Gauss maps in positive characteristic

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KAIST
Fyodor Zak

0 Introduction

This is a story of projective algebraic geometry in positive characteristic. I survey a history of studies on Gauss maps of projective varieties in positive characteristic, and state some recent results.

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This is a story of projective algebraic geometry in positive characteristic. I survey a history of studies on Gauss maps of projective varieties in positive characteristic, and state some recent results.

The theme of those studies is originally condensed into the following:

Problem ([Kleiman (1987)])

> "It would be good to have an example of a smooth curve X such that every tangent makes 2 or more contacts or to prove that such X do not exist."

S.Kleiman (with A.Thorup), "Intersection theory and enumerative geometry: A decade in review," in "Algbraic Geometry — Bowdoin 1985," S.J.Bloch (ed.) Proc. Symposia Pure Math. 46–Part 2 (1987), pp. 321–370.

TOC

$/k$ an algebraically closed field of characteristic $p \geq 0$

1 Examples and the definition of a Gauss map

Example 1 ([Wallace (1956)])

$$X : x^{p+1} + y^{p+1} + z^{p+1} = 0 \subseteq \mathbb{P}^2 \quad (p > 0)$$

- $T_P X = \{a^p x + b^p y + c^p z = 0\}$ for $P = (a : b : c) \in X$.
- the contact multiplicity: $i(X, T_P X; P) = p$ for a general $P \in X$.

  In fact, $X \cap T_P X = pP + Q$ with $Q := (a^{p^2} : b^{p^2} : c^{p^2})$.
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  In fact, $X \cap T_P X = pP + Q$ with $Q := (a^{p^2} : b^{p^2} : c^{p^2})$.
- the dual map is given by

$$\gamma : X \to \check{\mathbb{P}}^2; P = (a : b : c) \mapsto [T_P X] = (a^p : b^p : c^p),$$

where $\mathbb{P}^2 \supseteq \{\xi x + \eta y + \zeta z = 0\} \leftrightarrow (\xi : \eta : \zeta) \in \check{\mathbb{P}}^2$.

- the dual curve $X^* := \gamma(X) = \{[T_P X] \in \check{\mathbb{P}}^2 | P \in X\}$.

$$\rightsquigarrow \quad X^* = \{\xi^{p+1} + \eta^{p+1} + \zeta^{p+1} = 0\} \simeq X.$$

Why?

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  $\rightsquigarrow$ $X^* = \{\xi^{p+1} + \eta^{p+1} + \zeta^{p+1} = 0\} \simeq X$.

$$\begin{aligned} \because (a^p)^{p+1} &+ (b^p)^{p+1} + (c^p)^{p+1} \\ &= (a^{p+1})^p + (b^{p+1})^p + (c^{p+1})^p \\ &= (a^{p+1} + b^{p+1} + c^{p+1})^p = 0 \ (\because (a : b : c) \in X). \ \square \end{aligned}$$

  $\rightsquigarrow$ $\gamma =$ the Frobenius morphism of X ($= X^{**}$ via $\mathbb{P}^2 = \check{\check{\mathbb{P}}}^2$).
- $K(X)/K(X^*)$ purely inseparable of degree p.

Example 2 (strange curve)

$$X : y = x^p \subseteq \mathbb{A}^2 \quad (p > 0)$$

- $T_P X = \{y - a^p = 0\}$ for $P = (a, a^p) \in X$ ($\because \frac{dy}{dx} = px^{p-1} = 0$)

  $\rightsquigarrow$ all tangent lines are parallel to x-axis, i.e.,

  all proj tangent lines $\ni$ a common pt $(1 : 0 : 0) \in \mathbb{P}^2$,

  where $\mathbb{A}^2 = \{(x : y : 1))\} \subseteq \mathbb{P}^2$.

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- $K(X)/K(X^*)$ purely inseparable of degree p.
- Note that X smooth $\Leftrightarrow p = 2$.

Fact ([Lluis (1962)], [Samuel (1966)])

The only non-linear smooth strange curve X in $\mathbb{P}^N$ is
a conic in $p = 2$.

Definition (Gauss map)

For a non-linear algebraic variety X of dim n in $\mathbb{P}^N$,

$$\gamma : X \dashrightarrow \mathbb{G}(n, N); x \mapsto T_x X,$$

where

$T_x X \subseteq \mathbb{P}^N$ the proj tangent space to X at a smooth pt x.

Remark

$N = 2, n = 1 \Rightarrow \gamma =$ the dual map $X \dashrightarrow X^*$.

Remark For a generically finite γ,

$$\# \text{ of } \begin{pmatrix} \text{contacts of a general} \\ \text{proj tangent space} \end{pmatrix} \text{ as a set } = \underbrace{[K(X) : K(\gamma(X))]_s}_{\text{separable degree of } \gamma}$$

Problem ([Kleiman (1987)]) rephrased as:

$\exists$? a smooth proj curve X s.t. γ has separable degree > 1 or prove that such an X does not exist.

2 Gauss maps of projective curves

Remark For a proj curve X,

$p = 0 \Rightarrow \gamma$ is birational, as is classically well known.

Observation

In positive characteristic case
various strange phenomina on tangency have been observed, and seem to be caused by the inseparability of Gauss maps γ.

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2 Gauss maps of projective curves

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Observation

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various strange phenomina on tangency have been observed, and
seem to be caused by the inseparability of Gauss maps γ.

Fact ([K (1989)]) For a proj curve X,

- γ is separable $\Rightarrow \gamma$ is birational (classically known for $X \subseteq \mathbb{P}^2$).
- $p \neq 2 \Leftrightarrow \exists$ a $\underline{\text{proj embedding}}$ of X s.t. γ is birational.

$$\underline{\text{proj embedding}} \, \| \, \text{biregular embedding into some proj space}$$

biregular embedding into some proj space

Birational embeddings of smooth proj curves (possibly singular case)

Theorem 1 ([Wallace (1956)], [Kleiman (1986)])

$\forall$ curve $X' \subseteq \check{\mathbb{P}}^2$ in $p > 0$, $\forall s \geq 1$, $\forall r \geq 1$,
$\exists$ a curve $X \subseteq \mathbb{P}^2$ s.t.

$$\gamma(X) = X' \quad \& \quad \begin{cases} [K(X) : K(X')]_s = s, \\ [K(X) : K(X')]_i = p^r. \end{cases}$$

Theorem 2 ([K (1989)])

$\forall$ inseparable ext K/K' of function fields of dim 1,
$\exists$ a proj model $X \subseteq \mathbb{P}^N$ of K s.t.

$$K/K' = K(X)/K(\gamma(X)).$$

Remark

Those curves X in Theorems above are singular in most cases. So, it would be natural to assume the smoothness in the problem.

Rational curves

Proposition 3 ([K (1986)], [Rathmann (1987)])

$\forall$ inseparable finite extension $K(\mathbb{P}^1)/K'$ of function fields,
$\exists$ a smooth rational curve $X \subseteq \mathbb{P}^N$ s.t.

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Proof

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Proof

- Take $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ a finite morphism.
  $\leftrightarrow$ $K(\mathbb{P}^1)/K'$ a given insep finite extension ($\rightsquigarrow K' \simeq K(\mathbb{P}^1)$)
- embed its graph $\Gamma_f := \{(x, f(x)) | x \in \mathbb{P}^1\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$
  so that $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric surface.
- $X :=$ the image of Γ_f in $\mathbb{P}^3$.
  $\rightsquigarrow$ f is recovered as the Gauss map γ of X.
  $\because$ every fibre $\mathbb{P}^1 \times \{y\}$ is a line in $\mathbb{P}^3$,
  tangent to X because of the inseparability of f. $\Box$

Biregular embeddings of smooth proj curves (smooth case)

Question For a smooth proj curve X,

which subfield K' of the function field $K(X)$ shows up as $K(\gamma(X))$, the function field of the image of the Gauss map ι for a biregular embedding of X into some $\mathbb{P}^M$?

Definition (subfields given by Gauss maps)

For a smooth proj curve X, consider

$$\mathcal{K}' := \left\{ K(\gamma(X)) \subseteq K(X) \,\middle|\, \gamma \begin{array}{l}\text{the Gauss map of} \\ \text{a biregular embedding of } X\end{array} \right\}.$$

Remark

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Remark With this notation, the known results are described as follows:

- $p = 0 \Rightarrow \mathcal{K}' = \{K(X)\}$ for any X.
- $K' \in \mathcal{K}'$ & $K(X)/K'$ separable $\Rightarrow K' = K(X)$.
- $p \neq 2 \Leftrightarrow K(X) \in \mathcal{K}'$ for any X.
- For $X = \mathbb{P}^1$, $\mathcal{K}' \backslash \{K(\mathbb{P}^1)\} = \{K' \subseteq K(\mathbb{P}^1) | K(\mathbb{P}^1)/K'$ finite insep ext$\}$.

Elliptic curves

Theorem 4 ([K (1989)], [K (1991)])

If X is an ordinary elliptic curve, then

$$\mathcal{K}' \setminus \{K(X)\} = \left\{ K(X') \subseteq K(X) \;\middle|\; \begin{array}{l} X \to X' \text{ insep isogeny, and} \\ \hat{X} \leftarrow \hat{X}' \text{ separable cyclic} \end{array} \right\} = \left\{ K' \subseteq K(X) \;\middle|\; \begin{array}{l} K(X)/K' \text{ insep finite, and} \\ K'_s/K' \text{ cyclic of deg } \not\equiv 0 \bmod p \end{array} \right\},$$

where

$\hat{X} := \mathrm{Pic}^0 X$ the dual of X, and
K'_s the separable closure of K' in $K(X)$.

Recall that for an elliptic curve X in char $p > 0$,

$$X \text{ is said to be } \begin{cases} \text{ordinary} & \text{if } r = 1, \\ \text{supersingular} & \text{if } r = 0, \end{cases}$$

where r is the p-rank of X defined by

$$\#\ker(p_X : X \to X; x \mapsto p \cdot x) = p^r. \quad (\rightsquigarrow r = 0, 1)$$

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where

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K'_s the separable closure of K' in $K(X)$.

Theorem 5 ([K (1989)])

If X is a supersingular elliptic curve, then

$$\mathcal{K}' = \begin{cases} \{K(X)^2, K(X)^{2^2}\}, & \text{if } p = 2, \\ \{K(X), K(X)^p\}, & \text{if } p > 2. \end{cases}$$

Key observation

Theorem 6 ([K (1989)]) X a smooth proj curve.

For an inseparable finite extension $K(X)/K'$, TFAE:

1. $K' \in \mathcal{K}'$;
2. $\exists$ a vector bundle $\mathcal{E}$ of rank 2 on X' and $\exists$ an embedding $\varphi : X \hookrightarrow \mathbb{P}(\mathcal{E})$ s.t.

$$K(X)/K' = K(\varphi(X))/K(X'),$$

where X' a smooth proj model of K'.

$\leftsquigarrow$

$$\begin{tikzcd} X \arrow[r, hook, "\varphi"] \arrow[dr] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & X' \end{tikzcd}$$

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$$\leftsquigarrow \quad \boxed{\begin{tikzcd} X \arrow[r, hook, "\varphi"] \arrow[rd] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & X' \end{tikzcd}}$$

Remark The idea of the proof of (1) $\Leftarrow$ (2):

- embed $\mathbb{P}(\mathcal{E})$ into some $\mathbb{P}^M$ as a scroll, and
- consider $\iota : X \overset{\varphi}{\hookrightarrow} \mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}^M$, the composite morphism.

$\rightsquigarrow$ the image of any fibre of $\mathbb{P}(\mathcal{E}) \rightarrow X'$ is a line tangents to $\iota(X)$ in $\mathbb{P}^M$, and the Gauss map of $\iota(X) \underset{\text{bir}}{\sim} X \rightarrow X'$ defined via φ.

Curves with higher genus

Theorem 7 ([K (1989)])

X a smooth proj curve of genus $g \geq 2$.
$F : X \longrightarrow X'$ the Frobenius morphism.
Then TFAE:

1. $K(X)^p \in \mathcal{K}'$ (i.e., $\exists \iota : X \hookrightarrow \mathbb{P}^M$ with $\gamma \underset{\text{bir}}{\sim}$ the Frob morph);
2. $\exists$ a stable vector bundle $\mathcal{E}$ on X', and $\exists$ $\mathcal{L} \in \operatorname{Pic} X$ s.t.

$$F^*\mathcal{E} \simeq \mathcal{P}^1_X(\mathcal{L}).$$

where
$\mathcal{P}^1_X(\mathcal{L})$ the bundle of principal parts of first order of $\mathcal{L}$.

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Remark The idea of the proof of (1) $\Leftarrow$ (2):

Apply the Key Observation (Theorem 6) with the stable $\mathcal{E}$ above and

$$\varphi : X \simeq \mathbb{P}(\mathcal{L}) \hookrightarrow \mathbb{P}(\mathcal{P}^1_X(\mathcal{L})) \simeq \mathbb{P}(F^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E}),$$

which turns out to be an embedding.

$\exists$ a natural exact sequence,

$$0 \rightarrow \Omega^1_X \otimes \mathcal{L} \rightarrow \mathcal{P}^1(\mathcal{L}) \rightarrow \mathcal{L} \rightarrow 0.$$

Definition (Tango-Raynaud curve)

X is a Tango-Raynaud curve

$\underset{\text{def}}{\Leftrightarrow} \exists\, \mathcal{N} \in \operatorname{Pic} X'$ s.t.

- $F^*\mathcal{N} \simeq \Omega^1_X$
- $F^* : H^1(X', \mathcal{N}^\vee) \longrightarrow H^1(X, F^*\mathcal{N}^\vee)$ not injective

where $F : X \longrightarrow X'$ the Frobenius morphism.

Remark

A Tango-Raynaud curve $\rightsquigarrow$ $\exists$ alg surfaces for which the Kodaira vanishing theorem does not hold.

Example

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Remark

A Tango-Raynaud curve $\rightsquigarrow$

∃ alg surfaces for which the Kodaira vanishing theorem does not hold.

Example (Theorem 7) For a Tango-Raynaud curve X of $g \geq 2$,

$$(0 \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{E} \longrightarrow \mathcal{N} \longrightarrow 0) \leftrightarrow 0 \neq \xi \in \ker F^* \subseteq \operatorname{Ext}^1_{X'}(\mathcal{N}, \mathcal{O}_{X'})$$

$\uparrow$

satisfies (2) in Theorem 7.

In fact, $\mathcal{E}$ stable, and

$$F^*\mathcal{E} \simeq \mathcal{O}_X \oplus \Omega^1_X = \mathcal{P}^1_X(\mathcal{O}_X).$$

Generic injectivity of Gauss map

Theorem 8 ([K (1989)])

X a proj curve of genus g in $\mathbb{P}^N$.
g' the genus of the normalization of $\gamma(X)$.
Then:

X smooth or nodal $\Rightarrow g = g'$.

In particular:

$g \geq 2 \Rightarrow \gamma$ birational or purely inseparable, i.e., generically injective.

Theorem 9 ([K (1991)], [Kleiman-Piene (1991)])

X a proj curve of genus g in $\mathbb{P}^N$
$\nu := \deg \Omega^1_{\widetilde{X}/\mathbb{P}^N}$ the # of cusps of X ($\widetilde{X}$ the normalization of X)
Then:

$$\nu < 2g - 2 \Rightarrow \gamma \text{ generically injective.}$$

Theorem 10 ([Kleiman-Piene (1991)]) With the same notation as above,

s the separable degree of γ.
Then:

$$2s(g - g') \leq (s - 1)\nu.$$

In particular, $\nu = 0 \Rightarrow g = g'$.

Brief Summary (proj curves):

- possibly singular case:

$\forall$ insep finite ext K/K' of function fields of dim 1,
$\exists$ a proj model X of K s.t.

$$K/K' = K(X)/K(\gamma(X)).$$

possible to take X to be smooth for rational K.

- smooth case:

$$\left.\begin{array}{r}\text{rational curves}\\ \text{ordinary elliptic curves}\end{array}\right\} \Rightarrow \begin{array}{l}\exists\text{ a biregular embedding s.t.}\\ \text{the Gauss map has}\\ \text{separable degree} > 1\text{, i.e.,}\\ \text{not generically injective.}\end{array}$$

$$g(X) = g(\widetilde{\gamma(X)}).$$

$$\left.\begin{array}{r}\text{supersingular elliptic curves}\\ \text{higher genus curves}\end{array}\right\} \Rightarrow \begin{array}{l}\text{the Gauss map of}\\ \text{any biregular embedding is}\\ \text{generically injective.}\end{array}$$

3 Gauss maps of projective varieties

The **Gauss map** of a proj var X of dim n in $\mathbb{P}^N$ is

$$\gamma : X \dashrightarrow \mathbb{G}(n, N); x \mapsto T_x X.$$

Fact

- [Griffiths-Harris (1979)]: $p = 0 \Rightarrow$ a gen fibre of γ is linear.
- [Zak (1993?)]: X smooth $\Rightarrow \gamma$ is finite in arbitrary char.

Hence, X smooth in $p = 0 \Rightarrow \gamma$ is birational finite.

Generic injectivity of Gauss map

Theorem 11 ([Kleiman-Piene (1991)])

X a smooth complete intersection of dim n and degree ≥ 2.
Assume:
$\gamma(X)$ smooth.
Then:

$$c_n(X) = c_n(\gamma(X)).$$

Theorem 12 The generic injectivity of the Gauss map holds for

1. [Kleiman-Piene (1991)]: a smooth complete int of dim 2.
2. [K-Noma (1997)]: a smooth proj var with generically ample cotangent bundle.
3. [Noma (1997)]: a smooth weighted complete int of general type with dim ≥ 3.
4. [Noma (1997)]: a smooth proj var of dim 2 or 3 with μ-semistable tangent bundle.

Gauss map with non-trivial separable degree

Theorem 13 ([Noma (2001)])

1. $\exists$ examples of proj var with an embedding s.t. the Gauss map of separable degree > 1 as follows:
   - (a) a smooth var of non-general type (i.e., Kodaira dim $\kappa < n$).
   - (b) a normal var of general type with only isolated sing.
2. $\exists$ an embedding of a proj space $\mathbb{P}^n$ into some proj space s.t.

   the Gauss map has separable degree > 1 and
   its image $=$ a normal var of general type.

A new direction: a general fibre of Gauss map

Example ([Fukasawa (2005)])

$$X : wy^6 - (x^6 + y^6 + z^6)z = 0 \subseteq \mathbb{P}^3 \quad (p = 3)$$

$\rightsquigarrow$ A general fibre of the Gauss map is a conic!

Remark

This is the first known example of a proj var s.t.
a gen fibre of the Gauss map is non-linear with dim > 0!

Theorem 14 ([Fukasawa (2006)])

For a given proj var Y in $\mathbb{P}^n$ in characteristic $p > 0$,
$\exists$ a proj var X of dim n s.t.

a general fibre of the Gauss map $\underset{\text{PGL}}{\sim}$ Y.

4 Recent results

joint work with S.Fukasawa (深澤 知) and K.Furukawa (古川勝久)

FK:=Fukasawa-Kaji FFK:=Fukasawa-Furukawa-Kaji

Question

Which smooth proj var X has a $\left\{\begin{matrix}\text{birational}\\ \text{biregular}\end{matrix}\right\}$ embedding with insep Gauss map γ?

Birational embedding case

Theorem 15 ([FK])

$\forall$ alg function field K over k in $p > 0$
$\exists$ a proj model $X \subseteq \mathbb{P}^N$ of K s.t.

$\gamma \underset{\text{bir}}{\sim}$ the Frobenius morphism.

More precisely,

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More precisely,

X a proj var of dim n in $p > 0$.

1. For an integer r with $0 \leq r \leq n$, TFAE:
   (a) $\exists$ a birational embedding $X \dashrightarrow \mathbb{P}^N$ s.t.
   γ gen finite & $\operatorname{rk} \gamma = r$;
   (b) $(p, r) \neq (2, 1)$.
2.

Here $\operatorname{rk} \gamma := \operatorname{rk}(d_x\gamma : t_xX \to t_{\gamma(x)}\mathbb{G}(n, \mathbb{P}^N))$ at a general $x \in X$.

Birational embedding case

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More precisely,

X a proj var of dim n in $p > 0$.

1. For an integer r with $0 \leq r \leq n$, TFAE:
   (a) $\exists$ a birational embedding $X \dashrightarrow \mathbb{P}^N$ s.t.
   γ gen finite & $\mathrm{rk}\,\gamma = r$;
   (b) $(p, r) \neq (2, 1)$.
2. Moreover for any r satisfying (b) above,
   $\exists$ a birational embedding $X \dashrightarrow \mathbb{P}^N$ s.t.
   $K(X)/K(\gamma(X))$ purely insep of degree p^{n-r}.

Here $\mathrm{rk}\,\gamma := \mathrm{rk}(d_x\gamma : t_xX \to t_{\gamma(x)}\mathbb{G}(n, \mathbb{P}^N))$ at a general $x \in X$.

Corollary ([FK])

$\forall$ alg function field K over k in $p > 0$

$\exists$ a proj model $X \subseteq \mathbb{P}^N$ of K s.t. X is strange, i.e.,

$$\exists P \in \mathbb{P}^N, \forall \text{ smooth } x \in X, P \in T_x X$$

Biregular embedding case (smooth case)

We introduce an intrinsic property for a proj var X as follows:

(GMRZ) $\exists$ an embedding $X \hookrightarrow \mathbb{P}^M$ s.t. $\mathrm{rk}\,\gamma = 0$.

Remark

- A variety X satisfies (GMRZ) $\Rightarrow p > 0$.
  $\because$ any rational map in $p = 0$ is separable
  $\rightsquigarrow \mathrm{rk}\,\gamma = \dim \gamma(X) > 0$.
- A variety X satisfies (GMRZ) $\Leftrightarrow d\gamma \equiv 0$
  $\Leftrightarrow K(\gamma(X)) \subseteq K(X)^p$ in $K(X)$.

Example (GMRZ)

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Example (GMRZ)

A Fermat hypersurf X of degree $\equiv 1$ mod $p > 0$ in $\mathbb{P}^N$
satisfies (GMRZ).

$\because$ The Gauss map γ_0 for the embedding $X \hookrightarrow \mathbb{P}^N$
factors thru the Frobenius morphism.

Moreover we have

Theorem 16 ([FFK])

1. A **Segre** var $\prod_{1\le i\le r}\mathbb{P}^{n_i}$ $(r \ge 2, n_i \ge 1)$ satisfies (GMRZ)
   $\Leftrightarrow p = 2$ & $n_i = 1$ $(\forall i)$.
2. A **Grassman** var $\mathbb{G}(l, m)$ $(1 \le l < m)$ satisfies (GMRZ)
   $\Leftrightarrow l = 1$ or $l = m - 1$, i.e., a proj sp.
3. A smooth **quadric** hypersurf X in $\mathbb{P}^N$ $(N \ge 3)$ satisfies (GMRZ)
   $\Leftrightarrow p = 2$ & $N = 3$.
4. A smooth **cubic** hypersurf X in $\mathbb{P}^N$ $(N \ge 3)$ satisfies (GMRZ)
   $\Rightarrow p = 2$.

The results above follow from

Theorem 17 ([FFK])

X a proj var.
$f : \mathbb{P}^1 \longrightarrow X$ an unramified morphism.
$N_f := \ker(f^* : f^*\Omega^1_X \longrightarrow \Omega^1_{\mathbb{P}^1})^\vee$ the normal bundle of f.
Assume:

- X smooth along $f(\mathbb{P}^1)$.
- $N_f^\vee \simeq \bigoplus_{i \geq -1} \mathcal{O}_{\mathbb{P}^1}(i)^{r_i}$ $(r_i \geq 0)$.

Then:

1. X satisfies (GMRZ) $\Rightarrow$ $r_{i-1} = 0$ or $r_i = 0$ $(\forall i \geq 0)$.
2. Moreover:
   (a) $r_i > 0$ $(i \geq 0)$ $\Rightarrow$ $p = 2$ or $p | i + 1$.
   (b) $r_{-1} > 0 \Rightarrow$ $p | \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1) - 1$ $(\forall X \overset{\iota}{\hookrightarrow} \mathbb{P}^M$ s.t. $d\gamma \equiv 0)$.

Proof

Theorem 17 ([FFK])

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Proof (Sketch) Assume: $\operatorname{rk}\gamma = 0$ for $\iota : X \hookrightarrow \mathbb{P}^M$, and set $\mathcal{L} := \iota^*\mathcal{O}_{\mathbb{P}^M}(1)$.

$\Rightarrow \gamma \circ f : \mathbb{P}^1 \longrightarrow \mathbb{G}(n, \mathbb{P}^M)$ has rank zero.
$\Rightarrow$ the splitting type of $f^*\mathcal{P}^1_X(\mathcal{L}) \equiv \overrightarrow{0}$ mod p.
$\rightsquigarrow$ information on p and N_f.
$\because 0 \longrightarrow N_f^\vee \otimes f^*\mathcal{L} \longrightarrow f^*\mathcal{P}^1_X(\mathcal{L}) \longrightarrow \mathcal{P}^1_{\mathbb{P}^1}(f^*\mathcal{L}) \longrightarrow 0$ splits, &
the splitting type of $\mathcal{P}^1_{\mathbb{P}^1}(f^*\mathcal{L}) = \ldots$ □

Cubic hypersurfaces

Theorem 18 ([FFK])

A smooth cubic X in $\mathbb{P}^N$ $(N \geq 5)$ satisfies (GMRZ)
$\Leftrightarrow X \underset{\text{PGL}}{\sim}$ a Fermat hypersurface in $p = 2$.

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Theorem 19 ([FFK])

A smooth cubic X in $\mathbb{P}^N$ $(N \geq 3)$ in $p = 2$ satisfies $\underline{\operatorname{rk}\gamma_0 = 0}$
$\Leftrightarrow X \underset{\text{PGL}}{\sim}$ a Fermat hypersurface.

Remark

The hard part in our proof of Theorem 17 is to show:

$$\text{(GMRZ)} \Rightarrow \operatorname{rk}\gamma_0 = 0$$

for a smooth cubic $X \subseteq \mathbb{P}^N$, where we need $N \geq 5$.
The cases $N = 3, 4$ are open.

General hypersurfaces

Theorem 20 ([FFK])

A general hypersurf X of degree d in $\mathbb{P}^N$ with $3 \leq d \leq 2N-3$ satisfies (GMRZ)
$\Rightarrow p = 2$ & $d = 2N-3$.

Remark

$2N-3-d = \chi(N_{L/X})$ is the “expected dim” of $\{$lines $L \subseteq X\}$.

Proof

General hypersurfaces

Theorem 20 ([FFK])

A general hypersurf X of degree d in $\mathbb{P}^N$ with $3 \leq d \leq 2N-3$ satisfies (GMRZ)

$\Rightarrow p = 2$ & $d = 2N-3$.

Remark

$2N - 3 - d = \chi(N_{L/X})$ is the "expected dim" of $\{\text{lines } L \subseteq X\}$.

Proof

- The splitting type of N_f for lines $f(\mathbb{P}^1)$ on X $\rightsquigarrow p = 2$ & either $d = 2N-3$ or $\underline{d = N-1}$.
- To rule out the the latter case, assume $d = N-1$ and consider conics $f(\mathbb{P}^1)$ on X instead of lines.
- The point is to show: for a general conic $f(\mathbb{P}^1) \subseteq X$,

$$N_f \simeq \mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}^{N-4}.$$

$\rightsquigarrow N = 4$.

$\rightsquigarrow$ contradiction. $\square$

5 Geometry of rational curves on algebraic varieties

Definition

- A rational curve (or a morphism) $f : \mathbb{P}^1 \longrightarrow X$ is free $\underset{\text{def}}{\Leftrightarrow}$ f^*T_X is generated by its global sections.
- A free f is minimal (or standard) $\underset{\text{def}}{\Leftrightarrow}$ $f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \underbrace{\mathcal{O}_{\mathbb{P}^1}(1)^{a-2} \oplus \mathcal{O}_{\mathbb{P}^1}^{n-a+1}}_{N_f}$,

  where

  $a = \deg(-f^*K_X)$ & $n = \dim X$.

Remark

A family of mini free rat curves $\rightsquigarrow$

characterization of proj space & quadric hypersurface,
(Mori, Cho, Sato, Miyaoka, Araujo, Shepherd-Barron, Andreatta, Wiśniewski, Druel, Kovács, ...)

Fano var,
(Andreatta, Chierici, Occhetta, Araujo, ...)

VMRT (=var of minimal rational tangents),
(Hwang, Kebekus, Mok, Conde, ...)

...

One of the most fundamental results in $p = 0$ to guarantee the existence of minimal free rational curves would be as follows:

Fact ([Kollár, "Ratoinal Curves on Algebraic Varieties," (IV.2.10)])

For a smooth proj var X in $p = 0$,

$\exists$ a free rat curve on X $\Rightarrow$ $\exists$ a minimal free rat curve on X.

This is no longer true in positive char!

One of the most fundamental results in $p = 0$ to guarantee the existence of minimal free rational curves would be as follows:

Fact ([Kollár, "Ratoinal Curves on Algebraic Varieties," (IV.2.10)])

For a smooth proj var X in $p = 0$,

$\exists$ a free rat curve on $X \Rightarrow \exists$ a minimal free rat curve on X.

This is no longer true in positive char!

Theorem 21 ([FFK])

X a Fermat hypersurface of degree $d \equiv 1$ mod $p > 0$ in $\mathbb{P}^N$.
Then:

$N \geq 2d - 1 \Rightarrow X$ has a free rational curve but
no minimal free rational curve.

This basically follows from

Theorem 22 ([FFK])

X a proj var of dim n.
$f : \mathbb{P}^1 \longrightarrow X$ a minimal free rational curve s.t.
X smooth along $f(\mathbb{P}^1)$.
Assume:
X satisfies (GMRZ) with $\iota : X \hookrightarrow \mathbb{P}^M$.
Then, one of the following holds:

1. $\deg(-f^*K_X) = n + 1,$ $d > p$ & $p|d - 1$;
2. $\deg(-f^*K_X) = p = 2$ & $2|d$,

where $d := \deg f^*\iota^*\mathcal{O}_{\mathbb{P}^M}(1)$.

Both cases actually occur!

Example 1

1. A proj space $\mathbb{P}^n$ in $p > 0$ satisfies (GMRZ), and a line $L \subseteq \mathbb{P}^n$ is minimal free with

$$\deg(-K_{\mathbb{P}^n}|_L) = n + 1.$$

2. A Segre var $(\mathbb{P}^1)^n$ in $p = 2$ satisfies (GMRZ), and a fibre $L := \mathbb{P}^1 \times \{\text{a point}\} \subseteq (\mathbb{P}^1)^n$ is minimal free with

$$\deg(-K_{(\mathbb{P}^1)^n}|_L) = 2 = p.$$

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$$\deg(-K_{(\mathbb{P}^1)^n}|_L) = 2 = p.$$

Example 2

A Fermat cubic surface $X \subseteq \mathbb{P}^3$ in $p = 2$ satisfies (GMRZ).

1. A twisted cubic curve $C_3 \subseteq X$ is minimal free with
$$\deg(-K_X|_{C_3}) = 3 = 2 + 1.$$
2. A conic $C_2 \subseteq X$ is minimal free with
$$\deg(-K_X|_{C_2}) = 2 = p.$$

6 Final comments

Question For a proj var X in $\mathbb{P}^N$ in $p > 0$,

The Gauss map γ is separable.

$\overset{?}{\Rightarrow}$

$\Leftarrow$

a general fiber of γ is linear, e.g.,
γ is birational for smooth X.

6 Final comments

Question For a proj var X in $\mathbb{P}^N$ in $p > 0$,

The Gauss map γ is separable. $\overset{?}{\Rightarrow}$ $\Leftarrow$ a general fiber of γ is linear, e.g., γ is birational for smooth X.

[FK (2007)] $\Searrow$ if $n \leq 2$. [KP (1991)] $\Nearrow$ $\not\Swarrow$ if $n \geq 3$. $\left\{\begin{array}{l}\text{[K (2003)]}\\ \text{[F (2006)]}\end{array}\right.$

X is reflexive.

Definition $n := \dim X$

X is reflexive

$\underset{\text{def}}{\Leftrightarrow}$ $C(X) = C(X^*)$ via a natural isom $\mathbb{P}^N \times \check{\mathbb{P}}^N \simeq \check{\mathbb{P}}^N \times \check{\check{\mathbb{P}}}^N$,

$$(x, H) \leftrightarrow (H, x^{**})$$

where

$C(X) := \{(x, H) | T_xX \subseteq H, x \in X \text{ a smooth pt}\}^- \subseteq \mathbb{P}^N \times \check{\mathbb{P}}^N$
the conormal var of X,

$X^* := \mathrm{Im}(C(X) \xrightarrow[\text{2nd proj}]{} \check{\mathbb{P}}^N)$ the dual var of X.

Thank you for your attention!