arXiv:0807.4976v1 [math.AG] 31 Jul 2008

SYZYGY STRUCTURES OF INNER PROJECTIONS

KANGJIN HAN AND SIJONG KWAK

Abstract. Let $X \subset \mathbb{P}^n$ be a projective reduced scheme. If the truncated ideal $(I_X)_{\geq d}$ has only the simplest linear syzygies up to p-th step, then we say that X satisfies property $N_{d,p}$. When $d = 2, p = 2$, I_X is generated by quadrics and there are only linear relations on quadrics. So, property $N_{2,p}$ can be regarded as a generalization of property N_p due to Green-Lazarsfeld.

In this paper, we obtain some results on syzygy structures and geometric properties of *inner* projections by using the extended mapping cone construction for *not finitely* generated graded modules and the partial elimination ideal theory. In particular, for a reduced scheme X with the condition $N_{2,p}$, the inner projections from any smooth point of X satisfies at least property $N_{2,p-1}$. This uniform behavior looks unusual in a sense that linear syzygies of outer projections heavily depend on moving the center of projection in an ambient space [4],[14], [16]. Note that the syzygies of projected varieties from the singular point is more complicated.

Keywords: linear syzygies, regularity, projection, partial elimination ideals.

Contents

1. Introduction

Let X be a non-degenerate reduced closed subscheme in a projective space $\mathbb{P}^n = \mathbb{P}(V)$ defined over an algebraically closed field k and $R = k[x_0, \ldots, x_n]$ be the coordinate ring of $\mathbb{P}(V)$. For the homogeneous coordinate ring R/I_X of X, we have the unique minimal free resolution of R/I_X as R-modules as follows;

$$\cdots \to L_i \to L_{i-1} \to \cdots \to L_1 \to R \to R/I_X \to 0 \tag{1.1}$$

where $L_i = \bigoplus_j R(-i-j)^{\oplus \beta_{i,j}}$. The distribution of zeroes of graded Betti numbers $\beta_{i,j}$ in the Betti table gives the significant geometric information on X and many long standing conjectures on the vanishing of Koszul cohomology groups deeply link between the geometry and syzygies of a projective variety X. One of the interesting natural questions is to compare the syzygies of X and its projections as we move the center of the projection. This kind of question is closely related to the

The second author is partially supported by the SRC Program of Korea Science and Engineering Foundation(KOSEF)R11-2007-035-02001-0.

Eisenbud-Goto conjecture on Castelnuovo-Mumford regularity and some topics in classical algebraic geometry.

First of all, let us recall the definitions and known results. One can define that X(or R/I_X) satisfies property $N_{d,p}$ (cf. [8]) if one of the following conditions holds:

(a) $\beta_{i,j} = 0$ for $1 \leq i \leq p$ and all $j \geq d$ in the minimal free resolution (1.1);
(b) the minimal free resolution of $(I_X)_{\geq d}$ is linear until p-th step, namely,

$$\cdots \to R(-d-p+1)^{\oplus \beta_{p,d-1}} \to \cdots \to R(-d)^{\oplus \beta_{1,d-1}} \to R \to R/(I_X)_{\geq d} \to 0.$$

The case of $d = 2$ has been of particular interest. For $d = 2, p = 1$, I_X is generated by quadrics and $N_{2,2}$ means that I_X is generated by quadrics and there are only linear relations on quadrics. Note that property $N_{2,p}$ is the same as property N_p (defined by Green-Lazarsfeld) if the given variety is projectively normal.

Our main purpose is to study the homological, cohomological and geometric properties of projected varieties according to moving the center of projections. In paper [1], they studied outer projections of a given variety $X \subset \mathbb{P}^n$ and their syzygetic and geometric properties as we move the center of projections in an ambient space $\mathbb{P}^n \setminus X$. In this case, higher secant varieties of X play an important role to control the syzygies of outer projections.

In the present paper, we are mainly interested in the inner projections with the center in X. For an inner projection of X from the center $q = (1, 0, \ldots, 0) \in X$, letting $Y = \overline{\pi_q(X)} \subset \mathbb{P}^{n-1}$ be the Zariski-closure of $\pi_q(X)$ in $\mathbb{P}^{n-1}$ where $S = k[x_1, \ldots, x_n]$ is the projective coordinate ring of $\mathbb{P}^{n-1}$. However, R/I_X is not a finitely generated S-module.

For a complete embedding $X \subset \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^n$ with the condition N_p embedded by complete linear system of a very ample line bundle $\mathcal{L}$ on X, the inner projection $\pi_q(X)$ for a point $q \in X$ is embedded in $\mathbb{P}(W)$ where $W = H^0(\mathcal{L}(-q))$. Let S be the coordinate ring of $\mathbb{P}(W)$. Note that the inner projection $\pi_q : X \dashrightarrow \mathbb{P}^{n-1}$ is a rational map which is well-defined outside q. Let $\sigma : \mathrm{Bl}_q(X) \to X$ be a blowing up of X at q. Then one has the regular morphism $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to \mathbb{P}^{n-1}$ with the following diagram;

$$\begin{tikzcd}
\tilde{X} = \mathrm{Bl}_q(X) \arrow[d, "\sigma"'] \arrow[dr, "\tilde{\pi}_q"] & \\
X \subset \mathbb{P}^r \arrow[r, dotted, "\pi_q"'] & X' = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n-1}
\end{tikzcd}$$

If $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to \mathbb{P}^{r-1}$ is an embedding, then the exceptional divisor E is linearly embedded via $\widetilde{\pi_q}$ in $\mathbb{P}^{n-1}$, i.e. $\widetilde{\pi_q}(E) = \mathbb{P}^{\ell-1} \subset \mathbb{P}^{n-1}$, $\ell = \dim(X)$.

In a paper [4], They showed by using vector bundle techniques and Koszul cohomology methods, that if $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth irreducible variety with property $N_p, p \geq 1$, then for any $q \in X \setminus \mathrm{Trisec}(X)$, $\widetilde{\pi_q}(\mathrm{Bl}_q(X)) = \overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}(W)$ is smooth and satisfies property N_{p-1}, i.e. property N_{p-1} holds for $(\mathrm{Bl}_q(X), \sigma^*\mathcal{L} - E)$.

In this paper, we would like to extend this theorem to the general case, i.e. projective reduced irreducible varieties with the condition $N_{2,p}$. So, we need to construct the extended mapping cone construction for infinitely generated graded modules and to understand their syzygy structures.

Finally, we obtain some results on syzygy structures and geometric properties of *inner* projections by using the extended mapping cone construction and the partial elimination ideal theory. In particular, for a reduced scheme X with the condition $N_{2,p}$, the inner projections from any smooth point of X satisfies at least property $N_{2,p-1}$. This uniform behavior looks unusual in a sense that linear syzygies of outer projections heavily depend on moving the center of projection in an ambient space [4],[14], [16]. Note that the syzygies of a projected variety from the singular point look more complicated.

Acknowledgements The second author would like to thank Korea Institute of Advanced Study(KIAS) for supports and hospitality during his stay for a sabbatical year.

2. Extended mapping cone construction for infinitely generated modules

Generally, the mapping cone construction of the chain map between two complexes is a kind of extension of complexes respecting the given chain map. J. Ahn and S. Kwak pointed out some mapping cone constructions related to projections very useful to understand algebraic and geometric structures of projections [1] and using this, they showed some relations of geometric and cohomological properties between an original variety and a projected variety in outer projections. We can exploit this construction to study an inner projection and establish a general framework to explore every projection from this construction. Let us briefly review the mapping cone construction.

Mapping cone for projections 2.1.

- Let $W = k\langle x_1, \cdots, x_n\rangle \subset V = k\langle x_0, \cdots, x_n\rangle$ be vector spaces over k and $S = \mathrm{Sym}(W) = k[x_1, \ldots, x_n] \subset R = \mathrm{Sym}(V) = k[x_0, \ldots, x_n]$ be polynomial rings.
- Let M be a graded R-module (which is also a graded S-module) and $K^S_*(M)$ be the graded Koszul complex of M as follows:

$$0 \to \wedge^n W \otimes M \to \cdots \to \wedge^2 W \otimes M \to W \otimes M \to M \to 0$$

  whose graded components are $K^S_i(M)_{i+j} = \wedge^i W \otimes M_j$.
- Let $\mathbb{F}_*$, $\mathbb{G}_*$ be the Koszul complexes $K^S_*(M(-1))$, $K^S_*(M)$. Consider the chain map $\mu : \mathbb{F}_* \to \mathbb{G}_*$ induced by the multiplicative map $M(-1) \xrightarrow{\cdot x_0} M$, i.e. $\mu : \mathbb{F}_* = K^S_*(M(-1)) \xrightarrow{\cdot x_0} \mathbb{G}_* = K^S_*(M)$.

Then, we construct the mapping cone $(\mathtt{Cone}_*(\mu), \mathtt{d}_\mu)$ such that:

$$0 \longrightarrow \mathbb{G}_* \longrightarrow \mathtt{Cone}_*(\mu) \longrightarrow \mathbb{F}_*[-1] \longrightarrow 0. \tag{2.1}$$

,where $\mathtt{Cone}_*(\mu) = \mathbb{G}_* \bigoplus \mathbb{F}_*[-1]$, each grade part $\mathtt{Cone}_*(\mu)_{*+j}$ is $[\mathbb{G}_*]_{*+j} \bigoplus [\mathbb{F}_{*-1}]_{*+j} = \wedge^* W \otimes M_j \oplus \wedge^{*-1} W \otimes M(-1)_{j+1} = \wedge^* W \otimes M_j \oplus \wedge^{*-1} W \otimes M_j$. the differential $\mathtt{d}_\mu$ is given by

$$\mathtt{d}_\mu = \begin{pmatrix} \partial_{\mathbb{G}} & \mu \\ 0 & -\partial_{\mathbb{F}} \end{pmatrix},$$

where ∂ is the differential of Koszul complex.

From the exact sequence (2.1), we have a natural long exact sequence of Koszul homology and the following lemma:

Lemma 2.2. *Let M be a graded R-module. Then there exist a natural sequence*

$$\begin{aligned}\to \quad \mathrm{Tor}_i^S(M,k)_{i+j} \quad \to \quad H_i(\mathtt{Cone}_*(\mu))_{i+j} \quad \to \\ \mathrm{Tor}_{i-1}^S(M,k)_{i+j-1} \quad \xrightarrow{\delta} \quad \mathrm{Tor}_{i-1}^S(M,k)_{i+j} \quad \to\end{aligned}$$

and the connecting homomorphism δ is induced by the multiplication by x_0. And we have the following natural isomorphism:

$$\mathrm{Tor}_i^R(M,k)_{i+j} \simeq H_i(\mathtt{Cone}_*(\mu))_{i+j}.$$

Proof. Since $\mathbb{G}_*(\mathbb{F}_*$ also.) is the Koszul complex of M, their homology $H_i(\mathbb{G}_*)_{i+j}$ is $\mathrm{Tor}_i^S(M,k)_{i+j}$. And from (2.1), we get the long exact sequence as above. Let $K_*^R(M)$ be the Koszul complex of a graded R-module M. Then the graded component in degree $i+j$ of $K_i^R(M)$ is $K_i^R(M)_{i+j} = \wedge^i V \otimes M_j$. Note that $\wedge^i V \cong [x_0 \wedge (\wedge^{i-1} W)] \oplus \wedge^i W$. Hence we see that the Koszul complex $K_i^R(M)$ has the following canonical decomposition in each graded component:

$$K_i^R(M)_{i+j} \quad \cong \quad \begin{matrix} \wedge^i W \otimes M_j \\ \bigoplus \\ [x_0 \wedge (\wedge^{i-1} W)] \otimes M_j \end{matrix} \quad \cong \quad \mathtt{Cone}_i(\mu)_{i+j}. \tag{2.2}$$

Using the decomposition (2.2), we can verify that the following diagram is commutative:

$$\begin{tikzcd} K_i^R(M)_{i+j} \arrow[r, "\cong"] \arrow[d, "\partial"] & \mathtt{Cone}_i(\mu)_{i+j} \arrow[d, "d_\mu"] \\ K_{i-1}^R(M)_{i+j} \arrow[r, "\cong"] & \mathtt{Cone}_{i-1}(\mu)_{i+j} \end{tikzcd} \tag{2.3}$$

Therefore, we have a natural isomorphism $\mathrm{Tor}_i^R(M,k)_{i+j} \simeq H_i(\mathtt{Cone}_*(\mu))_{i+j}$. □

Because $\mathrm{Tor}^R(M,k)$ can be obtained by the homology of our mapping cone, we could take up our job about projections. Let's restate above lemma as the following useful Theorem.

Theorem 2.3. *Let $S = k[x_1, \dots, x_n] \subset R = k[x_0, x_1 \dots, x_n]$ be polynomial rings. For a graded R-module M, we have the following long exact sequence:*

$$\longrightarrow \mathrm{Tor}_i^S(M,k)_{i+j} \longrightarrow \mathrm{Tor}_i^R(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}^S(M,k)_{i+j-1} \longrightarrow$$

$$\xrightarrow{\delta} \mathrm{Tor}_{i-1}^S(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-1}^R(M,k)_{i+j} \longrightarrow \mathrm{Tor}_{i-2}^S(M,k)_{i+j-1} \xrightarrow{\delta} \quad \cdots$$

whose connecting homomorphism δ is the multiplicative map $\cdot x_0$.

Proof. It is clear from Lemma 2.2. □

This theorem 2.3 appears in [1] originally. We remark that this theorem is also true even for infinitely generated S-module M. For each degree piece of Tor, there exists a long exact sequence as above. This gives us an useful information about syzygies of every projections of projective varieties.

As a first step, we derive the following interesting interpretation about Tor's.

Corollary 2.4. *Let $I \subset R$ be a homogeneous ideal. Assume that I admits d-linear resolution ($d \geq 2$) up to p-th step for $p \geq 1$ (i.e. satisfies $N_{d,p}$). Then,*

(a) *A multiplication by x_0 induces a sequence of isomorphisms on $\mathrm{Tor}_i^S(R/I,k)_{i+j}$ except $j = \{d-2, d-1\}$ step for $1 \leq i \leq p-1$ ($p \geq 2$) as follows:*

$$\cdots \overset{\cdot x_0}{\overset{\sim}{\to}} \mathrm{Tor}_i^S(R/I,k)_{i+d-2} \overset{\cdot x_0}{\hookrightarrow} \mathrm{Tor}_i^S(R/I,k)_{i+d-1} \overset{\cdot x_0}{\twoheadrightarrow} \mathrm{Tor}_i^S(R/I,k)_{i+d} \overset{\cdot x_0}{\overset{\sim}{\to}} \cdots$$

(b) *And for $i = p$ case,*

$$\cdots \overset{\cdot x_0}{\twoheadrightarrow} \mathrm{Tor}_p^S(R/I,k)_{p+d-2} \overset{\cdot x_0}{\to} \mathrm{Tor}_p^S(R/I,k)_{p+d-1} \overset{\cdot x_0}{\twoheadrightarrow} \mathrm{Tor}_p^S(R/I,k)_{p+d} \overset{\cdot x_0}{\twoheadrightarrow} \cdots$$

Proof. (a) First, consider the exact sequence by Theorem 2.3 for $M = R/I$

$$\mathrm{Tor}_i^R(R/I,k)_{i+j} \to \mathrm{Tor}_{i-1}^S(R/I,k)_{i-1+j} \xrightarrow{\cdot x_0} \mathrm{Tor}_{i-1}^S(R/I,k)_{i-1+j+1} \to \mathrm{Tor}_{i-1}^R(R/I,k)_{i-1+j+1}$$

Note that $\mathrm{Tor}_i^R(R/I)_{i+j} = 0$ for $1 \leq i \leq p$ and $j \neq d-1$ by assumption that I is d-linear up to p-th step. So We have an isomorphism

$$\mathrm{Tor}_{i-1}^S(R/I,k)_{i-1+j} \xrightarrow{\cdot x_0} \mathrm{Tor}_{i-1}^S(R/I,k)_{i-1+j+1},$$

for $1 \leq i-1 \leq p-1$ and for all $j \notin \{d-2, d-1\}$. And we have an injection (resp. a surjection) for $j = d-2$ (resp. $j = d-1$).

(b) In case $i = p$ we know $\mathrm{Tor}_p^R(R/I)_{r+j} = 0$ for $j \neq d-1$. So we have vanishing Tor of the right hand side in the following sequence

$$\mathrm{Tor}_{p+1}^R(R/I,k)_{p+1+j} \to \mathrm{Tor}_p^S(R/I,k)_{p+j} \xrightarrow{\cdot x_0} \mathrm{Tor}_p^S(R/I,k)_{p+j+1} \to \mathrm{Tor}_p^R(R/I,k)_{p+j+1}$$

Therefore we get the desired surjection for $i = p$. □

Now we are going to mainly think about an *inner projection*, i.e. a projection of a variety X from the point q of X, and its effect to syzygies. Let's consider the preliminary settings.

Notations & Preliminaries 2.5. We are working on the following background:

- $R = k[x_0, \ldots, x_n] = \mathrm{Sym}(V)$ and $S = k[x_1, x_2 \ldots, x_n] = \mathrm{Sym}(W)$: two polynomial rings where $W \subset V$, $\mathrm{codim}(W,V) = 1$.
- (Betti number) $\beta_{i,j}^R(M) := \dim_k \mathrm{Tor}_i^R(M,k)_{i+j}$, $\beta_{i,j}^S(M) := \dim_k \mathrm{Tor}_i^S(M,k)_{i+j}$.
- Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be a non-degenerate projective variety and q be a point of X. We can assume $q = (1,0,\ldots,0) \in X$ (by suitable coordinate change)
- We consider inner projection $\pi_q : X \setminus \{q\} \subset \mathbb{P}^n \longrightarrow Y \subset \mathbb{P}^{n-1} = \mathbb{P}(W)$. Let Y be the closure of the image, $\overline{\pi_q(X \setminus \{q\})}$. From now on, we mainly focus on the syzygetic study of Y.

In fact, the inner projection of X from q is a rational map defined on $X \setminus \{q\}$, so we take Zariski closure of the image, Y. Geometrically, this is just adding points, which is the image of tangential projection from q, to $\pi_q(X \setminus \{q\})$. Algebraically, this process corresponds to the elimination of 1st variable x_0 of ideal I_X (so, $I_Y = I_X \bigcap S$).

$$\begin{array}{ccc} \pi_q : \quad X \setminus \{q\} & \longrightarrow & \pi_q(X \setminus \{q\}) \subset \mathbb{P}^{n-1} = \mathbb{P}(W) \\ \Big\downarrow \text{Blow up} & & \Big\downarrow \text{Zariski closure} \\ \mathrm{Bl}_q X & \longrightarrow & Y \subset \mathbb{P}^{n-1} = \mathbb{P}(W) \end{array}$$

If we blow up X at q, we may make the projection map to a morphism. Y is the very image of the morphism. But, in this paper, we are interested in the syzygies and its related cohomological, geometric properties of the image Y, instead of exploring how they are mapped(or embedded) to Y in $\mathbb{P}^{n-1}$. So we will keep on the view in which we consider $\pi_q : X \longrightarrow Y$ (in the sense of putting the tangent direction image together in mind) and Y is simply one given by elimination ideal of I_X algebraically.

Note that I_X(or R/I_X) is not finitely generated S-module (Since $q \in X$, there is no polynomial like $f = {x_0}^n + (other\,terms)$ in I_X). But we can still consider their S-module syzygies and they have an interesting syzygetic structure.

In general, I_X(or R/I_X) has the following S-module syzygy:

$$\begin{array}{cccccccc} & & & & & \bigoplus_{i=0}^{\infty} S(-i)\cdot {x_0}^i & \rightarrow & R/I_X \quad \rightarrow 0 \\ & S(-j-1)^{\beta_{j,1}} & & S(-2)^{\beta_{1,1}} & \nearrow & \uparrow & & \\ \cdots\rightarrow & \oplus S(-j-2)^{\beta_{j,2}} & \rightarrow\cdots\rightarrow & \oplus S(-3)^{\beta_{1,2}} & \rightarrow & I_X & \rightarrow & 0 \\ & \oplus S(-j-3)^{\beta_{j,3}} & & \oplus S(-4)^{\beta_{1,3}} & & \uparrow & & \\ & \cdots & & \cdots & & 0 & & \end{array}$$

If we assume that X is generated by quadrics(i.e. satisfies $N_{2,1}$), then we can say

$$I_X = (x_0 l_1 - q_1, \ldots, x_0 l_t - q_t, Q_1, \ldots, Q_s)$$

,where l_i is a linear form and q_i, Q_j are quadratic forms in $S = k[x_1, \ldots, x_n]$. We remark that we can also assume all $\{l_i\}$ are linearly independent, and all $\{q_i\}$ are distinct. And we know that t is the codim(X) and $\{l_i\}$ generate $(T_qX)^*$ if q is smooth point. Generally, t is equal to $n - \dim T_qX$. Let's think about S-module syzygy of I_X. First of all, in first syzygy module the quadric generators are $x_0l_1 - q_1, \ldots, x_0l_t - q_t, Q_1, \ldots, Q_s$. In case of cubic generators, $x_0(x_0l_i - q_i)$, x_0Q_j could be candidates. All $x_0(x_0l_i - q_i)$ are should be the cubic generators because they have ${x_0}^2$. But not all the x_0Q_j are the minimal cubic generator, because there would be a relation such that $L\cdot(x_0l_i - q_i) + L'\cdot Q_k = x_0Q_j$. Quartic is similar, but has more possibilities. Although we can do this kind of analysis for higher degree generators and next syzygy modules by manipulating concrete equations, it is very hard to get an essential information of S-module syzygy of I_X(or R/I_X).

On the other hand, we can derive a more systematic result for S-module syzygy of I_X(or R/I_X) from Tor-relations (Corollary 2.4) if we more assume about linear syzygies of X(i.e. $N_{2,p}$ for $p \geq 2$).

Proposition 2.6. *Let X be a non-degenerate projective variety in $\mathbb{P}^n = \mathbb{P}(V)$. Consider the inner projection $\pi_q : X \rightarrow Y \subset \mathbb{P}^{n-1}$. Then, we have the following results:*

(a) *If X satisfies property $N_{2,p}$ $(p \geq 2)$, then I_X has simple syzygies up to $(p-1)$-th step as follows:*

$$\to \begin{matrix} S(-(p-1)-1)^{\beta_{p-1,1}} \\ \oplus S(-(p-1)-2)^{\beta_{p-1,2}} \\ \oplus S(-(p-1)-3)^{\beta_{p-1,3}} \\ \cdots \end{matrix} \to \cdots \to \begin{matrix} S(-3)^{\beta_{2,1}} \\ \oplus S(-4)^{\beta_{2,2}} \\ \oplus S(-5)^{\beta_{2,3}} \\ \cdots \end{matrix} \to \begin{matrix} S(-2)^{\beta_{1,1}} \\ \oplus S(-3)^{\beta_{1,2}} \\ \oplus S(-4)^{\beta_{1,3}} \\ \cdots \end{matrix} \to I_X \to 0$$

in the sense of $\ldots = \beta_{i,-1} = \beta_{i,0} = 0$, $\beta_{i,1} \geq \beta_{i,2} = \beta_{i,3} = \ldots$ $(1 \leq i \leq p-1)$ *and* $\ldots = \beta_{p,-1} = \beta_{p,0} = 0$, $\beta_{p,1} \geq \beta_{p,2} \geq \beta_{p,3} \geq \ldots$ *($i = p$ case), where* $\beta_{m,n} = \beta^S_{m,n}(R/I_X)$.

(b) *When X satisfies property $N_{d,p}$ $(p \geq 2)$, then we have similar simple syzygies of $(I_X)_{\geq d}$ up to $(p-1)$-th step such that:*
$\ldots = \beta_{i,d-3} = \beta_{i,d-2} = 0$, $\beta_{i,d-1} \geq \beta_{i,d} = \beta_{i,d+1} = \ldots$ $(1 \leq i \leq p-1)$ *and* $\ldots = \beta_{p,d-3} = \beta_{p,d-2} = 0$, $\beta_{p,d-1} \geq \beta_{p,d} \geq \beta_{p,d+1} \geq \ldots$ *($i = p$ case).*

Proof. (a) This is a direct result of Corollary 2.4 when $d = 2$, since $\beta_{m,n}$ is $\dim_k \operatorname{Tor}^S_m(R/I_X, k)_{m+n}$.

(b) If X satisfies property $N_{d,p}$ $(p \geq 2)$, then $(I_X)_{\geq d}$ has d-linear syzygies up to p-th step. As above, apply Corollary 2.4 to $(I_X)_{\geq d}$. $\square$

From this result we could guess what the generator of each degree of syzygy modules are and how they are varied in Tor-module by multiplying x_0. For example, we can easily deduce $\beta_{1,1} = t + s$ if $I_X = (x_0 l_1 - q_1, \ldots, x_0 l_t - q_t, Q_1, \ldots, Q_s)$, because they are all minimal quadric generators. And we might set $\beta_{1,2} = t + s'$ $(s' \leq s)$ because $x_0(x_0 l_i - q_i)$ should be a cubic S-module generator, while $x_0 Q_j$ might not if there is a relation such that $L \cdot (x_0 l_i - q_i) + L' \cdot Q_k = x_0 Q_j$. And $\beta_{1,3} = t + s''$ $\ldots$ and so on.

Then, some natural questions arise at this point. Is it possible to know the syzygy of I_Y from the S-module syzygy of I_X? Is I_Y generated only by quadrics if so I_X is? If not, how about the case that X satisfies $N_{2,2}$? Possibly there appear some cubic generators like $l_i q_j - l_j q_i (= l_j \cdot (x_0 l_i - q_i) - l_i \cdot (x_0 l_j - q_j))$ in I_Y if $\{Q_k\}$ doesn't generate it. In next section, we will answer this questions completely by the S-module syzygies of I_X and the partial elimination ideals, $K_i(I_X)$.

3. Syzygy structures of inner projections

For a projective variety $X \subset \mathbb{P}^n$, property $N_{2,p}$ is a natural generalization of property N_p of M. Green. The following theorems show that property $N_{2,p}$ plays an important role to control defining equations and syzygies of projected varieties under inner projection.

Theorem 3.1. (inner projection of varieties satisfying $N_{2,p}$)
Let $X \subset \mathbb{P}^n$ be a non-degenerate projective variety satisfying property $N_{2,p}$ for some $p \geq 2$ and q be a smooth point of X. Consider the inner projection $\pi_q : X \to Y \subset \mathbb{P}^{n-1}$. Then the projected variety Y is cut out by quadrics and satisfies property $N_{2,p-1}$.

Before proving this theorem, let us explain the basic definition and information on the partial elimination ideals under projections. For $q = (1, 0, \cdots, 0, 0)$, consider a projection $\pi_q : X \to Y \subset \mathbb{P}^{n-1} = \mathrm{Proj}(S)$, $S = k[x_1, x_2, \ldots, x_n]$. For the degree lexicographic order, if $f \in I_X$ has leading term $\mathrm{in}(f) = x_0^{d_0} \cdots x_n^{d_n}$, we set $d_0(f) = d_0$, the leading power of x_0 in f. Then we can give the definition of partial elimination ideals, which was given by M. Green in [11].

Definition 3.2 ([11])**.** Let $I_X \subset R$ be a homogeneous ideal of X and let

$$\tilde{K}_i(I_X) = \bigoplus_{m \geq 0} \big\{ f \in (I_X)_m \mid d_0(f) \leq i \big\}.$$

If $f \in \tilde{K}_i(I_X)$, we may write uniquely $f = x_0^i \bar{f} + g$ where $d_0(g) < i$. Now we define $K_i(I_X)$ by the image of $\tilde{K}_i(I_X)$ in S_1 under the map $f \mapsto \bar{f}$ and we call $K_i(I_X)$ the i-th partial elimination ideal of I_X.

Remark 3.3. We can remark some properties of these ideals

- 0-th partial elimination ideal of I_X is
$$I_Y = \bigoplus_{m \geq 0} \big\{ f \in (I_X)_m \mid d_0(f) = 0 \big\} = I_X \cap S$$
i.e. I_Y is equal to $K_0(I_X) = \tilde{K}_0(I_X)$.
- And $\tilde{K}_i(I_X)$ is finitely generated S-module and there is a short exact sequence as graded S-modules

$$0 \to \frac{\tilde{K}_{i-1}(I_X)}{\tilde{K}_0(I_X)} \to \frac{\tilde{K}_i(I_X)}{\tilde{K}_0(I_X)} \to K_i(I_X)(-i) \to 0. \tag{3.1}$$

- We have the following filtration on partial elimination ideals of I_X:
$$I_Y = K_0(I_X) \subset K_1(I_X) \subset K_2(I_X) \subset \cdots \subset K_i(I_X) \subset \cdots \subset S.$$

Lemma 3.4. *Let $X \subset \mathbb{P}^n$ be a non-degenerate projective variety and $q = (1, 0, \ldots, 0)$ be a smooth point of X. Suppose that I_X is generated by quadrics(i.e. satisfies $N_{2,1}$) and write $I_X = (x_0 l_1 - q_1, \ldots, x_0 l_t - q_t, Q_1, \ldots, Q_s)$.*

(a) *$K_i(I_X)$ stabilizes at $i = 1$ step to an ideal $I_\Sigma = (l_1, \ldots, l_t)$ which defines a linear subspace of $\mathbb{P}^n$, the tangent space T_qX,*
$$\text{i.e.} \quad I_Y \subset I_\Sigma = K_1(I_X) = K_2(I_X) = \cdots = K_i(I_X) = \cdots$$
(b) *I_X/I_Y has simple S-module syzygies such that:*

$$0 \to \begin{array}{c} S(-t-1)^{b_t} \\ \oplus S(-t-2)^{b_t} \\ \oplus S(-t-3)^{b_t} \\ \cdots \end{array} \to \cdots \to \begin{array}{c} S(-3)^{b_2} \\ \oplus S(-4)^{b_2} \\ \oplus S(-5)^{b_2} \\ \cdots \end{array} \to \begin{array}{c} S(-2)^{b_1} \\ \oplus S(-3)^{b_1} \\ \oplus S(-4)^{b_1} \\ \cdots \end{array} \to I_X/I_Y \to 0,$$

where $b_i = \binom{t}{i}$.

Proof. (a) From the definition 3.2, we know that $(l_1, \ldots, l_t) \subset K_i(I_X)$ for $\forall\, i \geq 1$. And note that all $\bar{f} \in K_i(I_X)\,(i \geq 1)$ are also regarded as the defining equations of tangent cone of X at q, TC_qX, because they come from $f = x_0^i \bar{f} + g \in I_X$ s.t. $d_0(g) < i$. Since q is smooth, $T_qX = TC_qX$ and we know $(l_1, \ldots, l_t) = \sqrt{(l_1, \ldots, l_t)}$. Thus, $K_i(I_X) \subset (l_1, \ldots, l_t)$ and we can say that $K_i(I_X)$ stabilizes at $i = 1$ to I_Σ.

(b) Note that $I_X = \tilde{K}_\infty(I_X)$. From the exact sequence (3.1), we get $\frac{\tilde{K}_1(I_X)}{I_Y} \simeq K_1(I_X)(-1)$ so that they have linear syzygies $0 \to S(-t-1)^{b_t} \to \cdots \to S(-3)^{b_2} \to$

$S(-2)^{b_1} \to \frac{\tilde{K}_1(I_X)}{I_Y} \to 0$. Next $K_2(I_X)(-2) = K_1(I_X)(-2)$ has also linear syzygies $0 \to S(-t-2)^{b_t} \to \cdots \to S(-4)^{b_2} \to S(-3)^{b_1} \to K_2(I_X)(-2) \to 0$ and we have the following exact sequence from (3.1) again,

$$0 \to \frac{\tilde{K}_1(I_X)}{I_Y} \to \frac{\tilde{K}_2(I_X)}{I_Y} \to K_2(I_X)(-2) \to 0.$$

By the long exact sequence of Tor, we know that

$$0 \to \begin{array}{c} S(-t-1)^{b_t} \\ \bigoplus \\ S(-t-2)^{b_t} \end{array} \to \cdots \to \begin{array}{c} S(-3)^{b_2} \\ \bigoplus \\ S(-4)^{b_2} \end{array} \to \begin{array}{c} S(-2)^{b_1} \\ \bigoplus \\ S(-3)^{b_1} \end{array} \to \frac{\tilde{K}_2(I_X)}{I_Y} \to 0.$$

Using (3.1) sequence, we can compute the syzygy of $\frac{\tilde{K}_i(I_X)}{I_Y}$ in a similar manner for any i, and in the end we get the desired syzygy of $I_X/I_Y = \frac{\tilde{K}_\infty(I_X)}{I_Y}$. $\square$

Now we know the S-module syzygy of I_X and I_X/I_Y, so we are getting back to the proof of the main theorem.

Proof. **of Theorem 3.1**

First of all, we note that $\mathrm{Tor}_i^S(R/I_X, k)_j = \mathrm{Tor}_{i-1}^S(I_X, k)_j$, $\mathrm{Tor}_i^S(S/I_Y, k)_j = \mathrm{Tor}_{i-1}^S(I_Y, k)_j$. We will use this fact throughout the proof. And we have a basic short exact sequence of S-modules,

$$0 \to I_Y \to I_X \to \frac{I_X}{I_Y} \to 0$$

Since X satisfies $N_{2,p}$ $(p \geq 2)$ and q is a smooth point, each one has the following S-module syzygies from Proposition 2.6, Lemma 3.4 :

$$\begin{array}{cccccccccc}
 & & & & & & & 0 & \\
 & & & & & & & \downarrow & \\
\to & \begin{array}{c} S(-r)^{\beta'_{r-1,1}} \\ \oplus \\ \cdots \\ S(-c_{r-1})^{\beta'_{r-1,c_{r-1}-r+1}} \end{array} & \to \cdots \to & \begin{array}{c} S(-3)^{\beta'_{2,1}} \\ \oplus \\ \cdots \\ S(-c_2)^{\beta'_{2,c_2-2}} \end{array} & \to & \begin{array}{c} S(-2)^{\beta'_{1,1}} \\ \oplus \\ \cdots \\ S(-c_1)^{\beta'_{1,c_1-1}} \end{array} & \to & I_Y & \to 0 \\
 & & & & & & & \downarrow & \\
\to & \begin{array}{c} S(-p)^{\beta_{p-1,1}} \\ \oplus S(-p-1)^{a_{p-1}} \\ \oplus S(-p-2)^{a_{p-1}} \\ \cdots \end{array} & \to \cdots \to & \begin{array}{c} S(-3)^{\beta_{2,1}} \\ \oplus S(-4)^{a_2} \\ \oplus S(-5)^{a_2} \\ \cdots \end{array} & \to & \begin{array}{c} S(-2)^{\beta_{1,1}} \\ \oplus S(-3)^{a_1} \\ \oplus S(-4)^{a_1} \\ \cdots \end{array} & \to & I_X & \to 0 \\
 & & & & & & & \downarrow & \\
0 \to & \begin{array}{c} S(-t-1)^{b_t} \\ \oplus S(-t-2)^{b_t} \\ \oplus S(-t-3)^{b_t} \\ \cdots \end{array} & \to \cdots \to & \begin{array}{c} S(-3)^{b_2} \\ \oplus S(-4)^{b_2} \\ \oplus S(-5)^{b_2} \\ \cdots \end{array} & \to & \begin{array}{c} S(-2)^{b_1} \\ \oplus S(-3)^{b_1} \\ \oplus S(-4)^{b_1} \\ \cdots \end{array} & \to & I_X/I_Y & \to 0 \\
 & & & & & & & \downarrow & \\
 & & & & & & & 0 &
\end{array}$$

Next we show that $a_i = b_i = \binom{t}{i}$ $(1 \leq i \leq p-1)$ and $\beta_{i,j} = b_i$ for $i \geq p$ $j \gg 0$. Since I_Y is finitely generated S-module, for sufficiently large j, $\mathrm{Tor}_i^S(I_Y, k)_{i+j} = 0$ for all i. From the long exact sequence of Tor, we have an isomorphism

$$0 = \mathrm{Tor}_i^S(I_Y, k)_{i+j} \to \mathrm{Tor}_i^S(I_X, k)_{i+j} \xrightarrow{\sim} \mathrm{Tor}_i^S(I_X/I_Y, k)_{i+j} \to \mathrm{Tor}_{i-1}^S(I_Y, k)_{i+j} = 0$$

for $j \gg 0$. Because $\dim_k \mathrm{Tor}^S_{i-1}(I_X, k)_{i+j} = a_i$ and $\dim_k \mathrm{Tor}^S_{i-1}(I_X/I_Y, k)_{i+j} = b_i$ for $j > 0$, we get the result.

Now we are going to show that I_Y has no cubic generator. From the long exact sequence of Tor, we have

$$\mathrm{Tor}^S_1(I_X, k)_3 \to \mathrm{Tor}^S_1(I_X/I_Y, k)_3 \to \mathrm{Tor}^S_0(I_Y, k)_3 \to \mathrm{Tor}^S_0(I_X, k)_3 \to \mathrm{Tor}^S_0(I_X/I_Y, k)_3 \to 0$$

Since $a_1 = b_1$, it is enough to show that the map $\mathrm{Tor}^S_1(I_X, k)_3 \to \mathrm{Tor}^S_1(I_X/I_Y, k)_3$ is surjective. But we deduce this surjectivity from the following diagram:

$$\begin{array}{ccc} S(-3)^{\beta_{2,1}} & \longrightarrow & S(-3)^{b_2} \\ h\Big\downarrow \text{surj.} & & g\Big\downarrow \wr \\ S(-k)^{a_2} & \xrightarrow{\sim} & S(-k)^{b_2} \end{array}$$

,where h, g are induced by multiplications by x_0 and $k \gg 0$. We already know that h is surjective, because of $N_{2,2}$ (Prop. 2.6) and that g is an isomorphism (Lemma 3.4). Therefore, $S(-3)^{\beta_{2,1}} \to S(-3)^{b_2}$ is surjective and $\mathrm{Tor}^S_0(I_Y, k)_3 = 0$, no cubic generators in I_Y. Analogously, we get that I_Y has no higher generators, either.

More generally, let's show that $c_i \le i + 1$ ($1 \le i \le p - 1$) by same argument as above. This means Y satisfying $N_{2,p-1}$. At a preceding paragraph, we show $c_1 = 2$. So, suppose that $c_i \le i + 1$ for $i < n$ ($2 \le n \le p - 1$). We will show $c_n \le n + 1$. For any $c > n + 1$, we have

$$\mathrm{Tor}^S_n(I_X, k)_c \to \mathrm{Tor}^S_n(I_X/I_Y, k)_c \to \mathrm{Tor}^S_{n-1}(I_Y, k)_c \to \mathrm{Tor}^S_{n-1}(I_X, k)_c \to \mathrm{Tor}^S_{n-1}(I_X/I_Y, k)_c \to 0$$

The zero in the very right side comes from the induction hypothesis, $\mathrm{Tor}^S_{n-2}(I_Y, k)_c = 0$. Since $a_n = b_n$ (i.e. $\mathrm{Tor}^S_{n-1}(I_X, k)_c \xrightarrow{\sim} \mathrm{Tor}^S_{n-1}(I_X/I_Y, k)_c$), we get $\mathrm{Tor}^S_{n-1}(I_Y, k)_c = 0$ if we show $\mathrm{Tor}^S_n(I_X, k)_c \twoheadrightarrow \mathrm{Tor}^S_n(I_X/I_Y, k)_c$, surjective. But we also have the following diagram:

$$\begin{array}{ccc} S(-c)^{\beta_{n,-n+c}} & \xrightarrow{\phi} & S(-c)^{b_n} \\ h\Big\downarrow \text{surj.} & & g\Big\downarrow \wr \\ S(-k)^{a_n} & \xrightarrow{\sim} & S(-k)^{b_n} \end{array}$$

,where h, g are induced by multiplications by x_0 and $k \gg 0$. Since X satisfies $N_{2,n+1}$ ($n + 1 \le p$), h is a surjection. That means ϕ is surjective, as we wish. So $\mathrm{Tor}^S_{n-1}(I_Y, k)_c = 0$, $c_n \le n + 1$. Hence Y satisfies $N_{2,p-1}$. □

Remark 3.5. (Inner projection from a singular point) Let X be a non-degenerate projective variety and $q \in X$. If the point q is a singular, we could expect that the inner projection from q has more complicate aspects and fails to satisfy $N_{2,p-1}$ according to multiplicity of the singularity. On the other hand, it might be possible to satisfy $N_{2,p-1}$ if q is quite mild.

References

[1] J. Ahn, S. Kwak, *Algebra and geometry under projections and applications*, preprint (2008)

[2] A. Alzati and F. Russo, *On the k-normality of projected algebraic varieties*, Bull. Braz. Math. Soc. (N.S.) 33 (2002), no.1, 27–48.

[3] M. Brodmann and P. Schenzel, *Arithmetic properties of projective varieties of almost minimal degree*, J. Algebraic Geom. 16 (2007), 347–400.
[4] Y. Choi, S. Kwak and P-L Kang, *Higher linear syzygies of inner projections*, J. Algebra 305 (2006), 859–876.
[5] Y. Choi, S. Kwak and E. Park, *On syzygies of non-complete embedding of projective varieties*, Math. Zeitschrift 258, no. 2 (2008), 463-475.
[6] D. Eisenbud, *Commutative Algebra with a View toward Algebraic Geometry*, Springer-Verlag, New York, 1995.
[7] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Alg. 88 (1984), 89-133.
[8] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restriction linear syzygies: algebra and geometry*, Compositio Math. 141 (2005), 1460-1478.
[9] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Small schemes and varieties of minimal degree*, Amer. J. Math.
[10] D. Eisenbud, C. Huneke, B.Ulrich, *The regularity of Tor and graded Betti numbers* , Amer. J. Math. 128 (2006), no. 3, 573–605.
[11] M. Green, *Generic Initial Ideals*, in Six lectures on Commutative Algebra, (Elias J., Giral J.M., Miró-Roig, R.M., Zarzuela S., eds.), Progress in Mathematics **166**, Birkhäuser, 1998, 119–186.
[12] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo and the equationsndefining projective varieties*, Inv. Math. 72 (1983), 491-506.
[13] S. Kwak, *Castelnuovo regularity of smooth projective varietiesof dimension* 3 *ansd* 4, J. Alg. Geom. 7(1998), 195-206.
[14] S. Kwak and E. Park *Some effects of property* N_p *on the higher normality and defining equations of nonlinearly normal varieties*, J. Reine Angew. Math. 582 (2005), 87–105.
[15] R. Larzarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. 55(1987), no. 2,423-429.
[16] E. Park, *Projective curves of degree=codimension +2*, Math. Zeit. 256 (2007), 185–208.
[17] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction*, Compositio Math. 125(2001), no.3, 263-282.

Department of Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Gusung-dong, Yusung-Gu, Daejeon, Korea
E-mail address: `hankj@kaist.ac.kr`

Department of Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Gusung-dong, Yusung-Gu, Daejeon, Korea
E-mail address: `skwak@kaist.ac.kr`