COMBINATORIAL RIGIDITY OF 3-DIMENSIONAL SIMPLICICAL POLYTOPES

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A *simplicial polytope* P is said to be *cohomologically rigid* if its dual simple polytope P^* is cohomologically rigid, that is, P^* supports a quasitoric manifold M, and its combinatorial face structure is determined by $H^*(M)$. Simply, we say M is *over* a simplicial polytope P if P^* supports M.

Choi et al. [1] have shown that $H^*(M) \cong H^*(N)$ implies $\beta_{i,j}(P) = \beta_{i,j}(Q)$ for all i, j, where M (resp, N) is a quasitoric manifold over P (resp, Q) and $\beta_{i,j}(P)$ is the *bigraded Betti numbers* of P which are important invariant coming from combinatorial commutative algebra.

Now we define that a simplicial polytope is *combinatorially rigid* (or simply, *rigid*) if its combinatorial structure is determined by its bigraded Betti numbers. Note that if P is combinatorially rigid and if P supports a quasitoric manifold, then P is cohomologically rigid.

We find a necessary condition to be combinatorially rigid for 3-dimensional reducible simplicial polytopes and provide some rigid reducible simplicial polytopes.

Let T_4, C_8, O_6, D_{20} and I_{12} be the five Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron respectively. See Figures 1, 3 and 4 for the definition of $\xi_1(C_8)$, $\xi_2(C_8)$, $\xi_1(D_{20})$, $\xi_2(D_{20})$ and B_n, the bipyramid with n vertices.

Theorem 1. *Let P be a 3-dimensional simplicial polytope. If P is reducible and combinatorially rigid, then P is either $T_4\#T_4\#T_4$ or $P_1\#P_2$, where*

$$P_1 \in \{T_4, O_6, I_{12}\},$$
$$P_2 \in \{T_4, O_6, I_{12}, \xi_1(C_8), \xi_2(C_8), \xi_1(D_{20}), \xi_2(D_{20})\} \cup \{B_n : n \geq 7\}.$$

Note that B_n is defined for $n \geq 5$ and we have $B_5 = T_4\#T_4$ and $B_6 = O_6$.

In fact, $T_4\#T_4\#T_4$ is known to be rigid, see [1]. We also prove that $P_1\#P_2$ is rigid for some P_1 and P_2 in Theorem 1.

Theorem 2. *The following polytopes are combinatorially rigid:*

$$T_4\#T_4, T_4\#O_6, T_4\#I_{12}, T_4\#B_n, O_6\#O_6, O_6\#B_n,$$

where $n \geq 7$.

#	T_4	O_6	I_{12}	B_n, $n \geq 7$	$\xi_1(C_8)$	$\xi_2(C_8)$	$\xi_1(D_{20})$	$\xi_2(D_{20})$
T_4	rigid	rigid	rigid	rigid	?	?	?	?
O_6	-	rigid	?	rigid	?	?	?	?
I_{12}	-	-	?	?	?	?	?	?

We remark that the following proposition is essential for the proofs above; Let P be a prism which is a product of a k-gon and an interval. Let e be an edge of one of the two k-gons of P. Then we can obtain another simple polytope from P by 'cutting' the edge e. We will call such polytope *edge-cut-prism*. A *semi-bipyramid* is the dual of an edge-cut-prism. See Figure 2.

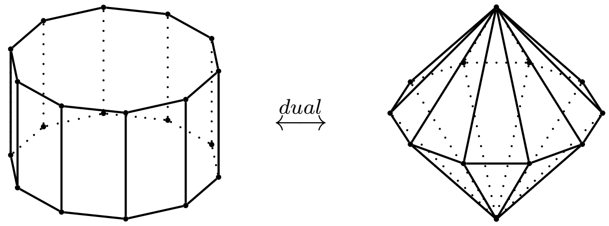

FIGURE 1. A prism and a bipyramid.

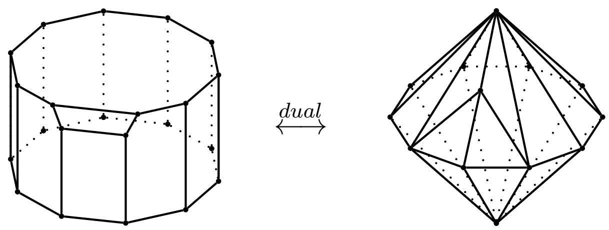

FIGURE 2. An edge-cut-prism and a semi-bipyramid.

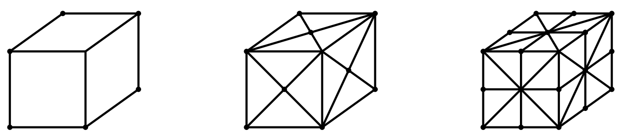

FIGURE 3. C_8, $\xi_1(C_8)$ and $\xi_2(C_8)$.

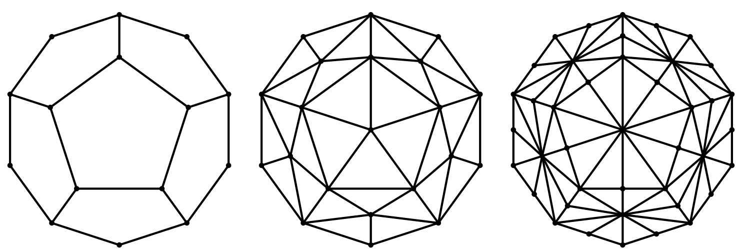

FIGURE 4. D_{20}, $\xi_1(D_{20})$ and $\xi_2(D_{20})$.

Proposition 3. *A bipyramid and a semi-bipyramid are rigid.*

We note that all 3-polytopes support quasitoric manifolds. Hence, the above all rigid polytopes are cohomologically rigid as well. This work is jointly with Jang Soo Kim.

REFERENCES

1. Suyoung Choi, Taras E. Panov, and Dong Youp Suh, *Toric cohomological rigidity of simple convex polytopes*, arXiv:0807.4800.

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