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Vector bundles over normal varieties trivialized by finite morphisms

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Abstract. Let Y be a normal and projective variety over an algebraically closed field k and V a vector bundle over Y. We prove that if there exist a k-scheme X and a finite surjective morphism $g : X \to Y$ that trivializes V then V is essentially finite.

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1. Introduction. Essentially finite vector bundles over a reduced, connected and proper scheme Y over a perfect field k have been defined by Nori [5,6]. They turn out to be those vector bundles V over Y which are trivialized by some principal G bundle $f : Z \to Y$ for a certain finite k-group scheme G (i.e., $f^*(V)$ is trivial). The aim of this paper is to prove the following

Theorem 1.1. (Cf. Theorem 2.1). *Let k be any algebraically closed field and Y a projective and normal variety over k. Assume there exist a projective variety X over k and a finite surjective morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then V is essentially finite.*

When Y is smooth then Theorem 1.1 is well known: it has first been proved by Parameswaran and Subramanian [7, Section 3], for $\dim(Y) = 1$ provided g is separable. Then it has been subsequently proved by Balaji and Parameswaran [1, Section 6] for Y smooth and projective of any dimension provided g is separable. Then finally Biswas and Dos Santos [2] have given a different proof: for any finite and surjective $g : X \to Y$, with Y smooth and projective over k, they first explain how to reduce to the case of curves ([2, Section 3]) by means of the Grothendieck–Lefschetz theorem for the S-fundamental group scheme, then in loc. cit. Section 4.2 they prove Theorem 1.1 for Y a smooth and projective curve and $g : X \to Y$ separable (the crucial point) and finally they prove the result for any g (loc. cit. Section 4.3) and Y a smooth and projective curve.

Our proof of Theorem 1.1 not only holds for Y normal but it is shorter, and we use neither Tannakian categories nor Grothendieck–Lefschetz theorem for the fundamental group scheme. The main argument of our proof is Lemma 2.5, which is of independent interest, where we prove that a finite and surjective morphism $g : X \to Y$ between normal and projective varieties over any algebraically closed field k, étale outside a closed set of codimension 2 in Y, factors through a Galois étale cover $g' : X' \to Y$ if and only if there exists a non trivial vector bundle V on Y such that $g^*(V)$ is trivial on X.

2. The theorem. Throughout the whole paper k will be an algebraically closed field and Y a normal and projective variety over k. Let us denote by $EF(Y)$ the neutral Tannakian category of essentially finite vector bundles over Y. The aim of this paper is to prove the following

Theorem 2.1. *Assume there exist a normal projective variety X over k and a finite surjective morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

It is clear that Theorem 1.1 can be easily deduced from Theorem 2.1 simply normalizing an irreducible component of X_{red} dominating Y.

Remark 2.2. This theorem holds in both zero and positive characteristic.

As pointed out in the introduction, the crucial point in the proof of Theorem 2.1 is to prove the statement for g separable (or generically étale, i.e., the extension $K(Y) \subset K(X)$ of the function fields induced by g is separable), and this will be the object of Lemma 2.4.

So first we consider the easier case where $g : X \to Y$ is purely inseparable (i.e., the extension $K(Y) \subset K(X)$ of their function fields is purely inseparable, which only occurs when $\mathrm{char}(k) > 0$) and then it will only remain to explain how to reduce to these two cases, the separable and purely inseparable ones.

Lemma 2.3. *Assume there exist a normal projective variety X over k and a finite, surjective, purely inseparable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

Proof. We are in the case $\mathrm{char}(k) = p > 0$. So let us denote by $F_X : X \to X$ and $F_Y : Y \to Y$ respectively the absolute Frobenius morphisms of X and Y. Since $K(Y) \subset K(X)$ is purely inseparable then there exists a positive integer n such that $K(X)^{(p^n)} \subset K(Y)$. This implies that there is a morphism $h : Y \to X$ such that $gh = F_Y^n$ (i.e. the Frobenius iterated n times) and $hg = F_X^n$. By assumption $g^*(V)$ is trivial on X, thus $h^*g^*(V) = (gh)^*(V) = (F_Y^n)^*(V)$ is trivial hence V is essentially finite (cf. [4, Section 2]). □

Lemma 2.4. *Assume there exist a normal projective variety X over k and a finite, surjective, separable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

Proof. We may assume that $K(X)$ is normal (then Galois) over $K(Y)$ with Galois group G (if it is not simply consider the normal closure of the extension $K(Y) \subset K(X)$).

Let $W := (g_*\mathcal{O}_X)_{\max}$ be the maximal semistable subsheaf of $g_*\mathcal{O}_X$ (i.e. the first term of the Harder–Narasimhan filtration of $g_*\mathcal{O}_X$, [1, Section 6]) then its slope $\mu(W) = \mu_{\max}(g_*\mathcal{O}_X) = 0$: indeed since there is at least the canonical morphism $\mathcal{O}_Y \to g_*\mathcal{O}_X$ then in particular we have

$$0 = \mu(\mathcal{O}_Y) \leq \mu_{\max}(g_*\mathcal{O}_X);$$

but g is separable then $g^*(W)$ is still semistable; now consider the isomorphism

$$\mathrm{Hom}_X(g^*(W), \mathcal{O}_X) \simeq \mathrm{Hom}_Y(W, g_*\mathcal{O}_X) \neq 0$$

from which we deduce $\mu(g^*(W)) \leq 0$ hence $\mu(W) \leq 0$ (recall that $\mu(W) = \mu(g^*(W))/\deg(g)$).

The coherent sheaf W is in general only torsion free over Y. But it is locally free if restricted to a big open subset $Y_0 \subset Y$, i.e., $\mathrm{codim}_Y(Y \backslash Y_0) \geq 2$. Let $W_0 := W_{|Y_0}$ denote the vector bundle over $Y_0, \mathrm{Sym}^*(W_0^*)$ the symmetric algebra of the dual of W_0, and consider $X_0 := \mathbf{Spec}(\mathrm{Sym}^*(W_0^*))$ with its canonical map $g_0 : X_0 \to Y_0$.

The vector bundle W_0 is strongly semistable of degree 0 over Y_0: indeed let us denote by F_{X_0} and F_{Y_0} respectively the absolute Frobenius morphisms of X_0 and Y_0, and assume W is not strongly semistable then there exists a subsheaf U of $F_{Y_0}^*(W)$ such that $\deg(U) > 0$. Let X_1 be the fiber product of $g_0 : X_0 \to Y_0$ and F_{Y_0}. It is an integral scheme. We denote by $\mathrm{pr} : X_1 \to X_0$ and $g_1 : X_1 \to Y_0$ the projections and also $h : X_0 \to X_1$ the map given by the universal property of the fiber product:

$$\begin{tikzcd}
X_0 \arrow[dr, "h"] \arrow[drr, bend left, "F_{X_0}"] \arrow[ddr, bend right, "g_0"'] & & \\
& X_1 \arrow[r, "pr"] \arrow[d, "g_1"'] & X_0 \arrow[d, "g_0"] \\
& Y_0 \arrow[r, "F_{Y_0}"] & Y_0.
\end{tikzcd}$$

Now $U \subseteq F_{Y_0}^*(g_{0*}(\mathcal{O}_{X_0})) = g_{1*}(pr^*(\mathcal{O}_{X_0})) = g_{1*}(\mathcal{O}_{X_1})$. But from $\mathcal{O}_{X_1} \hookrightarrow h_*(\mathcal{O}_{X_0})$ we obtain $g_{1*}(\mathcal{O}_{X_1}) \hookrightarrow g_{1*}(h_*(\mathcal{O}_{X_0})) = g_{0*}(\mathcal{O}_{X_0})$ the latter being semistable whence a contradiction. As a consequence we have a homomorphism of $\mathcal{O}_{Y_0}$-algebras $g_{0*}(\mathcal{O}_{X_0}) \simeq W_0$ (cf. also [1, Section 6]).

Since $g_{0*}(\mathcal{O}_{X_0})$ is semistable of slope 0 over Y_0 then X_0 is a Galois-étale cover over Y_0 ([1, Lemma 6.2]), the Galois group of g_0 still being G. Now let us fix some notations: recall that by assumption V is a vector bundle over Y such that $T := g^*(V)$ is trivial on X; we set $V_0 := V_{|Y_0}$ and $T_0 := g_0^*(V_0)$ so the latter is also trivial on X_0. Since g_0 is a Galois-étale cover, T_0 is a G-bundle on X_0. But X_0 is a big open set in X thus G acts on X and then G acts also on T. Since T is a G-bundle, we go on as follows: we have $X/G \simeq Y$ and the trivial bundle T on X descends to Y. So by Kempf's lemma (cf. e.g., [3, Théorème 2.3]), for all x in X, the stabilizer G_x acts trivially on the fibre T_x. But T is trivial and both X and X_0 have no global sections except constants, this means that there is a map

$$\rho : G \to GL(T_x) = GL_r$$

over X, where $r := \mathrm{rank}(T)$. Assume first that the map $\rho : G \to GL_r$ is injective. We already know that G acts freely on X_0. So let us take $x \in X \backslash X_0$: since G_x is a subgroup of G, G_x has to be trivial. This proves that G acts freely on X. So $g : X \to Y$ is a Galois-étale cover hence V is in $EF(Y)$. Up to now we have assumed ρ to be injective. If it is not, then just consider $H := G/ker(\rho)$ and $X' = X/(\ker(\rho))$, which is provided with a faithfull H-action and clearly $Y \simeq X'/H$. Hence $H \to GL_r$ is injective, V is trivial over X' and we proceed as before. □

From the previous discussion follows Lemma 2.5 which is of independent interest:

Lemma 2.5. *Let Y and X be normal and projective varieties and $f : X \to Y$ a finite and surjective morphism, étale outside a closed set of codimension 2 in Y, then we have proved that there exists a non-trivial vector bundle V on Y such that f^*V is trivial on X if and only if f factors through a Galois étale cover $f' : X' \to Y$.*

Proof. The Galois étale cover $f' : X' \to Y$ is the one constructed in the proof of Lemma 2.4. □

Remark 2.6. Let notations be as in Lemma 2.4 and its proof. In Lemma 2.5 we obtain the smallest Galois étale cover where V becomes trivial. Indeed $f' : X' \to Y$ determines and is determined by the kernel of $\rho : G \to GL_r$; if ρ is injective then $X' = X$. If ρ is not injective then $X' := X/\ker(\rho)$ is Galois étale over Y. It can happen that there are no Galois étale covers between X and X'. This happens if and only if $\mu_{\max} C < 0$, where C is the cokernel of $f'_*(\mathcal{O}_{X'}) \to f_*(\mathcal{O}_X)$.

We are now ready to prove the main result:
Proof of Theorem 2.1. if $\mathrm{char}(k) = 0$, then Lemma 2.4 is sufficient to conclude. So let us assume $\mathrm{char}(k) = p > 0$: if g is purely inseparable, then Lemma 2.3 is enough to conclude. Otherwise, if g is arbitrary, we argue as follows: again we may assume that $K(X)$ is normal over $K(Y)$ with Galois group G. It is known that $L := K(X)^G$ is a proper purely inseparable field extension of $K(Y)$ while $K(X)$ is separable over L, then Galois. Let Z be the integral closure of Y in L, then $g : X \to Y$ factors through the maps $s : X \to Z$ and $t : Z \to Y$ (i.e., $ts = g$) where $t : Z \to Y$ is purely inseparable and $s : X \to Z$ is separable. By Lemma 2.4 the vector bundle $W := t^*(V) \in EF(Z)$ because $s^*(W)$ is trivial on X. As we did for Lemma 2.3, there exists a morphism $h : Y \to Z$ such that $h^*t^*(V) = (th)^*(V) = (F_Y^n)^*(V)$ for some integer n; but $h^*(W) \in EF(Y)$ thus $(F_Y^n)^*(V) \in EF(Y)$ then there exists $m \geq n$ such that $(F_Y^m)^*(V)$ is Galois-étale trivial (i.e., there exists a Galois-étale cover $j : Y' \to Y$ such

that $j^*((F_Y^m)^*(V))$ is trivial on Y'), and that is enough to conclude that V is essentially finite on Y. $\square$

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