Total fractional colorings of graphs with large girth

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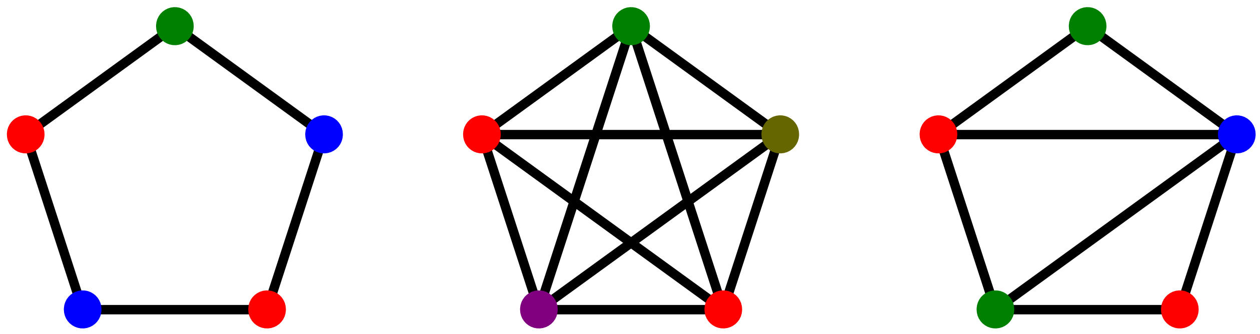
Overview

- Graph colorings—basic notions
- Fractional graph parameters
- Problem and our results
- Main proof idea

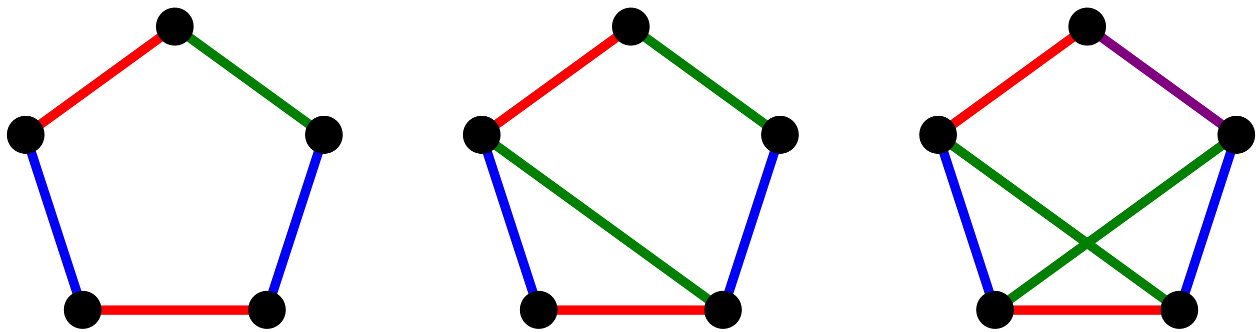
Vertex colorings



- two adjacent vertices must receive distinct colors
- chromatic number $\chi(G)$
- $\chi(G) \leq \Delta + 1$

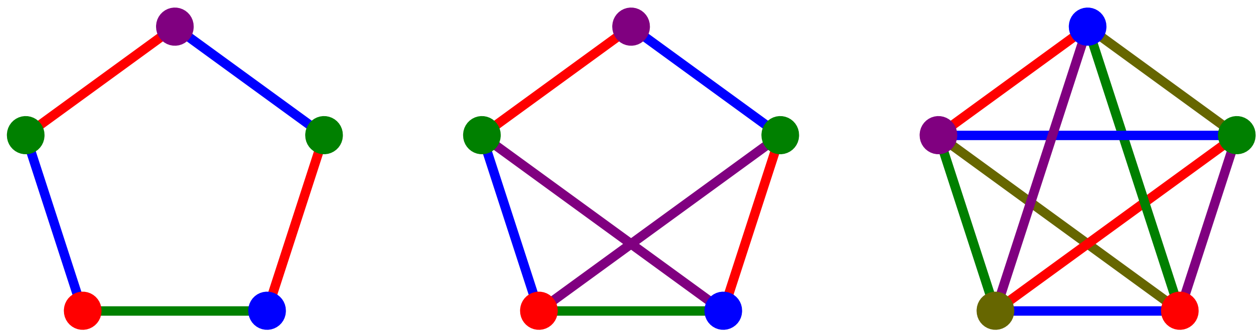
  Brooks' theorem (1941): $\chi(G) \leq \Delta$ for a connected graph G unless G is complete or an odd cycle

Edge colorings



- two incident edges must receive distinct colors
- chromatic index $\chi'(G)$
- Vizing's theorem (1964): $\chi'(G) \in \{\Delta, \Delta + 1\}$
  Holyer (1981): NP-complete to decide between the two values

Total colorings



- coloring of vertices and edges
  any two adjacent/incident elements must receive distinct colors
- total chromatic number $\chi_t(G)$
- Behzad's conjecture (1965): $\chi_t(G) \leq \Delta + 2$
  Molloy and Reed's bound (1998): $\chi_t(G) \leq \Delta + 10^{28}$

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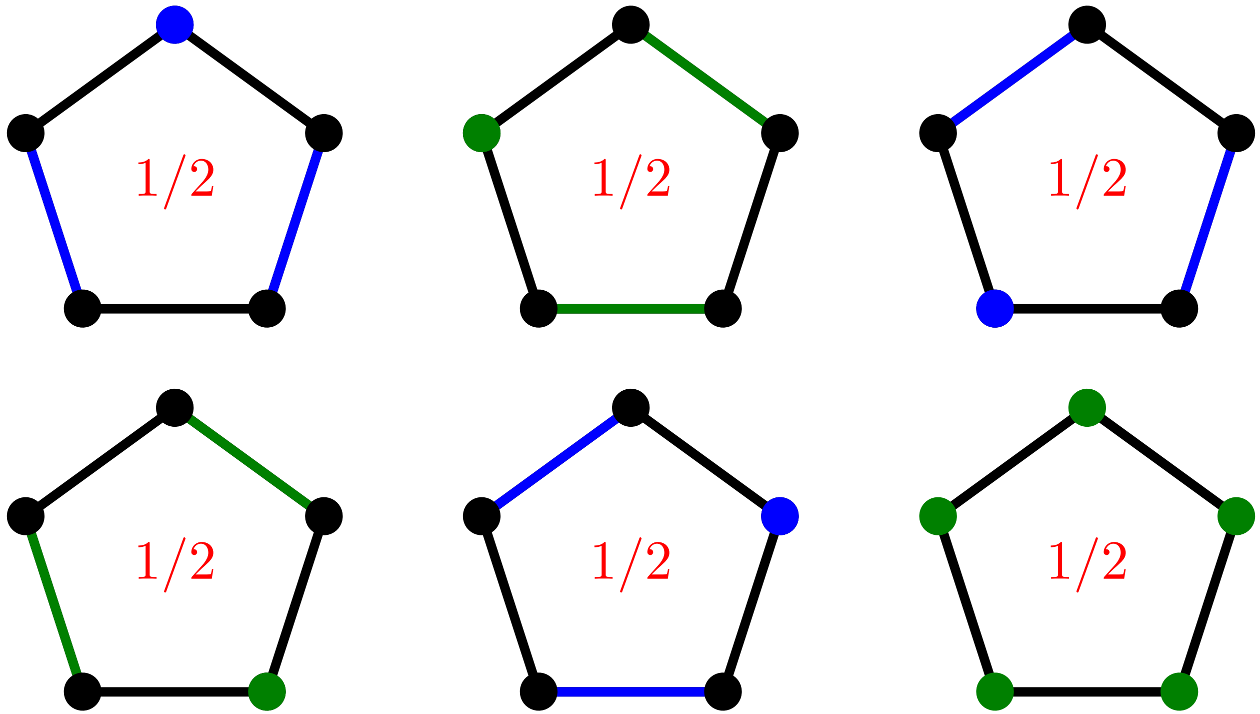
Fractional colorings

- coloring = partitioning into disjoint color classes
- fractional coloring
  each color class has weight, they need not be disjoint
  every element in classes of total weight at least one
  minimizing the total weightof all color classes
- total fractional chromatic number $\chi_f(G)$
- weights only zero or one $\Rightarrow$ coloring

Example



$\chi_{t,f} = 3$ and $\chi_t = 4$

Basic results

- clearly, $\chi_f(G) \leq \chi(G)$, $\chi'_f(G) \leq \chi'(G)$ and $\chi_{t,f}(G) \leq \chi_t(G)$
- the gap between $\chi_f(G)$ and $\chi(G)$ can be arbitrary
  $\chi(G)$ can be arbitrary and $\chi_f(G) \leq 2 + \varepsilon$
- $\chi_f(G) \leq 2.416$ if G is cubic and has large girth
  examples of cubic graphs with large girth with $\chi_f(G) \geq 2.196$

- $\chi'_f(G) = 3$ for cubic bridgeless graphs
- $\Delta \leq \chi'_f(G) \leq \Delta + \varepsilon$ for graphs with large girth

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Previous results

- Behzad's conjecture is fractionally true

  Kilakos and Reed (1993): $\chi_{t,f}(G) \le \Delta + 2$

- When does the equality hold?

  Ito, Kennedy and Reed (2009): only if G is K_{2n} or $K_{n,n}$

- What about large girth?

Reed's conjecture

Let Δ be an integer. For every $\varepsilon > 0$, there exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.

Our results

- Theorem (Kardoš, Král', Sereni):
  Let Δ be an integer. For every $\varepsilon > 0$, there exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.

- Theorem (Kaiser, King, Král'):
  Let $\Delta \in \{3, 4, 6, 8, \ldots\}$. There exists g such that every graph G with maximum degree Δ and girth at least g has total fractional chromatic number equal to $\Delta + 1$.

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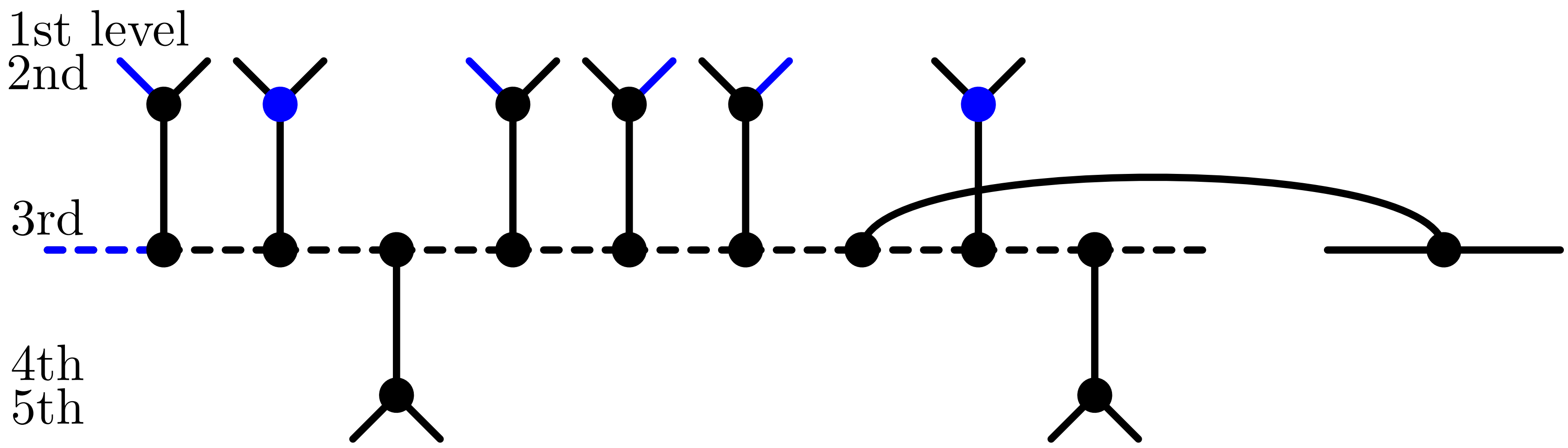
Decomposition into paths

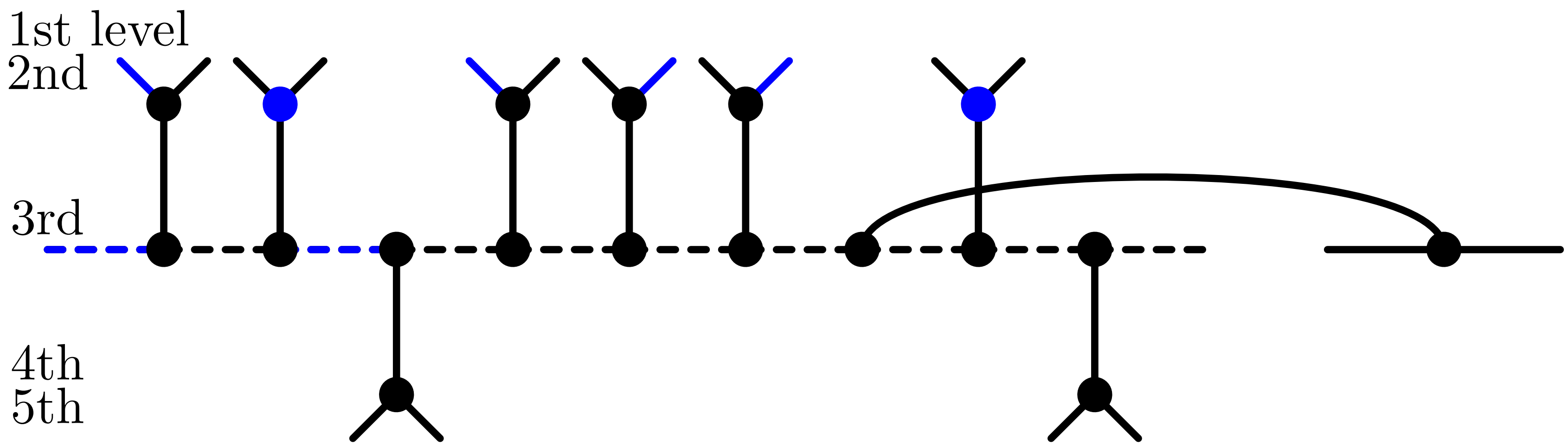
- $\chi_{t,f}(G) \leq \alpha \Leftrightarrow \exists$ probability distribution on independent sets, each element included with probability at least $1/\alpha$
- exposition restricted to cubic bridgeless case for simplicity (3 slides) $\alpha = 4 + \varepsilon$
- there exists a distribution on perfect matchings such that each edge included with probability $1/3$
- remove vertices at distance $g/100$ from the complementary 2-factor
  collection of paths of length at most $g/100$
  each edge included with probability at least $2/3 - 200/g$
  each vertex with probability $1 - 100/g$

Colorings by levels



- split paths into 10 levels
- sweep paths in the 1st level, then in the 2nd level, etc.
  include vertices and edges greedily into a total independent set
- if a vertex can be included, include it with probability $1 - \xi$

Colorings by levels



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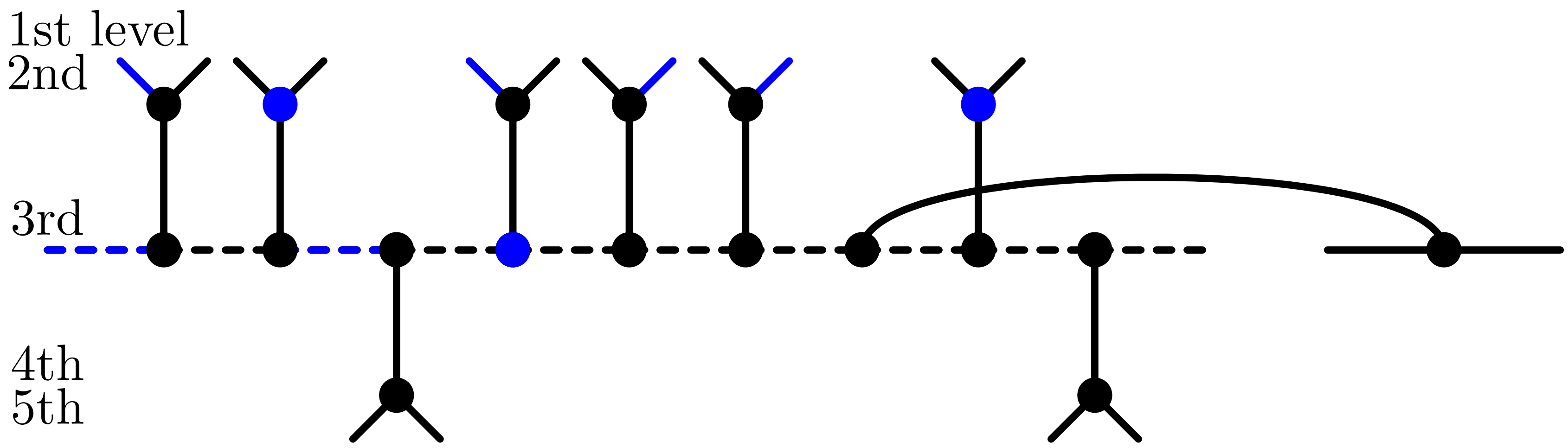

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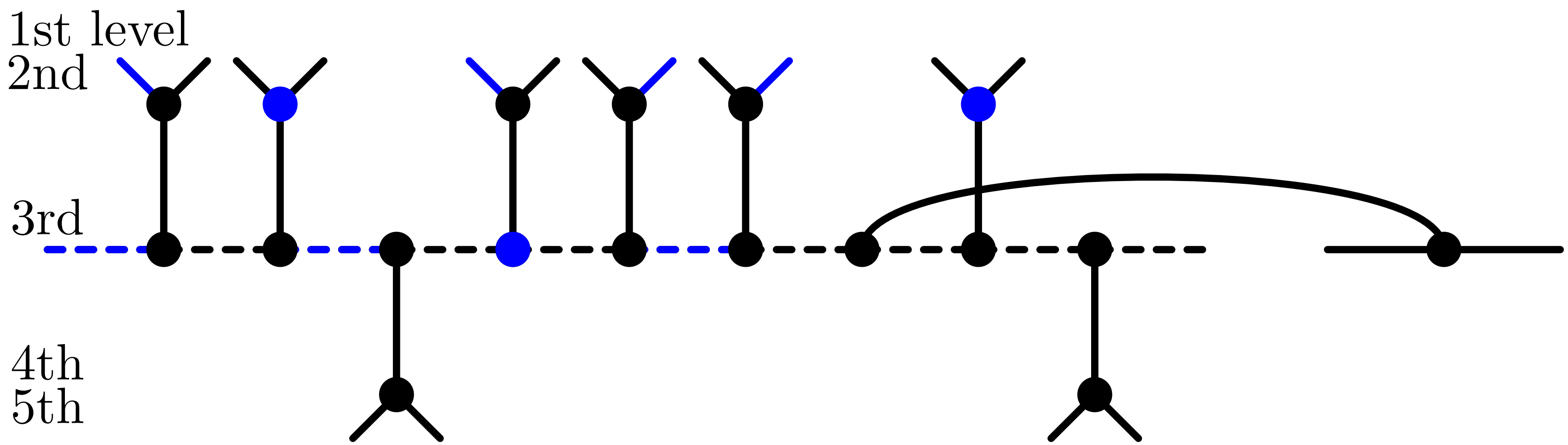

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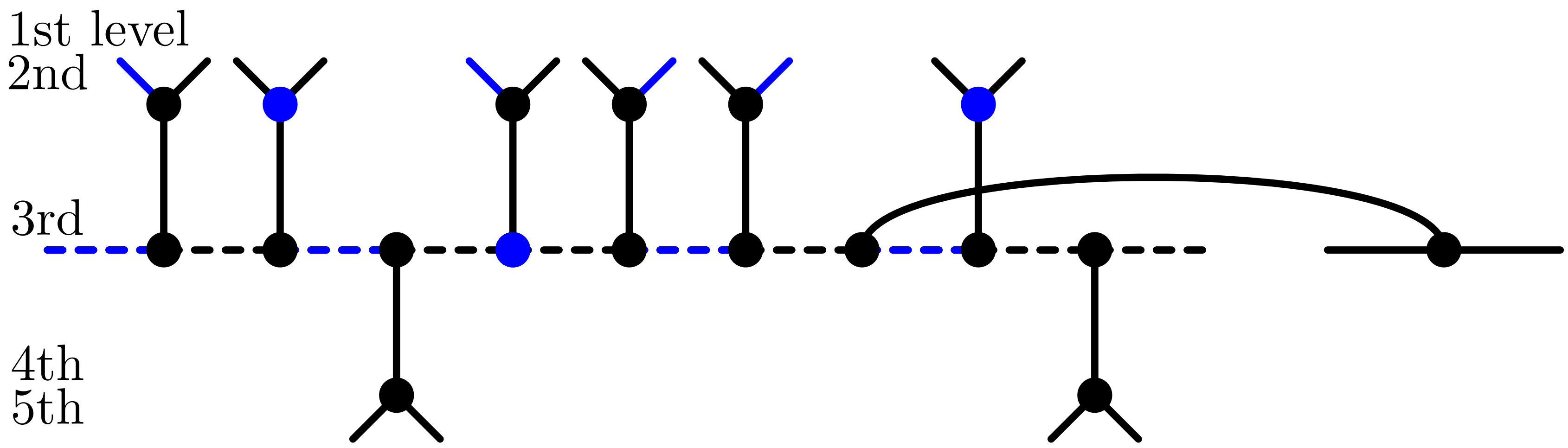

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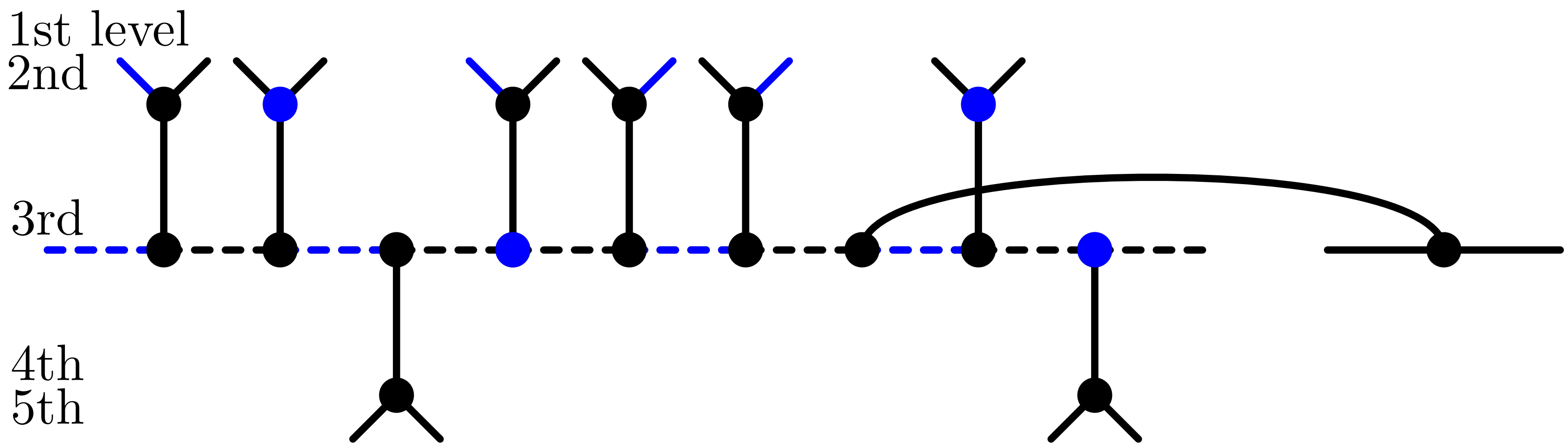

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Summary for cubic graphs

- levels guarantee mutual independence of neighbors for each path
- for a suitable choice of $\xi > 0$
  each vertex on a path included with probability $1/4$
  each edge in a path included with probability $3/8$
- considering the distribution over perfect matchings:
  each vertex included with probability $1/4 - 25/g$
  each edge included with probability $1/4 - 75/g$
- a better resolution of coloring conflicts at the ends of the paths needed to obtain $\chi_{t,f} = 4$ for large g

Differences for general graphs

- if $\Delta = 4, 6, 8, \ldots,$ uniform coverings by 2-factors exist
- if $\Delta = 5, 7, 9, \ldots,$ uniform coverings by 2-factors exist assuming the graph is cyclically $(\Delta - 1)$-edge-connected
- a stronger result on extending precolorings can be proven
- if the graph is not cyclically $(\Delta - 1)$-edge-connected, split the graph into two pieces, one inclusion-wise minimal the minimal piece is cyclically $(\Delta - 1)$-edge-connected induction on the bigger piece and extend to the smaller one

Thank you for your attention!