New interpretations for noncrossing partitions of classical types

Jang Soo Kim

University of Paris 7

Discrete Math Seminar, KAIST, January 21, 2010

Outline

- $\mathrm{NC}(n)$ was first studied by Kreweras in 1972.

Outline

- $\mathrm{NC}(n)$ was first studied by Kreweras in 1972.
- Bessis (2003), Brady and Watt (2002) defined $\mathrm{NC}(W)$ for each finite reflection group W such that $\mathrm{NC}(A_{n-1}) \cong \mathrm{NC}(n)$.

Outline

- $\mathrm{NC}(n)$ was first studied by Kreweras in 1972.
- Bessis (2003), Brady and Watt (2002) defined $\mathrm{NC}(W)$ for each finite reflection group W such that $\mathrm{NC}(A_{n-1}) \cong \mathrm{NC}(n)$.
- $\mathrm{NC}(B_n) \cong \mathrm{NC}(C_n)$ has a combinatorial model $\mathrm{NC}_B(n)$ (Reiner, 1997).

Outline

- $\mathrm{NC}(n)$ was first studied by Kreweras in 1972.
- Bessis (2003), Brady and Watt (2002) defined $\mathrm{NC}(W)$ for each finite reflection group W such that $\mathrm{NC}(A_{n-1}) \cong \mathrm{NC}(n)$.
- $\mathrm{NC}(B_n) \cong \mathrm{NC}(C_n)$ has a combinatorial model $\mathrm{NC}_B(n)$ (Reiner, 1997).
- $\mathrm{NC}(D_n)$ has a combinatorial model $\mathrm{NC}_D(n)$ (Athanasiadis and Reiner, 2004).

Outline

- $\mathrm{NC}(n)$ was first studied by Kreweras in 1972.
- Bessis (2003), Brady and Watt (2002) defined $\mathrm{NC}(W)$ for each finite reflection group W such that $\mathrm{NC}(A_{n-1}) \cong \mathrm{NC}(n)$.
- $\mathrm{NC}(B_n) \cong \mathrm{NC}(C_n)$ has a combinatorial model $\mathrm{NC}_B(n)$ (Reiner, 1997).
- $\mathrm{NC}(D_n)$ has a combinatorial model $\mathrm{NC}_D(n)$ (Athanasiadis and Reiner, 2004).
- **Main purpose** : Find interpretations for $\mathrm{NC}_B(n)$.

Noncrossing partitions

- $[n] := \{1, 2, \ldots, n\}$

Noncrossing partitions

- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

Noncrossing partitions

- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

- Each B_i is called **block**.

Noncrossing partitions

- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

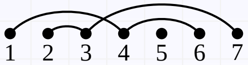
- Each B_i is called **block**.
- $\Pi(n)$: the set of partitions of $[n]$.

Noncrossing partitions

- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

- Each B_i is called **block**.
- $\Pi(n)$: the set of partitions of $[n]$.
- The **standard representation** of a partition $\{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$:

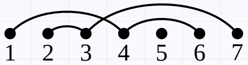


Noncrossing partitions

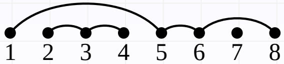
- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

- Each B_i is called **block**.
- $\Pi(n)$: the set of partitions of $[n]$.
- The **standard representation** of a partition $\{\{1, 4, 6\}, \{2, 3, 7\}, \{5\}\} \in \Pi(7)$:



- A **noncrossing partition** is a partition without crossing in its standard representation.

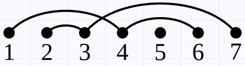


Noncrossing partitions

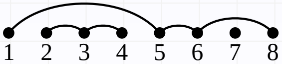
- $[n] := \{1, 2, \ldots, n\}$
- A **partition** π of $[n]$ is a collection $\{B_1, B_2, \ldots, B_r\}$ such that

$$B_1 \cup B_2 \cup \cdots \cup B_r = [n], \qquad B_i \neq \emptyset, \quad B_i \cap B_j = \emptyset \quad \forall i \neq j.$$

- Each B_i is called **block**.
- $\Pi(n)$: the set of partitions of $[n]$.
- The **standard representation** of a partition $\{\{1, 4, 6\}, \{2, 3, 7\}, \{5\}\} \in \Pi(7)$:



- A **noncrossing partition** is a partition without crossing in its standard representation.



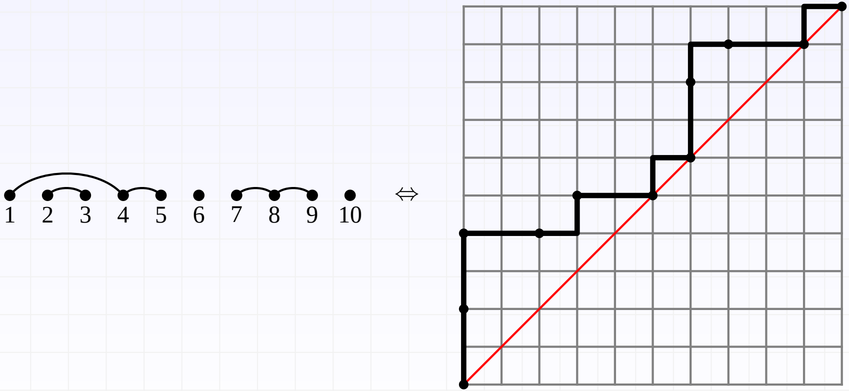
- $\mathrm{NC}(n)$: the set of noncrossing partitions of $[n]$.

The cardinality of $\mathrm{NC}(n)$

- The cardinality of $\mathrm{NC}(n)$ is the **Catalan number**$= \dfrac{1}{n+1}\dbinom{2n}{n}$.

The cardinality of $\mathrm{NC}(n)$

- The cardinality of $\mathrm{NC}(n)$ is the **Catalan number**= $\dfrac{1}{n+1}\dbinom{2n}{n}$.
- A bijection between $\mathrm{NC}(n)$ and **Dyck paths**:



Block size enumeration

- For $\pi \in \mathrm{NC}(n)$, **block size vector** of π is $(b; b_1, b_2, \ldots, b_n)$ where

Block size enumeration

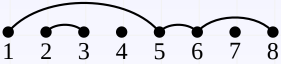
- For $\pi \in \mathrm{NC}(n)$, **block size vector** of π is $(b; b_1, b_2, \ldots, b_n)$ where
  - b is the number of blocks of π,

Block size enumeration

- For $\pi \in \mathrm{NC}(n)$, **block size vector** of π is $(b; b_1, b_2, \dots, b_n)$ where
  - b is the number of blocks of π,
  - b_i is the number of blocks of size i.

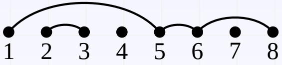
Block size enumeration

- For $\pi \in \mathrm{NC}(n)$, **block size vector** of π is $(b; b_1, b_2, \dots, b_n)$ where
  - b is the number of blocks of π,
  - b_i is the number of blocks of size i.
- The block size vector of the following is $(4; 2, 1, 0, 1, 0, \dots, 0)$:



Block size enumeration

- For $\pi \in \mathrm{NC}(n)$, **block size vector** of π is $(b; b_1, b_2, \ldots, b_n)$ where
  - b is the number of blocks of π,
  - b_i is the number of blocks of size i.
- The block size vector of the following is $(4; 2, 1, 0, 1, 0, \ldots, 0)$:



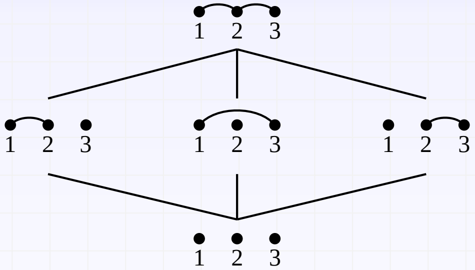
Theorem (Kreweras, 1972)

*The number of $\pi \in \mathrm{NC}(n)$ with the **block size vector** $(b; b_1, b_2, \ldots, b_n)$ is equal to*

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{b-1}.$$

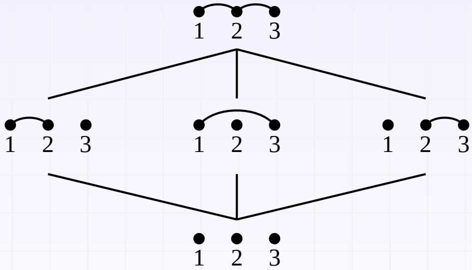
$\mathrm{NC}(n)$ as a poset

- $\pi_1 \leq \pi_2 \quad \Leftrightarrow \quad \pi_1$ is a **refinement** of π_2



NC(n) as a poset

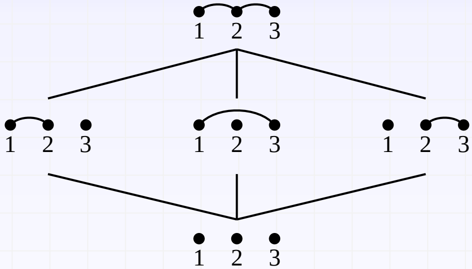
- $\pi_1 \leq \pi_2 \quad \Leftrightarrow \quad \pi_1$ is a **refinement** of π_2



- NC(n) is **graded** with $\mathrm{rank}(\pi) = n - \mathrm{bk}(\pi)$.

NC(n) as a poset

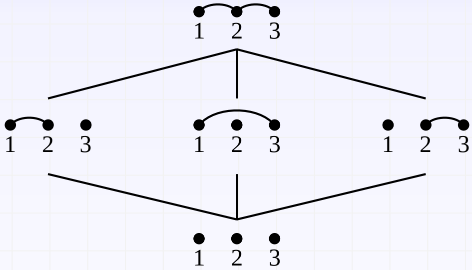
- $\pi_1 \leq \pi_2 \quad \Leftrightarrow \quad \pi_1$ is a **refinement** of π_2



- NC(n) is **graded** with $\mathrm{rank}(\pi) = n - \mathrm{bk}(\pi)$.
- A multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ has **rank jump vector** $(s_1, s_2, \ldots, s_{\ell+1})$ if

NC(n) as a poset

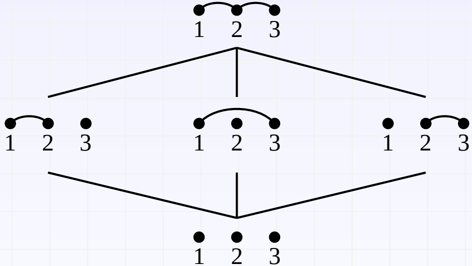
- $\pi_1 \leq \pi_2 \quad \Leftrightarrow \quad \pi_1$ is a **refinement** of π_2



- NC(n) is **graded** with $\mathrm{rank}(\pi) = n - \mathrm{bk}(\pi)$.
- A multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ has **rank jump vector** $(s_1, s_2, \ldots, s_{\ell+1})$ if

NC(n) as a poset

- $\pi_1 \le \pi_2 \quad \Leftrightarrow \quad \pi_1$ is a **refinement** of π_2



- NC(n) is **graded** with $\mathrm{rank}(\pi) = n - \mathrm{bk}(\pi)$.
- A multichain $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ has **rank jump vector** $(s_1, s_2, \ldots, s_{\ell+1})$ if

$$\begin{tikzcd}
\hat{0} \arrow[r, phantom, "\le"] \arrow[r, bend left, "\mathrm{rank} + s_1"] & \pi_1 \arrow[l, bend left, "\mathrm{bk} + s_1"] \arrow[r, phantom, "\le"] \arrow[r, bend left, "\mathrm{rank} + s_2"] & \pi_2 \arrow[l, bend left, "\mathrm{bk} + s_2"] & \le \quad \cdots \quad \le & \pi_{\ell-1} \arrow[r, phantom, "\le"] \arrow[r, bend left, "\mathrm{rank} + s_\ell"] & \pi_\ell \arrow[l, bend left, "\mathrm{bk} + s_\ell"] \arrow[r, phantom, "\le"] \arrow[r, bend left, "\mathrm{rank} + s_{\ell+1}"] & \hat{1} \arrow[l, bend left, "\mathrm{bk} + s_{\ell+1}"]
\end{tikzcd}$$

Chain enumeration

Theorem (Edelman, 1980)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ in $\mathrm{NC}(n)$ with ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

Chain enumeration

Theorem (Edelman, 1980)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ *in* $\mathrm{NC}(n)$ *with* ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- If $\pi \in \mathrm{NC}(n)$ has k blocks, then it is a multichain with rank jump vector $(n-k, k-1)$.

Chain enumeration

Theorem (Edelman, 1980)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ in $\mathrm{NC}(n)$ with ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- If $\pi \in \mathrm{NC}(n)$ has k blocks, then it is a multichain with rank jump vector $(n-k, k-1)$.
- The number of such π's is the **Narayana number**:

$$\frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Chain enumeration

Theorem (Edelman, 1980)

*The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ in $\mathrm{NC}(n)$ with **rank jump vector** $(s_1, s_2, \ldots, s_{\ell+1})$ is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- If $\pi \in \mathrm{NC}(n)$ has k blocks, then it is a multichain with rank jump vector $(n-k, k-1)$.
- The number of such π's is the **Narayana number**:

$$\frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- A **maximal chain** has rank jump vector $(\overbrace{1, 1, \ldots, 1}^{n-1})$.

Chain enumeration

Theorem (Edelman, 1980)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ *in* $\mathrm{NC}(n)$ *with* ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- If $\pi \in \mathrm{NC}(n)$ has k blocks, then it is a multichain with rank jump vector $(n-k, k-1)$.
- The number of such π's is the **Narayana number**:

$$\frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- A **maximal chain** has rank jump vector $(\overbrace{1, 1, \ldots, 1}^{n-1})$.
- The number of maximal chains is $\mathbf{n^{n-2}}$.

Chain enumeration

Theorem (Edelman, 1980)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ in $\mathrm{NC}(n)$ with ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- If $\pi \in \mathrm{NC}(n)$ has k blocks, then it is a multichain with rank jump vector $(n-k, k-1)$.
- The number of such π's is the **Narayana number**:

$$\frac{1}{n}\binom{n}{n-k}\binom{n}{k-1} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- A **maximal chain** has rank jump vector $(\overbrace{1, 1, \ldots, 1}^{n-1})$.
- The number of maximal chains is $\mathbf{n^{n-2}}$.
- Stanley (1996) : maximal chains in $\mathrm{NC}(n)$ $\Leftrightarrow$ **parking functions**.

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ *in* $\mathrm{NC}(n)$ *such that*

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ *in* $\mathrm{NC}(n)$ *such that*

- ***rank jump vector*** *is* $(s_1, s_2, \ldots, s_{\ell+1})$

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ *in* $\mathrm{NC}(n)$ *such that*

- ***rank jump vector*** *is* $(s_1, s_2, \ldots, s_{\ell+1})$
- ***block size vector*** *of* π_1 *is* $(b; b_1, b_2, \ldots, b_n)$

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_\ell$ in $\mathrm{NC}(n)$ such that

- ***rank jump vector*** *is $(s_1, s_2, \ldots, s_{\ell+1})$*
- ***block size vector*** *of π_1 is $(b; b_1, b_2, \ldots, b_n)$*

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ *in* $\mathrm{NC}(n)$ *such that*

- ***rank jump vector*** *is* $(s_1, s_2, \ldots, s_{\ell+1})$
- ***block size vector*** *of* π_1 *is* $(b; b_1, b_2, \ldots, b_n)$

is equal to

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- Krattenthaler and Müller proved analogous formulas for noncrossing partitions of type B and type D.

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ *in* $\mathrm{NC}(n)$ *such that*

- ***rank jump vector*** *is* $(s_1, s_2, \ldots, s_{\ell+1})$
- ***block size vector*** *of* π_1 *is* $(b; b_1, b_2, \ldots, b_n)$

is equal to

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- Krattenthaler and Müller proved analogous formulas for noncrossing partitions of type B and type D.
- In the introduction, they wrote that their theorems for type A and B seem amenable to combinatorial proofs, however, to find a combinatorial proof of their theorem for type D seems rather hopeless.

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ in $\mathrm{NC}(n)$ such that

- ***rank jump vector*** *is $(s_1, s_2, \ldots, s_{\ell+1})$*
- ***block size vector*** *of π_1 is $(b; b_1, b_2, \ldots, b_n)$*

is equal to

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- Krattenthaler and Müller proved analogous formulas for noncrossing partitions of type B and type D.
- In the introduction, they wrote that their theorems for type A and B seem amenable to combinatorial proofs, however, to find a combinatorial proof of their theorem for type D seems rather hopeless.
- Since noncrossing partitions of type D are special noncrossing partitions of type B, we need to understand type B very well.

K-M Theorem of type A

Theorem (Krattenthaler and Müller, 2007)

The number of multichains $\pi_1 \le \pi_2 \le \cdots \le \pi_\ell$ in $\mathrm{NC}(n)$ such that

- ***rank jump vector*** *is $(s_1, s_2, \ldots, s_{\ell+1})$*
- ***block size vector*** *of π_1 is $(b; b_1, b_2, \ldots, b_n)$*

is equal to

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

- Krattenthaler and Müller proved analogous formulas for noncrossing partitions of type B and type D.
- In the introduction, they wrote that their theorems for type A and B seem amenable to combinatorial proofs, however, to find a combinatorial proof of their theorem for type D seems rather hopeless.
- Since noncrossing partitions of type D are special noncrossing partitions of type B, we need to understand type B very well.
- **Main purpose** : Give a new interpretation for noncrossing partitions of type B

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \leq_T w \iff \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \leq_T w \iff \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c = s_1 s_2 \cdots s_n$,

$$\mathrm{NC}(W) \cong \mathrm{NC}(W, c) = \{w \in W : 1 \leq_T w \leq_T c\}.$$

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \leq_T w \iff \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c = s_1 s_2 \cdots s_n$,

$$\mathrm{NC}(W) \cong \mathrm{NC}(W, c) = \{w \in W : 1 \leq_T w \leq_T c\}.$$

Example

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \leq_T w \quad \iff \quad \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c = s_1 s_2 \cdots s_n$,

$$\mathrm{NC}(W) \cong \mathrm{NC}(W, c) = \{w \in W : 1 \leq_T w \leq_T c\}.$$

Example

- Let $W = A_{n-1}$.

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \leq_T w \iff \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c = s_1 s_2 \cdots s_n$,

$$\mathrm{NC}(W) \cong \mathrm{NC}(W, c) = \{w \in W : 1 \leq_T w \leq_T c\}.$$

Example

- Let $W = A_{n-1}$.
- $(1, 2, \cdots, n)$ is a Coxeter element.

Definition of noncrossing partitions

- (W, S) : a finite Coxeter system, $T = \{wsw^{-1} : s \in S, w \in W\}$
- $\ell_T(w) := \min\{k : w = t_1 t_2 \cdots t_k, t_i \in T\}$
- $u \le_T w \iff \ell_T(w) = \ell_T(u) + \ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c = s_1 s_2 \cdots s_n$,

$$\mathrm{NC}(W) \cong \mathrm{NC}(W, c) = \{w \in W : 1 \le_T w \le_T c\}.$$

Example

- Let $W = A_{n-1}$.
- $(1, 2, \cdots, n)$ is a Coxeter element.
- $\pi \in \mathrm{NC}(A_{n-1})$ if the elements of each cycle of π is in increasing order and if π is a noncrossing partition.

Definition of noncrossing partitions

- (W,S) : a finite Coxeter system, $T=\{wsw^{-1} : s\in S, w\in W\}$
- $\ell_T(w) := \min\{k : w = t_1t_2\cdots t_k, t_i\in T\}$
- $u\le_T w \iff \ell_T(w)=\ell_T(u)+\ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c=s_1s_2\cdots s_n$,

$$\mathrm{NC}(W)\cong \mathrm{NC}(W,c)=\{w\in W : 1\le_T w\le_T c\}.$$

Example

- Let $W=A_{n-1}$.
- $(1,2,\cdots,n)$ is a Coxeter element.
- $\pi\in \mathrm{NC}(A_{n-1})$ if the elements of each cycle of π is in increasing order and if π is a noncrossing partition.
- $\pi=(1,5,6,8)(2,3,4)(7)\leftrightarrow \pi=\{\{1,5,6,8\},\{2,3,4\},\{7\}\}$

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

- $\pi = \{\{1, 4, 6\}, \{2, 3, 7\}, \{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \le i < j \le n\}$$

- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n

$$\{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}$$

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.
$$\{x_i - x_j = 0 : 1 \le i < j \le n\}$$
- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to
$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$
- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n
$$\{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}$$
- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to
$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$
$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \le i < j \le n\}$$

- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n

$$\{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}$$

- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to

$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$

$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Definition

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n

$$\{x_i \pm x_j = 0 : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\}$$

- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to

$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$

$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Definition

- A **partition of type B_n** is a partition of $[\pm n] = \{1,2,\ldots,n,-1,-2,\ldots,-n\}$ such that

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.
$$\{x_i - x_j = 0 : 1 \le i < j \le n\}$$
- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to
$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$
- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n
$$\{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}$$
- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to
$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$
$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Definition

- A **partition of type B_n** is a partition of $[\pm n] = \{1,2,\dots,n,-1,-2,\dots,-n\}$ such that
  1. $B \in \pi \Rightarrow -B = \{-x : x \in B\} \in \pi$,

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \leq i < j \leq n\}$$

- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n

$$\{x_i \pm x_j = 0 : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\}$$

- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to

$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$

$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Definition

- A **partition of type B_n** is a partition of $[\pm n] = \{1,2,\ldots,n,-1,-2,\ldots,-n\}$ such that
  1. $B \in \pi \Rightarrow -B = \{-x : x \in B\} \in \pi$,
  2. there is at most one block, called **zero block**, which satisfies $B = -B$.

Partitions of type B

- $\pi \in \Pi(n)$: an intersection of reflecting hyperplanes of the Coxeter group A_{n-1}.

$$\{x_i - x_j = 0 : 1 \le i < j \le n\}$$

- $\pi = \{\{1,4,6\},\{2,3,7\},\{5\}\} \in \Pi(7)$ corresponds to

$$\{\mathbf{x} \in \mathbb{R}^7 : x_1 = x_4 = x_6, \quad x_2 = x_3 = x_7\}.$$

- $\pi \in \Pi_B(n)$: an intersection of reflecting hyperplanes of type B_n

$$\{x_i \pm x_j = 0 : 1 \le i < j \le n\} \cup \{x_i = 0 : 1 \le i \le n\}$$

- $\{\mathbf{x} \in \mathbb{R}^8 : x_1 = -x_3 = x_6, \quad x_2 = x_4 = 0, \quad x_5 = x_8\}$ corresponds to

$$\{\pm\{1,-3,6\},\{2,4,-2,-4\},\pm\{5,8\},\pm\{7\}\}$$

$$= \{\{1,-3,6\},\{-1,3,-6\},\{2,4,-2,-4\},\{5,8\},\{-5,-8\},\{7\},-\{7\}\}$$

Definition

- A **partition of type B_n** is a partition of $[\pm n] = \{1,2,\dots,n,-1,-2,\dots,-n\}$ such that
  1. $B \in \pi \Rightarrow -B = \{-x : x \in B\} \in \pi$,
  2. there is at most one block, called **zero block**, which satisfies $B = -B$.
- $\Pi_B(n)$: the set of partitions of type B_n

The number of elements in $\Pi_B(n)$

Theorem (K., 2009)

We have

$$\#\Pi_B(n) = \sum_{k=1}^{n} S(n,k)t_{k+1},$$

where $S(n,k)$ is the Stirling number of the second kind, t_n : the number of involutions of $[n]$

The number of elements in $\Pi_B(n)$

Theorem (K., 2009)

We have

$$\#\Pi_B(n) = \sum_{k=1}^{n} S(n,k) t_{k+1},$$

where $S(n,k)$ is the Stirling number of the second kind, t_n : the number of involutions of $[n]$

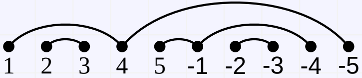
- Compare with

$$\#\Pi(n) = \sum_{k=1}^{n} S(n,k).$$

Combinatorial models for $\mathrm{NC}(B_n)$

Definition

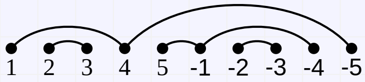
A **noncrossing partition of type B_n** is a partition $\pi \in \Pi_B(n)$ without crossing in the standard representation of π with respect to the order $1, 2, \ldots, n, -1, -2, \ldots, -n$.



Combinatorial models for $\mathrm{NC}(B_n)$

Definition

A **noncrossing partition of type B_n** is a partition $\pi \in \Pi_B(n)$ without crossing in the standard representation of π with respect to the order $1, 2, \ldots, n, -1, -2, \ldots, -n$.

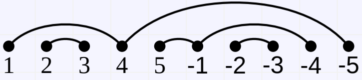


- $\mathrm{NC}_B(n)$: the set of noncrossing partitions of type B_n.

Combinatorial models for $\mathrm{NC}(B_n)$

Definition

A **noncrossing partition of type B_n** is a partition $\pi \in \Pi_B(n)$ without crossing in the standard representation of π with respect to the order
$1, 2, \ldots, n, -1, -2, \ldots, -n.$

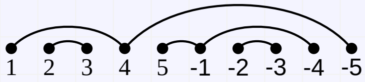


- $\mathrm{NC}_B(n)$: the set of noncrossing partitions of type B_n.
- $\mathrm{NC}_B(n) \cong \mathrm{NC}(B_n)$

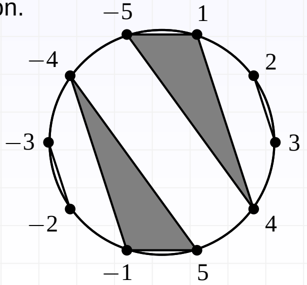
Combinatorial models for $\mathrm{NC}(B_n)$

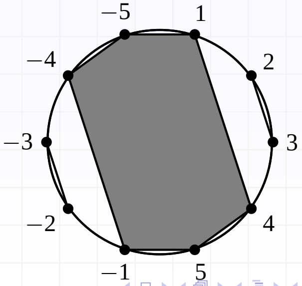
Definition

A **noncrossing partition of type B_n** is a partition $\pi \in \Pi_B(n)$ without crossing in the standard representation of π with respect to the order $1, 2, \ldots, n, -1, -2, \ldots, -n$.

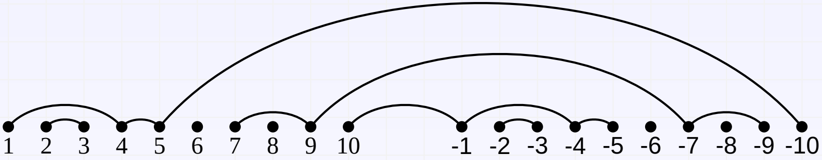


- $\mathrm{NC}_B(n)$: the set of noncrossing partitions of type B_n.
- $\mathrm{NC}_B(n) \cong \mathrm{NC}(B_n)$
- The circular representation of $\pi \in \mathrm{NC}_B(n)$ is invariant under 180° rotation.

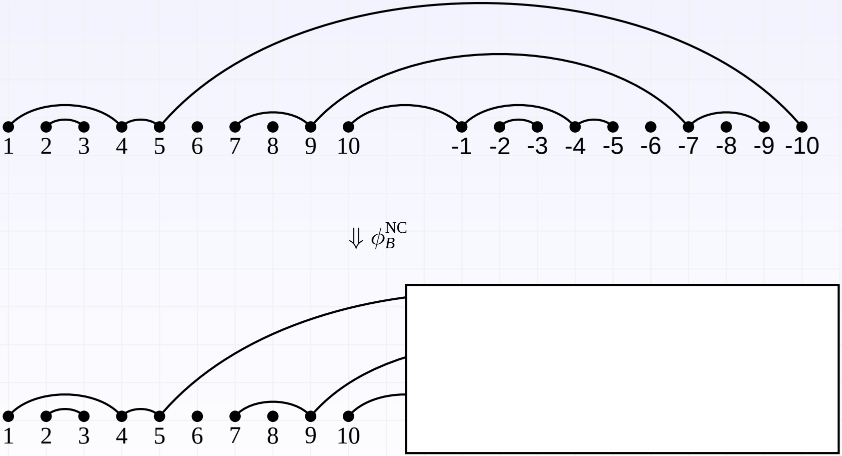




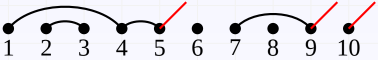
A very simple map



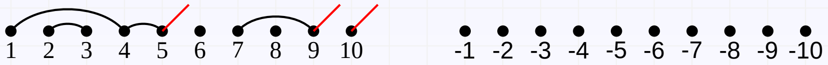
A very simple map



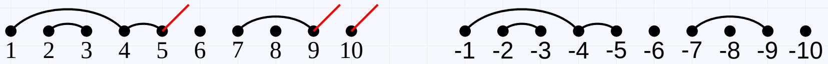
The inverse map



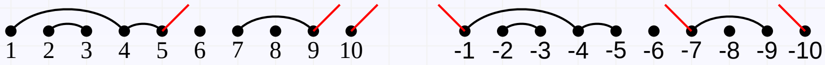
The inverse map



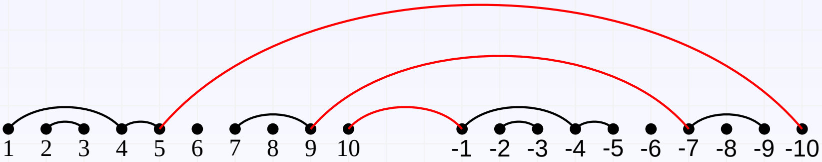
The inverse map



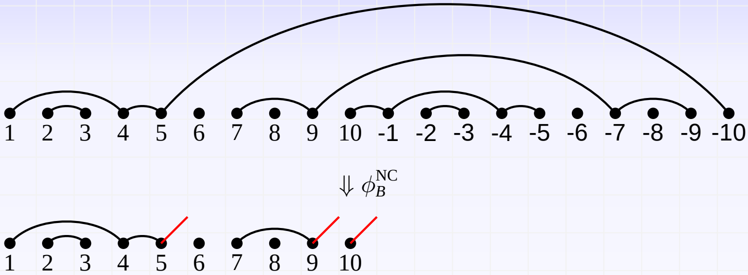
The inverse map



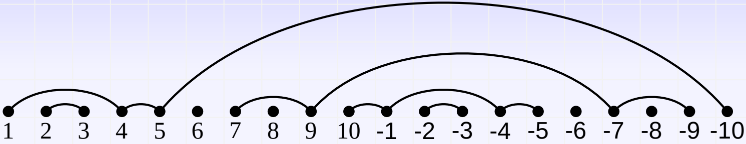
The inverse map



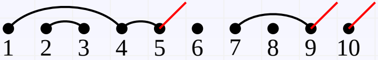
$\mathrm{NC}^{\mathrm{NN}}(n)$: First Interpretation for $\mathrm{NC}_B(n)$



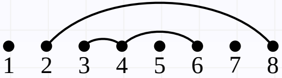
$\mathrm{NC}^{\mathrm{NN}}(n)$: First Interpretation for $\mathrm{NC}_B(n)$



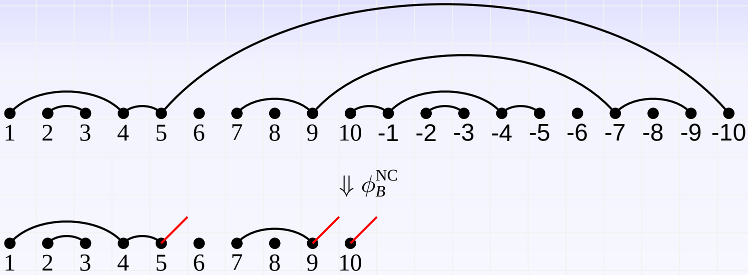
$$\Downarrow \phi_B^{\mathrm{NC}}$$



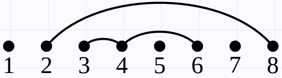
- A block B of $\pi \in \Pi(n)$ is **nested** if there is an edge (i,j) with $i < \min(B) \leq \max(B) < j$.



$\mathrm{NC}^{\mathrm{NN}}(n)$: First Interpretation for $\mathrm{NC}_B(n)$

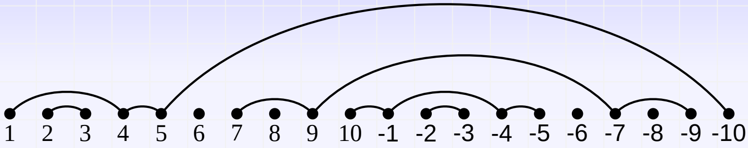


- A block B of $\pi \in \Pi(n)$ is **nested** if there is an edge (i,j) with $i < \min(B) \leq \max(B) < j$.

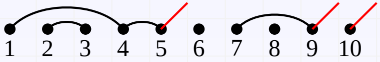


- Otherwise, it is called **nonnested**.

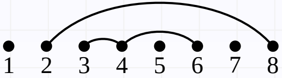
$\mathrm{NC}^{\mathrm{NN}}(n)$: First Interpretation for $\mathrm{NC}_B(n)$



$$\Downarrow \phi_B^{\mathrm{NC}}$$

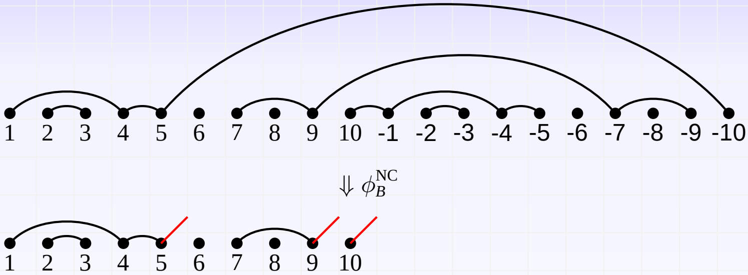


- A block B of $\pi \in \Pi(n)$ is **nested** if there is an edge (i, j) with $i < \min(B) \le \max(B) < j$.

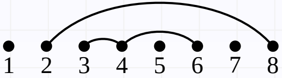


- Otherwise, it is called **nonnested**.
- $\mathrm{NC}^{\mathrm{NN}}(n) = \{(\sigma, X) : \sigma \in \mathrm{NC}(n),\ X$ is a set of **nonnested blocks** of $\sigma\}$

$\mathrm{NC}^{\mathrm{NN}}(n)$: First Interpretation for $\mathrm{NC}_B(n)$



- A block B of $\pi \in \Pi(n)$ is **nested** if there is an edge (i, j) with $i < \min(B) \leq \max(B) < j$.



- Otherwise, it is called **nonnested**.
- $\mathrm{NC}^{\mathrm{NN}}(n) = \{(\sigma, X) : \sigma \in \mathrm{NC}(n),\ X$ is a set of **nonnested blocks** of $\sigma\}$

Proposition (K., 2009)

The map $\phi_B^{\mathrm{NC}} : \mathrm{NC}_B(n) \to \mathrm{NC}^{\mathrm{NN}}(n)$ *is a bijection.*

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

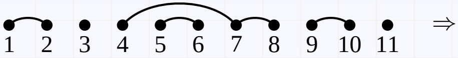
- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:

$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:

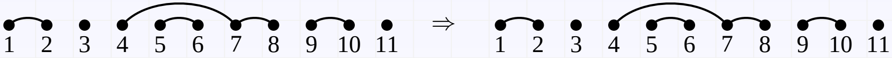
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



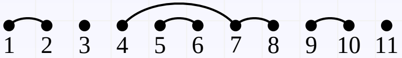
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:

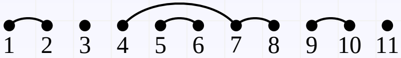


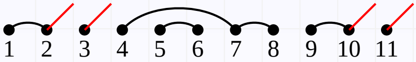
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



$\Rightarrow$

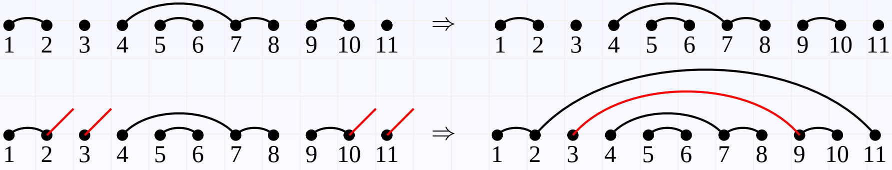




$\Rightarrow$

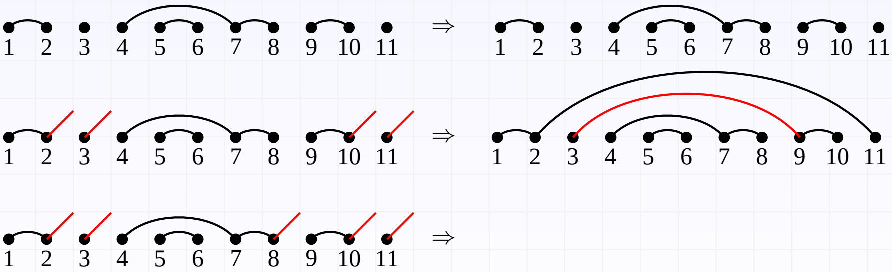
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



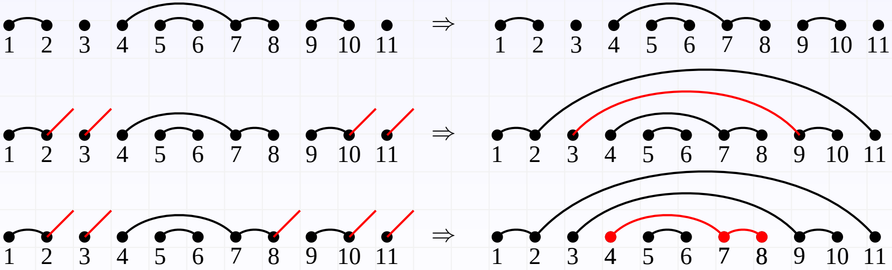
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



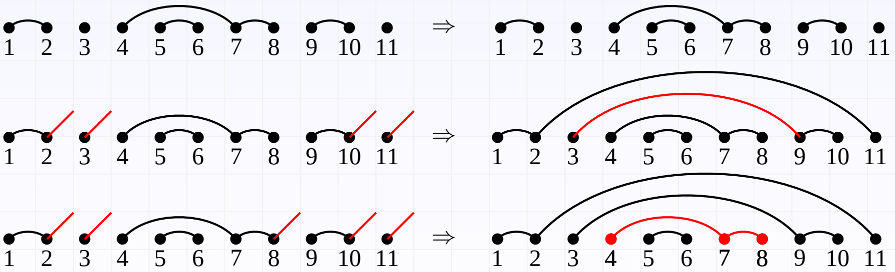
$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



$\mathfrak{B}(n)$: Second Interpretation for $\mathrm{NC}_B(n)$

- $\mathfrak{B}(n)$: the set of (σ, y) such that
  - $\sigma \in \mathrm{NC}(n)$ and
  - y is either
    1. **the emptyset** $\emptyset$,
    2. an **edge** of σ,
    3. a **block** of σ.
- Note that $\mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1]$.
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma, X)$ as follows:



Proposition (K., 2009)

$\psi_B = \varphi_B \circ \phi_B^{\mathrm{NC}} : \mathrm{NC}_B(n) \to \mathfrak{B}(n)$ *is a bijection.*

Enumeration of $\mathrm{NC}_B(n)$ with fixed block sizes

Theorem (1972, Kreweras)

The number of $\pi \in \mathrm{NC}(n)$ with ***block size vector*** $(b; b_1, b_2, \ldots, b_n)$ *is equal to*

$$\frac{1}{b}\binom{b}{b_1, b_2, \ldots, b_n}\binom{n}{b-1}.$$

Enumeration of $\mathrm{NC}_B(n)$ with fixed block sizes

Theorem (1972, Kreweras)

The number of $\pi \in \mathrm{NC}(n)$ *with* ***block size vector*** $(b; b_1, b_2, \dots, b_n)$ *is equal to*

$$\frac{1}{b}\binom{b}{b_1, b_2, \dots, b_n}\binom{n}{b-1}.$$

Enumeration of $\mathrm{NC}_B(n)$ with fixed block sizes

Theorem (1972, Kreweras)

The number of $\pi \in \mathrm{NC}(n)$ *with* ***block size vector*** $(b; b_1, b_2, \dots, b_n)$ *is equal to*

$$\frac{1}{b}\binom{b}{b_1, b_2, \dots, b_n}\binom{n}{b-1}.$$

Using the interpretation $\mathfrak{B}(n)$ for $\mathrm{NC}_B(n)$, we can easily prove the following.

Theorem (1998, Athanasiadis)

The number of $\pi \in \mathrm{NC}_B(n)$ *with* ***block size vector*** $(b; b_1, b_2, \dots, b_n)$ *is equal to*

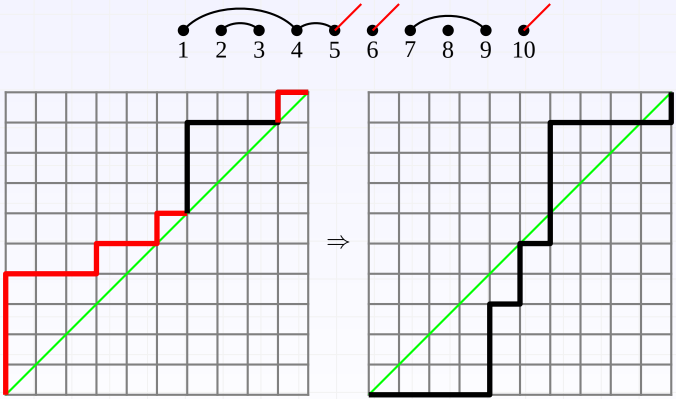
$$\binom{b}{b_1, b_2, \dots, b_n}\binom{n}{b-1}.$$

Lattice paths

- $\mathrm{LP}(n)$: the set of lattice paths from $(0,0)$ to (n,n)

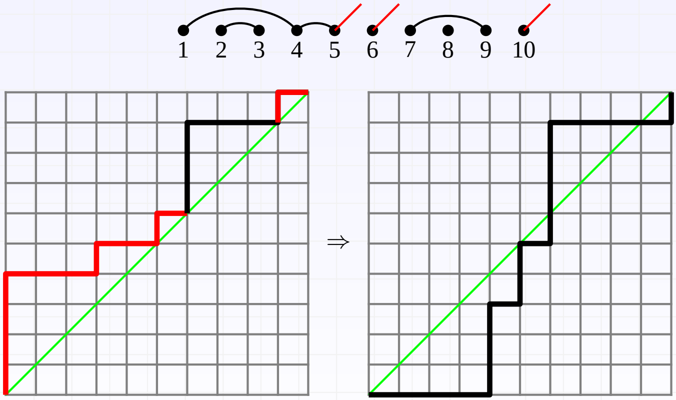
Lattice paths

- $\mathrm{LP}(n)$: the set of lattice paths from $(0,0)$ to (n,n)
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $g(\sigma, X) \in \mathrm{LP}(n)$ as follows.



Lattice paths

- $\mathrm{LP}(n)$: the set of lattice paths from $(0,0)$ to (n,n)
- For $(\sigma, X) \in \mathrm{NC}^{\mathrm{NN}}(n)$, define $g(\sigma, X) \in \mathrm{LP}(n)$ as follows.



Proposition (K., 2009)

The map $g : \mathrm{NC}^{\mathrm{NN}}(n) \to \mathrm{LP}(n)$ *is a bijection.*

Lattice paths

Corollary (K., 2009)

The following gives ***a new bijective proof*** *of* $\#\,\mathrm{NC}(n) = \frac{1}{n+1}\binom{2n}{n}$:

$$\begin{tikzcd}
\mathrm{NC}_B(n) \arrow[r, dash] & \mathrm{NC}^{\mathrm{NN}}(n) \arrow[r, dash] \arrow[d, dash] & \mathfrak{B}(n) = \mathrm{NC}(n) \times [n+1] \\
& \mathrm{LP}(n) &
\end{tikzcd}$$

k-divisible noncrossing partitions

- π is **k-divisible** if each block of π is of size divisible by k

k-divisible noncrossing partitions

- π is **k-divisible** if each block of π is of size divisible by k
- $\mathrm{NC}^{(k)}(n)$: the poset of k-divisible noncrossing partitions of $[kn]$

k-divisible noncrossing partitions

- π is ***k*-divisible** if each block of π is of size divisible by k
- $\mathrm{NC}^{(k)}(n)$: the poset of k-divisible noncrossing partitions of $[kn]$
- The cardinality of $\mathrm{NC}^{(k)}(n)$ is the **Fuss-Catalan number**:

$$\#\,\mathrm{NC}^{(k)}(n) = \frac{1}{kn+1}\binom{(k+1)n}{n}$$

Armstrong's conjecture

- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the** 180° **rotation** in the circular representation.

Armstrong's conjecture

- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the 180° rotation** in the circular representation.
- $\widetilde{\mathrm{NC}}^{(k)}(2n) \cong \mathrm{NC}_B^{(k)}(n)$

Armstrong's conjecture

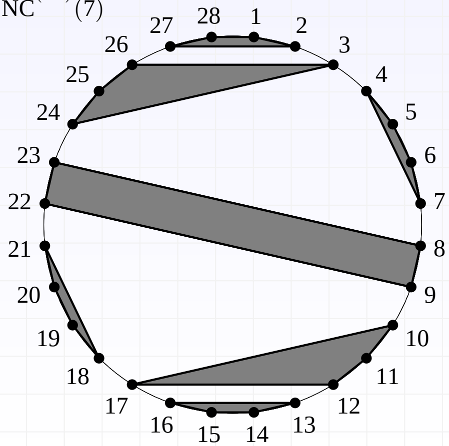
- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the 180° rotation** in the circular representation.
- $\widetilde{\mathrm{NC}}^{(k)}(2n) \cong \mathrm{NC}_B^{(k)}(n)$
- $\widetilde{\mathrm{NC}}^{(2k+1)}(2n+1) = \emptyset$

Armstrong's conjecture

- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the 180° rotation** in the circular representation.
- $\widetilde{\mathrm{NC}}^{(k)}(2n) \cong \mathrm{NC}_B^{(k)}(n)$
- $\widetilde{\mathrm{NC}}^{(2k+1)}(2n+1) = \emptyset$
- $\widetilde{\mathrm{NC}}^{(2k)}(2n+1) =?$

Armstrong's conjecture

- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the $180°$ rotation** in the circular representation.
- $\widetilde{\mathrm{NC}}^{(k)}(2n) \cong \mathrm{NC}_B^{(k)}(n)$
- $\widetilde{\mathrm{NC}}^{(2k+1)}(2n+1) = \emptyset$
- $\widetilde{\mathrm{NC}}^{(2k)}(2n+1) =$?
- An element in $\widetilde{\mathrm{NC}}^{(2\cdot 2)}(7)$



Armstrong's conjecture

Conjecture (Armstrong, 2006)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1), \ell) = \binom{n + \ell(2kn + k)}{n}.$$

Armstrong's conjecture

Conjecture (Armstrong, 2006)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1), \ell) = \binom{n+\ell(2kn+k)}{n}.$$

- Let $1 \leq r < k$.

Armstrong's conjecture

Conjecture (Armstrong, 2006)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1), \ell) = \binom{n + \ell(2kn+k)}{n}.$$

- Let $1 \leq r < k$.
- $\pi \in \mathrm{NC}(kn+r)$ is **augmented k-divisible** if all but one blocks of π have sizes divisible by k.

Armstrong's conjecture

Conjecture (Armstrong, 2006)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1), \ell) = \binom{n + \ell(2kn+k)}{n}.$$

- Let $1 \leq r < k$.
- $\pi \in \mathrm{NC}(kn+r)$ is **augmented k-divisible** if all but one blocks of π have sizes divisible by k.
- $\mathrm{NC}^{(2k)}(n; r)$: the poset of augmented k-divisible noncrossing partitions of $[kn+r]$

Armstrong's conjecture

Conjecture (Armstrong, 2006)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1),\ell) = \binom{n+\ell(2kn+k)}{n}.$$

- Let $1 \leq r < k$.
- $\pi \in \mathrm{NC}(kn+r)$ is **augmented k-divisible** if all but one blocks of π have sizes divisible by k.
- $\mathrm{NC}^{(2k)}(n;r)$: the poset of augmented k-divisible noncrossing partitions of $[kn+r]$

Theorem (K., 2009)

Let n and k be positive integers. Then

$$\widetilde{\mathrm{NC}}^{(2k)}(2n+1) \cong \mathrm{NC}^{(2k)}(n;k).$$

Armstrong's conjecture

Corollary (K., 2009)

The number of multichains in $\widetilde{\mathrm{NC}}^{(2k)}(2n+1)$ *with* ***rank jump vector*** $(s_1, s_2, \ldots, s_{\ell+1})$ *is equal to*

$$\binom{n}{s_1}\binom{2kn+k}{s_2}\cdots\binom{2kn+k}{s_{\ell+1}}.$$

Armstrong's conjecture

Corollary (K., 2009)

The number of multichains in $\widetilde{\mathrm{NC}}^{(2k)}(2n+1)$ *with* ***rank jump vector*** $(s_1, s_2, \dots, s_{\ell+1})$ *is equal to*

$$\binom{n}{s_1}\binom{2kn+k}{s_2}\cdots\binom{2kn+k}{s_{\ell+1}}.$$

Corollary (K., 2009)

Armstrong's conjecture is ***true****!*

Nonnesting partitions

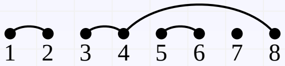
- Postnikov defined nonnesting partitions for each classical Coxeter group.

Nonnesting partitions

- Postnikov defined nonnesting partitions for each classical Coxeter group.
- Similarly we can interpret $\mathrm{NN}_B(n)$ in terms of $\mathrm{NN}(n)$.

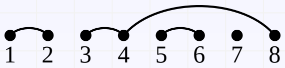
Nonnesting partitions

- Postnikov defined nonnesting partitions for each classical Coxeter group.
- Similarly we can interpret $\mathrm{NN}_B(n)$ in terms of $\mathrm{NN}(n)$.
- A block B of $\pi \in \Pi(n)$ is **aligned** if there is an edge (i, j) with $\max(B) < i$.



Nonnesting partitions

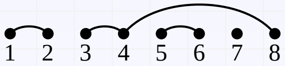
- Postnikov defined nonnesting partitions for each classical Coxeter group.
- Similarly we can interpret $\mathrm{NN}_B(n)$ in terms of $\mathrm{NN}(n)$.
- A block B of $\pi \in \Pi(n)$ is **aligned** if there is an edge (i, j) with $\max(B) < i$.



- Otherwise, it is called **nonaligned**.

Nonnesting partitions

- Postnikov defined nonnesting partitions for each classical Coxeter group.
- Similarly we can interpret $\mathrm{NN}_B(n)$ in terms of $\mathrm{NN}(n)$.
- A block B of $\pi \in \Pi(n)$ is **aligned** if there is an edge (i,j) with $\max(B) < i$.



- Otherwise, it is called **nonaligned**.
- Let

$$\mathrm{NN}^{\mathrm{NA}}(n) = \{(\sigma, X) : \sigma \in \mathrm{NN}(n),\ X \text{ is a set of } \textbf{nonaligned blocks} \text{ of } \sigma\}.$$

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$
- $\mathrm{NN}_B(n) \leftrightarrow \mathrm{NN}^{\mathrm{NA}}(n) \leftrightarrow \mathrm{NC}^{\mathrm{NA}}(n)$

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$
- $\mathrm{NN}_B(n) \leftrightarrow \mathrm{NN}^{\mathrm{NA}}(n) \leftrightarrow \mathrm{NC}^{\mathrm{NA}}(n)$

Theorem (K., 2009)

There is an involution on $\mathrm{NC}(n)$ exchanging ***nonnested blocks*** *and* ***nonaligned blocks****.*

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$
- $\mathrm{NN}_B(n) \leftrightarrow \mathrm{NN}^{\mathrm{NA}}(n) \leftrightarrow \mathrm{NC}^{\mathrm{NA}}(n)$

Theorem (K., 2009)

*There is an involution on $\mathrm{NC}(n)$ exchanging **nonnested blocks** and **nonaligned blocks**.*

Corollary (K., 2009)

For any integers i and j,

$$\begin{aligned} & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = i, \mathrm{na}(\pi) = j\} \\ = \quad & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = j, \mathrm{na}(\pi) = i\}. \end{aligned}$$

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$
- $\mathrm{NN}_B(n) \leftrightarrow \mathrm{NN}^{\mathrm{NA}}(n) \leftrightarrow \mathrm{NC}^{\mathrm{NA}}(n)$

Theorem (K., 2009)

There is an involution on $\mathrm{NC}(n)$ exchanging ***nonnested blocks*** *and* ***nonaligned blocks****.*

Corollary (K., 2009)

For any integers i and j,

$$\begin{aligned} & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = i, \mathrm{na}(\pi) = j\} \\ = \; & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = j, \mathrm{na}(\pi) = i\}. \end{aligned}$$

Corollary (K., 2009)

The following is a block size preserving bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$.

$$\mathrm{NC}_B(n) \;\text{———}\; \mathrm{NC}^{\mathrm{NN}}(n) \;\text{———}\; \mathrm{NC}^{\mathrm{NA}}(n) \;\text{———}\; \mathrm{NN}^{\mathrm{NA}}(n) \;\text{———}\; \mathrm{NN}_B(n)$$

A bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$

- $\mathrm{NC}_B(n) \leftrightarrow \mathrm{NC}^{\mathrm{NN}}(n)$
- $\mathrm{NN}_B(n) \leftrightarrow \mathrm{NN}^{\mathrm{NA}}(n) \leftrightarrow \mathrm{NC}^{\mathrm{NA}}(n)$

Theorem (K., 2009)

*There is an involution on $\mathrm{NC}(n)$ exchanging **nonnested blocks** and **nonaligned blocks**.*

Corollary (K., 2009)

For any integers i and j,

$$\begin{aligned} & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = i, \mathrm{na}(\pi) = j\} \\ = \ & \#\{\pi \in \mathrm{NC}(n) : \mathrm{nn}(\pi) = j, \mathrm{na}(\pi) = i\}. \end{aligned}$$

Corollary (K., 2009)

The following is a block size preserving bijection between $\mathrm{NC}_B(n)$ and $\mathrm{NN}_B(n)$.

$$\mathrm{NC}_B(n) \text{ --- } \mathrm{NC}^{\mathrm{NN}}(n) \text{ --- } \mathrm{NC}^{\mathrm{NA}}(n) \text{ --- } \mathrm{NN}^{\mathrm{NA}}(n) \text{ --- } \mathrm{NN}_B(n)$$

- Our bijection is different from that of Fink and Giraldo.

Final remarks

- We have found two interpretations for $\mathrm{NC}_B(n)$.

Final remarks

- We have found two interpretations for $\mathrm{NC}_B(n)$.
- Many interesting properties of $\mathrm{NC}(n)$ can be translated to $\mathrm{NC}_B(n)$ using our interpretations.

Final remarks

- We have found two interpretations for $\mathrm{NC}_B(n)$.
- Many interesting properties of $\mathrm{NC}(n)$ can be translated to $\mathrm{NC}_B(n)$ using our interpretations.

Thank you for your attention!