut}_0(X)$ denote the identity component of $\mathrm{Aut}(X)$ consisting of automorphisms homotopically equivalent to the identity. In a recent paper [13], Zhang investigated a question of finiteness of $N(G)$ for solvable subgroups $G$ of $\mathrm{Aut}(X)$. As a consequence, he proved that if $r(G) = n-1 = 2$ and $\mathrm{Aut}_0(X)$ is trivial, then $N(G)$ is finite ([13], Theorem 1.1 (3)). As a consequence of Theorem 1.1 and its approach, we significantly extend the result of Zhang, which holds only for complex dimension equal to 3, to an arbitrary complex dimension $n$.

Our second main result of this paper which affirmatively and also completely answers to Question 2.18 in [12] is

**Theorem 1.4.** *Let $X$ be a compact Kähler manifold of complex dimension $n$, and let $G$ be a connected solvable subgroup of the automorphism group* $\mathrm{Aut}(X)$*. Assume that the rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$ and that* $\mathrm{Aut}_0(X)$ *is trivial. Then $N(G)$ is a finite set.*

If $G$ is further assumed to be abelian in Theorem 1.4, it was shown in [4] that, even without the triviality of $\mathrm{Aut}_0(X)$, $N(G)$ is finite (see also [9], Proposition 2.2). However, it is known as in Example 4.5 of [4] that there is an abelian variety $X$ of complex dimension $n$ with a solvable subgroup $G$ of $\mathrm{Aut}(X)$ such that $N(G) = \mathrm{Aut}_0(X) \cong X$ and the rank $r(G) = n-1$ (see also [13], Remark 1.3 (1)).

In fact, it is easy to construct an example of a solvable subgroup which is not abelian by using an abelian variety as above. More precisely, let $E = \mathbb{C}/(\mathbb{Z}+\sqrt{-1}\mathbb{Z})$ and $X = E^n$. Then we let $C_4 = \langle\sqrt{-1}\rangle$ act diagonally on the $n$-dimensional abelian variety $X$. Next, set $G = C_4 \rtimes T$ be the semi-direct product of the cyclic group $C_4$ and the group $T$ of translations of $X$. Then clearly $G$ is not abelian, but $G^{(1)} = [G, G] = T$ and so $G^{(2)} = [G^{(1)}, G^{(1)}] = 0$. Thus $G$ is indeed a solvable subgroup of the whole automorphism group $\mathrm{Aut}(X)$. Consequently, once again we stress that a solvable subgroup of the automorphism group of a compact Kähler manifold is *not* necessarily abelian.

We organize this paper as follows. In Section 2, we first collect some basic facts which are relavent to the proof of Theorem 1.3, and then construct a homomorphism from a solvable subgroup of automorphisms to the abelian group $(\mathbf{R}^m, +)$. Here one of the key technical ingredients is a theorem of Birkhoff-Perron-Frobenius in [1] (or a theorem of Lie-Kolchin type in [7]). In Section 3, we give a detailed proof of Theorem 1.3. Finally, Section 4 is devoted to giving a proof of Theorem 1.4.

## 2 Theorem of Birkhoff-Perron-Frobenius and its Applications

The goal of this section is to set up some preliminary results necessary for the proof of our main Theorem 1.3. As mentioned earlier, one of the key ingredients is a theorem of Birkhoff-Perron-Frobenius in [1] (or more generally a theorem of Lie-Kolchin type established in [7]).

Let $V$ be a finite dimensional real vector space and let $V_{\mathbf{C}}$ be its complexification. For a solvable group $G$, let $\rho : G \to GL(V_{\mathbf{C}})$ be a complex linear representation of $G$. Then we take the Zariski closure, denoted by $Z$, of $\rho(G)$ in $GL(V_{\mathbf{C}})$. Let $Z_0$ be the connected component of the identity in $Z$ and let $G_0 = \rho^{-1}(Z_0)$. Since $G$ is a solvable group, the group $Z_0$ is conjugate to a group of upper triangular matrices whose determinant is non-zero. Let $N(G_0)$ be the subgroup of $G_0$ whose elements are defined by the statement that $f$ is an element of $N(G_0)$ if and only if all eigenvalues of $f$ on $V_{\mathbf{C}}$ are equal to 1. Now observe that $N(G_0)$ is a normal subgroup of $G_0$ and that the abelian group $G_0/N(G_0)$ embeds into $(\mathbf{C}^*)^{\dim V_{\mathbf{C}}}$. Hence the rank of an abelian group $G_0/N(G_0)$ should be finite and, moreover, bounded from above by $\dim V_{\mathbf{C}}$. In order to obtain a sharp upper bound on the rank of $G_0/N(G_0)$, however, considerably more refined arguments need to be involved as in our paper. From now on, we shall denote by $G$ the group $G_0$.

Then we will need the following lemma whose proof is simple (e.g., see [7], Lemma 2.5).

**Lemma 2.1.** *Let $Z_0$ be a connected solvable subgroup of $GL(V_{\mathbf{C}})$. Then the eigenvalues of every element of the commutator subgroup $[Z_0, Z_0]$ of $Z_0$ are all equal to* 1.

Since $G$ is solvable, there exists a derived series of $G$, as follows.

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \quad \cdots \quad \triangleright G^{(k)} \triangleright G^{(k+1)} = \{\mathrm{id}\},$$

where $G^{(i+1)}$ is a normal subgroup of $G^{(i)}$ and $G^{(i+1)}$ is the commutator subgroup $[G^{(i)}, G^{(i)}]$ of $G^{(i)}$ $(0 \leq i \leq k)$. Let $A = G^{(k)}$. Then $A$ is an abelian subgroup of $G$, and clearly $A$ is a subset of $[G, G]$. Thus, by Lemma 2.1, every element of $A$ has all the eigenvalues equal to 1.

Recall that if $C$ is a subset of a real vector space $V$, then $C$ is said to be a *strictly convex closed cone* of $V$ if $C$ is closed in $V$, closed under addition and multiplication by a non-negative scalar, and contains no 1-dimensional linear space. In this paper, we also need the following theorem of Birkhoff-Perron-Frobenius in [1].

**Theorem 2.2.** *Let $C$ be a non-trivial strictly convex closed cone of $V$ with non-empty interior in $V$. Then any element $f$ of $GL(V)$ such that $f(C) \subset C$ has an eigenvector $v_f$ in $C$ whose eigenvalue is the spectral radius $\rho(f)$ of $f$ in $V$.*

In fact, if we use the subgroup $A$ of $[G, G]$ and Lemma 2.1, we obtain a stronger version for connected solvable subgroups of $GL(V)$ as in [7] which is called the *theorem of Lie-Kolchin type for a cone.* For more precise statement of Theorem 2.3, see Theorem 2.1 in [7].

**Theorem 2.3.** *Let $V$ be a finite dimensional real vector space, and let $C \neq \{0\}$ be a strictly convex closed cone of $V$. Let $G$ be a connected solvable subgroup of $GL(V)$ such that $G(C) \subset C$. Then there exists a nonzero vector in $C \backslash \{0\}$ which spans a one-dimensional subcone of $C$ invariant under $G$.*

From now on, let $X$ be a compact connected Kähler manifold of complex dimension $n$ as before, and let $V$ denote the Dolbeault cohomology group $H^{1,1}(X, \mathbf{R})$. In this paper, we will apply the above general discussion to a solvable subgroup $G$ of $\mathrm{Aut}(X)$ acting on $V_{\mathbf{C}} = V \otimes \mathbf{C} = H^{1,1}(X, \mathbf{R}) \otimes \mathbf{C}$. Since every element of $[G, G]$ has all the eigenvalues equal to 1 (i.e., every element of $[G, G]$ is *unipotent*) and $A$ is a subset of $[G, G]$, every element of $A$ also has all the eigenvalues equal to 1. In other words, this says that every element of $A$ is of null entropy.

Now let $\mathcal{K}$ denote the Kähler cone in the Dolbeault cohomology group $H^{1,1}(X, \mathbf{R})$. Then $\mathcal{K}$ is the cone of strictly positive smooth $(1,1)$-forms in $H^{1,1}(X, \mathbf{R})$, and it is a strictly convex open cone in $H^{1,1}(X, \mathbf{R})$ whose closure $\bar{\mathcal{K}}$ is also a strictly convex closed cone such that $\bar{\mathcal{K}} \cap -\bar{\mathcal{K}} = \{0\}$.

With these understood, we have the following corollary which is an immediate consequence of Theorem 2.3 and its proof in [7].

**Corollary 2.4.** *Let $G$ be a connected solvable group of automorphisms of a compact Kähler manifold, and let $f_0 \in G$. Then the following properties hold:*

(a) *For all $f \in G$, there exist a non-zero class $c_{f_0}$ in $\bar{\mathcal{K}}$ and a positive real number $\chi(f) \leq \rho(f)$ such that $c_{f_0}$ spans a one-dimensional subcone of $C$ invariant under $G$, i.e.,*

$$f^*(c_{f_0}) = \chi(f) c_{f_0},$$

*and such that $\chi(f_0)$ is greater than or equal to $1$.*

(b) *If $f \in G$ is of null entropy, then $\chi(f)$ is exactly equal to $1$.*

*Remark* 2.5. This corollary is a generalization of Lemma 4.1 in [4]. That is, if $G$ is abelian and $f_0$ is of positive entropy, then the statement holds to be true with $\chi(f_0)$ replaced by the spectral radius $\rho(f_0)$ of $f_0$ greater than 1.

*Proof.* To prove (a), we simply take $C := \bar{\mathcal{K}}$ in order to apply Theorem 2.3. Then it follows from Theorem 2.3 of Lie-Kolchin type (or [7], Theorem 2.1 and its proof) that there exists a non-zero eigenvector $c_{f_0} \in C = \bar{\mathcal{K}}$ for $G$ which spans a one-dimensional subcone of $C$ invariant under all of $G$. That is, we have $f^*(c_{f_0}) = \chi(f) c_{f_0}$ with a positive real number $\chi(f) \leq \rho(f)$. Moreover, the proof of Theorem 2.3 (or [7], Theorem 2.1) actually shows that given an element $f_0 \in G$, $\chi(f_0)$ is always taken to be a positive real number greater than or equal to 1. This completes the proof of Corollary 2.4 (a).

For the proof of (b), note first that if $f$ is of null entropy, then so is $f^{-1}$. Hence, if $\chi(f)$ is less than 1, it follows from $(f^{-1})^*(c_{f_0}) = \chi(f)^{-1} c_{f_0}$ that $f^{-1}$ cannot be of null entropy. This contradicts the choice of $f$. Note also that, again by the choice of $f$, $\chi(f)$ cannot be greater than 1. This completes the proof of Corollary 2.4 (b). □

To give a proof of Theorem 1.3, we need one more notation.

**Definition 2.6.** Let $\tau = (\tau(f))_{f \in G} \in \mathbf{R}^G$, and let $\Gamma_\tau$ be the cone of classes $c$ in $\bar{\mathcal{K}}$ such that

$$f^*(c) = \exp(\tau(f)) c$$

for all $f \in G$.

Then we set

$$F = \{\tau \in \mathbf{R}^G \,|\, \Gamma_\tau \neq \{0\}\}.$$

If $\Gamma_\tau \neq \{0\}$, then $\exp(\tau(f))$ is an eigenvalue of $f^*$ on $V = H^{1,1}(X, \mathbf{R})$. Since $V$ is finite dimensional, clearly $F$ must be finite. So, let $\tau_1, \tau_2, \cdots, \tau_m$ be all the elements of a finite set $F$. We then define a map $\pi : G \to \mathbf{R}^m$ given by

$$\pi : G \to \mathbf{R}^m, \quad f \mapsto (\tau_1(f), \tau_2(f), \ldots, \tau_m(f)). \tag{2.1}$$

It is not difficult to show that the following lemma holds:

**Lemma 2.7.** (a) *The integer $m$ satisfies the inequality*

$$m \leq h_1 := \dim H^{1,1}(X, \mathbf{R}).$$

(b) *The map $\pi$ is always a homomorphism into the abelian group $(\mathbf{R}^m, +)$. In particular, the image $\pi(G)$ is also abelian.*

*Proof.* For the proof of (a), since $\tau_1, \tau_2, \cdots, \tau_m$ are all distinct, there exists an element $f_0 \in G$ such that $\tau_i(f_0) \neq \tau_j(f_0)$ for $1 \leq i < j \leq m$. Thus $f_0^*$ on $V$ has at most $m$ distinct eigenvalues. Since the number of eigenvalues is clearly less than or equal to the dimension of $V$, $m$ should be less than or equal to the dimension of $V$ that is equal to $h_1$ in our case. This completes the proof of (a).

For the proof of (b), it suffices to prove that $\pi$ is a homomorphism. To do so, for each $c_i \in \Gamma_{\tau_i}$ observe first that

$$(f \circ g)^* c_i = \exp(\tau_i(f) + \tau_i(g)) c_i.$$

Hence we have $\pi(f \circ g) = \pi(f) + \pi(g)$ for all $f$ and $g$ in $G$, which means that $\pi$ is a group homomorphism. This completes the proof of Lemma 2.7. □

# 3 Proof of Theorem 1.3

In this section we give a proof of Theorem 1.3. The proof of this section is essentially an adaptation of the proof of the Principal Theorem by Dinh and Sibony in [4]. For the sake of reader's convenience, however, we shall show how to prove Theorem 1.3 relatively in detail. See [4] for more details.

First we need the following key technical lemma ([4], Lemma 4.3) from the paper [4] of Dinh and Sibony. Assume that $X$ is a compact Kähler manifold of dimension $n$, as before.

**Lemma 3.1.** *Let $c, c'$, and $c_i$ be the non-zero classes in $\bar{\mathcal{K}}$, $1 \leq i \leq t \leq n-2$, and let $f \in \mathrm{Aut}(X)$. Assume that there exist two distinct positive real constants $\lambda$ and $\lambda'$ such that*

$$f^*(c_1 \wedge \cdots \wedge c_t \wedge c) = \lambda c_1 \wedge \cdots \wedge c_t \wedge c,$$
$$f^*(c_1 \wedge \cdots \wedge c_t \wedge c') = \lambda' c_1 \wedge \cdots \wedge c_t \wedge c'.$$

*Assume also that* $c_1 \wedge \cdots \wedge c_t \wedge c \neq 0$ *and* $c_1 \wedge \cdots \wedge c_t \wedge c \wedge c' = 0$*. Then we have*

$$c_1 \wedge \cdots \wedge c_t \wedge c' = 0.$$

*Proof.* This lemma has nothing to do with a solvable subgroup of automorphisms of $X$. So we are done by Lemma 4.3 in [4]. $\square$

Lemma 3.1 will play an essential role in the proofs of Lemma 3.2, Theorem 3.5, Proposition 3.6, and Theorem 1.4.

As the proof of Theorem 3.5 below shows, if the rank $\tilde{r}$ of the image of the homomorphism $\pi$ defined in (2.1) is greater than or equal to $n-1$, it will be enough to use the homomorphism $\pi$ in order to prove Theorem 1.3. On the other hand, if $\tilde{r}$ is less than or equal to $n-2$, it turns out that we need some more homomorphisms from $G$ to $\mathbf{R}$ other than $\tau_i$'s, in order to obtain an injective homomorphism from $G/N(G)$. The following lemma provides such additional homomorphisms. Here we adapt a variation of some arguments originated from [4] (see also [12]).

**Lemma 3.2.** *Let* $\tilde{r}$ *denote the rank of the image of the homomorphism* $\pi$ *defined in* (2.1)*. Then the following properties hold:*

(a) *There exist non-zero classes* $c_j$ $(j = 1, 2, \cdots, \tilde{r})$ *in* $\bar{\mathcal{K}}$ *such that*

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}} \neq 0, \text{ and } f^*(c_j) = \exp(\tau_j(f))c_j \text{ for all } f \in G,$$

*where* $\tau_j : G \to \mathbf{R}$ *is a homomorphism.*

(b) *Assume that* $\tilde{r} \leq n-2$*. Then there exist additional non-zero classes* $c_{\tilde{r}+j}$ *in* $\bar{\mathcal{K}}$ *and homomorphisms* $\tilde{\tau}_{\tilde{r}+j} : G \to \mathbf{R}$ $(j = 1, \cdots, n - \tilde{r} - 1)$ *such that*

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1} \neq 0$$

*and such that for all* $f \in G$

$$\begin{aligned} &f^*(c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}} \wedge c_{\tilde{r}+1} \wedge \cdots \wedge c_{\tilde{r}+j}) \\ &= \exp(\tau_1(f)) \cdots \exp(\tau_{\tilde{r}}(f)) \exp(\tilde{\tau}_{\tilde{r}+1}(f)) \cdots \exp(\tilde{\tau}_{\tilde{r}+j}(f)) \\ &\cdot c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}} \wedge c_{\tilde{r}+1} \wedge \cdots \wedge c_{\tilde{r}+j}. \end{aligned} \tag{3.1}$$

*Remark* 3.3. By the way of construction, it is obvious that that each subcone of $V$ spanned by a non-zero class $c_{\tilde{r}+j}$ in $\bar{\mathcal{K}}$ in the statement of Lemma 3.2 (b) is not necessarily invariant under $G$. However, it follows from Corollary

2.4 (b) that those additional classes $c_{\tilde{r}+j}$ $(j = 1, 2, \cdots, n - \tilde{r} - 1)$ as well as $c_1, c_2, \cdots, c_{\tilde{r}}$ are all invariant under $N(G)$. That is, all of

$$\tau_1(f), \cdots, \tau_{\tilde{r}}(f), \tilde{\tau}_{\tilde{r}+1}(f), \cdots, \tilde{\tau}_{n-1}(f)$$

are zero for all $f \in N(G)$. This fact will play a crucial role later. In particular, it enables us to prove Theorem 1.4.

*Proof of Lemma 3.2.* For the proof of (a), we assume without loss of generality that the first $\tilde{r}$ coordinates of the map $\pi$ generate the image of the map $\pi$. Let us denote by $\tau_1, \cdots, \tau_{\tilde{r}}$ such $\tilde{r}$ coordinates. Let $c_i$ be a non-zero class in $\Gamma_{c_i}$ $(1 \le i \le \tilde{r})$. Then for any $I = \{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, \tilde{r}\}$, set

$$c_I = c_{i_1} \wedge \cdots \wedge c_{i_{k-1}} \wedge c_{i_k}.$$

To prove (a), it suffices to show that $c_I \neq 0$ for any subset $I$ of $\{1, 2, \cdots, \tilde{r}\}$, and we prove it only for the case of $I = \{1, 2, \cdots, k\}$, since other cases are similar. Indeed, if $k = 1$, by construction we have $c_1 \neq 0$ and so there is nothing to prove. For $k \ge 2$, we want to prove $c_1 \wedge c_2 \wedge \cdots \wedge c_k \neq 0$ by contradiction. So suppose that

$$\begin{aligned} &c_1 \wedge \cdots \wedge c_{k-2} \wedge c_{k-1} \neq 0, \quad c_1 \wedge \cdots \wedge c_{k-2} \wedge c_k \neq 0, \text{ and} \\ &c_1 \wedge \cdots \wedge c_{k-1} \wedge c_k = 0. \end{aligned}$$

We then apply Lemma 3.1 for $t = k - 2$, $c = c_{k-1}$, and $c' = c_k$. By Lemma 3.1, it is easy to see that $\tau_{k-1}(f) = \tau_k(f)$ for all $f \in G$. But, this implies that the image of $\pi$ lies in the hyperplane $\{x_{k-1} = x_k\}$, which contradicts the choice of $\tau_i$'s. This completes the proof of (a).

For the proof of (b), we continue to use the notations used in the proof of (a). Then consider the induced action of $G$ on the real subspace $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$ of $H^{\tilde{r}+1, \tilde{r}+1}(X, \mathbf{R})$. Here $c_1, \cdots, c_{\tilde{r}}$ are the non-zero classes that have been obtained in (a) above. Note that $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$ spans $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$.

Note also that $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$ is a strictly convex closed cone in $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$ that is invariant under $G$. To see it, we first make use of an elementary argument to show that $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$ is closed in $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$. Indeed, let $\{\alpha_j\}_{j=1}^{\dim \bar{\mathcal{K}}}$ be a basis for the vector space spanned by $\bar{\mathcal{K}}$. Then extend it to a basis of $H^{1,1}(X, \mathbf{R})$, denoted by $\{\alpha_j\}_{j=1}^{h_1}$. For the sake of simplicity, we assume that

$$\{c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \alpha_j\}_{j=1}^{h_1}$$

is a basis for $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$. Now, let

$$\{c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \sum_{j=1}^{\dim \bar{\mathcal{K}}} x_j^{(k)} \alpha_j\}_{k=1}^{\infty}, \quad x_j^{(k)} \in \mathbf{R}_{\geq 0}$$

be a sequence converging to an element

$$\beta = c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \sum_{j=1}^{h_1} x_j \alpha_j \in c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$$

for some $x_j \in \mathbf{R}$, with respect to the standard topology. Then it is clear that each $x_j^{(k)}$ converges to $x_j$ as $k$ goes to $\infty$. So all of the coefficients $x_j$'s, except possibly for $1 \leq j \leq \dim \bar{\mathcal{K}}$, are zero. That is, $\beta$ is an element of $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$. This finishes the proof of the claim.

The fact that $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$ is a strictly convex cone in $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$ which is invariant under $G$ is well-known (see, e.g., [12], Lemma 2.3 (1)).

Next, by Theorem 2.3 of Lie-Kolchin type applied to $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge \bar{\mathcal{K}}$ in $c_1 \wedge \cdots \wedge c_{\tilde{r}} \wedge H^{1,1}(X, \mathbf{R})$, we obtain a non-zero class $c_{\tilde{r}+1}$ in $\bar{\mathcal{K}}$ such that

$$\begin{aligned} & f^*(c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}+1}) \\ & = \exp(\tau_1(f)) \exp(\tau_2(f)) \cdots \exp(\tilde{\tau}_{\tilde{r}+1}(f)) c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}+1} \end{aligned}$$

for some functions $\tilde{\tau}_{\tilde{r}+1} : G \to \mathbf{R}$. Finally, in order to obtain the rest of the non-zero classes $c_{\tilde{r}+j} \in \bar{\mathcal{K}}$ satisfying the equation (3.1) one may apply the mathematical induction, which is now fairly straightforward. So we leave it to the reader.

As in the proof of Lemma 2.7, it is also easy to show that each $\tilde{\tau}_{\tilde{r}+j}$ $(j = 1, 2, \cdots, n - \tilde{r} - 1)$ is a homomorphism. This completes the proof of (b) and so Lemma 3.2. □

With these preliminaries, we next give a definition of the homomorphism $\Pi$ which will be used in Theorem 3.5.

**Definition 3.4.** Let $\tilde{r}$ denote the rank of the image of the homomorphism $\pi$. From now on, we assume without loss of generality that the first $\tilde{r}$ coordinates of the map $\pi$ generate the image of the map $\pi$. Let us denote by $\tau_1, \cdots, \tau_{\tilde{r}}$ such $\tilde{r}$ coordinates.

(a) If $\tilde{r} \geq n - 1$, a homomorphism $\Pi : G \to \mathbf{R}^{n-1}$ is just defined to be the homomorphism $\pi$ in (2.1) composed by the canonical projection onto the first $n - 1$ factors.

(b) On the other hand, if $1 \leq \tilde{r} \leq n-2$, by using the additional homomorphisms $\tilde{\tau}_{\tilde{r}+j}$ $(j = 1, 2, \cdots, n-\tilde{r}-1)$ constructed in Lemma 3.2 (b) we define a homomorphism $\Pi$ as follows.

$$\Pi : G \to \mathbf{R}^{n-1}, \quad f \mapsto (\tau_1(f), \cdots, \tau_{\tilde{r}}(f), \tilde{\tau}_{\tilde{r}+1}(f), \cdots, \tilde{\tau}_{n-1}(f)).$$

We are now in a position to prove Theorem 3.5, which also gives a proof of Theorem 1.3.

**Theorem 3.5.** *Let $G$ be a connected solvable group of automorphisms of a compact Kähler manifold $X$ of complex dimension $n$, and let $N(G)$ be the normal subgroup of $G$ defined as in Theorem 1.3. Then the induced homomorphism $\Pi : G/N(G) \to \mathbf{R}^{n-1}$ is injective and its image is discrete. In particular, $G/N(G)$ is a free abelian group of rank $\leq n-1$.*

*Proof.* The proof is divided into three steps.

*Step 1*: We shall first show that $\Pi(G/N(G))$ is discrete in the additive group $\mathbf{R}^{n-1}$ with the standard topology. To do so, it suffices to prove that $(0)_{i=1}^{n-1}$ is an isolated point in the image of $\Pi$ with the induced topology from $\mathbf{R}^{n-1}$. This is because the map $\Pi$ is a homomorphism.

First we deal with the case when $\tilde{r} \geq n-1$. Fix an arbitrary positive real number $\delta$. Then consider the set of all elements $f \in G/N(G)$ satisfying the following condition:

$$|\tau_j(f)| < \delta, \quad j = 1, 2, \cdots, n-1. \tag{3.2}$$

By Theorem 2.2 of Birkhoff-Perron-Frobenius, there exist non-zero classes $\kappa_i$ $(i = 1, 2)$ in the closure of the Kähler cone such that $f^*(\kappa_i) = \lambda_i \kappa_i$ with $\lambda_1 = \rho(f) > 1$ and $\lambda_2^{-1} = \rho(f^{-1}) > 1$. Moreover, it follows from Lemma 3.2 (a) that there exist non-zero classes $c_j \in \bar{\mathcal{K}}$ $(j = 1, 2, \cdots, n-1)$ such that

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \neq 0 \text{ and } f^*(c_j) = \exp(\tau_j(f)) c_j$$

for all $f \in G$. Thus we have

$$\begin{aligned} &c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge \kappa_i = f^*(c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge \kappa_i) \\ &= \exp(\tau_1(f)) \cdots \exp(\tau_{n-1}(f)) \lambda_i (c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge \kappa_i), \end{aligned}$$

where in the first equality we used the fact that $\deg(f) = 1$.

If $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge \kappa_i \neq 0$ for all $i = 1, 2$, then we have

$$\exp(\tau_1(f)) \cdots \exp(\tau_{n-1}(f)) \lambda_i = 1.$$

Thus we have $1 < \lambda_1 = \lambda_2 < 1$, which is a contradiction.

Next, we assume that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge \kappa_1 = 0$ by changing the role of $f^{-1}$ by $f$ and vice versa, if necessary. If $c_1 \wedge \kappa_1 = 0$, then it follows from Corollary 3.2 of [4] that $c_1$ is parallel to $\kappa_1$. So we have

$$\rho(f) = \lambda_1 = \exp(\tau_1(f)) < e^{\delta}.$$

On the other hand, if $c_1 \wedge \kappa_1 \neq 0$, then we let $l$ be the minimal integer $\geq 2$ such that $c_1 \wedge c_2 \wedge \cdots \wedge c_l \wedge \kappa_1 = 0$. It then follows from the definition of $l$ that $c_1 \wedge c_2 \wedge \cdots \wedge c_{l-1} \wedge \kappa_1 \neq 0$, and so clearly $c_1 \wedge c_2 \wedge \cdots \wedge c_{l-2} \wedge c_{l-1} \neq 0$ and $c_1 \wedge c_2 \wedge \cdots \wedge c_{l-2} \wedge \kappa_1 \neq 0$. Furthermore, by its construction and the fact that $l \leq n-1 \leq \tilde{r}$, we have

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{l-1} \wedge c_l \neq 0.$$

Note that the following two identities hold:

$$\begin{aligned}
&f^*(c_1 \wedge \cdots \wedge c_{l-1} \wedge \kappa_1) \\
&= \exp(\tau_1(f)) \exp(\tau_2(f)) \cdots \exp(\tau_{l-1}(f)) \lambda_1 c_1 \wedge \cdots \wedge c_{l-1} \wedge \kappa_1, \\
&f^*(c_1 \wedge \cdots \wedge c_{l-1} \wedge c_l) \\
&= \exp(\tau_1(f)) \exp(\tau_2(f)) \cdots \exp(\tau_l(f)) c_1 \wedge \cdots \wedge c_{l-1} \wedge c_l.
\end{aligned} \tag{3.3}$$

Now, by applying Lemma 3.1 to the equations in (3.3) with the roles of $t = l-1$, $c = \kappa_1$ and $c' = c_l$, we have

$$\begin{aligned}
&\exp(\tau_1(f)) \exp(\tau_2(f)) \cdots \exp(\tau_{l-1}(f)) \lambda_1 \\
&= \exp(\tau_1(f)) \exp(\tau_2(f)) \cdots \exp(\tau_{l-1}(f)) \exp(\tau_l(f)).
\end{aligned}$$

Thus we obtain

$$\lambda_1 = \rho(f) = \exp(\tau_l(f)) < e^{\delta}.$$

Since $\rho(f^{\pm 1})$ is always less than or equal to $\rho(f^{\mp})^{n-1}$ as observed in Section 1, we should have $\rho(f^{-1}) \leq \exp((n-1)\delta)$. For all $f \in G$ satisfying (3.2), the absolute values of all the eigenvalues of $f^*|_{H^{1,1}(X,\mathbf{C})}$ are therefore bounded from above by $\exp((n-1)\delta)$.

Finally, let $\Psi_f(x)$ be the characteristic polynomial of $f^*$ on $H^{1,1}(X, \mathbf{C})$. Since $G$ can be regarded as a subgroup of $\mathrm{GL}(H^2(X, \mathbf{Z}))$, $\Psi_f(x)$ can also be assumed to be a polynomial with integer coefficients. Recall that the absolute values of all the eigenvalues of $f^*|_{H^{1,1}(X,\mathbf{C})}$ of $f$ satisfying (3.2) are shown to be bounded from above by $\exp((n-1)\delta)$. Thus the coefficients of the characteristic polynomials $\Psi_f(x)$ of all such $f^*$'s are also all bounded.

This implies that there are only finitely many such characteristic polynomials, so the set of the vectors $\pi(f)$ for all $f$ satisfying (3.2) is finite. As a consequence, we see that the zero vector $(0)_{i=1}^{n-1}$ is isolated in $\pi(G/N(G))$ and the image $\pi(G/N(G))$ is discrete. This completes the proof of *Step 1*.

*Step 2*: We next show that the kernel of $\pi$ coincides with $N(G)$. To see it, suppose that there is an element $f$ of positive entropy with $\pi(f) = (0)_{i=1}^{n-1} \in \mathbf{R}^{n-1}$. Then it follows from the exactly same arguments as in *Step 1* that there exists a $\tau_l(f)$ for some $1 \le l \le n-1$ such that $\rho(f) = \exp(\tau_l(f)) > 1$. But, this contradicts the choice of $f$. So the kernel of the map $\pi$ is just $N(G)$, and the induced map $\pi : G/N(G) \to \mathbf{R}^{n-1}$ is actually injective.

For the case of $\tilde{r} \ge n-1$, this completes the proof that $G/N(G)$ is free abelian of rank $n-1$, and, in fact, in this case $\tilde{r}$ should be equal to $n-1$.

*Step 3*: The proof of other cases when $\tilde{r} \le n-2$ is completely similar. But this time we need to use $\tau_1, \cdots, \tau_{\tilde{r}}, \tilde{\tau}_{\tilde{r}+1}, \cdots, \tilde{\tau}_{n-1}$ instead of $\tau_1, \cdots, \tau_{n-1}$. We leave its detailed proof to a reader.

This completes the proof of Theorem 3.5. $\square$

As another interesting consequence of the injectivity of $\Pi$ in Theorem 3.5, the following proposition holds, which is now straightforward from Proposition 4.4 of [4].

**Proposition 3.6.** *Let $G$ be a connected solvable subgroup of the automorphism group $\mathrm{Aut}(X)$ and let $N(G)$ be the normal subgroup of null entropy, as in Theorem 1.3. Let $r$ denote the rank $r(G)$ of the quotient group $G/N(G)$. Then the following properties hold:*

(a) *Let $h_k$ be the real dimension of the cohomology group $H^{k,k}(X, \mathbf{R})$. Assume that $r = n-1$. Then $h_k$ satisfies*

$$h_k \ge \binom{n-1}{k}, \quad 1 \le k \le n-1.$$

*In addition, if $k$ divides $n-1$ as well, then we have $h_k \ge \binom{n-1}{k} + 1$.*

(b) *There exist $(r+1)$ non-zero classes $c_1, \ldots, c_{r+1}$ in $\bar{\mathcal{K}}$ such that*

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{r+1} \ne 0.$$

*Proof.* To prove the proposition, it suffices to notice that the proof of Proposition 4.4 of [4] works also for solvable subgroups of automorphisms. The reason is essentially due to the fact that there exists an induced homomorphism $\Pi : G/N(G) \to \mathbf{R}^{n-1}$ which is injective by Theorem 3.5. To be a bit more precise, for the proof we need to use Theorem 2.3 and its Corollary 2.4 (or Theorem 2.2) instead of Lemma 4.1 in [4] as well as Lemma 3.1 (refer to the proof of Theorem 1.4 in Section 4). But this does not make any significant difference in the proof, since in any case $\Pi(G/N(G))$ cannot be contained in a finite union of hyperplanes in $\mathbf{R}^r$. This completes the proof of Proposition 3.6. $\square$

Note that Theorem 2.2 instead of Theorem 2.3 can also be used to obtain Proposition 3.6 (b). But then the last non-zero class $c_n$ may not be invariant under $N(G)$. So $c_1, c_2, \cdots, c_{n-1}$, and $c_n$ obtained in that way cannot be used for the proof of Theorem 1.4. See the proof of Theorem 1.4 in Section 4 for some details.

As an interesting consequence of Proposition 3.6 (b), we see that $r + 1$ is less than or equal to $n$, due to the dimensional reason of $X$. Therefore $r$ should be less than or equal to $n - 1$. This proves Theorem 1.3 (a), once again.

# 4 Proof of Theorem 1.4

The aim of this section is to give a proof of Theorem 1.4. For the proof, it should be pointed out that it is enough to use the homomorphism $\pi$ instead of $\Pi$.

To begin with the proof, assume first that the rank $r(G)$ of $G/N(G)$ is equal to $n-1$. Then we claim that there exist non-zero classes $c_1, \cdots, c_n$ in the closure $\bar{\mathcal{K}}$ of the Kähler cone $\mathcal{K}$ such that

(i) $f^*c_i = c_i$ for all $1 \le i \le n$ and all $f \in N(G)$,

(ii) $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \neq 0$.

The proof of the claim essentially follows from that of Proposition 3.6 (b). Indeed, by Remark 3.3 we have $f^*c_i = c_i$ for all $1 \le i \le n-1$ and all $f \in N(G)$. Since the image of the injective homomorphism $\pi : G/N(G) \to \mathbf{R}^{n-1}$ spans $\mathbf{R}^{n-1}$, there exists an $f_0 \in G/N(G)$ such that all of $\tau_j(f_0)$'s $(1 \le j \le n-1)$ are negative.

Applying Theorem 2.3 of Lie-Kolchin type for a cone (or Corollary 2.4 (a)) to the closure $\bar{\mathcal{K}}$ of the Kähler cone, we see that there exists a non-zero class $c_n \in \bar{\mathcal{K}}$ such that $c_n$ spans a one-dimensional subcone of $C$ invariant under $G$ and such that

$$f_0^*(c_n) = \exp(\tau_n(f_0))c_n$$

for a non-negative real number $\tau_n(f_0)$. In particular, this implies that $\tau_j(f_0)$ is not equal to $\tau_n(f_0)$ for all $1 \le j \le n-1$ and, by Corollary 2.4 (b), $f^*(c_n) = c_n$ for all $f \in N(G)$. But then, it follows from Lemma 3.2 (a) that $c_1 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}$ is a non-zero class. Furthermore, by repeating the standard argument in the first step of the proof of Theorem 3.5, it is easy to show that $c_1 \wedge \cdots \wedge c_{n-2} \wedge c_n$ is also a non-zero class.

Now we want to show that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \ne 0$. So suppose that, on the contrary, $c_1 \wedge c_2 \wedge \cdots \wedge c_n = 0$. Note that we have

$$\begin{aligned} &f_0^*(c_1 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}) \\ &= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_{n-1}(f_0)) c_1 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}, \\ &f_0^*(c_1 \wedge \cdots \wedge c_{n-2} \wedge c_n) \\ &= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_n(f_0)) c_1 \wedge \cdots \wedge c_{n-2} \wedge c_n. \end{aligned} \tag{4.1}$$

Applying Lemma 3.1 to the equations in (4.1) for $f_0$ with $t = n-2$, $c = c_{n-1}$, and $c' = c_n$, we obtain

$$\begin{aligned} &\exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_{n-1}(f_0)) \\ &= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_n(f_0)). \end{aligned}$$

This implies that $0 > \tau_{n-1}(f_0) = \tau_n(f_0) \ge 0$, which is a contradiction. Thus we have

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \ne 0.$$

This finishes the proof of the claim.

Next, let $c = c_1 + c_2 + \cdots + c_{n-1} + c_n$. Then clearly $f^*(c) = c$ for all $f$ in $N(G)$, but not in the whole of $G$. Since $c_1 \wedge c_2 \wedge \cdots \wedge c_n \ne 0$, it is also clear that $c^n \ne 0$. It is known by a theorem of Demailly and Paun in [3] that the Kähler cone $\mathcal{K}$ is connected and that every class of $\mathcal{K}$ is characterized by the condition $\int_X c^n > 0$. By construction, $c$ lies in $\bar{\mathcal{K}}$ and $c^n \ne 0$. Thus we may assume by using the theorem of Demailly and Paun above that $c$ lies in the Kähler cone $\mathcal{K}$. That is, we may assume that $c$ is a Kähler class.

Finally, let $\mathrm{Aut}_c(X)$ denote the automorphism group preserving the Kähler class $c$. Then $N(G)$ is a subgroup of $\mathrm{Aut}_c(X)$. Recall that the quotient group $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is a finite group by a theorem of Liberman ([9], Proposition 2.2). So, if $\mathrm{Aut}_0(X)$ is trivial, then $\mathrm{Aut}_c(X)$ should be finite. This implies that $N(G)$ is also finite, which completes the proof of Theorem 1.4.

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## References

[1] G. Birkhoff, *Linear transformations with invariant cones*, Amer. Math. Monthly **74** (1967), 274–276.

[2] S. Cantat, *Groupes de transformations birationnelles du plan*, preprint; available at http://perso.univ-rennes1.fr/serge.cantat/Articles/cremona_court.pdf.

[3] J. P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. Math **159** (2004), 1247–1274; arXiv:math/0105176v2.

[4] T.-C. Dinh and N. Sibony, *Groupes commutatifs d'automorphismes d'une variété kählérienne compacte*, Duke Math. J. **123** (2004), 311–328; arXiv:math.DS/0212081v1.

[5] M. Gromov, *Entropy, homology, and semialgebraic geometry*, Astérisque **145–146** (1987), 225–240.

[6] V. Guedj, *Ergodic properties of rational mappings with large topological degrees*, Annals of Math. **161** (2005), 1589–1607.

[7] J. Keum, K. Oguiso, and D.-Q. Zhang, *Conjecture of Tits type for complex varieties and theorem of Lie-Kolchin type for a cone*, Math. Res. Lett. **16** (2009), 133–148; arXiv:math/0703103.

[8] J.-H. Kim, *Solvable automorphism groups of a compact Kähler manifold*, preprint (2007); arXiv:0712.0438v1.

[9] D. Lieberman, *Compactness of the Chow scheme: applications to automorphisms and deformations of Kahler manifolds*, Fonctions de Plusieurs Variables Complex III, Lecture Notes in Math. **670** (1978), Springer Verlag, 140–186.

[10] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.

[11] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.

[12] D.-Q. Zhang, *A theorem of Tits type for compact Kähler manifolds*, Invent. Math. **176** (2009), 449–459; arXiv:0805.4114.

[13] D.-Q. Zhang, *Automorphism groups of positive entropy on minimal projective varieties*, to appear in Advances in Mathematics; arXiv:1004.4781v1.

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