Graphs with many ± 1 or $\pm\sqrt{2}$ eigenvalues

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&
POSTECH

Preliminaries

The spectrum of a graph

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- The **spectrum** of a graph G, denoted by $\mathrm{Spec}(G)$, is the set of eigenvalues of G, together with their multiplicities.

$(v,\ k,\ \lambda)$-designs

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- Let $X = \{x_1, \ldots, x_v\}$, and $\mathcal{B} = \{B_1, \ldots, B_v\}$ be k-subsets (blocks) of X. The pair $(X, \mathcal{B})$ is called a **$(v,\ k,\ \lambda)$-design** if each two distinct B_i, B_j $(1 \leqslant i, j \leqslant v)$ intersect in λ elements; and $0 \leqslant \lambda < k < v - 1$.

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- Each combinatorial design is completely determined by its corresponding **incidence matrix**; this is the $(0,1)$-matrix $A = (a_{ij})$ defined by taking $a_{ij} = 1$ if $x_j \in B_i$ and $a_{ij} = 0$ if $x_j \notin B_i$.

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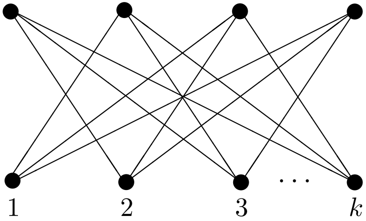
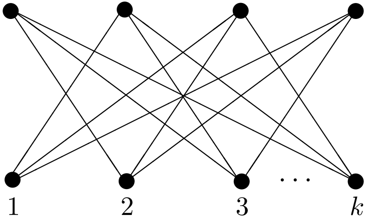
Examples



$\mathcal{L}_{k,k}$

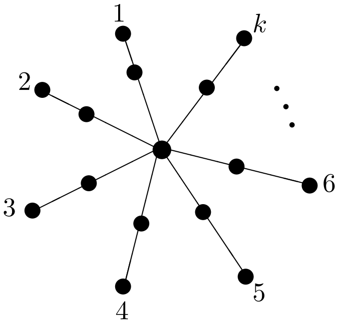
Examples



$\mathcal{L}_{k,k}$

$$\mathrm{Spec}(\mathcal{L}_{k,k}) = \left\{ \pm(k-1),\ (\pm 1)^{k-1} \right\}$$

Examples



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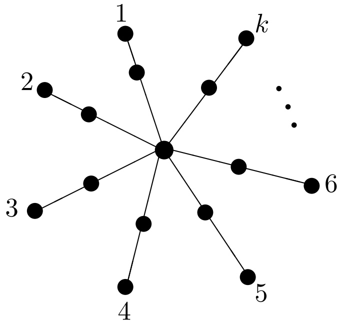

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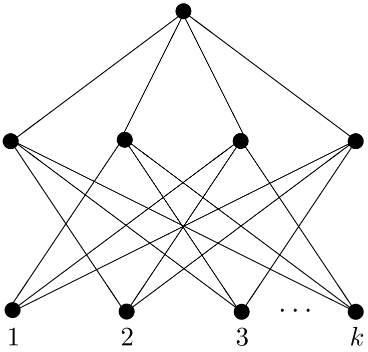

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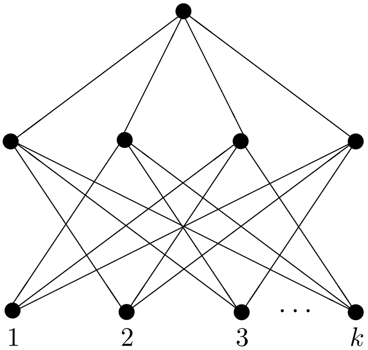

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$$\mathrm{Spec}(\mathcal{H}_{k,k+1}) = \left\{ \pm\sqrt{k^2 - k + 1},\ 0,\ (\pm 1)^{k-1} \right\}$$

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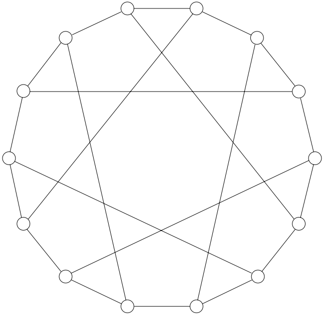

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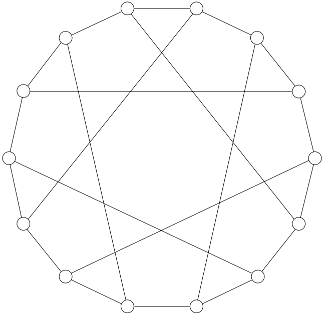

The Heawood graph

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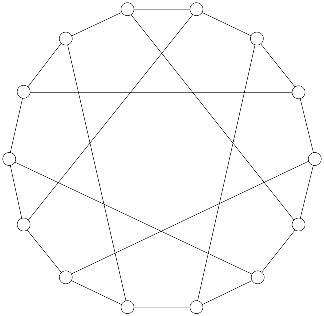

The Heawood graph
The incidence graph of the Fano plane

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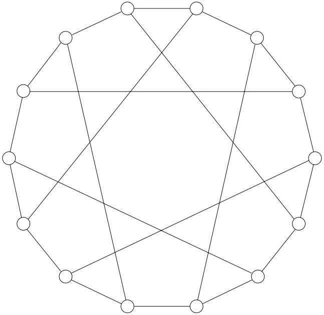

The Heawood graph
The incidence graph of the Fano plane

$$\mathrm{Spec}(\mathrm{Heawood}) = \left\{ \pm 3, (\pm\sqrt{2})^6 \right\}$$

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Multiplicative designs were introduced by Ryser (1942), and have been studied by Bridges, Mena, Host (1980's), and recently by van Dam and Spence (2004).

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Definition

A **pseudo $(v,\ k,\ \lambda)$-design** is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B} = \{B_1, \ldots, B_{v-1}\}$ is a collection of k-subsets (blocks) of X such that each two distinct B_i, B_j intersect in λ elements; and $0 < \lambda < k < v - 1$.

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Developed by O. Marrero, H.J. Ryser, and D.R. Woodall, etc.

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Types of pseudo designs

A pseudo (v, k, λ)-design is called **primary** if $v\lambda \neq k^2$ and is called **nonprimary** when $v\lambda = k^2$. It follows that in a nonprimary pseudo design, $v = 2k$. Thus a pseudo (v, k, λ)-design is nonprimary if and only if $v = 4\lambda$ and $k = 2\lambda$. In fact, the existence of a nonprimary pseudo (v, k, λ)-design is equivalent to existence of a Hadamard design:

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Theorem (Marrero 1974)

The incidence matrix of a given pseudo $(4\lambda, 2\lambda, \lambda)$-design can always be obtained from the incidence matrix A of a $(4\lambda - 1, 2\lambda - 1, \lambda - 1)$-design by adjoining one column of all 1's to A and then possibly complementing some rows of A.

Primary pseudo $(v,\, k,\, \lambda)$-designs

Theorem (Marrero 1974)

The incidence matrix A of a primary pseudo $(v,\, k,\, \lambda)$-design $\mathcal{D}$ can be obtained from the incidence matrix of a $(\bar{v},\, \bar{k},\, \bar{\lambda})$-design whenever $\mathcal{D}$ satisfies one of the following arithmetical conditions on its parameters.

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(iv) If $k = 2\lambda$, then A is obtained from the incidence matrix B of a $(v,\, k,\, \lambda)$-design as follows: a row is discarded from B and then the k' columns of B which had a 1 in the discarded row are complemented (0's and 1's are interchanged in these columns).

Type (i)

Graphs with $(\pm 1)^{\frac{n-2}{2}} \subset \mathrm{Spec}(G)$

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  $\Rightarrow G = K_{\frac{n}{2},\frac{n}{2}}$ minus a perfect matching (i.e., $\mathcal{L}_{\frac{n}{2},\frac{n}{2}}$).

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- If G is not regular $\Rightarrow$ G is the incidence graph of a so called "non-symmetric uniform multiplicative design" $\Rightarrow$ **(van Dam & Spence, 2004)** G has the adjacency matrix of the form

$$\begin{pmatrix} O & N \\ N^\top & O \end{pmatrix},$$

where

$$N = \begin{pmatrix} J_3 - I_3 & J_3 \\ O_3 & J_3 - I_3 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \mathbf{1}^\top \\ \mathbf{1} & I_4 \end{pmatrix}.$$

Theorem

Let G be a connected graph of order n with

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Then G is either $\mathcal{L}_{\frac{n}{2},\frac{n}{2}}$ or on the graph the graph G_1 and G_2.

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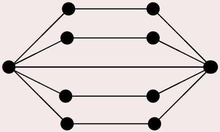
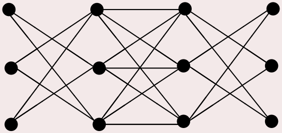


G_1



G_2

Type (ii)

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- G is bipartite of order $n = 2k + 1$ with five distinct eigenvalues;
- The vertices in the smaller part of G have the same degree d;
- G is the incidence graph of a pseudo $(k, d, d-1)$-design.

Pseudo (v, k, λ)-design with $k = \lambda + 1$

Pseudo $(v,\, k,\, \lambda)$-design with $k = \lambda + 1$

Theorem

Let $\mathcal{D}$ be a pseudo $(v,\, k,\, \lambda)$-design with $k = \lambda + 1$. Then $\mathcal{D}$ is obtained from a

$$(v-1,\, 1,\, 0)\text{-design or } (v-1,\, v-2,\, v-3)\text{-design}$$

by either adding an isolated point or a point which belongs to all of the blocks.

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$$\Updownarrow$$

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Graphs with $(\pm 1)^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

- $(v-1,\ 1,\ 0)$-design with a point added to all of its blocks

$$\Updownarrow$$

the graph $\mathcal{S}_v$

- $(v-1,\ v-2,\ v-3)$-design with a point added to all of its blocks

$$\Updownarrow$$

the graph $\mathcal{H}_{\frac{v-1}{2},\frac{v+1}{2}}$

Graphs with $(\pm 1)^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

Theorem

Let G be a connected graph of order n. If $(\pm 1)^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$, then G is either $\mathcal{S}_n$ or $\mathcal{H}_{\frac{n-1}{2},\frac{n+1}{2}}$.

Graphs with $(\pm 1)^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

Corollary

The graph $\mathcal{H}_{k,k+1}$ is DS (i.e., determined by its spectrum).

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The graph $\mathcal{S}_{2k+1}$ is DS if $k \notin S$, where

$$S = \{\ell^2 - 1 \mid \ell \in \mathbb{N}\} \cup \{\ell^2 - \ell \mid \ell \in \mathbb{N}\}.$$

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$$S = \{\ell^2 - 1 \mid \ell \in \mathbb{N}\} \cup \{\ell^2 - \ell \mid \ell \in \mathbb{N}\}.$$

Moreover, for $k \in S$ we have

- $\mathcal{S}_{17}$ has exactly two cospectral mates which are $\mathcal{L}_{3,3} \cup 5K_2 \cup K_1$ and $G_1 \cup 3K_2 \cup K_1$;
- $\mathcal{S}_{31}$ has exactly two cospectral mates which are $\mathcal{L}_{4,4} \cup 11K_2 \cup K_1$ and $G_2 \cup 9K_2 \cup K_1$;
- if $k = \ell^2 - 1$ and $k \neq 8, 15$, $\mathcal{S}_{2k+1}$ has exactly one cospectral mate which is $\mathcal{L}_{\ell,\ell} \cup (k-\ell)K_2 \cup K_1$;
- if $k = \ell^2 - \ell$, $\mathcal{S}_{2k+1}$ has exactly one cospectral mate which is $\mathcal{H}_{\ell,\ell+1} \cup (k-\ell)K_2$.

Type (iii)

Graphs with $(\pm\sqrt{2})^{\frac{n-2}{2}} \subset \mathrm{Spec}(G)$

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Let G be a connected graph of order n. If $(\pm\sqrt{2})^{\frac{n-2}{2}} \subset \mathrm{Spec}(G)$, then G has an adjacency matrix of the form

$$\begin{pmatrix} O & N \\ N^\top & O \end{pmatrix},$$

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- incidence matrix of the complement of the Fano plane;
- $$\begin{pmatrix} N_1 & J_7 \\ O_7 & N_2 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & \mathbf{1}^\top & \mathbf{1}^\top \\ \mathbf{1} & I_5 & I_5 \\ \mathbf{1} & I_5 & J_5 - I_5 \end{pmatrix},$$

  where N_1 and N_2 are the incidence matrices of the Fano plane and (7, 4, 2)-design, respectively.

Type (iv)

Graphs with $(\pm\sqrt{2})^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

Let G be a connected graph of order n with

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Graphs with $(\pm\sqrt{2})^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

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Then,

- G is bipartite of order $n = 2k + 1$ with five distinct eigenvalues;

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- G is bipartite of order $n = 2k + 1$ with five distinct eigenvalues;
- The vertices in the smaller part of G have the same degree d;
- G is the incidence graph of a pseudo $(k,\, d,\, d-2)$-design.

Pseudo $(v,\, k,\, \lambda)$-design with $k = \lambda + 2$

Theorem

Let $\mathcal{D}$ be a pseudo $(v,\, k,\, \lambda)$-design with $k = \lambda + 2$. Then $\mathcal{D}$

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Let $\mathcal{D}$ be a pseudo $(v,\, k,\, \lambda)$-design with $k = \lambda + 2$. Then $\mathcal{D}$

- is obtained by omitting one block either from the unique (7, 4, 2)-design or the unique (7, 3, 1)-design (Fano plane);

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- is obtained by omitting one block either from the unique $(7,\, 4,\, 2)$-design or the unique $(7,\, 3,\, 1)$-design (Fano plane);
- or it is one of the

$$
\begin{aligned}
\mathcal{D}_1 =& \{1238,\, 1458,\, 1678,\, 3568,\, 2478,\, 3468,\, 2568\},\\
\mathcal{D}_2 =& \{4567,\, 1458,\, 1678,\, 2478,\, 2568,\, 3578,\, 3468\},\\
\mathcal{D}_3 =& \{4567,\, 2367,\, 1678,\, 3578,\, 2478,\, 3468,\, 2568\},\\
\mathcal{D}_4 =& \{4567,\, 1458,\, 1678,\, 3578,\, 1356,\, 1257,\, 2568\},\\
\mathcal{D}_5 =& \{4567,\, 1458,\, 1678,\, 3578,\, 1356,\, 3468,\, 1347\},\\
\mathcal{D}_6 =& \{1238,\, 2367,\, 2345,\, 3578,\, 1356,\, 3468,\, 1347\},\\
\mathcal{D}_7 =& \{4567,\, 2367,\, 2345,\, 3578,\, 2478,\, 1257,\, 1347\}.
\end{aligned}
$$

Theorem

Let G be a connected graph of order n. If the spectrum of G contains $(\pm\sqrt{2})^{\frac{n-3}{2}}$, then G is the incidence graph of one of the following 9 pseudo designs:

Graphs with $(\pm\sqrt{2})^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

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Graphs with $(\pm\sqrt{2})^{\frac{n-3}{2}} \subset \mathrm{Spec}(G)$

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- the unique pseudo (7, 3, 1)-design;
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Theorem

Let G be a connected graph of order n. If the spectrum of G contains $(\pm\sqrt{2})^{\frac{n-3}{2}}$, then G is the incidence graph of one of the following 9 pseudo designs:

- the unique pseudo (7, 3, 1)-design;
- the unique pseudo (7, 4, 2)-design; or
- one of the seven pseudo (8, 4, 2)-designs $\mathcal{D}_1, \ldots, \mathcal{D}_7$.

Thank You!