Subscheme methods for nodal curves

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Outline

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$\overline{\mathcal{M}}_{\mathrm{g}}$: Objectives

- Classical focus: study one smooth curve X (say over $\mathbb{C}$)
- Modern Objective: study simultaneously *all* curves of given genus g and their families.
- 'Suffices' to study the universal family (stack) $\mathcal{M}_g$.
- For meaningful global (e.g. enumerative) results, need to compactify.
- Good compactification: $\overline{\mathcal{M}}_{\mathrm{g}}$ = parameter space for stable curves, i.e. nodal curves with finite automorphism group.
- Studying $\overline{\mathcal{M}}_{\mathrm{g}}$ is equivalent to simultaneously studying all families of stable curves.

Methods

- Classical: For single smooth curve X, use divisors on X ($\sim$, essentially, finite subschemes)
- Modern: In $\overline{\mathcal{M}}_{\mathrm{g}}$ situation, usual methods nowadays is GW theory: study moduli spaces $\mathcal{M}_g(Y)$ of maps $X \to Y$ for fixed Y (e.g. $Y = \mathbb{P}^n$).
- Our purpose here (postmodern ? neoclassical ?): adapt divisor methods to the setting of stable curves.
- NB: On a singular curves, ideals are no longer locally principal.
- To get a compact parameter space, must work with *subschemes* rather than invertible sheaves.

Smudgy curves

Work with stack of stable **smudgy curves**

$$\mathcal{M}_g^{[m]} = \{\, (X, z) : X \text{ nodal of genus } g, \\ z \subset X \text{ arbitrary subscheme of length } m, \\ |\mathrm{Aut}(X, z)| < \infty \,\}$$

NB: X not necessarily stable, but e.g. for each smooth rational component $C \subset X$, $C \cap (z \cup \mathrm{sing}(X))$ has length ≥ 3.
Smudgy is 'intrinsic' (easier ?) counterpart to 'extrinsic' GW.
To study $\mathcal{M}_g^{[m]}$, work 'one family at a time',
study the associated relative Hilbert scheme.

Outline

Setting

Fix a 'nice' family of nodal curves:

$$\pi : X \to B, X_b = \pi^{-1}(b).$$

May assume, e.g. X is smooth.
Associate to this the *relative Hilbert scheme of degree m*:

$$X_B^{[m]} \to B$$

Paramterizes pairs (b, z) where $b \in B, \quad z \subset X_b \quad$ a length-m subscheme.

Hilb basics

- Universal property: $X_B^{[m]}$ represents the functor

$$S \mapsto \left\{ \begin{matrix} Z & \hookrightarrow & X_S & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ S & = & S & \to & B \end{matrix} \quad Z/S \text{ finite flat of degree } m \right\}$$

- Universal subscheme: $Z_m/X_B^{[m]} \subset X_B^{[m]} \times_B X$.
- Tautological bundles: given $L =$ vector bundle on X, get bundles $\Lambda_m(L)$ of rank $m.\mathrm{rk}(L)$ on $X_B^{[m]}$.

Hilb and geometry

Typically, geometric applications of Hilb proceed via the $\Lambda_m(L)$.

Example

Let $\omega = \omega_{X/B}$, the relative canonical bundle.
Let $\mathbb{E} = \pi_*(\omega)$, the *Hodge* bundle.
Over $X_B^{[m]}$, $\exists$ evaluation map ('Brill-Noether')

$$\phi_m : (\pi^{[m]})^*(\mathbb{E}) \to \Lambda_m(\omega).$$

For X/B smooth, by Riemann-Roch:

$$\{(b,z) : \mathrm{rk}(\phi_m(b,z)) \le m - r\} = \{(b,z) : h^0(z) \ge r + 1\}.$$

Kempf-Kleiman-Laksov (ca. 1970): use this to prove existence of special divisors on smooth curves.

Hilb: structure

We study Hilb via the cycle (or 'Hilb-to-Sym') map

$$c : X_B^{[m]} \to X_B^{(m)}$$

where $X_B^{(m)}$ = relative symmetric product.

- $X_B^{[m]}$ is 'virtually smooth' i.e. smooth if X is (whereas $X_B^{(m)}$ has non-$\mathbb{Q}$-Gorenstein singularities)
  Thus $\overline{\mathcal{M}}_g^{[m]}$ is a smooth stack.
- c_m is a small blowup, supported over the cycles with multiplicity ≥ 2 at nodes. In fact, c_m is the blowup of the discriminant (Weil) divisor

$$D^m = \text{ locus of nonreduced cycles.}$$

- Fibre over $m\theta, \theta =$ fibre node (locally $xy = 0$) is

$$C_1^m \cup ... \cup C_{m-1}^m, \text{ where}$$
$$C_i^m \simeq \mathbb{P}^1 = \{(x^i - ay^{m-i}), a \in \mathbb{C}^*\}, \cup \{(x^i, y^{m+1-i}), (x^{i+1}, y^{m-i})\}$$

Node scrolls defined

Get 'discriminant polarization' on $X_B^{[m]}$:

$$\Gamma^{(m)} = \frac{1}{2} c_m^{-1}(D^m)$$

Main 'new' geometric object in Hilb (rel Sym): node (poly)scroll, globalizing the C_i^m:
let
$\theta_1, ..., \theta_r =$ collection of nodes;
$n_1, ..., n_r =$ multiplicities;
$1 \leq j_k < n_k, k = 1, ..., r$
$X^{\theta.}/T =$ normalization at $\theta.$ of subfamily with nodes $(\theta.)$.
Have node polysroll (scroll, if $r = 1$)

$$F_{j.}^{n.,m}(\theta.; X/B) \to (X^{\theta.})_T^{[m - \sum n_j]}$$

It is a $(\mathbb{P}^1)^r$ bundle.

Node scrolls: structure

Polyscroll is 'iterated scroll', so suffices to consider scrolls. Note X^{θ}_{T} is endowed with distinguished sections θ_x, θ_y, so T carries cotangent classes $\psi_x = \theta_x^*(\omega_{X^\theta/T}), \psi_y$.

Node scrolls: structure

Theorem

There is a polarized isomorphism

$$F_j^{m,n}(\theta) \simeq \mathbb{P}(\mathcal{O}(-D_j^n(\theta)) \oplus \mathcal{O}(-D_{j+1}^n(\theta)))$$

(polarized: $\Gamma^{(m)}$ on left $\quad\leftrightarrow\quad$ $\mathcal{O}(1)$ on right)
where

$$\begin{aligned} D_j^n(\theta) = & -\binom{n-j+1}{2}\psi_x - \binom{j}{2}\psi_y \\ & + (n-j+1)[m-n]_*\theta_x + j[m-n]_*\theta_y \\ & + \Gamma^{[m-n]}. \end{aligned}$$

Outline

Tautological classes

- Significant classes on Hilb:
  Chern classes $c.(\Lambda_m(L)), L =$ vector bundle on X.
- 'Meaningful' numbers are polynomials in those.
- Objective: compute all polynomials in the Chern classes.
- Idea: use induction on m, via the *flag Hilbert scheme* $X_B^{[m,m-1]}$, parametrizing flags $z_{m-1} \subset z_m$.

Splitting principle

Have diagram

$$\begin{array}{ccccc} & & X_B^{[m,m-1]} & & \\ & p_m \swarrow & \downarrow p_{m-1} & \searrow p & \\ X_B^{[m]} & & X_B^{[m-1]} & & X \end{array}$$

Theorem (Splitting principle, recursive version)

On $X_B^{[m,m-1]}$, we have

$$p_m^* c(\Lambda_m(L)) = c(p_{m-1}^* \Lambda_{m-1}(L)) c(p^* L(-p_m^* \Gamma^{(m)} + p_{m-1}^* \Gamma^{(m-1)}))$$

Discriminant module

This program is realized in 2 main steps.
Step 1 involves the *tautological module* $T^m(X/B)$:
$T^m(X/B)$ is a $H^\cdot(\mathrm{Sym}^m(X)) = \mathrm{Sym}^m(H^\cdot(X))$-module ($\mathbb{Q}$-coefficients).
Generators:

- diagonal cycles Γ_μ, $\mu =$ partition of m
- node scrolls and polyscrolls F (as above)
- node sections $F.\Gamma^{(m)}$

+ their 'twists' (by classes from X).

Discriminant module cont.

Theorem (Tautological module is Discriminant module)

Via intersection product, $T^m(X/B)$ admits an explicit $\mathbb{Q}[\Gamma^{(m)}]$-module structure.

In particular,

- $\Gamma^{(m)}.\Gamma_\mu$ is an explicit linear combination of diagonals (standard) and node scrolls (new).
- $(\Gamma^{(m)})^n \in T^m(X/B)$ and is an explicit linear combination of generators.

Step 2 involves the transfer map

$$p_*q^* : H^{\cdot}(X_B^{[m-1]}) \to H^{\cdot}(X_B^{[m]})$$

via the natural correspondence

$$\begin{array}{ccccc} & & H^{\cdot}(X_B^{[m,m-1]}) & & \\ & q \swarrow & & \searrow p & \\ H^{\cdot}(X_B^{[m-1]}) & & & & H^{\cdot}(X_B^{[m]}) \end{array}$$

Transfer cont.

Theorem (Transfer)

$p_* q^*$ is compatible with an explicit map

$$\tau_m : T^{m-1}(X/B) \to T^m(X/B)$$

The combination of the Discriminant Module and Transfer Theorems complete the program of computing all polynomials in the tautological classes $c.(\Lambda_m(L))$.

Outline

Modular Subvarieties

Consider canonically defined subvarieties in the smudgy moduli $\overline{\mathcal{M}}_g^{[m]}$ and their images $V \subset \overline{\mathcal{M}}_{\mathrm{g}}$.

Problem

Compute the fundamental class $[V]$ *(or* $[V]_{\mathrm{vir}}$*, if necessary), preferably in the Mumford ring (generated by the kappa classes).*

Example

$\overline{\mathcal{M}}^r_{g,d}$ = closure of locus of smooth curves carrying a g^r_d

Hyperelliptic locus

We focus on

Subexample ($r = 1, d = 2$)

$\overline{H}_g =$ closure of locus of smooth hyperelliptic curves.

'Simple' yet challenging.

Brill Noether modified

Starting point: degree-2 Brill-Noether, extended over $\overline{\mathcal{M}}_g^{[2]}$:

$$\phi : \mathbb{E} \to \Lambda_2(\omega)$$

Over the interior $\mathcal{M}_g^{[2]}$, the determinantal locus $D_2(\phi)$ coincides with the '2-point hyperelliptic locus'.
Over the boundary, it is very poorly behaved.
The boundary is $\delta = \delta_0 + \delta_+ = \delta_0 + \sum_{i>0} \delta_i$.

Boundary

$\delta_0 =$ closure of locus of irreducible curves,
$\delta_i =$ locus of curve of the form $C_i \cup_\theta C_{\mathfrak{g}-i}$.
Look first at the easier part δ_+.

δ_+: separating nodes

The curves in δ_+ have a *separating node* θ.
θ separates X in 2 connected components or 'sides' ${}_{\mathrm{L}}X(\theta), {}_{\mathrm{R}}X(\theta)$.
X is 'limit-hyperelliptic' iff

$$({}_{\mathrm{L}}X, {}_{\mathrm{L}}\theta), ({}_{\mathrm{R}}X, {}_{\mathrm{R}}\theta) \in \{(\textbf{hyperelliptic curve}, \textbf{Weierstrass point})\}.$$

The Brill-Noether map drops rank

- on the locus subschemes meeting θ;
- if ${}_{\mathrm{L}}X$ **or** ${}_{\mathrm{R}}X$ is hyperelliptic, on hyperelliptic involution pairs

Quantifier problem: change 'or' to 'and'.

Modifying Brill Noether: separated boundary

Locally on δ_0, the Hodge bundle splits:

$$\mathbb{E} = {}_{\mathrm{L}}\mathbb{E} \oplus {}_{\mathrm{R}}\mathbb{E}$$

Sections in ${}_{\mathrm{L}}\mathbb{E}$ vanish on the opposite side ${}_{\mathrm{R}}X$, and vice versa.
There are corank-1 subbundles

$${}_{\mathrm{L}}\mathbb{E}^0 \subset {}_{\mathrm{L}}\mathbb{E}, \quad {}_{\mathrm{R}}\mathbb{E}^0 \subset {}_{\mathrm{R}}\mathbb{E}$$

of sections vanishing *twice* on the opposite side.
On $X_B^{[2]}$, have divisors

$${}_{\mathrm{L}}D = ({}_{\mathrm{L}}X)_B^{[2]}, {}_{\mathrm{R}}D = ({}_{\mathrm{R}}X)_B^{[2]}$$

Modifying Brill Noether: separated boundary, II

Do elementary modification on $\mathbb{E}$, pulled back to $X_B^{[2]}$, with respect to the data

$$({}_{\mathrm{L}}\mathbb{E}, {}_{\mathrm{L}}D), ({}_{\mathrm{R}}\mathbb{E}, {}_{\mathrm{R}}D)$$

Then again with respect to

$$({}_{\mathrm{L}}\mathbb{E}^0, {}_{\mathrm{L}}D), ({}_{\mathrm{R}}\mathbb{E}^0, {}_{\mathrm{R}}D)$$

Get a new bundle $\mathbb{E}^+$, with 'modified BN':

$$\phi^+ : \mathbb{E}^+ \to \Lambda_2(\omega).$$

Modifying Brill Noether: separated boundary, III

Can show: off δ_0, the determinantal locus $D_2(\phi^+)$ is

$$D_2(\phi^+) = \overline{\mathcal{HE}^2} \cup R_\theta$$

$R_\theta =$ locus of subschemes supported in θ.
Virtual fundamental class $[D_2(\phi^+)]_{\mathrm{vir}}$ computable by Porteous.
Contribution of R_θ computable by Fulton-MacPherson residual intersection theory.
This yields $[\overline{\mathcal{HE}^2}] \mod \delta_0$.

Next step: δ_0.

separating binodes

Problem with δ_0 is with the *reducible* curves (codim 2 locus in $\overline{\mathcal{M}}_g$),
i.e. those with a (properly) *separating binode* $\underline{\theta} = (\theta_1, \theta_2)$.
These split X in a right and left side (depending on $\underline{\theta}$:

$$X = {}_{\mathrm{L}}X \cup_{\underline{\theta}} {}_{\mathrm{R}}X$$

For now, assume ${}_{\mathrm{L}}X, {}_{\mathrm{R}}X$ irreducible.

$$\omega_X = \omega_{{}_{\mathrm{L}}X}(\underline{\theta}) \cup \omega_{{}_{\mathrm{R}}X}(\underline{\theta})$$

$$\mathbb{E}_X = {}_{\mathrm{L}}\mathbb{E} \oplus {}_{\mathrm{R}}\mathbb{E}$$

ω_X is never very ample, so the Brill-Noether map ϕ or ϕ^+ always drops rank.
Correct boundary notion: X is 'limit-hyperelliptic' iff ${}_{\mathrm{L}}X, {}_{\mathrm{R}}X$ both hyperelliptic with $\underline{\theta}$ an involution pair.

Modifying Hilb

Want to modify the Hodge bundle $\mathbb{E}$ to trim degeneracy of ϕ.
Problem: Issue occurs in codimension 2 (or more).
Solution: blow up.
Problem: what to blow up ?
Solution: look for loci where ϕ is actually zero on a subbundle of $\mathbb{E}$ like ${}_{\mathrm{L}}\mathbb{E}, {}_{\mathrm{R}}\mathbb{E}$.
These are ${}_{\mathrm{L}}X^{[2]}, {}_{\mathrm{R}}X^{[2]} \subset X_B^{[2]}$.
Blow these up, to divisors ${}_{\mathrm{L}}D, {}_{\mathrm{R}}D \subset X_B^{\{2\}} = X_B^{\{2\}}(\underline{\theta})$.
$X_B^{\{2\}}$ is the *azimuthal* Hilbert scheme (depending on $\underline{\theta}$).

Modifying Brill Noether

On $X_B^{\{2\}}$, can do elementary modification of $\mathbb{E}$

$$
{}_{\mathrm{L}}\mathbb{E} \rightsquigarrow {}_{\mathrm{L}}\mathbb{E}(-{}_{\mathrm{R}}D)
$$
$$
{}_{\mathrm{R}}\mathbb{E} \rightsquigarrow {}_{\mathrm{R}}\mathbb{E}(-{}_{\mathrm{L}}D)
$$

Get new bundle and map (depending on $\underline{\theta}$):

$$
\phi^{+} : \mathbb{E}^{+} \to \Lambda_2(\omega)
$$

Behaviour of this at given subscheme is the max of

$$
\omega_{{}_{\mathrm{L}}X}(2\underline{\theta}) \cup \omega_{{}_{\mathrm{R}}X}, \omega_{{}_{\mathrm{L}}X}(\underline{\theta}) \cup \omega_{{}_{\mathrm{R}}X}(\underline{\theta}), \omega_{{}_{\mathrm{L}}X} \cup \omega_{{}_{\mathrm{R}}X}(2\underline{\theta})
$$

$\therefore$ drops rank on right locus.

Multiple binodes

Next problem: multiple separating binodes.
ok as long as they are disjoint.
Then, loci to blow up are transverse.
Next situation: when separating binodes bunch up into *polyseparators* Θ

$\Theta =$ node set where every pair separates.

$\exists$ nice way to blowup the loci ${}_{\mathrm{R}}X(\underline{\theta})_{B}^{[2]}$ for all separating binodes.
Then modify Hodge correspondingly.

The ultimate

Ultimately, for the modified Hodge and Brill-Noether ,

$$\phi^{+} : \mathbb{E}^{+} \to \Lambda_2(\omega)$$

degeneracy locus =

$$\text{closure of hyperelliptics} + \bigcup_{\theta} R(\theta)$$

(union over all separating nodes).