SEMI-TOPOLOGIZATION IN MOTIVIC HOMOTOPY THEORY AND APPLICATIONS

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Abstract. In this paper, we study semi-topological cohomology theories such as semi-topological K-theory and morphic cohomology from the perspective of the motivic homotopy theory. Using some motivic descent theorems for Morel's $\mathbb{A}^1$-B.G. motivic spectra, we construct a triangulated endo-functor on the stable motivic homotopy category $\mathcal{SH}(\mathbb{C})$, which we call *homotopy semi-topologization*.

Using this functor we show that the semi-topological K-theory of Friedlander-Walker and the morphic cohomology of Friedlander-Lawson are representable in $\mathcal{SH}(\mathbb{C})$. The homotopy semi-topologization functor allows us to define a semi-topological analogue of the algebraic cobordism of Voevodsky. We show that there is a Hopkins-Morel type spectral sequence, which relates the morphic cohomology with the semi-topological cobordism, and this spectral sequence degenerates with the rational coefficients. We also show that the Voevodsky algebraic cobordism coincides with the semi-topological cobordism with finite coefficients.

1. Introduction

The motivic homotopy theory of algebraic varieties was pioneered by Morel and Voevodsky in [33]. In this theory, smooth schemes over a fixed noetherian base scheme S are treated as representable simplicial presheaves on the category of smooth schemes over S. Using suitable closed model structures on these simplicial presheaves, one defines the unstable and the stable motivic homotopy categories over S. These homotopy categories are analogous to the unstable and stable homotopy categories of topological spaces. The motivic homotopy theory has had some outstanding applications, notably, in the proofs of the Milnor conjecture and the Bloch-Kato conjecture.

The goal of this paper is to study some cohomology theories such as semi-topological K-theory and morphic cohomology from the perspective of the motivic homotopy theory. The semi-topological K-theory of algebraic varieties was introduced by Friedlander and Walker [10]. This theory interpolates between the algebraic and the topological K-theories of complex algebraic varieties. The morphic cohomology was introduced by Friedlander and Lawson [5], and it interpolates between the motivic and the singular cohomology theories of complex algebraic varieties.

It is known that the algebraic K-theory and the motivic cohomology of algebraic varieties are representable in the stable motivic homotopy category over S. We show in this paper that the semi-topological K-theory and the morphic cohomology of complex algebraic varieties are also representable in the stable motivic homotopy category.

In order to prove these results, we use the idea of Friedlander and Walker [9] of defining the semi-topological K-theory as a certain kind of *semi-topologization* of the algebraic K-theory spectrum. It turns out that this semi-topologization can be defined as a functor on the categories of presheaves on smooth schemes over $\mathbb{C}$ of simplicial sets and spectra. Our first main result in this paper is that the semi-topologization of

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the presheaves induces a derived endo-functor on the stable motivic homotopy category $\mathcal{SH}(\mathbb{C})$, which is compatible with the triangulated structure on $\mathcal{SH}(\mathbb{C})$.

In the process, we run into several technical issues. This is mainly because the semi-topologization functor as is does not respect motivic weak-equivalences of the presheaves. So, we need to take a detour.

The key fact is that the semi-topologization process gets along well with *object-wise* weak-equivalences between presheaves of spectra via the recognition principle of Friedlander and Walker [12]. Our basic motivic descent theorem (Theorem 3.1.5) shows that among "$\mathbb{A}^1$-B.G." presheaves, all motivic weak-equivalences are object-wise weak-equivalences. This class of $\mathbb{A}^1$-B.G. presheaves was introduced by Morel [35, Definition A.5]. See Definition 3.1.1. The motivic descent theorem is the motivic analogue of the Nisnevich descent theorem of Morel and Voevodsky [33] for B.G. simplicial sheaves.

We extend the above motivic descent theorem to (s,p)-bispectra in Theorem 3.4.3. These (s,p)-bispectra can be seen as T-spectrum objects in the category of presheaves of spectra, where T is the pointed motivic space $(\mathbb{P}^1, \infty)$. After proving some more intermediate results, we achieve our goal of constructing the *homotopy semi-topologization* triangulated endo-functor on $\mathcal{SH}(\mathbb{C})$. We establish several properties of homotopy semi-topologization, and we prove the representability of the semi-topological K-theory and the morphic cohomology in $\mathcal{SH}(\mathbb{C})$.

Using the homotopy semi-topologization, we define a semi-topological analogue of the algebraic cobordism theory of Voevodsky [47], and then construct the semi-topological analogue of the spectral sequence of Hopkins and Morel. This spectral sequence relates the morphic cohomology of a smooth scheme over $\mathbb{C}$ to its semi-topological cobordism. We also show that the semi-topological cobordism coincides with the algebraic cobordism with finite coefficients.

Summary of results: The following is a summary of our main results. The full statements of these results and the underlying notations and definitions can be found in the main body of the text.

Theorem 1.0.1 (Motivic descent theorems). *A motivic weak-equivalence $E \to F$ of $\mathbb{A}^1$-B.G. presheaves of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$ is an object-wise weak-equivalence.*

A stable motivic weak-equivalence $E \to F$ between $\mathbb{A}^1$-B.G. motivic Ω_T-bispectra (see Definition 3.4.2) on $\mathbf{Sm}_S$ is a T-level-wise object-wise weak-equivalence, i.e., each $f_n : E_n \to F_n$ is an object-wise weak-equivalence of presheaves of spectra on $\mathbf{Sm}_S$.

Theorem 1.0.2 (Homotopy semi-topologization). *There exists a triangulated endo-functor* **host** $: \mathcal{SH}(\mathbb{C}) \to \mathcal{SH}(\mathbb{C})$, *which coincides with the Friedlander-Walker semi-topologization functor on $\mathbb{A}^1$-B.G. motivic Ω_T-bispectra.*

Theorem 1.0.3 (Representability and spectral sequence). *The semi-topological K-theory and the morphic cohomology are representable in $\mathcal{SH}(\mathbb{C})$.*

There is a semi-topological cobordism $\mathrm{MGL}_{\mathbf{sst}}$ *as a cohomology theory on $\mathbf{Sm}_{\mathbb{C}}$, with a natural transformation* $\mathrm{MGL} \to \mathrm{MGL}_{\mathbf{sst}}$ *that becomes an isomorphism with finite coefficients. For any $X \in \mathbf{Sm}_{\mathbb{C}}$ and $n \geq 0$, there is a spectral sequence*

$$E_2^{p,q}(n) = L^{n-q}H^{p-q}(X) \underset{\mathbb{Z}}{\otimes} \mathbb{L}^q \Rightarrow \mathrm{MGL}_{\mathbf{sst}}^{p+q,n}(X),$$

and this spectral sequence degenerates after tensoring with $\mathbb{Q}$.

It turns out that many of the expected properties (some of which are known for the semi-topological K-theory) such as Nisnevich Mayer-Vietoris sequence, localization and strong homotopy invariance are immediate consequences of Theorem 1.0.3. See Theorem 8.2.1.

Outline of the paper: Here is an outline of the structure of the paper. We recollect notations, definitions and the constructions of various motivic homotopy categories in § 2. In § 3, we recall the notion of $\mathbb{A}^1$-B.G. presheaves of (pointed) simplicial sets and spectra. Our main results in this section are the motivic descent theorems: Theorem 3.1.5 and Theorem 3.4.3. In § 4 and § 5, we recall the definition of the semi-topologization functor. In § 6, we prove some technical results about the semi-topologization functor and construct the *homotopy semi-topologization* as a triangulated endo-functor on $\mathcal{SH}(\mathbb{C})$. In § 7 and § 8, we prove the representability theorems for the semi-topological K-theory and the morphic cohomology in $\mathcal{SH}(\mathbb{C})$. In § 9, we study the homotopy semi-topologization of the motivic Thom spectrum MGL of Voevodsky, and prove the semi-topological analogue of the Hopkins-Morel spectral sequence with some consequences.

Conventions and notations: When S is a noetherian scheme of finite Krull dimension, an S-scheme is a separated scheme of finite type over S. We denote the category of S-schemes by $\mathbf{Sch}_S$. Its full subcategory of smooth schemes over S will be denoted by $\mathbf{Sm}_S$. When $S = \mathrm{Spec}\,(k)$ for a field k, we shall often write $\mathbf{Sch}_k$ and $\mathbf{Sm}_k$ for $\mathbf{Sch}_S$ and $\mathbf{Sm}_S$, respectively. A variety over k is a reduced k-scheme, not necessarily quasi-projective. The category of k-varieties will be denoted by $\mathbf{Var}_k$.

We let **Set** be the category of sets, **Spc** be the category of simplicial sets and $\mathbf{Spc}_\bullet$ be the category of pointed simplicial sets. We let **Spt** be the category of Bousfield-Friedlander spectra [2], which is also recalled in Section 2.2. We shall denote the homotopy categories of $\mathbf{Spc}_\bullet$ and **Spt** by $\mathbf{HoSpc}_\bullet$ and $\mathcal{SH}$, respectively. The set of maps $K \to L$ in $\mathbf{Spc}_\bullet$ will be denoted by $\mathrm{Hom}_\bullet(K, L)$. The set of homotopy classes of maps $K \to L$ in $\mathbf{Spc}_\bullet$ or **Spt** will be denoted by $\pi(K, L)$, and the set of maps $K \to L$ in $\mathbf{HoSpc}_\bullet$ or in $\mathcal{SH}$ will be denoted by $[K, L]$.

The symbol Δ is used for the following cases, and we hope no confusion may arise. First, Δ denotes the category whose objects are $[n] := \{0, \cdots, n\}$ for $n \geq 0$ and the morphisms $[m] \to [n]$ are nondecreasing set functions. The notation $\Delta[n]$ denotes the simplicial set $\mathrm{Hom}_{\mathbf{Set}}(-, [n])$ given by the Yoneda embedding. The notation Δ^n_{top} denotes the topological n-simplex $\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | 0 \leq t_i \leq 1, \sum_i t_i = 1\}$, while Δ^n denotes the algebraic n-simplex $\mathrm{Spec}\,(k[t_0, \cdots, t_n]/\sum_i t_i = 1)$.

2. Recollection of motivic homotopy theory

In this section, we recall basic definitions in motivic homotopy theory to fix the terminologies. The original reference is [33] but we often adapt the presentations given in [19] and [34]. Throughout the Sections 2 and 3, we let S denote a fixed noetherian scheme of finite Krull dimension.

2.1. **Motivic spaces.** We regard an object of **Spc** as a *space*. An object of $\mathbf{Spc}_\bullet$ is regarded as a *pointed space*.

A *motivic space over* S is a simplicial presheaf on $\mathbf{Sm}_S$ and a *pointed motivic space over* S is a pointed simplicial presheaf on $\mathbf{Sm}_S$. We let $\mathbf{Spc}(S)$ and $\mathbf{Spc}_\bullet(S)$ be the categories of unpointed and pointed motivic spaces over S, respectively. A presheaf of sets on $\mathbf{Sm}_S$ will be regarded as a motivic space over S of simplicial dimension zero. Each $X \in \mathbf{Sm}_S$ is seen as a motivic space over S by Yoneda embedding, that we still denote by X. By X_+, we mean the pointed motivic space $X \coprod S$. Each (pointed) space K also defines a (pointed) motivic space over S as a constant presheaf on $\mathbf{Sm}_S$.

When U is a pointed motivic space over S, we have the suspension functor $\Sigma_U : \mathbf{Spc}_\bullet(S) \to \mathbf{Spc}_\bullet(S)$ that sends E to $E \wedge U$. For $U = S^1, (\mathbb{G}_m, \{1\})$ and $T = (\mathbb{P}^1, \infty)$, we write Σ_U as Σ_s, Σ_t, and Σ_T, respectively.

Given $E, F \in \mathbf{Spc}(S)$, we let $\mathcal{H}om(E, F)$ denote the internal hom presheaf of simplicial sets. For $E, F \in \mathbf{Spc}_\bullet(S)$, the pointed internal hom presheaf will be denoted by $\mathcal{H}om_\bullet(E, F)$. The simplicial hom set from E to F will be denoted by $\mathbf{S}(E, F)$. There is a bijection

$$\mathbf{S}(E \wedge F, G) \simeq \mathbf{S}(E, \mathcal{H}om_\bullet(F, G)) \ \text{ for } \ E, F, G \in \mathbf{Spc}_\bullet(S). \tag{2.1.1}$$

For a pointed motivic space E, we shall denote the functor $\mathcal{H}om_\bullet(E, -)$ on $\mathbf{Spc}_\bullet$ by $\Omega_E(-)$. For $E = S^1$ (resp. $(\mathbb{G}_m, 1)$), $\Omega_E(-)$ will be often written as $\Omega_s(-)$ (resp. $\Omega_t(-)$).

2.2. **S^1-stable motivic homotopy category.** Recall from [19, Theorem 1.1] that the category $\mathbf{Spc}(S)$ of motivic spaces over S is a proper simplicial cellular closed model category, where a map $f : E \to F$ is a *Nisnevich local weak-equivalence* if the induced maps on all Nisnevich stalks $E_x \to F_x$ are weak-equivalences of simplicial sets, while *cofibrations* are monomorphisms, and *Nisnevich fibrations* are defined in terms of the right lifting property with respect to all trivial cofibrations. One defines a similar model structure on $\mathbf{Spc}_\bullet(S)$ by forgetting the base points. Their homotopy categories, obtained by formally inverting the Nisnevich local weak-equivalences, are denoted by $\mathcal{H}^{Nis}(S)$ and $\mathcal{H}^{Nis}_\bullet(S)$. When E and F are pointed motivic spaces, we let $[E, F]_{Nis}$ denote the set $\mathrm{Hom}_{\mathcal{H}^{Nis}_\bullet(S)}(E, F)$. See [19] and [33] for more details on these model structures.

We now recall the construction of S^1-stable motivic homotopy category from [19] and [34]. A *spectrum*, also called an *S^1-spectrum*, is a sequence of pointed simplicial sets $(E_0, E_1, \cdots)$ with the structure morphisms $S^1 \wedge E_n \xrightarrow{\sigma_n} E_{n+1}$ of pointed simplicial sets. The category of spectra is denoted by $\mathbf{Spt}$, and the category of presheaves of spectra on $\mathbf{Sm}_S$ is denoted by $\mathbf{Spt}(S)$. For simplicity, a presheaf of spectra on $\mathbf{Sm}_S$ will be called a *motivic spectrum* (over S), or often a *spectrum on* $\mathbf{Sm}_S$.

2.2.1. *Nisnevich model structure on motivic spectra over S.* Recall that a morphism $f : E \to F$ of motivic spectra is an *object-wise weak-equivalence* if for each $U \in \mathbf{Sm}_S$, the map $f(U) : E(U) \to F(U)$ is a weak-equivalence of spectra (*i.e.*, the induced maps on the stable homotopy groups are isomorphisms).

A morphism $f : E \to F$ of motivic spectra is said to be a *Nisnevich local weak-equivalence* if for each $U \in \mathbf{Sm}_S$ and $x \in U$, the induced map $f_x : E_x \to F_x$ on the Nisnevich stalks is a weak-equivalence of spectra. A map $f : E \to F$ of motivic spectra is a *cofibration* if f_0 is a monomorphism and $E_{n+1} \coprod_{S^1 \wedge E_n} S^1 \wedge F_n \to F_{n+1}$ is a monomorphism of motivic spaces for each $n \geq 0$. This is equivalent to saying that the maps $E_n \to F_n$ and $S^1 \wedge (F_n/E_n) \to F_{n+1}/E_{n+1}$ are monomorphisms of motivic spaces for each $n \geq 0$. A *Nisnevich fibration* between motivic spectra is a map with the right lifting property with respect to all trivial cofibrations.

Observe that giving a cofibration $E \to F$ of motivic spectra is equal to giving cofibrations $E(U) \to F(U)$ of spectra. A morphism $E \to F$ of motivic spectra is a Nisnevich local weak-equivalence if and only if the induced map of Nisnevich sheaves associated to the presheaves $U \mapsto \pi_n(E(U))$, $U \mapsto \pi_n(F(U))$, is an isomorphism for all $n \in \mathbb{Z}$.

Theorem 2.2.1. ([18, Theorem 2.34], [34, Lemma 2.3.6]) *The above classes of Nisnevich local weak-equivalences, cofibrations and Nisnevich fibrations define a proper simplicial closed model structure on* $\mathbf{Spt}(S)$. *An object E is cofibrant if and only if the structure maps $\sigma_n : S^1 \wedge E_n \to E_{n+1}$ are monomorphisms and an object F is Nisnevich fibrant if* *and only if each F_n is a Nisnevich fibrant motivic space and the adjoint of the structure maps $\widetilde{\sigma}_n : E_n \to \Omega^1_s E_{n+1}$ are Nisnevich local weak-equivalences.*

It follows from Theorem 2.2.1 that for every $E \in \mathbf{Spc}_\bullet(S)$, the motivic spectrum $\Sigma^\infty_s E = (E, \Sigma^1_s E, \Sigma^2_s E, \cdots)$ is cofibrant. The homotopy category of $\mathbf{Spt}(S)$ with respect to the *Nisnevich local injective model structure* defined above will be denoted by $\mathcal{SH}^{Nis}_{S^1}(S)$. For the motivic spectra E and F, the set $\mathrm{Hom}_{\mathcal{SH}^{Nis}_{S^1}(S)}(E, F)$ will be denoted by $[E, F]_{Nis}$.

2.2.2. *Motivic model structure on motivic spectra over S.* The homotopy category of $\mathbf{Spc}_\bullet(S)$ with respect to the motivic model structure (see [19]) will be denoted by $\mathcal{H}_\bullet(S)$, and the set of all maps from E to F in $\mathcal{H}_\bullet(S)$ will be denoted by $[E, F]_{\mathbb{A}^1}$. Below we recall from [34, § 4], the motivic model structure on $\mathbf{Spt}(S)$, which is similar to that on $\mathbf{Spc}_\bullet(S)$.

A motivic spectrum Z is said to be $\mathbb{A}^1$-*local* if for any motivic spectrum E, the projection map $E \wedge \mathbb{A}^1_+ \to E$ induces an isomorphism of abelian groups $[E, Z]_{Nis} \xrightarrow{\simeq} [E \wedge \mathbb{A}^1_+, Z]_{Nis}$. A morphism $f : E \to F$ of motivic spectra is called an S^1-*stable motivic weak-equivalence* if for each $\mathbb{A}^1$-local Z, the induced map $f^* : [F, Z]_{Nis} \to [E, Z]_{Nis}$ is an isomorphism. We often say that f is a *motivic weak-equivalence* of motivic spectra, for simplicity.

The classes of motivic weak-equivalences, cofibrations (as defined in §2.2.1) and *motivic fibrations* given by the right lifting property with respect to all trivial cofibrations, define a closed model structure on $\mathbf{Spt}(S)$, called the S^1-*stable motivic model structure*. This model structure is the left localization of the Nisnevich local injective model structure with respect to the class of maps $E \wedge \mathbb{A}^1_+ \to E$ for $E \in \mathbf{Spt}(S)$. It follows from [15, Proposition 3.4] and Theorem 2.2.1 that the motivic model structure on $\mathbf{Spt}(S)$ is left proper and simplicial. In fact, it is proper. A motivic spectrum is motivic fibrant if and only if it is Nisnevich fibrant and $\mathbb{A}^1$-local. The homotopy category of $\mathbf{Spt}(S)$ with respect to the S^1-stable motivic model structure will be denoted by $\mathcal{SH}_{S^1}(S)$.

The S^1-stable motivic model structure on $\mathbf{Spt}(S)$ is equivalent to the one obtained by stabilizing the motivic model structure on $\mathbf{Spc}_\bullet(S)$ with respect to the simplicial suspension Σ_s, as described in [19, Theorem 1.1]. It follows that a motivic spectrum E is motivic fibrant if and only if it is level-wise motivic fibrant in the motivic model structure on $\mathbf{Spc}_\bullet(S)$ and each map $E_n \to \Omega_s E_{n+1}$ is a motivic weak-equivalence.

It follows from [34, Proposition 3.1.1] that $\mathcal{SH}_{S^1}(S)$ is a triangulated category, where the shift functor $E \mapsto E[1]$ is given by Σ_s. The abelian group of all maps from E to F in $\mathcal{SH}_{S^1}(S)$ will be denoted by $[E, F]_{\mathbb{A}^1}$.

Note that we have a pair of adjoint functors $\Sigma^\infty_s : \mathbf{Spc}_\bullet(S) \leftrightarrow \mathbf{Spt}(S) : Ev^0_s$ given by $\Sigma^\infty_s(E) = (E, \Sigma_s E, \Sigma^2_s E, \cdots)$ and $Ev^0_s(F) = F_0$. The functor Σ^∞_s clearly preserves cofibrations. Moreover, as there are natural isomorphisms

$$\begin{aligned} [\Sigma^\infty_s E[p], F]_{Nis} &\simeq \operatorname*{colim}_{n \geq -p} [S^{n+p} \wedge E, F_n]_{Nis}, \\ [\Sigma^\infty_s E[p], F]_{\mathbb{A}^1} &\simeq \operatorname*{colim}_{n \geq -p} [S^{n+p} \wedge E, F_n]_{\mathbb{A}^1}, \end{aligned} \tag{2.2.1}$$

for $E \in \mathbf{Spc}_\bullet(S), F \in \mathbf{Spt}(S)$ and $p \in \mathbb{Z}$ (*cf.* [47, Theorem 5.2]), the functor Σ^∞_s also preserves motivic weak-equivalences. In particular, the pair $(\Sigma^\infty_s, Ev^0_s)$ forms a Quillen pair of adjoint functors and one gets adjoint functors $\Sigma^\infty_s : \mathcal{H}_\bullet(S) \leftrightarrow \mathcal{SH}_{S^1}(S) : \mathbf{R}Ev^0_s$.

2.3. **Stable motivic homotopy category.** The stable motivic homotopy category $\mathcal{SH}(S)$ of motivic spaces over S was first constructed by Voevodsky in [47]. This category can now be constructed from several different models. We give a brief review of two different constructions of $\mathcal{SH}(S)$. Recall that for $F \in \mathbf{Spt}(S)$ and $E \in \mathbf{Spc}_\bullet(S)$, we let $\Sigma_E F$ denote the motivic spectrum $(F_0 \wedge E, F_1 \wedge E, \cdots)$. For $E = S^1$, the spectrum $\Sigma_E F$ will be denoted by $\Sigma_s F$.

2.3.1. *(s,p)-bispectra model.* Recall from [26, § 8] that an *(s,p)-bispectrum* over S is a collection $E = \{E_{m,n} | m, n \geq 0\}$ of pointed motivic spaces over S together with the horizontal bonding maps $\Sigma_s E_{m,n} \to E_{m+1,n}$ and the vertical bonding maps $\Sigma_T E_{m,n} \to E_{m,n+1}$ such that the horizontal and the vertical bonding maps commute. We regard it as a sequence $(E_0, E_1, \cdots)$ of motivic spectra with the bonding maps $\epsilon : \Sigma_T E_n \to E_{n+1}$, where E_n is the motivic spectrum $E_{*,n} := (E_{0,n}, E_{1,n}, \cdots)$. We let $\mathbf{Spt}_{(s,p)}(S)$ denote the category of (s,p)-bispectra over S.

Given an (s,p)-bispectrum E and $p, q \in \mathbb{Z}$, we define $\pi_{p,q}(E)$ to be the presheaf

$$U \mapsto \operatorname*{colim}_n \operatorname{Hom}_{\mathcal{SH}_{S^1}(S)}(\Sigma_s^{p-2q} \Sigma_T^{q+n} \Sigma_s^\infty U_+, E_n). \tag{2.3.1}$$

A morphism $f : E \to F$ of (s,p)-bispectra over S is called a *stable motivic weak-equivalence* if the induced morphism $f_* : \pi_{p,q}(E) \to \pi_{p,q}(F)$ of presheaves is a stalk-wise isomorphism of abelian groups on $(\mathbf{Sm}_S)_{Nis}$. We often drop the word *stable* for simplicity, and call it just a *motivic weak-equivalence* of (s,p)-bispectra.

There is a closed model structure on $\mathbf{Spt}_{(s,p)}(S)$ (*cf.* [16, § 3], [26, § 8.2]), whose weak-equivalences are the stable motivic weak-equivalences. This model structure on $\mathbf{Spt}_{(s,p)}(S)$ is called the *stable motivic model structure*, or simply the motivic model structure. It follows from [16, Proposition 1.14] that this model structure is obtained as a Bousfield localization of the level-wise model structure on $\mathbf{Spt}_{(s,p)}(S)$ in which weak-equivalences (fibrations) are T-level-wise S^1-stable motivic weak-equivalences (motivic fibrations) of motivic spectra, and $E \to F$ is a *cofibration* if the maps $E_0 \to F_0$ and

$$E_{n+1} \coprod_{\Sigma_T E_n} \Sigma_T F_n \to F_{n+1}$$

are cofibrations in the S^1-stable motivic model structure on $\mathbf{Spt}(S)$ for all $n \geq 0$. The stable motivic model structure on $\mathbf{Spt}_{(s,p)}(S)$ is proper and simplicial. It follows from [16, Theorem 3.4] that an (s,p)-bispectrum E is stable motivic fibrant if and only if each E_n is S^1-stable motivic fibrant and the adjoint of the bonding maps $\widetilde{\epsilon} : E_n \to \Omega_T E_{n+1}$ are S^1-stable motivic weak-equivalences for all $n \geq 0$.

2.3.2. *T-spectra model.* A *T-spectrum* E over S is a collection $(E_0, E_1, \cdots)$ of pointed motivic spaces over S together with the bonding maps $\Sigma_T E_n \to E_{n+1}$. The category of T-spectra over S is denoted by $\mathbf{Spt}_T(S)$. As in the case of (s,p)-bispectra, we define for $p, q \in \mathbb{Z}$, the presheaf $\pi_{p,q}(E)$ on $\mathbf{Sm}_S$ by

$$(\pi_{p,q}(E))(U) = \operatorname*{colim}_n \operatorname{Hom}_{\mathcal{H}_\bullet(S)}(\Sigma_s^{p-2q} \Sigma_T^{q+n} U_+, E_n).$$

There is a proper simplicial closed model structure on $\mathbf{Spt}_T(S)$ in which $E \to F$ is a weak-equivalence (called a *stable motivic weak-equivalence of T-spectra*) if the induced map $f_* : \pi_{p,q}(E) \to \pi_{p,q}(F)$ is a stalk-wise isomorphism of abelian groups on $(\mathbf{Sm}_S)_{Nis}$. This model structure is obtained as a Bousfield localization of the model structure on $\mathbf{Spt}_T(S)$ in which weak-equivalences (fibrations) are level-wise motivic weak-equivalences (motivic fibrations) of pointed motivic spaces.

Given a motivic spectrum E, let $\Omega_s^\infty E := \operatorname*{colim}_m \Omega_s^m E_m$. For a T-spectrum E, let $\Omega_T^\infty E := \operatorname*{colim}_m \Omega_T^m E_m$.

A T-spectrum $E = (E_0, E_1, \cdots)$ defines an (s,p)-bispectrum $\mathcal{E} := (\Sigma_s^\infty E_0, \Sigma_s^\infty E_1, \cdots)$ by taking the term-wise simplicial infinite suspensions. Conversely, given an (s,p)-bispectrum $\mathcal{F} = (F_0, F_1, \cdots)$, we obtain a T-spectrum $F = (\Omega_s^\infty F_0, \Omega_s^\infty F_1, \cdots)$. The correspondence

$$\Sigma_s^\infty : \mathbf{Spt}_T(S) \leftrightarrow \mathbf{Spt}_{(s,p)}(S) : \Omega_s^\infty \tag{2.3.2}$$

induces an equivalence between the stable homotopy categories of T-spectra and (s,p)-bispectra. We use the notation $\mathcal{SH}(S)$ for the common homotopy category.

For any pointed motivic space $X \in \mathbf{Spc}_\bullet(S)$, one associates the infinite T-suspension spectrum of X, defined by $\Sigma_T^\infty X := (X, \Sigma_T X, \Sigma_T^2 X, \cdots)$, with the identity bonding maps $T \wedge T^{\wedge(n-1)} \wedge X \to T^n \wedge X$. We can also extend the suspension operations $\Sigma_T, \Sigma_s, \Sigma_t$ to $\mathbf{Spt}_{(s,p)}(S)$ and $\mathbf{Spt}_T(S)$ in the obvious way. The category $\mathcal{SH}(S)$ is a triangulated category in which the shift functor $E \mapsto E[1]$ is given by the functor Σ_s. On this category, the functors $\Sigma_T, \Sigma_s, \Sigma_t$ are all auto-equivalences. For (s,p)-bispectra E and F, we shall denote $\mathrm{Hom}_{\mathcal{SH}(S)}(E,F)$ by $[E,F]_{\mathbb{A}^1}$.

There is a Quillen pair of adjoint functors $\Sigma_T^\infty : \mathbf{Spt}(S) \leftrightarrow \mathbf{Spt}_{(s,p)}(S) : \Omega_T^\infty$ given by $\Sigma_s^\infty(E) = (E, \Sigma_T E, \Sigma_T^2 E, \cdots)$ and $\Omega_T^\infty(F) = (\Omega_T^\infty F_{0,*}, \Omega_T^\infty F_{1,*}, \cdots)$. This yields an adjoint pair of derived functors

$$\Sigma_T^\infty : \mathcal{SH}_{S^1}(S) \leftrightarrow \mathcal{SH}(S) : \mathbf{R}\Omega_T^\infty. \tag{2.3.3}$$

Given $F \in \mathbf{Spt}_{(s,p)}(S)$, one has $\mathbf{R}\Omega_T^\infty(F) = \Omega_T^\infty(\widetilde{F}) = \widetilde{F}_0$, where $F \to \widetilde{F}$ is a stable motivic fibrant replacement of F and $\widetilde{F}_0$ is the motivic spectrum given by $(\widetilde{F})_{*,0} = ((\widetilde{F})_{0,0}, (\widetilde{F})_{1,0}, \cdots)$.

2.3.3. *Cohomology theories.* Finally, given any E and $F \in \mathcal{SH}(S)$, the E-cohomology $E^{a,b}(F)$ of F for $a, b \in \mathbb{Z}$ is defined by $E^{a,b}(F) := [F, \Sigma^{a,b} E]_{\mathbb{A}^1}$, where $\Sigma^{a,b} E = \Sigma_s^{a-b} \Sigma_t^b E$. In particular, if $X \in \mathbf{Sm}_S$, we have $\Sigma_T^\infty X_+ \in \mathcal{SH}(S)$ so that we define

$$E^{a,b}(X) := E^{a,b}(\Sigma_T^\infty X_+) = [\Sigma_T^\infty X_+, \Sigma^{a,b} E]_{\mathbb{A}^1} = [\Sigma_T^\infty \Sigma_s^\infty X_+, \Sigma_s^{a-2b} \Sigma_T^b E]_{\mathbb{A}^1}. \tag{2.3.4}$$

3. Motivic descent for $\mathbb{A}^1$-B.G. presheaves

Recall that a (pre)sheaf of simplicial sets (pointed simplicial sets, spectra) E on $\mathbf{Sm}_S$ is said to have the *B.G. property* (named after Brown and Gersten who first used this concept for the Zariski topology) if every elementary (distinguished) Nisnevich square (see [33, Definition 3.1.5]) in $\mathbf{Sm}_S$

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{j} & X \end{array} \tag{3.0.5}$$

is converted into a homotopy Cartesian square of simplicial sets (pointed simplicial sets, spectra) after applying E. A presheaf with the B.G. property is called a B.G. presheaf. An important technical result in the $\mathbb{A}^1$-homotopy theory of Morel and Voevodsky is the following *Nisnevich descent theorem* ([33, Proposition 3.1.16]):

Theorem 3.0.1. *A (pre)sheaf of simplicial sets (pointed simplicial sets, spectra) E on $\mathbf{Sm}_S$ is B.G. if and only if every Nisnevich fibrant replacement $f : E \to F$ is an object-wise weak-equivalence. A Nisnevich local weak-equivalence $E \to F$ of B.G. (pre)sheaves of simplicial sets (pointed simplicial sets, spectra) is an object-wise weak-equivalence.*

We remark that [33] proved this result for sheaves. The presheaf version and the analogous result for the presheaves of spectra can be found in [19, Theorem 1.3, Corollary 1.4]. This result is important because it gives a necessary and sufficient condition for a Nisnevich fibrant replacement to be an object-wise weak-equivalence. It is a crucial ingredient in many applications in motivic homotopy theory.

3.1. **Motivic descent theorem.** Recall from § 2.2.2 that a motivic fibrant replacement $E \to F$ is a fibrant replacement of E in the motivic model structure. Our goal in this section is to establish a necessary and sufficient condition for a motivic fibrant replacement of a motivic space and a motivic spectrum to be an object-wise weak-equivalence. We shall follow this up with several applications. It turns out that, as we will show, for this purpose the following concept introduced by Morel ([35, Definition A.5]) is most appropriate.

Definition 3.1.1. Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$. We say that

(1) E is $\mathbb{A}^1$-*weak-invariant* if the map $E(X) \to E(X \times \mathbb{A}^1)$, induced by the projection $X \times \mathbb{A}^1 \to X$, is a weak-equivalence for all $X \in \mathbf{Sm}_S$.
(2) E is $\mathbb{A}^1$-*B.G.* if it is B.G. and $\mathbb{A}^1$-weak-invariant.
(3) E is quasi-fibrant (resp. motivic quasi-fibrant) if every Nisnevich fibrant (resp. motivic fibrant) replacement $E \to F$ of E is an object-wise weak-equivalence.

Theorem 3.0.1 shows that E is B.G. if and only if E is quasi-fibrant. We wish to prove its motivic analogue, which we will call the *motivic descent theorem*. We begin with the following series of deductions.

Lemma 3.1.2. *Let $X \in \mathbf{Sm}_S$.*

(1) *If $F \in \mathbf{Spc}_\bullet(S)$ is Nisnevich fibrant, then for $p \geq 0$, we have a bijection*
$$[S^p \wedge X_+, F]_{Nis} \simeq \pi_p(F(X)).$$
If $F \in \mathbf{Spt}(S)$ is Nisnevich fibrant, then for $p \in \mathbb{Z}$, we have a bijection
$$[\Sigma_s^\infty X_+[p], F]_{Nis} \simeq \pi_p(F(X)).$$
(2) *If $F \in \mathbf{Spc}_\bullet(S)$ is motivic fibrant, then $p \geq 0$, we have a bijection*
$$[S^p \wedge X_+, F]_{\mathbb{A}^1} \simeq \pi_p(F(X)).$$
If $F \in \mathbf{Spt}(S)$ is motivic fibrant, then for $p \in \mathbb{Z}$, we have a bijection
$$[\Sigma_s^\infty X_+[p], F]_{\mathbb{A}^1} \simeq \pi_p(F(X)).$$

Proof. For $X \in \mathbf{Sm}_S$, the functors $ev_X : \mathbf{Spc}_\bullet(S) \leftrightarrow \mathbf{Spc}_\bullet : sm_X$ given by $(ev_X : F \mapsto F(X))$ and $(sm_X : K \mapsto K \wedge X_+)$ form a Quillen pair with respect to the Nisnevich local injective model structure and motivic model structure on $\mathbf{Spc}_\bullet(S)$. In particular, their derived functors induce an adjoint pair of functors on the homotopy categories. The first isomorphism of (1) follows immediately from this if $F \in \mathbf{Spc}_\bullet(S)$ is Nisnevich fibrant and the first isomorphism of (2) follows if $F \in \mathbf{Spc}_\bullet(S)$ is motivic fibrant. The second isomorphisms of (1) and (2) follow from the first set of isomorphisms by applying Theorem 2.2.1 and (2.2.1). □

The following result follows immediately from Theorem 3.0.1 and [34, Lemma 4.1.4].

Lemma 3.1.3. *Let E be a B.G. presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$.*

(1) *Let $E \to E'$ be a Nisnevich fibrant replacement. Then E is $\mathbb{A}^1$-weak-invariant if and only if so is E'.*

(2) *E is $\mathbb{A}^1$-weak-invariant if and only if E is $\mathbb{A}^1$-local.*

Lemma 3.1.4. *Let E be an $\mathbb{A}^1$-B.G. presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$. Let $f : E \to F$ be a motivic fibrant replacement. Then f is also a Nisnevich fibrant replacement of E.*

Proof. We consider the case of presheaves of spectra as the other cases are similar. Since F is Nisnevich fibrant (*cf.* § 2.2.2) and since the cofibrations in the motivic model structure are Nisnevich cofibrations, we only have to show that f is a Nisnevich local weak-equivalence.

We factor f as a composition $E \xrightarrow{g} E' \xrightarrow{f'} F$, where g is a Nisnevich trivial cofibration (in particular, motivic trivial cofibration) and f' is a Nisnevich fibration. It follows from the 2-out-of-3 axiom that f' is a motivic weak-equivalence. We need to show that f' is a Nisnevich local weak-equivalence.

Since F is Nisnevich fibrant and f' is a Nisnevich fibration, it follows that E' is Nisnevich fibrant. In particular, g defines a Nisnevich fibrant replacement of E. Moreover, it follows from our assumption and Lemma 3.1.3 that E' is $\mathbb{A}^1$-local. We conclude that E' is motivic fibrant. It follows now from Lemma 3.1.2 that $f' : E' \to F$ is in fact an object-wise weak-equivalence. In particular, f' is a Nisnevich local weak-equivalence. □

Theorem 3.1.5 (Motivic descent theorem)**.** *Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$. Then E is $\mathbb{A}^1$-B.G. if and only if it is motivic quasi-fibrant. Furthermore, a motivic weak-equivalence $E \to F$ of $\mathbb{A}^1$-B.G. presheaves of simplicial sets (pointed simplicial sets, spectra) is an object-wise weak-equivalence.*

Proof. Suppose first that E is a presheaf of spectra which is motivic quasi-fibrant. Let $f : E \to E'$ be a motivic fibrant replacement. Then E' is Nisnevich fibrant (thus B.G.) and $\mathbb{A}^1$-local. It follows from Lemma 3.1.3 that E' is an $\mathbb{A}^1$-B.G. presheaf of spectra. Since E is motivic quasi-fibrant, $f : E \to E'$ is an object-wise weak-equivalence, thus a Nisnevich local weak-equivalence. So, f is a Nisnevich fibrant replacement, which is an object-wise weak-equivalence. So E is B.G. by Theorem 3.0.1, and it is $\mathbb{A}^1$-weak-invariant by Lemma 3.1.3. We conclude that E is $\mathbb{A}^1$-B.G.

Suppose next that E is an $\mathbb{A}^1$-B.G. presheaf of spectra. Let $f : E \to E'$ be a motivic fibrant replacement. It follows from Lemma 3.1.4 that f is also a Nisnevich fibrant replacement. The assertion that f is an object-wise weak-equivalent follows now from Theorem 3.0.1. Thus, E is motivic quasi-fibrant.

To prove the second assertion for a motivic weak-equivalence $f : E \to F$ of $\mathbb{A}^1$-B.G. presheaves, we can form a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ E' & \xrightarrow{f'} & F' \end{array} \tag{3.1.1}$$

where the vertical arrows are motivic fibrant replacements. They are object-wise weak-equivalences by the first part. It follows from the 2-out-of-3 axiom that f' is a motivic weak-equivalence. In this case, we have shown in the proof of Lemma 3.1.4 that f' is an object-wise weak-equivalence. We have just shown that the two vertical arrows are also object-wise weak-equivalences. It follows that f is an object-wise weak-equivalences. This proves the theorem for the presheaves of spectra. The cases of presheaves of (pointed) simplicial sets are proven in a similar way. □

3.2. **Consequences of the motivic descent theorem.** We now list some consequences of Theorem 3.1.5 which will shall use several times in this paper.

Corollary 3.2.1. *The isomorphisms in Lemma 3.1.2(1) hold for all B.G. pointed motivic spaces (spectra). The isomorphisms in Lemma 3.1.2(2) hold for all $\mathbb{A}^1$-B.G. pointed motivic spaces (spectra).*

Corollary 3.2.2. *Let $\{E_i\}$ be a filtered family of $\mathbb{A}^1$-B.G. presheaves of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_S$ indexed by a right filtering category Λ. Then $\mathrm{colim}_{i\in\Lambda}\ E_i$ is motivic quasi-fibrant.*

Proof. This follows directly form Theorem 3.1.5 and the fact that $\mathrm{colim}_{i\in\Lambda}\ E_i$ is $\mathbb{A}^1$-B.G. by the proof of [34, Corollary 4.2.7]. □

Definition 3.2.3. For a motivic spectrum $E = (E_0, E_1, \cdots)$ over S, let $E\{n\}$ denote the motivic spectrum $(E_n, E_{n+1}, \cdots)$. Let $m \geq -1$. We say that E is an *object-wise (resp. motivic) Ω_s-spectrum above level m* if the map $\widetilde{\sigma} : E_n \to \Omega_s E_{n+1}$ is an object-wise (resp. motivic) weak-equivalence for each $n > m$. An object-wise (resp. motivic) Ω_s-spectrum above level $m = -1$ will be called an object-wise (resp. motivic) Ω_s-spectrum.

Corollary 3.2.4. *Let E be an $\mathbb{A}^1$-B.G. motivic spectrum over S, and let $f : E \to F$ be a motivic fibrant replacement. Then the following hold:*

1. *For each $m, n, p \geq 0$, the map $\Omega_s^m F_n \to \Omega_s^{m+p} F_{n+p}$ is an object-wise weak-equivalence.*
2. *For each $m, n \geq 0$, the motivic spectrum $\Omega_s^m F\{n\}$ is S^1-stable motivic fibrant.*
3. *For each $n > m$, the map $E_n \to F_n$ is an object-wise weak-equivalence if E is an object-wise Ω_s-spectrum above level m.*

Proof. Since a motivic spectrum is S^1-stable motivic fibrant if and only if it is level-wise motivic fibrant and a motivic Ω_s-spectrum, it follows that each F_n is a motivic fibrant pointed motivic space. Since S^1 is cofibrant, we see that each $\Omega_s^m F_n$ is also motivic fibrant and the map $F_n \to \Omega_s F_{n+1}$ is a motivic weak-equivalence. In particular, the map $\mathbf{R}\Omega_s^m F_n \to \mathbf{R}\Omega_s^{m+p} F_{n+p}$ is a motivic isomorphism. Since all presheaves $\Omega_s^m F_n$ are motivic fibrant, it follows that the map $\Omega_s^m F_n \to \Omega_s^{m+p} F_{n+p}$ is a motivic weak-equivalence for all $m, n, p \geq 0$. It follows now from Lemma 3.1.2 that this map is in fact an object-wise weak-equivalence. This proves (1).

Since we have just shown that each $\Omega_s^m F_n$ is motivic fibrant and the map $\Omega_s^m F_{n+p} \to \Omega_s^{m+1} F_{n+p+1}$ is a motivic weak-equivalence, it follows that $\Omega_s^m F\{n\}$ is S^1-stable motivic fibrant, proving (2).

For (3), we first apply Theorem 3.1.5 to deduce that $f : E \to F$ is an object-wise stable weak-equivalence. For $n > m, p \geq 0$ and $X \in \mathbf{Sm}_S$, we get now a sequence of isomorphisms

$$\begin{aligned} \pi_p\left(E_n(X)\right) \ &\simeq^1\ \operatorname*{colim}_{q\to\infty} \pi_{p+q}\left(E_{n+q}(X)\right) \ \simeq\ \pi_{p-n}\left(E(X)\right) \\ &\simeq\ \pi_{p-n}\left(F(X)\right) \ \simeq^2\ \pi_p\left(F_n(X)\right), \end{aligned} \tag{3.2.1}$$

where $\simeq^1$ holds because E is an object-wise Ω_s-spectrum above level m, and $\simeq^2$ holds because F is an object-wise Ω_s-spectrum. This completes the proof. □

3.3. **$\mathbb{A}^1$-B.G. property of motivic spaces and motivic spectra.** Our next goal is to characterize the $\mathbb{A}^1$-B.G. property of a motivic spectrum E in terms of the similar property of its levels. What we show below is that two conditions are related when E is a motivic Ω_s-spectrum. The following well known adjoint property of the pointed internal hom functors will be used in what follows.

Lemma 3.3.1. *Given any $E \in \mathbf{Spc}_\bullet(S)$, $K \in \mathbf{Spc}_\bullet$ and $U \in \mathbf{Sm}_S$, there is a canonical bijection of pointed simplicial sets:*

$$\mathcal{H}om_\bullet(K, E)(U) \xrightarrow{\simeq} \mathrm{Hom}_\bullet(K, E(U)).$$

It follows from Lemma 3.3.1 that for any $E \in \mathbf{Spc}_\bullet(S)$ and $U \in \mathbf{Sm}_S$, there is a canonical isomorphism $(\Omega_s E)(U) \simeq \Omega_s(E(U))$. Since $\mathrm{Hom}_\bullet(S^1, -)$ preserves weak-equivalences and fibration sequences in $\mathbf{Spc}_\bullet$, we conclude at once the following.

Corollary 3.3.2. *The functor $\Omega_s(-)$ preserves object-wise weak-equivalences, B.G. property, and $\mathbb{A}^1$-weak-invariance of pointed motivic spaces. It preserves motivic weak-equivalences of $\mathbb{A}^1$-B.G. pointed motivic spaces. If E is an $\mathbb{A}^1$-B.G. pointed motivic space, then natural map $\Omega_s E \to \mathbf{R}\Omega_s E$ is an isomorphism in $\mathcal{H}_\bullet(S)$.*

Proof. The first statement is obvious from Lemma 3.3.1 and the second statement follows from the first and Theorem 3.1.5. To see the last statement, just take a motivic fibrant replacement $E \to E'$, apply the second statement, and use the isomorphism $\Omega_s E' \xrightarrow{\simeq} \mathbf{R}\Omega_s E'$. □

A motivic spectrum E will be called *level-wise* $\mathbb{A}^1$*-B.G.* if each E_n is $\mathbb{A}^1$-B.G.

Corollary 3.3.3. *Let $f : E \to F$ be a level-wise motivic weak-equivalence of level-wise $\mathbb{A}^1$-B.G. motivic spectra. If E is a motivic Ω_s-spectrum, then so is F.*

Proof. This is an immediate consequence of Theorem 3.1.5 and Corollary 3.3.2. □

Lemma 3.3.4. *Let $f : E \to F$ be a morphism of level-wise $\mathbb{A}^1$-B.G. motivic Ω_s-spectra on $\mathbf{Sm}_S$. Then f is an S^1-stable motivic weak-equivalence if and only if each $f_n : E_n \to F_n$ is an object-wise weak-equivalence.*

Proof. Suppose that $f : E \to F$ is an S^1-stable motivic weak-equivalence of level-wise $\mathbb{A}^1$-B.G. motivic Ω_s-spectra. Let us fix $n, p \geq 0$ and $U \in \mathbf{Sm}_S$. Since E and F are level-wise $\mathbb{A}^1$-B.G., we can apply Corollary 3.2.1 to get

$$\begin{array}{rcl} \pi_p(E_n(U)) & \simeq & [S^p \wedge U_+, E_n]_{\mathbb{A}^1} \ \simeq^1 [S^p \wedge U_+, \Omega_s^{m-n} E_m]_{\mathbb{A}^1} \\ & \simeq^2 & [S^p \wedge U_+, \mathbf{R}\Omega_s^{m-n} E_m]_{\mathbb{A}^1} \ \simeq^3 [S^{m+p-n} \wedge U_+, E_m]_{\mathbb{A}^1} \\ & \simeq^4 & \operatorname*{colim}_{m \to \infty} [S^{m+p-n} \wedge U_+, E_m]_{\mathbb{A}^1} \ \simeq^5 [\Sigma_s^\infty U_+[p-n], E]_{\mathbb{A}^1}. \end{array} \tag{3.3.1}$$

The isomorphism $\simeq^1$ in (3.3.1) follows from our assumption that E is a motivic Ω_s-spectrum, $\simeq^2$ follows from Corollary 3.3.2, $\simeq^3$ follows from the adjointness, $\simeq^4$ follows because the integer $m \gg 0$ is arbitrary and $\simeq^5$ follows from (2.2.1). By the same argument, we get $\pi_p(F_n(U)) \simeq [\Sigma_s^\infty U_+[p-n], F]_{\mathbb{A}^1}$. Since f is an S^1-stable motivic weak-equivalence, we conclude that the map $f_n : E_n \to F_n$ is an object-wise weak-equivalence. The other direction in the statement of the lemma is obvious. □

Corollary 3.3.5. *Let E be a level-wise $\mathbb{A}^1$-B.G. motivic Ω_s-spectrum. Then E is motivic quasi-fibrant.*

Proof. Consider an S^1-stable motivic fibrant replacement of E. Since an S^1-stable motivic fibrant motivic spectrum is a level-wise motivic fibrant motivic Ω_s-spectrum, the corollary follows directly from Lemma 3.3.4 and Theorem 3.1.5. □

3.4. **Motivic descent for (s, p)-bispectra.** Given an open or a closed immersion of schemes $A \subseteq B$ in $\mathbf{Sm}_S$, let $\Omega_{B/A}(-)$ denote the functor $E \mapsto \Omega_{B/A} E = (\Omega_{B/A} E_0, \Omega_{B/A} E_1, \cdots)$ on $\mathbf{Spt}(S)$, where $\Omega_{B/A} F = \mathcal{H}om_\bullet(B/A, F)$ is the object-wise fiber of the map $\mathcal{H}om(B, F) \to$

$\mathcal{H}om(A, F)$ (*cf.* [19, Corollary 1.10]) for $F \in \mathbf{Spc}_\bullet(S)$. In particular, there is an object-wise fiber sequence of presheaves of spectra

$$\Omega_{B/A}E \to E_B \to E_A, \tag{3.4.1}$$

where $E_B(X) := E(B \times X) = \mathcal{H}om(B, E)(X)$. Recall (*cf.* [19, Corollary 3.2]) that given an object-wise fiber sequence of presheaves of spectra as in (3.4.1), the map $E_B/(\Omega_{B/A}E) \to E_A$ is an object-wise stable weak-equivalence.

The natural isomorphism $S^1 \wedge E_X \to (S^1 \wedge E)_X$ for any $E \in \mathbf{Spc}_\bullet(S)$ and any $X \in \mathbf{Sm}_S$ together with (3.4.1) show that there is a natural map $S^1 \wedge \Omega_{B/A}E_n \to \Omega_{B/A}(S^1 \wedge E_n)$ for any $E \in \mathbf{Spt}(S)$ and $n \geq 0$. Composing this map with the bonding map $\Omega_{B/A}(S^1 \wedge E_n) \to \Omega_{B/A}(E_{n+1})$, we see that $E \to \Omega_{B/A}E$ is an endo-functor on $\mathbf{Spt}(S)$. Moreover, there is a natural bijection $\mathrm{Hom}_{\mathbf{Spt}(S)}(\Sigma_{B/A}E, F) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Spt}(S)}(E, \Omega_{B/A}F)$.

The following analogue of Corollary 3.3.2 for motivic spectra is immediate from Theorem 3.1.5 and (3.4.1).

Lemma 3.4.1. *The functor $\Omega_{B/A}(-)$ preserves object-wise weak-equivalences, B.G. property, and $\mathbb{A}^1$-weak-invariance of motivic spectra. It preserves motivic weak-equivalences of $\mathbb{A}^1$-B.G. motivic spectra. If E is an $\mathbb{A}^1$-B.G. motivic spectrum, then the natural map $\Omega_{B/A}E \to \mathbf{R}\Omega_{B/A}E$ is an isomorphism in $\mathcal{SH}_{S^1}(S)$.*

If $f : E \to F$ is an S^1-stable motivic weak-equivalence of $\mathbb{A}^1$-B.G. motivic spectra, then $\Omega_{B/A}f : \Omega_{B/A}E \to \Omega_{B/A}F$ is also an S^1-stable motivic weak-equivalence.

Recall from § 2.3.1 that an (s,p)-bispectrum $E = (E_{m,n})_{m,n\geq 0}$ can be regarded as a sequence of motivic spectra $E = (E_0, E_1, \cdots)$ with the bonding maps $\tau : \Sigma_T E_n = T \wedge E_n \to E_{n+1}$ for each $n \geq 0$.

Definition 3.4.2. For an (s,p)-bispectrum E, we say that

1. E is a *motivic Ω_T-bispectrum* if the adjoint of the bonding maps, $\widetilde{\tau} : E_n \to \Omega_T E_{n+1}$ are motivic weak-equivalences of motivic spectra over S for all $n \geq 0$.
2. E is $\mathbb{A}^1$-*B.G.* if each E_n is an $\mathbb{A}^1$-B.G. motivic spectrum for all $n \geq 0$.

Theorem 3.4.3. *Let $f : E \to F$ be a stable motivic weak-equivalence of $\mathbb{A}^1$-B.G. motivic Ω_T-bispectra on $\mathbf{Sm}_S$. Then f is a T-level-wise object-wise weak-equivalence, i.e., each $f_n : E_n \to F_n$ is an object-wise weak-equivalence.*

Proof. Suppose that $f : E \to F$ is a morphism of $\mathbb{A}^1$-B.G. motivic Ω_T-bispectra on $\mathbf{Sm}_S$ which is stable motivic weak-equivalence. Let us fix $n \geq 0, p \in \mathbb{Z}$ and $U \in \mathbf{Sm}_S$. Since E is (T-level-wise) $\mathbb{A}^1$-B.G., we can apply Corollary 3.2.1 to get

$$\begin{aligned} \pi_p(E_n)(U) &\simeq [\Sigma_s^\infty U_+[p], E_n]_{\mathbb{A}^1} \simeq^1 [\Sigma_s^\infty U_+[p], \Omega_T^{m-n}E_m]_{\mathbb{A}^1} \\ &\simeq^2 [\Sigma_s^\infty U_+[p], \mathbf{R}\Omega_T^{m-n}E_m]_{\mathbb{A}^1} \simeq^3 [\Sigma_T^{m-n}\Sigma_s^{p+n-n}\Sigma_s^\infty U_+, E_m]_{\mathbb{A}^1} \\ &\simeq^4 \operatorname*{colim}_{m\to\infty} [\Sigma_T^{m-n}\Sigma_s^{p+n-n}\Sigma_s^\infty U_+, E_m]_{\mathbb{A}^1} \simeq^5 \pi_{p-2n,-n}(E)(U). \end{aligned} \tag{3.4.2}$$

The isomorphism $\simeq^1$ above follows from our assumption that E is a motivic Ω_T-bispectrum, $\simeq^2$ follows from Lemma 3.4.1, $\simeq^3$ follows from the adjointness, $\simeq^4$ follows because the integer $m \gg 0$ is arbitrary and $\simeq^5$ follows from (2.3.1). By the same argument, we get $\pi_p(F_n(U)) \simeq \pi_{p-n,-n}(F)(U)$.

It follows from our assumption and (2.3.1) that the map $\pi_p(E_n) \to \pi_p(F_n)$ induces an isomorphisms between the associated Nisnevich sheaves for all $p \in \mathbb{Z}$. In particular, $f_n : E_n \to F_n$ is a Nisnevich local weak-equivalence, and hence an S^1-stable motivic weak-equivalence. Since these are $\mathbb{A}^1$-B.G. motivic spectra, it follows from Theorem 3.1.5 that f_n is an object-wise weak-equivalence. □

Corollary 3.4.4. *Let E be an (s,p)-bispectrum over S and let $f : E \to E'$ be a stable motivic fibrant replacement. Then, E is an $\mathbb{A}^1$-B.G. motivic Ω_T-bispectrum if and only if f is a T-level-wise object-wise weak-equivalence.*

Proof. The forward direction is obvious by Theorem 3.4.3. For the backward direction, note that each level $E_n \to E'_n$ is an object-wise weak-equivalence, with E'_n is motivic fibrant, so that each E_n is $\mathbb{A}^1$-B.G. by Theorem 3.1.5. It only remains to see that E is a motivic Ω_T-bispectrum. This is an easy consequence of Lemma 3.4.1. $\square$

4. Singular semi-topologization

For the rest of this paper, our base scheme is $S = \mathrm{Spec}\,(\mathbb{C})$. For a complex algebraic variety U, we shall denote the associated complex analytic space by U^{an}. This is the space $U(\mathbb{C})$ with the complex analytic topology. In this section, we recall the singular semi-topologization (=**sst**) functor of Friedlander-Walker from [13, Definition 10].

4.1. **Definition and basic properties.**

We recall the **sst**-process by Friedlander-Walker, that transforms a presheaf on $\mathbf{Sch}_{\mathbb{C}}$ into another presheaf on $\mathbf{Sch}_{\mathbb{C}}$.

4.1.1. *Simplicial category.* Let T be a topological space, and let $\mathbf{Var}_{\mathbb{C}}$ be the category of complex varieties. Let $(T|\mathbf{Var}_{\mathbb{C}})$ be the category whose objects are the pairs (f, U), where $U \in \mathbf{Var}_{\mathbb{C}}$ and $f : T \to U^{an}$ is a continuous map. A morphism from (f, U) to (g, V) is a morphism $h : U \to V$ of varieties such that the induced continuous map $h^{an} : U^{an} \to V^{an}$ satisfies $h^{an} \circ f = g$. One checks that $(T|\mathbf{Var}_{\mathbb{C}})$ is a small cofiltered category.

We are particularly interested in the case when T is the topological n-simplex Δ^n_{top} for $n \geq 0$. Recall that $\Delta^\bullet_{\mathrm{top}} = \{\Delta^n_{\mathrm{top}}\}_{n\geq 0}$ is a cosimplicial space with the natural cofaces ∂^i and the codegeneracies s^i.

Definition 4.1.1. For $n > 0$ and $0 \leq i \leq n$, define $\partial_i : (\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}} \to (\Delta^{n-1}_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$ by

$$(\Delta^n_{\mathrm{top}} \xrightarrow{f} U^{an}) \mapsto (\Delta^{n-1}_{\mathrm{top}} \xrightarrow{\partial^i} \Delta^n_{\mathrm{top}} \xrightarrow{f} U^{an}).$$

For $n \geq 0$ and $0 \leq i \leq n$, define $s_i : (\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}} \to (\Delta^{n+1}_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$ by

$$(\Delta^n_{\mathrm{top}} \xrightarrow{f} U^{an}) \mapsto (\Delta^{n+1}_{\mathrm{top}} \xrightarrow{s^i} \Delta^n_{\mathrm{top}} \xrightarrow{f} U^{an}).$$

Using the cosimplicial identities of $\Delta^\bullet_{\mathrm{top}}$ involving ∂^i and s^i, one checks easily that $(\Delta^\bullet_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$ is a simplicial category.

4.1.2. *Realization and diagonal spectra associated to a simplicial spectrum.* We briefly review the diagonal and the realization of a simplicial spectrum, *i.e.*, a simplicial object in **Spt**. Recall first that for a bisimplicial set A_{**}, the *realization* $|A|$ is the simplicial set obtained by taking the coequalizer of the diagram

$$\coprod_{(\alpha:[n]\to[k])\in\Delta^{\mathrm{op}}} A_n \times \Delta[k] \quad \rightrightarrows \quad \coprod_{n\geq 0} A_n \times \Delta[n], \tag{4.1.1}$$

where the components of the two morphisms are $(\alpha, x, t) \mapsto (x, \alpha^*(t))$ and $(\alpha, x, t) \mapsto (\alpha_*(x), t)$. If A_{**} is a simplicial object in the category of pointed simplicial sets, then $|A|$ is obtained by replacing $A_n \times \Delta[k]$ by $A_n \wedge (\Delta[k])_+$ in (4.1.1). The *diagonal* $\mathrm{diag}\, A$ is the composite $\Delta^{\mathrm{op}} \xrightarrow{\delta} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{A_{**}} \mathbf{Set}$. It is known ([2, Proposition B.1]) that there is a natural isomorphism of simplicial sets $\mathrm{diag}\, A \to |A|$.

If $E : \Delta^{\rm op} \to \mathbf{Spt}$ is a simplicial spectrum, then its realization spectrum $|E|$ is defined exactly as in (4.1.1), where $A_n \times \Delta[k]$ is replaced by $E(\Delta[n]) \wedge (\Delta[k])_+$. A simplicial spectrum E is also a spectrum object in the category of pointed bisimplicial sets, *i.e.*, E is a sequence $(E^0_{**}, E^1_{**}, \cdots)$, where each E^n_{**} is a pointed bisimplicial set and there are bonding maps $S^1 \wedge E^n_{**} \to E^{n+1}_{**}$. In particular, the spectrum $|E|$ is a spectrum such that $|E|_n$ is the pointed simplicial set given by $|E^n_{**}|$ and the induced bonding maps $S^1 \wedge |E^n_{**}| \to |E^{n+1}_{**}|$.

The diagonal of E is defined as the spectrum $\operatorname{diag} E$ such that $(\operatorname{diag} E)_n = \operatorname{diag}(E^n_{**})$. We have the map $S^1 \wedge E^n_p \to E^{n+1}_p$ for each $n, p \geq 0$, where $E^n_p = (E(\Delta[p]))_n$. This is equivalent to the maps of pointed sets $(S^1)_i \wedge E^n_{p,i} \to E^{n+1}_{p,i}$ for $i, n, p \geq 0$. Hence, we get the maps $(S^1)_p \wedge E^n_{p,p} \to E^{n+1}_{p,p}$ and these give rise to the bonding maps $S^1 \wedge (\operatorname{diag} E)_n \to (\operatorname{diag} E)_{n+1}$ of the spectrum $\operatorname{diag} E$. It follows from the case of bisimplicial sets that for a simplicial spectrum E, there is a natural isomorphism $\operatorname{diag} E \xrightarrow{\simeq} |E|$.

If E is a presheaf of simplicial spectra on $\mathbf{Sch}_S$ or $\mathbf{Sm}_S$, we define its realization and the diagonal presheaves of spectra by defining these presheaves object-wise. We conclude that for a simplicial presheaf of spectra E on $\mathbf{Sch}_S$ or $\mathbf{Sm}_S$, the map $\operatorname{diag} E \xrightarrow{\simeq} |E|$ is an isomorphism of presheaves of spectra. Thus, from now on, we won't distinguish the diagonal and the realization for simplicial presheaves of spectra.

4.1.3. *Semi-topologization.*

Definition 4.1.2 (Friedlander-Walker [13])**.** Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra, complexes of abelian groups) on $\mathbf{Sch}_{\mathbb{C}}$. Let $X \in \mathbf{Sch}_{\mathbb{C}}$ and let T be a topological space. Define $E(T \times X) := \operatorname{colim}_{(f,U) \in (T|\mathbf{Var}_{\mathbb{C}})^{\rm op}} E(U \times X)$.

Let $E(\Delta^\bullet_{\rm top} \times X)$ be the bisimplicial set (simplicial pointed simplicial set, simplicial spectrum, simplicial object in the category of complexes of abelian groups) $\{E(\Delta^n_{\rm top} \times X)\}_{n \geq 0}$. Let $E^{\mathbf{sst}}(X)$ be the realization $|E(\Delta^\bullet_{\rm top} \times X)|$ (or the diagonal), which is a simplicial set (pointed simplicial set, spectrum, complex of abelian groups). This $E^{\mathbf{sst}}$ is a presheaf of simplicial sets (pointed simplicial sets, spectra, complexes of abelian groups) on $\mathbf{Sch}_{\mathbb{C}}$. We call it *the (singular) semi-topologization of E.*

There is a natural morphism of presheaves $\tau_E : E \to E^{\mathbf{sst}}$ on $\mathbf{Sch}_{\mathbb{C}}$ and this defines a natural transformation $\tau : \mathrm{Id} \to (-)^{\mathbf{sst}}$ of functors on presheaves on $\mathbf{Sch}_{\mathbb{C}}$.

The following elementary result about the semi-topologization for presheaves on $\mathbf{Sch}_{\mathbb{C}}$ will be used often in this text.

Lemma 4.1.3. *Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sch}_{\mathbb{C}}$ and let $X \in \mathbf{Sch}_{\mathbb{C}}$. Define a presheaf E_X on $\mathbf{Sch}_{\mathbb{C}}$ by $E_X(U) := E(U \times X)$. Then, there is a natural identification $(E_X)^{\mathbf{sst}} = (E^{\mathbf{sst}})_X$.*

Proof. It follows from the definitions that for $U \in \mathbf{Sch}_{\mathbb{C}}$,

$$\begin{array}{rcl} (E^{\mathbf{sst}})_X(U) & = & E^{\mathbf{sst}}(X \times U) \;\; = \;\; |\big\{E(\Delta^n_{\rm top} \times X \times U)\big\}_{n\geq 0}| \\ & = & |\left\{ \operatorname*{colim}_{(f,C) \in \mathcal{I}^{\rm op}_n} E(C \times X \times U) \right\}_{n \geq 0}| \;\; = \;\; |\left\{ \operatorname*{colim}_{(f,C) \in \mathcal{I}^{\rm op}_n} E_X(C \times U) \right\}_{n \geq 0}| \\ & = & |\big\{E_X(\Delta^n_{\rm top} \times U)\big\}_{n \geq 0}| \;\; = \;\; (E_X)^{\mathbf{sst}}(U). \end{array}$$

□

4.2. **$\mathbb{A}^1$-weak-invariance of semi-topologization.** When E is a presheaf on $\mathbf{Sm}_{\mathbb{C}}$, it is well-known that the realization of $E(\Delta^\bullet \times -)$ is $\mathbb{A}^1$-weak-invariant, where Δ^n is the algebraic n-simplex. (*cf.* [7, Proposition 7.2], [8, p. 150]). In this subsection we shall prove the $\mathbb{A}^1$-weak-invariance of its semi-topological analogue.

Theorem 4.2.1 ($\mathbb{A}^1$-weak-invariance)**.** *Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sch}_{\mathbb{C}}$. Let $X \in \mathbf{Sch}_{\mathbb{C}}$ and let $\pi : \mathbb{A}^1 \times X \to X$ be the projection. Then $\pi^* : E^{\mathbf{sst}}(X) \to E^{\mathbf{sst}}(\mathbb{A}^1 \times X)$ is a weak-equivalence of simplicial sets (pointed simplicial sets, spectra). In other words, $E^{\mathbf{sst}}$ is $\mathbb{A}^1$-weak-invariant.*

Proof. **Step 1:** We first show that the two morphisms $E^{\mathbf{sst}}(\mathbb{A}^1 \times X) \rightrightarrows E^{\mathbf{sst}}(X)$ induced by the maps $i_0 \times \mathrm{Id}_X, i_1 \times \mathrm{Id}_X : X \rightrightarrows \mathbb{A}^1 \times X$ are simplicially homotopic and they induce the same homomorphisms in homotopy groups, where $i_0, i_1 : \mathrm{Spec}\,(\mathbb{C}) \to \mathbb{A}^1$ map the singleton to the points $\{0\}, \{1\}$. Its proof is almost identical to that of [7, Lemma 7.1], except that we need to replace Δ^n suitably by Δ^n_{top}.

Given any $n \geq 0$ and a continuous map $h : \Delta^n_{\mathrm{top}} \to U^{an}$ in $(\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$, let h^* denote the canonical map $E(U \times X) \to E(\Delta^n_{\mathrm{top}} \times X)$. Let $I_\bullet$ be the simplicial set corresponding to the poset $\{0 < 1\}$, *i.e.*, n-simplices are $I_\bullet$ are nondecreasing sequences $j_0 \leq j_1 \leq \cdots \leq j_n$ with $j_k \in \{0 < 1\}$. Note that $I = |I_\bullet| = [0, 1]$. Choose any continous function $g : I \to (\mathbb{A}^1)^{an}$ that sends the end points of I to the points $\{0\}, \{1\} \in \mathbb{A}^1$, respectively. We have an explicit simplicial homotopy

$$H : I \times |E(\Delta^\bullet_{\mathrm{top}} \times \mathbb{A}^1 \times X)| = |I_\bullet \times E(\Delta^\bullet_{\mathrm{top}} \times \mathbb{A}^1 \times X)| \to |E(\Delta^\bullet_{\mathrm{top}} \times X)|$$

defined as follows: let $j \in I_n$ and let $m \in E(\Delta^n_{\mathrm{top}} \times \mathbb{A}^1 \times X)$ be represented by a class $m_h \in E(U \times \mathbb{A}^1 \times X)$ under a continuous map $h : \Delta^n_{\mathrm{top}} \to U^{an}$ in $(\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$. We write $m = h^*(m_h)$ for simplicity. Consider $f_j : \Delta^n_{\mathrm{top}} \to \Delta^n_{\mathrm{top}} \times I$ defined to be the linear morphism sending the k-th vertex $v_k \in \Delta^n_{\mathrm{top}}$ to $v_k \times j_k \in \Delta^n_{\mathrm{top}} \times I$. Define the continuous composition $h_j : \Delta^n_{\mathrm{top}} \xrightarrow{f_j} \Delta^n_{\mathrm{top}} \times I \xrightarrow{h \times g} U^{an} \times (\mathbb{A}^1)^{an} = (U \times \mathbb{A}^1)^{an}$. We define $H_n(j \times m) \in E(\Delta^n_{\mathrm{top}} \times X)$ to be the class of $h_j^*(m_h)$. Then $\{H_n\}$ gives the desired simplicial homotopy.

Step 2: We now show that the map $\pi^* : E^{\mathbf{sst}}(X) \to E^{\mathbf{sst}}(\mathbb{A}^1 \times X)$ given by the projection $\pi : \mathbb{A}^1 \times X \to X$ is a weak-equivalence. This part of the proof is almost identical to that of [7, Proposition 7.2].

The composition $(i_0 \times \mathrm{Id}_X)^* \circ \pi^* : E^{\mathbf{sst}}(X) \to E^{\mathbf{sst}}(\mathbb{A}^1 \times X) \to E^{\mathbf{sst}}(X)$ is the identity on $E^{\mathbf{sst}}(X)$. For the composition in the opposite order, we have:

Claim. *$\pi^* \circ (i_0 \times \mathrm{Id}_X)^*$ is simplicially homotopic to the identity map on $E^{\mathbf{sst}}(\mathbb{A}^1 \times X)$.*

Let $m : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the morphism given by $(a, b) \mapsto ab$. Let $m_X := m \times \mathrm{Id}_X$. One sees that for the compositions (where $l = 0$ and 1),

$$m_X \circ (i_l \times \mathrm{Id}_{\mathbb{A}^1 \times X}) : \mathbb{A}^1 \times X \rightrightarrows \mathbb{A}^1 \times \mathbb{A}^1 \times X \to \mathbb{A}^1 \times X,$$

we have the identities

$$\begin{cases} m_X \circ (i_0 \times \mathrm{Id}_{\mathbb{A}^1 \times X}) = (i_0 \times \mathrm{Id}_X) \circ \pi, \\ m_X \circ (i_1 \times \mathrm{Id}_{\mathbb{A}^1 \times X}) = \mathrm{Id}_{\mathbb{A}^1 \times X}, \end{cases}$$

which imply that

$$\begin{cases} (i_0 \times \mathrm{Id}_{\mathbb{A}^1 \times X})^* \circ m_X^* = \pi^* \circ (i_0 \times \mathrm{Id}_X)^*, \\ (i_1 \times \mathrm{Id}_{\mathbb{A}^1 \times X})^* \circ m_X^* = \mathrm{Id}_{E^{\mathbf{sst}}(\mathbb{A}^1 \times X)}. \end{cases}$$

But, by **Step 1** (with X replaced by $\mathbb{A}^1 \times X$), the maps $(i_l \times \mathrm{Id}_{\mathbb{A}^1 \times X})^* : E^{\mathbf{sst}}(\mathbb{A}^1 \times \mathbb{A}^1 \times X) \to E^{\mathbf{sst}}(\mathbb{A}^1 \times X)$ for $l = 0, 1$ are simplicially homotopic. Hence $\pi^* \circ (i_0 \times \mathrm{Id}_X)^*$ and $\mathrm{Id}_{E^{\mathbf{sst}}(\mathbb{A}^1 \times X)}$ are simplicially homotopic by [15, Corollary 9.5.4], proving the Claim. This shows that π^* is a simplicial homotopy equivalence (in the sense of [15, Definition 9.5.8]) so that by [15, Proposition 9.5.16], the map π^* is a weak-equivalence. $\square$

4.3. **Semi-topologization and $\wedge$-product.** In this section, we study the behavior of the **sst**-process with respect to various operations related to the $\wedge$-product. This is used later in the study of semi-topologization of T-spectra and (s, p)-bispectra.

Recall that for two pointed simplicial sets A and B (all base points are denoted by $\star$), their wedge product is defined as

$$A \wedge B := (A \times B)/(A \vee B),$$

where the join $A \vee B$ is the union $(A \times \star) \cup (\star \times B)$ in $A \times B$.

For two presheaves E and F of pointed simplicial sets on a category $\mathcal{C}$, the presheaf $E \wedge F$ on $\mathcal{C}$ is defined object-wise as $(E \wedge F)(U) = E(U) \wedge F(U)$. In particular, one still has the formula $E \wedge F = (E \times F)/(E \vee F)$. When F is a presheaf of spectra on $\mathbf{Sch}_{\mathbb{C}}$ while E is as above, we can define $E \wedge F$ level-wise, namely, $E \wedge F = (E \wedge F_0, E \wedge F_1, \cdots)$.

Proposition 4.3.1. *Let E, F, F' be presheaves of pointed simplicial sets on $\mathbf{Sch}_{\mathbb{C}}$. Then we have the following identities, where* (1), (3) *are identities in the category of presheaves of (unpointed) simplicial sets on $\mathbf{Sch}_{\mathbb{C}}$.*

1. $(E \times F)^{\mathbf{sst}} = E^{\mathbf{sst}} \times F^{\mathbf{sst}}$.
2. $(E \vee F)^{\mathbf{sst}} = E^{\mathbf{sst}} \vee F^{\mathbf{sst}}$.
3. *If $F \subset F'$, then $(F'/F)^{\mathbf{sst}} = F'^{\mathbf{sst}}/F^{\mathbf{sst}}$.*
4. $(E \wedge F)^{\mathbf{sst}} = E^{\mathbf{sst}} \wedge F^{\mathbf{sst}}$.
5. *When E is as above and F is a presheaf of spectra on $\mathbf{Sch}_{\mathbb{C}}$, we have $(E \wedge F)^{\mathbf{sst}} = E^{\mathbf{sst}} \wedge F^{\mathbf{sst}}$.*

Proof. Let $X \in \mathbf{Sch}_{\mathbb{C}}$ be a fixed scheme. For (1), recall that the product $E \times F$ is defined object-wise as $(E \times F)(U) := E(U) \times F(U)$ for each $U \in \mathbf{Sch}_{\mathbb{C}}$. In particular, $(E \times F)(U \times X) = E(U \times X) \times F(U \times X)$.

Now, over the objects $(f : \Delta^n_{\text{top}} \to U^{an})$ of the filtered category $(\Delta^n_{\text{top}} | \mathbf{Var}_{\mathbb{C}})^{\text{op}}$, we take the filtered colimit. Note that products are finite limits, and by [29, § IX.2 Theorem 1] finite limits commute with filtered colimits. Thus, $(E \times F)(\Delta^n_{\text{top}} \times X) = E(\Delta^n_{\text{top}} \times X) \times F(\Delta^n_{\text{top}} \times X)$. Taking the diagonals, this immediately implies (1).

For (2), a similar argument works. Namely, for each $U \in \mathbf{Sch}_{\mathbb{C}}$, we have a push-out (a colimit) diagram

$$\begin{array}{ccc} \star \times \star & \longrightarrow & \star \times F(U \times X) \\ \big\downarrow & & \big\downarrow \\ E(U \times X) \times \star & \longrightarrow & (E \vee F)(U \times X) \end{array}$$

and we take the filtered colimits over the objects $(f : \Delta^n_{\text{top}} \to U^{an})$ of $(\Delta^n_{\text{top}} | \mathbf{Var}_{\mathbb{C}})^{\text{op}}$. Colimits commute among themselves (by Fubini theorem for colimits, see [29, § IX.8]) so that we deduce that $(E \vee F)(\Delta^n_{\text{top}} \times X) = E(\Delta^n_{\text{top}} \times X) \vee F(\Delta^n_{\text{top}} \times X)$. This implies (2), by taking the diagonals.

For (3), for presheaves $F \subset F'$, the quotient presheaf F'/F satisfies $(F'/F)(U) = F'(U)/F(U)$ for $U \in \mathbf{Sch}_{\mathbb{C}}$. In particular, $F'(U \times X)/F(U \times X)$ is the push-out

$$\begin{array}{ccc} F(U \times X) & \longrightarrow & F'(U \times X) \\ \big\downarrow & & \big\downarrow \\ \star & \longrightarrow & F'(U \times X)/F(U \times X). \end{array}$$

Taking the filtered colimits over $(\Delta^n_{\text{top}} | \mathbf{Var}_{\mathbb{C}})^{\text{op}}$ and then the diagonals, we deduce (3). Now, (4) follows from (1) - (3)

For (5), since the limits and colimits of spectra are all defined level-wise, this part follows from (4). This completes the proof. $\square$

5. Semi-topologization of presheaves on smooth schemes

In this section, we discuss how one can define semi-topologization for presheaves defined on $\mathbf{Sm}_{\mathbb{C}}$ and study its properties. Some of the properties are analogous to those of the semi-topologization functor on presheaves on $\mathbf{Sch}_{\mathbb{C}}$.

5.1. **Presheaves on smooth schemes.** For a presheaf E of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sch}_{\mathbb{C}}$, we used the categories $(\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$ to define $E^{\mathbf{sst}}$. If E is defined only on $\mathbf{Sm}_{\mathbb{C}}$ *a priori*, then one may want either to extend the functor F to all of $\mathbf{Sch}_{\mathbb{C}}$, or to shrink the indexing categories to, say, $(\Delta^n_{\mathrm{top}}|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}$.

Both raise some issues. Extension of a functor from $\mathbf{Sm}_{\mathbb{C}}$ to $\mathbf{Sch}_{\mathbb{C}}$ is not unique so that one may face a well-definedness problem. On the other hand, the inclusion $(\Delta^n_{\mathrm{top}}|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}) \hookrightarrow (\Delta^n_{\mathrm{top}}|\mathbf{Var}_{\mathbb{C}})^{\mathrm{op}}$ is not cofinal. Furthermore, the indexing categories $(\Delta^n_{\mathrm{top}}|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}$ are not filtered, so that the colimits over them may have poorer properties.

To avoid such problems, we shall use only a particular sort of functorial extension process (that we call the *artificial extension*) to obtain a presheaf on $\mathbf{Sch}_{\mathbb{C}}$, and then we apply the $\mathbf{sst}$-process as defined in Section 4. More generally, we prove that there is a functorial way to extend presheaves on $\mathbf{Sm}_{\mathbb{C}}$ of objects in a cocomplete category $\mathcal{M}$ to presheaves on $\mathbf{Sch}_{\mathbb{C}}$.

Definition 5.1.1. Let $W \in \mathbf{Sch}_{\mathbb{C}}$. Consider objects of the form (f, X), where $X \in \mathbf{Sm}_{\mathbb{C}}$ and $f : W \to X$ is a morphism of $\mathbb{C}$-schemes. Given two pairs (f, X) and (g, Y), with $X, Y \in \mathbf{Sm}_{\mathbb{C}}$, a morphism ψ from (f, X) to (g, Y) is defined to be a morphism of $\mathbb{C}$-schemes $\psi : X \to Y$ such that $\psi \circ f = g$. Let $(W|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}$ be the category of equivalence classes of the pairs (f, X) with the above morphisms. This is a small category.

Definition 5.1.2. Let E be a presheaf on $\mathbf{Sm}_{\mathbb{C}}$ of objects in a cocomplete category $\mathcal{M}$. For $W \in \mathbf{Sch}_{\mathbb{C}}$, define *the artificial extension* $\bar{E}$ of E by

$$\bar{E}(W) := \operatorname*{colim}_{(f,X)\in(W|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}} E(X).$$

Note that if $W \in \mathbf{Sm}_{\mathbb{C}}$, then $(W|\mathbf{Sm}_{\mathbb{C}})^{\mathrm{op}}$ has the terminal object (Id_W, W) so that $\bar{E}(W) = E(W)$. It is easy to verify that given $\phi : W \to W'$ in $\mathbf{Sch}_{\mathbb{C}}$, the assignment $(f : W' \to X) \mapsto (f \circ \phi : W \to X)$ makes $\bar{E}$ a presheaf on $\mathbf{Sch}_{\mathbb{C}}$ of objects in $\mathcal{M}$. One verifies that the artificial extension process defines a functor on the category of functors

$$\mathbf{ext} : \mathrm{Funct}(\mathbf{Sm}^{\mathrm{op}}_{\mathbb{C}}, \mathcal{M}) \to \mathrm{Funct}(\mathbf{Sch}^{\mathrm{op}}_{\mathbb{C}}, \mathcal{M}). \tag{5.1.1}$$

In the opposite direction, we also have the restriction functor $\mathbf{rest} : \mathrm{Funct}(\mathbf{Sch}^{\mathrm{op}}_{\mathbb{C}}, \mathcal{M}) \to \mathrm{Funct}(\mathbf{Sm}^{\mathrm{op}}_{\mathbb{C}}, \mathcal{M})$ and it is clear that $\mathbf{rest} \circ \mathbf{ext} = \mathrm{Id}$. However, the reader should be warned that there is a natural transformation $\mathbf{ext} \circ \mathbf{rest} \to \mathrm{Id}$, which is far from being an isomorphism in general.

Definition 5.1.3. Let E be a presheaf of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_{\mathbb{C}}$. We define its *semi-topologization* as the presheaf $(\mathbf{ext}(E))^{\mathbf{sst}}|_{\mathbf{Sm}_{\mathbb{C}}} = \bar{E}^{\mathbf{sst}}|_{\mathbf{Sm}_{\mathbb{C}}} = \mathbf{rest} \circ \mathbf{sst} \circ \mathbf{ext}(E)$ on $\mathbf{Sm}_{\mathbb{C}}$. The resulting presheaf will be denoted by $E^{\mathbf{sst}}$.

Immediately from Theorem 4.2.1, we get the following:

Proposition 5.1.4. *Let E be a presheaf of spectra on $\mathbf{Sm}_{\mathbb{C}}$. Then, $E^{\mathbf{sst}}$ is $\mathbb{A}^1$-weak-invariant.*

5.2. **The recognition principle.** The semi-topologization of presheaves of simplicial sets (pointed simplicial sets, spectra) on $\mathbf{Sm}_{\mathbb{C}}$ defines a natural transformation of functors $\mathrm{Id} \to (-)^{\mathbf{sst}}$. In order to deal with the semi-topologization for presheaves of spectra on $\mathbf{Sm}_{\mathbb{C}}$, the following *recognition principle* of Friedlander-Walker in [13, Theorem 11] is a very important technical tool. We recall it in the form we use:

Theorem 5.2.1 (Friedlander-Walker recognition principle)**.** *Let E and F be presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$ and let $f : E \to F$ be a morphism of presheaves, which is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. Then $|E(\Delta^\bullet_{\rm top})| \to |F(\Delta^\bullet_{\rm top})|$ is a weak-equivalence.*

The recognition principle will be often used in this paper in the following form.

Theorem 5.2.2.
(1) *If $f : E \to F$ is a morphism of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$, which is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$, then $f^{\mathbf{sst}} : E^{\mathbf{sst}} \to F^{\mathbf{sst}}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$.*
(2) *If $f : E \to F$ is a morphism of presheaves of spectra on $\mathbf{Sm}_{\mathbb{C}}$, which is an object-wise weak-equivalence, then $f^{\mathbf{sst}} : E^{\mathbf{sst}} \to F^{\mathbf{sst}}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$.*

Proof. We first prove (1). It follows from our assumption that for any $X \in \mathbf{Sm}_{\mathbb{C}}$, the map $E_X \to F_X$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. It follows from Theorem 5.2.1 that the map $(E_X)^{\mathbf{sst}}(\mathrm{Spec}\,(\mathbb{C})) \to (F_X)^{\mathbf{sst}}(\mathrm{Spec}\,(\mathbb{C}))$ is a weak-equivalence. We can now apply Lemma 4.1.3 to conclude that the map $E^{\mathbf{sst}}(X) \to F^{\mathbf{sst}}(X)$ is a weak-equivalence.

To prove (2), notice that our assumption implies that the map $\bar{E} \to \bar{F}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. It follows from (1) that the map $\bar{E}^{\mathbf{sst}}(X) \to \bar{F}^{\mathbf{sst}}(X)$ is a weak-equivalence for $X \in \mathbf{Sm}_{\mathbb{C}}$. But this is equivalent to saying that the map $E^{\mathbf{sst}}(X) \to F^{\mathbf{sst}}(X)$ is a weak-equivalence. □

Remark 5.2.3. We remark that if E is a presheaf of spectra on $\mathbf{Sch}_{\mathbb{C}}$ (or on $\mathbf{Sm}_{\mathbb{C}}$) that is already $\mathbb{A}^1$-weak-invariant on $\mathbf{Sm}_{\mathbb{C}}$, then $\mathbb{A}^1$-weak-invariance of $E^{\mathbf{sst}}$ on $\mathbf{Sm}_{\mathbb{C}}$ is a trivial consequence of Theorem 5.2.2 without resorting to Theorem 4.2.1 or Proposition 5.1.4. Indeed, for such E, the morphism $E \to E_{\mathbb{A}^1}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. Hence, by Theorem 5.2.2, so is $E^{\mathbf{sst}} \to (E_{\mathbb{A}^1})^{\mathbf{sst}}$. By Lemma 4.1.3, this means $E^{\mathbf{sst}}$ is $\mathbb{A}^1$-weak-invariant on $\mathbf{Sm}_{\mathbb{C}}$. For the purpose of this paper, this is enough.

5.3. **Commutativity between the loop space and the sst-functors.** Recall that for a map $f : E \to F$ of presheaves of pointed simplicial sets on $\mathbf{Sm}_{\mathbb{C}}$ or $\mathbf{Sch}_{\mathbb{C}}$, the fiber $\mathrm{fib}(f)$ is defined via the Cartesian square

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] & E \arrow[d, "f"] \\ \star \arrow[r] & F. \end{tikzcd} \tag{5.3.1}$$

The sequence of maps $\mathrm{fib}(f) \to E \xrightarrow{f} F$ will be called a fiber sequence. The reader should be warned that this is not same as a homotopy fiber sequence unless f is a fibration.

Recall from § 3.4 that given an open or a closed immersion of schemes $A \subseteq B$ in $\mathbf{Sm}_{\mathbb{C}}$, the functor $\Omega_{B/A}(-)$ on $\mathbf{Spt}(\mathbb{C})$ is by definition $E \mapsto \Omega_{B/A}E = (\Omega_{B/A}E_0, \Omega_{B/A}E_1, \cdots)$, where $\Omega_{B/A}E_n = \mathcal{H}om_\bullet(B/A, E_n) = \mathrm{fib}\,(\mathcal{H}om(B, E_n) \to \mathcal{H}om(A, E_n))$. There is thus an object-wise fiber sequence of presheaves of spectra

$$\Omega_{B/A}E \to E_B \to E_A, \tag{5.3.2}$$

where $E_B(X) = E(B \times X) = \mathcal{H}om(B, E)(X)$.

For any $B \in \mathbf{Sm}_{\mathbb{C}}$, the natural map $\mathcal{H}om(B, E) \to \mathcal{H}om(B, E^{\mathbf{sst}})$ defines a natural transformation $\mathcal{H}om(B, E)^{\mathbf{sst}} \to \mathcal{H}om(B, E^{\mathbf{sst}})$ and using (5.3.2), we see that there is a natural transformation of functors $(\Omega_{B/A}(-))^{\mathbf{sst}} \to \Omega_{B/A}((-)^{\mathbf{sst}})$. As another application of Theorem 5.2.2, we prove the following.

Proposition 5.3.1. *Given $A \subseteq B$ in $\mathbf{Sm}_{\mathbb{C}}$ and $E \in \mathbf{Spt}(\mathbb{C})$, there is an object-wise weak-equivalence $(\Omega_{B/A}E)^{\mathbf{sst}} \to \Omega_{B/A}(E^{\mathbf{sst}})$ of motivic spectra over $\mathbb{C}$.*

Proof. Let $\bar{E}$ denote the artificial extension of E to $\mathbf{Sch}_{\mathbb{C}}$ and let $\bar{E}_B$ denote the presheaf of spectra $\bar{E}_B(X) = \bar{E}(B \times X)$ on $\mathbf{Sch}_{\mathbb{C}}$. Let $\Omega_{B/A}\bar{E}$ denote the object-wise fiber of the map $\bar{E}_B \to \bar{E}_A$. Since the restriction of $\bar{E}_B$ on $\mathbf{Sm}_{\mathbb{C}}$ coincides with E_B (since B is smooth), we see that there is a fiber sequence of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$

$$\Omega_{B/A}\bar{E} \to \bar{E}_B \to \bar{E}_A, \tag{5.3.3}$$

which restricts to (5.3.2) on $\mathbf{Spt}(\mathbb{C})$.

We also notice that there is a map of presheaves of spectra $\overline{E_B} \to \bar{E}_B$ on $\mathbf{Sch}_{\mathbb{C}}$, which is natural in B, and which is an isomorphism on $\mathbf{Sm}_{\mathbb{C}}$ (both are isomorphic to E_B). Using (5.3.3), we see that there is a map of presheaves of spectra $\mathrm{fib}\left(\overline{E_B} \to \overline{E_A}\right) \to \Omega_{B/A}\bar{E}$ on $\mathbf{Sch}_{\mathbb{C}}$ which is an isomorphism on $\mathbf{Sm}_{\mathbb{C}}$. Set $\widetilde{\Omega_{B/A}}E = \mathrm{fib}\left(\overline{E_B} \to \overline{E_A}\right)$. We conclude that there is a commutative diagram of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$

$$\begin{tikzcd}
\overline{\Omega_{B/A}E} \arrow[r] \arrow[d] & \overline{E_B} \arrow[r] \arrow[d, equal] & \overline{E_A} \arrow[d, equal] \\
\widetilde{\Omega_{B/A}}E \arrow[r] \arrow[d] & \overline{E_B} \arrow[r] \arrow[d] & \overline{E_A} \arrow[d] \\
\Omega_{B/A}\bar{E} \arrow[r] & \bar{E}_B \arrow[r] & \bar{E}_A,
\end{tikzcd} \tag{5.3.4}$$

where the bottom two rows are object-wise fiber sequences and all vertical arrows are isomorphisms on $\mathbf{Sm}_{\mathbb{C}}$. Let $u : \overline{\Omega_{B/A}E} \to \Omega_{B/A}\bar{E}$ denote the composite vertical arrow on the left.

Since filtered colimits commute with fiber products (in fact, they commute with all finite limits), the above diagram gives us for every $n \geq 0$, a commutative diagram of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$

$$\begin{tikzcd}
\widetilde{\Omega_{B/A}}E(\Delta^n_{\mathrm{top}} \times -) \arrow[r] \arrow[d] & \overline{E_B}(\Delta^n_{\mathrm{top}} \times -) \arrow[r] \arrow[d] & \overline{E_A}(\Delta^n_{\mathrm{top}} \times -) \arrow[d] \\
\Omega_{B/A}\bar{E}(\Delta^n_{\mathrm{top}} \times -) \arrow[r] & \bar{E}_B(\Delta^n_{\mathrm{top}} \times -) \arrow[r] & \bar{E}_A(\Delta^n_{\mathrm{top}} \times -),
\end{tikzcd} \tag{5.3.5}$$

where the two rows are object-wise fiber sequences.

Since the two rows in (5.3.5) are object-wise fiber sequences and since the fiber of a map of spectra is defined level-wise, taking the diagonals of these maps of simplicial spectra as in § 4.1.2, we get a commutative diagram of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$

$$\begin{tikzcd}
(\widetilde{\Omega_{B/A}}E)^{\mathbf{sst}} \arrow[r] \arrow[d] & (\overline{E_B})^{\mathbf{sst}} \arrow[r] \arrow[d] & (\overline{E_A})^{\mathbf{sst}} \arrow[d] \\
(\Omega_{B/A}\bar{E})^{\mathbf{sst}} \arrow[r] & (\bar{E}_B)^{\mathbf{sst}} \arrow[r] & (\bar{E}_A)^{\mathbf{sst}},
\end{tikzcd} \tag{5.3.6}$$

where the two rows are object-wise fiber sequences. Since each vertical arrow in (5.3.4) is a morphism between two presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$ which is an isomorphism on $\mathbf{Sm}_{\mathbb{C}}$, we conclude from Theorem 5.2.2 that each vertical arrow in (5.3.6) is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$.

It is immediate from the definition and Lemma 4.1.3 that the map $(\bar{E}_B)^{\mathbf{sst}} \to (\bar{E}^{\mathbf{sst}})_B = (E^{\mathbf{sst}})_B$ is an isomorphism on $\mathbf{Sm}_{\mathbb{C}}$, and the same is true for E_A. Composing the vertical maps in (5.3.6) with these isomorphisms and using the identification $(E_B)^{\mathbf{sst}} = (\overline{E_B})^{\mathbf{sst}}$, we get a commutative diagram of presheaves of spectra on $\mathbf{Sm}_{\mathbb{C}}$

$$\begin{tikzcd} (\widetilde{\Omega_{B/A} E})^{\mathbf{sst}} \arrow[r] \arrow[d] & (E_B)^{\mathbf{sst}} \arrow[r] \arrow[d] & (E_A)^{\mathbf{sst}} \arrow[d] \\ (\Omega_{B/A} \bar{E})^{\mathbf{sst}} \arrow[r] & (E^{\mathbf{sst}})_B \arrow[r] & (E^{\mathbf{sst}})_A, \end{tikzcd} \tag{5.3.7}$$

where the two rows are object-wise fiber sequences and the vertical arrows are object-wise weak-equivalences in $\mathbf{Sm}_{\mathbb{C}}$.

Finally, in the sequence of maps $(\overline{\Omega_{B/A} E})^{\mathbf{sst}} \xrightarrow{u^{\mathbf{sst}}} (\Omega_{B/A}\bar{E})^{\mathbf{sst}} \to \Omega_{B/A}(\bar{E}^{\mathbf{sst}})$, the second map is an isomorphism since the bottom row of (5.3.7) is an object-wise fiber sequence. The composite map is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$ by Theorem 5.2.2 since the map $\overline{\Omega_{B/A} E} \to \Omega_{B/A}\bar{E}$ is an isomorphism on $\mathbf{Sm}_{\mathbb{C}}$, as shown in (5.3.4). It follows from the 2-out-of-3 axiom that $u^{\mathbf{sst}}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. Since $(\overline{\Omega_{B/A} E})^{\mathbf{sst}} = (\Omega_{B/A} E)^{\mathbf{sst}}$ and $\Omega_{B/A}(\bar{E}^{\mathbf{sst}}) = \Omega_{B/A}(E^{\mathbf{sst}})$, we conclude the proof of the proposition. $\square$

Applying Proposition 5.3.1 to $t = (\mathbb{G}_m, 1)$ and $T = (\mathbb{P}^1, \infty)$, we conclude:

Corollary 5.3.2. *Given $E \in \mathbf{Spt}(\mathbb{C})$, there are object-wise weak-equivalences $(\Omega_t E)^{\mathbf{sst}} \to \Omega_t(E^{\mathbf{sst}})$ and $(\Omega_T E)^{\mathbf{sst}} \to \Omega_T(E^{\mathbf{sst}})$ of motivic spectra over $\mathbb{C}$.*

Remark 5.3.3. Using the fact that $\Omega_{S^1} E(X) = \Omega_{S^1}(E(X))$ for any presheaf of spectra E on $\mathbf{Sch}_{\mathbb{C}}$ and that $\Omega_{S^1}\bar{E} \cong \overline{\Omega_{S^1} E}$ for any presheaf of spectra E on $\mathbf{Sm}_{\mathbb{C}}$, it is relatively easy to show that the map $(\Omega_{S^1} E)^{\mathbf{sst}} \to \Omega_{S^1}(E^{\mathbf{sst}})$ is an isomorphism for any presheaf of spectra E on $\mathbf{Sm}_{\mathbb{C}}$.

6. Homotopy semi-topologization

In this section, we prove that the classes of B.G. and $\mathbb{A}^1$-B.G. presheaves of spectra remain invariant under semi-topologization. We prove similar results for (s, p)-bispectra. As a consequence, we will define the *homotopy semi-topologization* (**host**) functor on the level of motivic homotopy categories.

6.1. **B.G. property and semi-topologization.** We begin with the following elementary results.

Lemma 6.1.1. *Let E be a simplicial spectrum such that each $E_p := E(\Delta[p])$ for $p \geq 0$ is a cofibrant spectrum. Then* $\operatorname{diag} E$ *is also cofibrant.*

Proof. According to Theorem 2.2.1, we need to show that the map $S^1 \wedge (\operatorname{diag} E)_n \to (\operatorname{diag} E)_{n+1}$ of pointed simplicial sets is a monomorphism for each $n \geq 0$. Recall that for a monomorphism $A \to B$ of simplicial sets, B/A is the simplicial set given by $(B/A)_n = B_n/A_n$ and for two pointed simplicial sets A, B, one has $(A \times B)_n = A_n \times B_n$ and $(A \wedge B)_n = A_n \wedge B_n$. Thus we need to show that the map of pointed sets $(S^1)_p \wedge (\operatorname{diag} E)_{n,p} \to (\operatorname{diag} E)_{n+1,p}$ is a monomorphism for each $n, p \geq 0$.

However, $(\operatorname{diag} E)_{n,p}$ is same as $E^n_{p,p}$ for each $n, p \geq 0$ (see § 4.1.2) and $(S^1)_i \wedge E^n_{p,i} \to E^{n+1}_{p,i}$ is a monomorphism for all $p, i \geq 0$ since each E_p is given to be cofibrant. In particular, the map $(S^1)_p \wedge (\operatorname{diag} E)_{n,p} \to (\operatorname{diag} E)_{n+1,p}$ is a monomorphism. $\square$

Lemma 6.1.2. *Let $f : E \to F$ be a morphism of simplicial spectra such that each map $f_p : E_p \to F_p$ (see Lemma 6.1.1) for $p \geq 0$ is a cofibration of spectra. Then the map* $\operatorname{diag} E \to \operatorname{diag} F$*, thus $|E| \to |F|$ as well, is also a cofibration.*

Proof. Recall that a cofibration of spectra is, in particular, a level-wise monomorphism of simplicial sets. So, the map $f : E \to F$ is a level-wise monomorphism of bisimplicial sets. This also shows that the map $\operatorname{diag} E \to \operatorname{diag} F$ is a level-wise monomorphism of simplicial sets. According to § 2.2.1 and Theorem 2.2.1, we need to show that the spectrum $\operatorname{diag} F/\operatorname{diag} E$ is cofibrant, where $(\operatorname{diag} F/\operatorname{diag} E)_n = (\operatorname{diag} F)_n/(\operatorname{diag} E)_n$.

We set $G = F/E$, where $G^n_{p,q} = F^n_{p,q}/E^n_{p.q}$. Since $(S^1 \wedge F^n_p)/(S^1 \wedge E^n_p) \simeq S^1 \wedge (F^n_p/E^n_p)$ for each $n, p \geq 0$, we see that G is a simplicial spectrum. Furthermore, it follows from the definition that

$$(\operatorname{diag} G)_{n,p} = G^n_{p,p} = F^n_{p,p}/E^n_{p,p} = (\operatorname{diag} F)_{n,p}/(\operatorname{diag} E)_{n,p} = (\operatorname{diag} F/\operatorname{diag} E)_{n,p}.$$

We conclude that $\operatorname{diag} G = \operatorname{diag} F/\operatorname{diag} E$. Hence by Lemma 6.1.1, it suffices to show that G_p is a cofibrant spectrum for each $p \geq 0$. However, the spectrum G_p is same as the spectrum F_p/E_p and our assumption that $E_p \to F_p$ is a cofibration, implies that F_p/E_p is cofibrant. This proves the lemma. □

Since Nisnevich or motivic cofibrations between presheaves of spectra on $\mathbf{Sm}_S$ are exactly object-wise cofibrations, the following is an immediate consequence of Lemma 6.1.2.

Corollary 6.1.3. *Let $f : E \to F$ be a morphism of presheaves of simplicial spectra on $\mathbf{Sch}_{\mathbb{C}}$ such that each map $f_p : E_p \to F_p$ for $p \geq 0$ is an object-wise cofibration. Then the map $\operatorname{diag} E \to \operatorname{diag} F$, thus $|E| \to |F|$ as well, is an object-wise cofibration.*

If $f : E \to F$ is a morphism of presheaves of simplicial spectra on $\mathbf{Sm}_{\mathbb{C}}$ such that each map $f_p : E_p \to F_p$ for $p \geq 0$ is a Nisnevich (motivic) cofibration of presheaves of spectra, then $\operatorname{diag} E \to \operatorname{diag} F$, thus $|E| \to |F|$ as well, is a Nisnevich (motivic) cofibration.

Proposition 6.1.4. *Let $E \xrightarrow{f} F \xrightarrow{g} G$ be an object-wise homotopy cofiber sequence of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$. Then*

$$E^{\mathbf{sst}} \xrightarrow{f^{\mathbf{sst}}} F^{\mathbf{sst}} \xrightarrow{g^{\mathbf{sst}}} G^{\mathbf{sst}}$$

is an object-wise homotopy cofiber sequence on $\mathbf{Sm}_{\mathbb{C}}$.

Proof. Recall from [2, §A 2] that saying that $E \xrightarrow{f} F \xrightarrow{g} G$ is an object-wise homotopy cofiber sequence, is equivalent to saying that there is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "f'"] \arrow[d, equal] & F' \arrow[r, "g'"] \arrow[d, "h"] & F'/E \arrow[d, "p"] \\ E \arrow[r, "f"] & F \arrow[r, "g"] & G \end{tikzcd} \tag{6.1.1}$$

of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$ such that f' is an object-wise cofibration and h and p are object-wise weak-equivalences. It follows from Theorem 5.2.2 that $h^{\mathbf{sst}}$ and $p^{\mathbf{sst}}$ are object-wise weak-equivalences on $\mathbf{Sm}_{\mathbb{C}}$. Using Proposition 4.3.1(3), it only remains to show that the map $E^{\mathbf{sst}} \xrightarrow{f'^{\mathbf{sst}}} F'^{\mathbf{sst}}$ is an object-wise cofibration. Equivalently, we need to show that the map $\operatorname{diag}(\widetilde{E}) \to \operatorname{diag}(\widetilde{F'})$ is a cofibration, where $\widetilde{E}$ is the presheaf of simplicial spectra on $\mathbf{Sch}_{\mathbb{C}}$ defined by $\widetilde{E}(\Delta[p])(-) = E(\Delta^p_{\text{top}} \times -)$ (see Definition 4.1.2) and $\widetilde{F'}$ is defined similarly.

Since $\widetilde{E}(\Delta[p]) \to \widetilde{F'}(\Delta[p])$ is a filtered colimit of object-wise cofibrations, it follows from [32, Proposition 3.2] that this map is an object-wise cofibration for each $p \geq 0$. Hence, by Corollary 6.1.3, the map $\operatorname{diag}(\widetilde{E}) \to \operatorname{diag}(\widetilde{F'})$ is an object-wise cofibration. This finishes the proof. □

Theorem 6.1.5. *Let E be a presheaf of spectra (complexes of abelian groups) on $\mathbf{Sm}_{\mathbb{C}}$. If E is B.G., then so is $E^{\mathbf{sst}}$. If E is $\mathbb{A}^1$-B.G., then so is $E^{\mathbf{sst}}$.*

Proof. We shall prove this result for presheaves of spectra as the other one is a special case by replacing a chain complex of abelian groups by the associated Eilenberg-MacLane spectrum via Dold-Kan.

We prove the first statement. Via the artificial extension in Definition 5.1.2, we regard E as a presheaf on $\mathbf{Sch}_{\mathbb{C}}$. Given $X \in \mathbf{Sm}_{\mathbb{C}}$, we define a presheaf E_X on $\mathbf{Sch}_{\mathbb{C}}$ by letting $E_X(Y) := E(X \times Y)$ for $Y \in \mathbf{Sch}_{\mathbb{C}}$. Given a Nisnevich square as in (3.0.5), where all of X, U, V, W are in $\mathbf{Sm}_{\mathbb{C}}$ with $W = U \times_X V$, we have a commutative diagram of presheaves of spectra on $\mathbf{Sch}_{\mathbb{C}}$

(6.1.2)
$$\begin{tikzcd} E_X \arrow[r, "j_1"] \arrow[d, "j_2"'] & E_U \arrow[d, "h_1"] \\ E_V \arrow[r, "h_2"] & E_W. \end{tikzcd}$$

This is object-wise homotopy Cartesian on $\mathbf{Sm}_{\mathbb{C}}$ because E is B.G. by the given assumption. Equivalently, this square of presheaves of spectra is object-wise homotopy co-Cartesian on $\mathbf{Sm}_{\mathbb{C}}$.

Let G_1 and G_2 denote the object-wise homotopy cofibers of the morphisms j_1 and h_2, respectively. Then the above square is object-wise homotopy co-Cartesian on $\mathbf{Sm}_{\mathbb{C}}$ if and only if the map $G_1 \xrightarrow{h} G_2$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$. Hence, we deduce from Theorem 5.2.2 that $G_1^{\mathbf{sst}} \xrightarrow{h^{\mathbf{sst}}} G_2^{\mathbf{sst}}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$.

On the other hand, by Proposition 6.1.4 there is a commutative diagram

(6.1.3)
$$\begin{tikzcd} E_X^{\mathbf{sst}} \arrow[r, "j_1^{\mathbf{sst}}"] \arrow[d, "j_2^{\mathbf{sst}}"'] & E_U^{\mathbf{sst}} \arrow[r] \arrow[d, "h_1^{\mathbf{sst}}"] & G_1^{\mathbf{sst}} \arrow[d, "h^{\mathbf{sst}}"] \\ E_V^{\mathbf{sst}} \arrow[r, "h_2^{\mathbf{sst}}"] & E_W^{\mathbf{sst}} \arrow[r] & G_2^{\mathbf{sst}}, \end{tikzcd}$$

where the rows are object-wise homotopy cofiber sequences of presheaves of spectra on $\mathbf{Sm}_{\mathbb{C}}$. Since $h^{\mathbf{sst}}$ is an object-wise weak-equivalence on $\mathbf{Sm}_{\mathbb{C}}$, we deduce that the left square in (6.1.3) is object-wise homotopy co-Cartesian on $\mathbf{Sm}_{\mathbb{C}}$. Equivalently, this square is object-wise homotopy Cartesian on $\mathbf{Sm}_{\mathbb{C}}$. Evaluating this at Spec $(\mathbb{C})$ and applying Lemma 4.1.3, it follows that

(6.1.4)
$$\begin{tikzcd} E^{\mathbf{sst}}(X) \arrow[r, "j_1^{\mathbf{sst}}"] \arrow[d, "j_2^{\mathbf{sst}}"'] & E^{\mathbf{sst}}(U) \arrow[d, "h_1^{\mathbf{sst}}"] \\ E^{\mathbf{sst}}(V) \arrow[r, "h_2^{\mathbf{sst}}"] & E^{\mathbf{sst}}(W) \end{tikzcd}$$

is a homotopy Cartesian square of spectra.

The second statement follows from the first and Remark 5.2.3 (or Proposition 5.1.4). This proves the theorem. □

Applying Theorems 3.1.5, 5.2.2 and 6.1.5, we conclude:

Corollary 6.1.6. *If $f : E \to F$ is an S^1-stable motivic weak-equivalence of $\mathbb{A}^1$-B.G. motivic spectra on $\mathbf{Sm}_{\mathbb{C}}$, then $f^{\mathbf{sst}} : E^{\mathbf{sst}} \to F^{\mathbf{sst}}$ is an object-wise weak-equivalence of $\mathbb{A}^1$-B.G. motivic spectra on $\mathbf{Sm}_{\mathbb{C}}$.*

Corollary 6.1.7. *There exists an endo-functor* **host** : $\mathcal{SH}_{S^1}(\mathbb{C}) \to \mathcal{SH}_{S^1}(\mathbb{C})$*, which coincides with the* **sst***-functor on* $\mathbb{A}^1$*-B.G. motivic spectra up to isomorphism.*

Proof. We have already seen that $\mathbf{sst} : \mathbf{Spt}(\mathbb{C}) \to \mathbf{Spt}(\mathbb{C})$ is a functor and it follows from Theorem 6.1.5 that this functor preserves $\mathbb{A}^1$-B.G. motivic spectra. Since an S^1-stable motivic fibrant motivic spectrum is Nisnevich fibrant and $\mathbb{A}^1$-local, it is $\mathbb{A}^1$-B.G. by Lemma 3.1.3. It follows from Corollary 6.1.6 that the **sst**-functor takes a trivial motivic fibration between S^1-stable motivic fibrant motivic spectra into an S^1-stable motivic weak-equivalence. Thus, by [15, Proposition 8.4.8] there exists a right derived endo-functor $\mathbf{host} : \mathcal{SH}_{S^1}(\mathbb{C}) \to \mathcal{SH}_{S^1}(\mathbb{C})$, which coincides with the **sst**-functor on $\mathbb{A}^1$-B.G. motivic spectra up to isomorphism. $\square$

6.2. **Semi-topologization of (s,p)-bispectra.** Recall that an (s,p)-bispectrum E is given by a sequence of motivic spectra $(E_0, E_1, \cdots)$ together with the maps $\epsilon_n : T \wedge E_n \to E_{n+1}$ for each $n \geq 0$. The map ϵ_n yields a map $(T \wedge E_n)^{\mathbf{sst}} \to E^{\mathbf{sst}}_{n+1}$, which is equal to the map $T^{\mathbf{sst}} \wedge E^{\mathbf{sst}}_n \to E^{\mathbf{sst}}_{n+1}$ by Proposition 4.3.1. Composing this with the natural map $T \wedge E^{\mathbf{sst}}_n \to T^{\mathbf{sst}} \wedge E^{\mathbf{sst}}_n$, we get the natural map $T \wedge E^{\mathbf{sst}}_n \to E^{\mathbf{sst}}_n$ that we call $\epsilon^{\mathbf{sst}}_n$. This gives an (s,p)-bispectrum $E^{\mathbf{sst}} := (E^{\mathbf{sst}}_0, E^{\mathbf{sst}}_1, \cdots)$. It is easy to see that given a morphism $f : E \to F$ of (s,p)-bispectra, there is a commutative diagram in $\mathbf{Spt}_{(s,p)}(\mathbb{C})$:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow & & \downarrow \\ E^{\mathbf{sst}} & \xrightarrow{f^{\mathbf{sst}}} & F^{\mathbf{sst}}. \end{array} \tag{6.2.1}$$

Recall from Definition 3.4.2 that an (s,p)-bispectrum $E = (E_0, E_1, \cdots)$ is called $\mathbb{A}^1$-B.G. if each E_n is an $\mathbb{A}^1$-B.G. motivic spectrum and, E is called a motivic Ω_T-bispectrum if the adjoints of the bonding maps $\widetilde{\epsilon}_n : E_n \to \Omega_T E_{n+1}$ are S^1-stable motivic weak-equivalences for all $n \geq 0$. The following result for (s,p)-bispectra follows from the analogous result for motivic spectra proven before.

Theorem 6.2.1. *The functor* $\mathbf{sst} : \mathbf{Spt}_{(s,p)}(\mathbb{C}) \to \mathbf{Spt}_{(s,p)}(\mathbb{C})$ *satisfies the following properties:*

(1) *The class of* $\mathbb{A}^1$*-B.G.* (s,p)*-bispectra is closed under the* **sst***-functor.*
(2) *The class of* $\mathbb{A}^1$*-B.G. motivic* Ω_T*-bispectra is closed under the* **sst***-functor.*
(3) *If* $f : E \to F$ *is a stable motivic weak-equivalence of* $\mathbb{A}^1$*-B.G. motivic* Ω_T*-bispectra, then* $f^{\mathbf{sst}} : E^{\mathbf{sst}} \to F^{\mathbf{sst}}$ *is a* T*-level-wise object-wise weak-equivalence of* $\mathbb{A}^1$*-B.G. motivic* Ω_T*-bispectra.*

Proof. The first part follows directly from Theorem 6.1.5. To prove the second part, let E be an $\mathbb{A}^1$-B.G. motivic Ω_T-bispectrum. We first use Lemma 3.4.1 to deduce that each $\Omega_T E_n$ is an $\mathbb{A}^1$-B.G. motivic S^1-spectrum. Using this, Theorem 3.1.5 and Corollary 6.1.6, we see that the map $E^{\mathbf{sst}}_n \to (\Omega_T E_{n+1})^{\mathbf{sst}}$ is an object-wise weak-equivalence. It follows now from Corollary 5.3.2 that the map $(E^{\mathbf{sst}})_n \to \Omega_T((E^{\mathbf{sst}})_{n+1})$ is an object-wise weak-equivalence, thus an S^1-stable motivic weak-equivalence. The third part follows by combining the first two parts with Theorems 3.4.3 and 5.2.2. $\square$

Before we get to the final result of this section, we consider one more situation. Recall that for a morphism $f : E \to F$ in $\mathbf{Spt}_{(s,p)}(\mathbb{C})$, the cone $C(f)$ is defined in terms of the push-out square

$$\begin{tikzcd} E \arrow[r, "\mathrm{Id}\wedge 0"] \arrow[d, "f"] & E \wedge \Delta[1] \arrow[d, "\bar{f}"] \\ F \arrow[r, "\widetilde{f}"] & C(f), \end{tikzcd} \tag{6.2.2}$$

where $\Delta[1]$ is pointed by one. Collapsing of F to the base point of $\Sigma_s E = E \wedge S^1$ and using the quotient map $E \wedge \Delta[1] \to E \wedge S^1$, there is a unique map $\delta_f : C(f) \to \Sigma_s E$. This gives rise to a sequence of maps

$$E \xrightarrow{f} F \xrightarrow{\widetilde{f}} C(f) \xrightarrow{\delta_f} \Sigma_s E. \tag{6.2.3}$$

Lemma 6.2.2. *Let $f : E \to F$ be a morphism of (s, p)-bispectra over $\mathbb{C}$, and consider the associated push-out square* (6.2.2)*. Then, the following is also a push-out square:*

$$\begin{tikzcd} E^{\mathbf{sst}} \arrow[r, "\mathrm{Id}\wedge 0"] \arrow[d, "f^{\mathbf{sst}}"'] & E^{\mathbf{sst}} \wedge \Delta[1] \arrow[d, "\bar{f}^{\mathbf{sst}}"] \\ F^{\mathbf{sst}} \arrow[r, "\widetilde{f}^{\mathbf{sst}}"] & (C(f))^{\mathbf{sst}}. \end{tikzcd} \tag{6.2.4}$$

Proof. For any given presheaf G of pointed simplicial sets (resp. spectra) on $\mathbf{Sm}_{\mathbb{C}}$, the presheaf $\bar{G}$ denotes its artificial extension on $\mathbf{Sch}_{\mathbb{C}}$ as in Definition 5.1.2. This naturally extends (s, p)-bispectra over $\mathbf{Sm}_{\mathbb{C}}$ to (s, p)-bispectra over $\mathbf{Sch}_{\mathbb{C}}$, defined in the obvious way. Recall that the push-out of a diagram of presheaves of (s, p)-bispectra is defined object-wise, and one has $\overline{E \wedge \Delta[1]} \simeq \bar{E} \wedge \Delta[1]$.

We know that the push-out diagram (6.2.2) is same as the colimit of the finite diagram $(F \leftarrow E \to E \wedge \Delta[1])$. Since the artificial extension is defined in terms of a colimit and since the colimits commute among themselves (*cf.* [29, § IX.8]), we see that (6.2.2) remains a push-out square if we replace the underlying presheaves by their artificial extensions. So, we may assume that the underlying presheaves of E and F are defined on $\mathbf{Sch}_{\mathbb{C}}$. The commutativity of two colimits with each other also implies that the diagram

$$\begin{tikzcd} E(\Delta^n_{\mathrm{top}} \times -) \arrow[r, "\mathrm{Id}\wedge 0"] \arrow[d, "f"'] & E(\Delta^n_{\mathrm{top}} \times -) \wedge \Delta[1] \arrow[d, "\bar{f}"] \\ F(\Delta^n_{\mathrm{top}} \times -) \arrow[r, "\widetilde{f}"] & C(f)(\Delta^n_{\mathrm{top}} \times -) \end{tikzcd} \tag{6.2.5}$$

remains a push-out square for each $n \geq 0$. We next use the adjointness isomorphism of the hom sets

$$\mathrm{Hom}_{\mathcal{C}}\left(X \wedge \Delta[k]_+, Y\right) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}\left(X, \mathcal{H}om_{\bullet}(\Delta[k]_+, Y)\right) \tag{6.2.6}$$

in the category $\mathcal{C}$ of presheaves of (s, p)-bispectra on $\mathbf{Sch}_{\mathbb{C}}$, and deduce that (6.2.5) remains a push-out square when we take the smash product of its vertices with $\Delta[k]_+$ for any $k \geq 0$. Since the coequalizer of a diagram is a colimit and the colimits commute among themselves, it follows now from (4.1.1) that the following is a push-out square:

$$\begin{tikzcd} E^{\mathbf{sst}} \arrow[r, "\mathrm{Id}\wedge 0"] \arrow[d, "f^{\mathbf{sst}}"'] & (E \wedge \Delta[1])^{\mathbf{sst}} \arrow[d, "\bar{f}^{\mathbf{sst}}"] \\ F^{\mathbf{sst}} \arrow[r, "\widetilde{f}^{\mathbf{sst}}"] & (C(f))^{\mathbf{sst}}. \end{tikzcd} \tag{6.2.7}$$

We now use Proposition 4.3.1(4) and the isomorphism $\Delta[1] \xrightarrow{\simeq} (\Delta[1])^{\mathbf{sst}}$ to conclude that (6.2.4) is a push-out square. $\square$

Theorem 6.2.3. *There exists an endo-functor* $\mathbf{host} : \mathcal{SH}(\mathbb{C}) \to \mathcal{SH}(\mathbb{C})$*, which coincides with the* $\mathbf{sst}$*-functor on* $\mathbb{A}^1$*-B.G. motivic* Ω_T*-bispectra up to isomorphism. Moreover,* $\mathbf{host}$ *is a triangulated endo-functor on the triangulated category* $\mathcal{SH}(\mathbb{C})$*.*

Proof. It follows from Theorem 6.2.1 that $\mathbf{sst}$ is an endo-functor on $\mathbf{Spt}_{(s,p)}(\mathbb{C})$ which preserves the $\mathbb{A}^1$-B.G. property of (s,p)-bispectra. It is known (*cf.* [34, Lemma 2.3.8]) that the functor $\Sigma_T : \mathbf{Spt}(\mathbb{C}) \to \mathbf{Spt}(\mathbb{C})$ preserves stable motivic weak-equivalences and cofibrations. Hence, Σ_T is a left Quillen endo-functor with the right adjoint $\Omega_T : \mathbf{Spt}(\mathbb{C}) \to \mathbf{Spt}(\mathbb{C})$.

An (s,p)-bispectrum $E = (E_0, E_1, \cdots)$ is stable motivic fibrant if and only if it is a motivic Ω_T-bispectrum and it is T-level-wise S^1-stable motivic fibrant (*cf.* [16, Definition 3.1, Theorem 3.4]). In particular, a stable motivic fibrant (s,p)-bispectrum is an $\mathbb{A}^1$-B.G. motivic Ω_T-bispectrum.

It follows from Theorem 6.2.1 that the $\mathbf{sst}$-functor on $\mathbf{Spt}_{(s,p)}(\mathbb{C})$ takes a trivial stable motivic fibration between stable motivic fibrant (s,p)-bispectra into a stable motivic weak-equivalence. We can now apply [15, Proposition 8.4.8] to get a right derived endo-functor $\mathbf{host} : \mathcal{SH}(\mathbb{C}) \to \mathcal{SH}(\mathbb{C})$, which coincides with the $\mathbf{sst}$-functor on $\mathbb{A}^1$-B.G. motivic Ω_T-bispectra up to isomorphism.

We now check that $\mathbf{host} : \mathcal{SH}(\mathbb{C}) \to \mathcal{SH}(\mathbb{C})$ is a triangulated endo-functor. Since $\mathbf{host}$ preserves finite coproducts and products in the triangulated category $\mathcal{SH}(\mathbb{C})$, it is an additive functor. The shift $E \mapsto E[1]$ on $\mathcal{SH}(\mathbb{C})$ is given by the functor $E \mapsto \Sigma_s E$. It follows from Proposition 4.3.1 and the isomorphism $S^1 \xrightarrow{\simeq} (S^1)^{\mathbf{sst}}$ that $\mathbf{host}$ commutes with the shift functor.

Recall that a distinguished triangle in $\mathcal{SH}(\mathbb{C})$ is the one which is isomorphic to a sequence of the form (6.2.3) for a map $E \xrightarrow{f} F$ in $\mathbf{Spt}_{(s,p)}(\mathbb{C})$ (*cf.* [40, § 2.3]). It follows from Lemma 6.2.2 and the isomorphism $(\Sigma_s E)^{\mathbf{sst}} \simeq \Sigma_s E^{\mathbf{sst}}$ that given a distinguished triangle such as in (6.2.3), the sequence of maps

$$E^{\mathbf{sst}} \xrightarrow{f^{\mathbf{sst}}} F^{\mathbf{sst}} \to (C(f))^{\mathbf{sst}} \to \Sigma_s E^{\mathbf{sst}}$$

also forms a distinguished triangle in $\mathcal{SH}(\mathbb{C})$. This shows that $\mathbf{host}$ is a triangulated endo-functor. $\square$

Definition 6.2.4. For the rest of this paper, we shall call the functor $\mathbf{host}$ of Corollary 6.1.7 and Theorem 6.2.3 by the name *homotopy semi-topologization* functor. For any E in $\mathcal{SH}_{S^1}(\mathbb{C})$ or $\mathcal{SH}(\mathbb{C})$, we shall denote $\mathbf{host}(E)$ by $E^{\mathbf{host}}$.

7. Representing semi-topological K-theory in $\mathcal{SH}(\mathbb{C})$

The purpose of this section is to prove that the semi-topological K-theory of Friedlander-Walker [9] is representable in $\mathcal{SH}(\mathbb{C})$ by the semi-topologization of an (s,p)-bispectrum model of the algebraic K-theory. We begin with the definition of the semi-topological K-theory from [9].

7.1. **Semi-topological K-theory.** The semi-topological K-theory of a complex quasi-projective variety X acts as a bridge between the algebraic and the topological K-theories of X. It helps one to understand and compare the algebraic and the topological K-theories. This theory was defined by Friedlander and Walker in [10] in terms of the stable homotopy groups of an infinite loop space. This loop space is constructed out of the stabilization of the analytic space of algebraic morphisms from the given variety to the Grassmann varieties. Friedlander and Walker in [9] gave another definition of the semi-topological K-theory of a complex variety X as

$$K^{\mathbf{sst}}_p(X) := \pi_p \left(|\mathcal{K}(\Delta^{\bullet}_{\rm top} \times X)|\right) \tag{7.1.1}$$

for $p \in \mathbb{Z}$, where $\mathcal{K}(-)$ is the presheaf of connective spectra on $\mathbf{Sch}_{\mathbb{C}}$ that represents the Quillen K-theory. They showed in [9, Theorem 1.4] that this definition of the semi-topological K-theory coincides with the original one in [10] for projective weakly normal varieties, so that (7.1.1) is considered as the correct definition of the semi-topological K-theory. The above papers and the subsequent papers ([4], [12], [13]) studied various properties of the semi-topological K-theory defined in (7.1.1).

7.2. **Algebraic K-theory as a motivic spectrum.** Recall that the assignment $X \mapsto \mathcal{K}(X)$ of Quillen with values in the category of pointed simplicial sets is in general only a pseudo-presheaf on $\mathbf{Sm}_{\mathbb{C}}$. However, using the notion of big vector bundles, Jardine [20] (see also [21]) constructed a presheaf of spectra on $\mathbf{Sm}_{\mathbb{C}}$ that represents the algebraic K-theory. Jardine's construction can be summarized as follows and it will be still denoted by $\mathcal{K}$. The $\mathbb{A}^1$-B.G. property of $\mathcal{K}$ is well-known and can be found, for example, in [45, Proposition 6.8, Theorem 10.8].

Theorem 7.2.1. ([20, Theorem 5, Proposition 9]) *There is a presheaf $\mathcal{K}$ of spectra on $\mathbf{Sm}_{\mathbb{C}}$ such that for $X \in \mathbf{Sm}_{\mathbb{C}}$, $\mathcal{K}(X)$ represents the algebraic K-theory of X. This is a presheaf of Ω_s-spectra above level zero and there are smash product morphisms $\mathcal{K}_i \wedge \mathcal{K}_j \to \mathcal{K}_{i+j}$ which commute with the bonding maps of $\mathcal{K}$. Furthermore, $\mathcal{K}$ is an $\mathbb{A}^1$-B.G. presheaf of spectra on $\mathbf{Sm}_{\mathbb{C}}$.*

We shall use Jardine's model for the algebraic K-theory in what follows. Recall that for a presheaf of spectra $E = (E_0, E_1, \cdots)$ on $\mathbf{Sm}_{\mathbb{C}}$, $E\{n\}$ denotes the presheaf of spectra $(E_n, E_{n+1}, \cdots)$.

Let $\mathcal{K} \xrightarrow{f} \mathcal{K}^{\rm fib} \xleftarrow{g} \mathcal{K}^{\rm cf}$ be the morphisms in $\mathbf{Spt}(\mathbb{C})$, where f is an S^1-stable motivic fibrant replacement of $\mathcal{K}$ and g is an S^1-stable motivic cofibrant replacement of $\mathcal{K}^{\rm fib}$. Since $\mathcal{K}^{\rm fib}$ is motivic fibrant and g is a motivic fibration, it follows that $\mathcal{K}^{\rm cf}$ is motivic cofibrant-fibrant. Moreover, it follows from Theorem 2.2.1 and Corollary 3.2.4 that each $\mathcal{K}^{\rm cf}\{n\}$ is motivic cofibrant-fibrant. It also follows from Theorem 7.2.1 and Corollary 3.2.4 that the maps $\mathcal{K}_n \to \mathcal{K}^{\rm fib}_n \leftarrow \mathcal{K}^{\rm cf}_n$ are object-wise weak-equivalences for each $n \geq 1$.

Using the product structure on $\mathcal{K}$ as in Theorem 7.2.1, we obtain a morphism of motivic spectra $\mathcal{K}^{\rm cf} \wedge \mathcal{K}^{\rm cf}_1 \to \mathcal{K}^{\rm cf}\{1\}$ in the homotopy category of motivic spectra. This is equivalent to a morphism $\mathcal{K}^{\rm cf} \to \mathbf{R}\Omega_{\mathcal{K}^{\rm cf}_1}\mathcal{K}^{\rm cf}\{1\} \simeq \Omega_{\mathcal{K}^{\rm cf}_1}\mathcal{K}^{\rm cf}\{1\}$ in the homotopy category. Since $\mathcal{K}^{\rm cf}$ is cofibrant and $\Omega_{\mathcal{K}^{\rm cf}_1}\mathcal{K}^{\rm cf}\{1\}$ is fibrant, this map lifts to a map of motivic spectra in $\mathbf{Spt}(\mathbb{C})$. Taking the adjoint of this map, we conclude that there is a morphism of motivic spectra

$$\phi : \mathcal{K}^{\rm cf} \wedge \mathcal{K}^{\rm cf}_1 \to \mathcal{K}^{\rm cf}\{1\}. \tag{7.2.1}$$

Note that $\mathcal{K}^{\rm cf}$ is a cofibrant-fibrant motivic spectrum model for the algebraic K-theory. The above product is a part of the structure of a presheaf of ring spectra on $\mathcal{K}^{\rm cf}$ which yields a ring structure on $K_*(X)$ for each $X \in \mathbf{Sm}_{\mathbb{C}}$.

7.3. **Algebraic K-theory as an (s,p)-bispectrum.** It is well-known that the above product structure on the presheaf of spectra $\mathcal{K}^{\rm cf}$ allows one to construct a T-spectrum that represents the algebraic K-theory in $\mathcal{SH}(\mathbb{C})$. For a detailed construction of this, we refer the reader to [21]. In order to prove the representability of the semi-topological K-theory in $\mathcal{SH}(\mathbb{C})$, we lift this T-spectrum to an (s,p)-bispectrum. We recycle the construction of the T-spectrum in [47, § 6.2] to construct the following (s,p)-bispectrum model for the algebraic K-theory. An (s,t)-bispectrum model for the algebraic K-theory is constructed in [39].

Lemma 7.3.1. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$, the following hold.*

(1) *For $p \geq m \geq 0$, there is a natural isomorphism $[\Sigma_s^p X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \xrightarrow{\simeq} K_{p-m}(X)$, and a split exact sequence*
$$0 \to [\Sigma_s^p \Sigma_T X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \to K_{p-m}(\mathbb{P}^1_X) \to K_{p-m}(X) \to 0.$$
(2) *For $0 \leq p < m$, $[\Sigma_s^p X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} = [\Sigma_s^p \Sigma_T X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} = 0$.*

Proof. Given any $p \geq 0$, the cofiber sequence in $\mathcal{SH}_{S^1}(\mathbb{C})$ (*cf.* [40, Lemma 2.16])
$$\Sigma_s^\infty \Sigma_s^p X_+ \to \Sigma_s^\infty \Sigma_s^p (\mathbb{P}^1_X)_+ \to \Sigma_s^\infty \Sigma_s^p \Sigma_T X_+ \tag{7.3.1}$$
and Lemma 3.1.2 give us a long exact sequence
$$\cdots \to [\Sigma_s^\infty \Sigma_s^p \Sigma_T X_+, \mathcal{K}^{\mathrm{cf}}]_{\mathbb{A}^1} \to K_p(\mathbb{P}^1_X) \xrightarrow{i_0^*} K_p(X) \to \cdots,$$
where the last map is split by the pull-back via the projection $X \times \mathbb{P}^1 \to X$. The first part of the lemma follows easily from this and the adjoint isomorphisms
$$[\Sigma_s^\infty A, \mathcal{K}^{\mathrm{cf}}]_{\mathbb{A}^1} \xrightarrow{\simeq} [A, \mathcal{K}_0^{\mathrm{cf}}]_{\mathbb{A}^1} \xrightarrow{\simeq} [A, \Omega_s^m \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \xrightarrow{\simeq} [\Sigma_s^m A, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1}$$
for any $A \in \mathbf{Spc}_\bullet(\mathbb{C})$. Notice here that $\mathcal{K}^{\mathrm{cf}}$ and $\mathcal{K}_m^{\mathrm{cf}}$ are all motivic (hence object-wise) fibrant and $\mathcal{K}^{\mathrm{cf}}$ is a motivic Ω_s-spectrum.

To prove the first part of (2), we first use Lemma 3.1.2 and Corollary 3.2.4 to conclude that for any $X \in \mathbf{Sm}_{\mathbb{C}}$, there are isomorphisms
$$[\Sigma_s^p X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \simeq \pi_p(\mathcal{K}_m^{\mathrm{cf}}(X)) \simeq \pi_{p-m}(\mathcal{K}^{\mathrm{cf}}(X)),$$
where the last term is zero if $p - m < 0$ since $\mathcal{K}^{\mathrm{cf}}(X)$ is a connective spectrum.

To prove the second part of (2), we use (7.3.1) to get an exact sequence
$$\begin{aligned}[\Sigma_s^\infty \Sigma_s^{p+1} (\mathbb{P}^1_X)_+, \mathcal{K}^{\mathrm{cf}}\{m\}]_{\mathbb{A}^1} \to [\Sigma_s^\infty \Sigma_s^{p+1} X_+, \mathcal{K}^{\mathrm{cf}}\{m\}]_{\mathbb{A}^1} \to [\Sigma_s^\infty \Sigma_s^p \Sigma_T X_+, \mathcal{K}^{\mathrm{cf}}\{m\}]_{\mathbb{A}^1} \\ \to [\Sigma_s^\infty \Sigma_s^p (\mathbb{P}^1_X)_+, \mathcal{K}^{\mathrm{cf}}\{m\}]_{\mathbb{A}^1}.\end{aligned}$$
By Corollary 3.2.4 and the adjointness, this exact sequence is equivalent to
$$[\Sigma_s^{p+1} (\mathbb{P}^1_X)_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \to [\Sigma_s^{p+1} X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \to [\Sigma_s^p \Sigma_T \wedge X_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1} \to [\Sigma_s^p (\mathbb{P}^1_X)_+, \mathcal{K}_m^{\mathrm{cf}}]_{\mathbb{A}^1}.$$
It follows from (1) and the first part of (2) that the first map in this exact sequence is surjective and the last term is zero if $0 \leq p < m$. Hence the third term must be zero. □

Recall that there is a ring isomorphism $K_0(\mathbb{C})[t]/(t-1)^2 \simeq K_0(\mathbb{P}^1_{\mathbb{C}})$ and it follows from Lemma 7.3.1 that the element $(t-1) = ([\mathcal{O}(1)] - [\mathcal{O}])$ defines a unique element $th \in [S^1 \wedge T, \mathcal{K}_1^{\mathrm{cf}}]_{\mathbb{A}^1}$, called the *Thom class*. Since $S^1 \wedge T$ is cofibrant and $\mathcal{K}_1^{\mathrm{cf}}$ is motivic fibrant, this yields a morphism of motivic spaces $S^1 \wedge T \to \mathcal{K}_1^{\mathrm{cf}}$. Equivalently, there is a morphism $\theta : T \to \Omega_s \mathcal{K}_1^{\mathrm{cf}}$ in $\mathbf{Spc}_\bullet(\mathbb{C})$.

Definition 7.3.2. We define an (s,p)-bispectrum $\mathcal{K}^{\mathrm{alg}} = \{\mathcal{K}^{\mathrm{alg}}_{m,n}\}$ as the sequence
$$\mathcal{K}^{\mathrm{alg}} := (\mathcal{K}^{\mathrm{cf}}, \Omega_s^1 \mathcal{K}^{\mathrm{cf}}\{1\}, \Omega_s^2 \mathcal{K}^{\mathrm{cf}}\{2\}, \cdots)$$
of motivic spectra with the following bonding maps: we apply the natural map $\Omega_s A \wedge B \to \Omega_s(A \wedge B)$ for $A \in \mathbf{Spt}(\mathbb{C}), B \in \mathbf{Spc}_\bullet(\mathbb{C})$ repeatedly to get the maps of motivic spectra
$$\begin{aligned}\Omega_s^n \mathcal{K}^{\mathrm{cf}}\{n\} \wedge T \to \Omega_s^n(\mathcal{K}^{\mathrm{cf}}\{n\} \wedge T) \xrightarrow{\Omega_s^n(\mathrm{Id} \wedge \theta)} \Omega_s^n(\mathcal{K}^{\mathrm{cf}}\{n\} \wedge \Omega_s \mathcal{K}_1^{\mathrm{cf}}) \to \Omega_s^{n+1}(\mathcal{K}^{\mathrm{cf}}\{n\} \wedge \mathcal{K}_1^{\mathrm{cf}}) \\ \xrightarrow{\Omega_s^{n+1}(\mathrm{Id} \wedge \phi)} \Omega_s^{n+1} \mathcal{K}^{\mathrm{cf}}\{n+1\}.\end{aligned}$$

Proposition 7.3.3. *The (s,p)-bispectrum $\mathcal{K}^{\mathrm{alg}}$ on $\mathbf{Sm}_{\mathbb{C}}$ is an $\mathbb{A}^1$-B.G. motivic Ω_T-bispectrum, and it represents the algebraic K-theory in $\mathcal{SH}(\mathbb{C})$.*

Proof. Since $\mathcal{K}^{\mathrm{alg}}_{*,n} = \Omega^n_s \mathcal{K}^{\mathrm{cf}}\{n\}$ for each $n \geq 0$, it follows from Corollary 3.2.4 that $\mathcal{K}^{\mathrm{alg}}$ satisfies the $\mathbb{A}^1$-B.G. property.

To show that $\mathcal{K}^{\mathrm{alg}}$ is a motivic Ω_T-bispectrum, it suffices to show that the map $\mathcal{K}^{\mathrm{alg}}_{m,n} \to \Omega_T \mathcal{K}^{\mathrm{alg}}_{m,n+1}$ between two motivic fibrant pointed motivic spaces is a motivic weak-equivalence for all $m, n \geq 0$. For this, it suffices to show using Corollary 3.2.4 and Lemma 3.1.2 that for every $X \in \mathbf{Sm}_{\mathbb{C}}$ and $p \geq 0$, the induced map $[\Sigma^p_s X_+, \Omega^n_s \mathcal{K}^{\mathrm{cf}}_{m+n}]_{\mathbb{A}^1} \to [\Sigma^p_s X_+, \Omega_T \Omega^{n+1}_s \mathcal{K}^{\mathrm{cf}}_{m+n+1}]_{\mathbb{A}^1}$ is an isomorphism. Equivalently, it suffices to show that the map $[\Sigma^p_s X_+, \Omega^n_s \mathcal{K}^{\mathrm{cf}}_{m+n}]_{\mathbb{A}^1} \to [\Sigma^p_s \Sigma_T X_+, \Omega^{n+1}_s \mathcal{K}^{\mathrm{cf}}_{m+n+1}]_{\mathbb{A}^1}$ is an isomorphism. Using Corollary 3.2.4, this is equivalent to showing that the map

$$[\Sigma^p_s X_+, \mathcal{K}^{\mathrm{cf}}_m]_{\mathbb{A}^1} \to [\Sigma^p_s \Sigma_T X_+, \mathcal{K}^{\mathrm{cf}}_m]_{\mathbb{A}^1} \tag{7.3.2}$$

is an isomorphism. However, it follows from Lemma 7.3.1 and the definition of the Thom class that, for $0 \leq p < m$ both these terms are zero, and for $p \geq m \geq 0$ this map is just the multiplication by the Thom class on the groups

$$K_{p-m}(X) \to K_{p-m}(\mathbb{P}^1_X, \{\infty\} \times X),$$

which is an isomorphism by the projective bundle formula. The representability now follows using Theorems 3.4.3, 7.2.1 and Corollary 3.2.1 (see the proof of Theorem 7.4.3). ∎

7.4. **Representability of semi-topological K-theory.** Using the (s,p)-bispectrum model for the algebraic K-theory, we now prove the representability of the semi-topological K-theory in motivic homotopy categories.

Proposition 7.4.1. *Let $\mathcal{K}$ denote the algebraic K-theory motivic spectrum on $\mathbf{Sm}_{\mathbb{C}}$ as in Theorem 7.2.1. For any $X \in \mathbf{Sm}_{\mathbb{C}}$ and $p \in \mathbb{Z}$, there is a natural isomorphism*

$$K^{\mathbf{sst}}_p(X) \xrightarrow{\simeq} [\Sigma^\infty_s X_+[p], \mathcal{K}^{\mathbf{sst}}]_{\mathbb{A}^1}.$$

In particular, the semi-topological K-theory is representable in $\mathcal{SH}_{S^1}(\mathbb{C})$.

Proof. This is a direct consequence of Corollary 3.2.1, Theorems 6.1.5, 7.2.1, and (7.1.1). ∎

Corollary 7.4.2. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$ and $p \geq 0$, there is a natural isomorphism*

$$K^{\mathbf{sst}}_p(X) \xrightarrow{\simeq} [\Sigma^p_s X_+, \mathbf{R}Ev_0 \mathcal{K}^{\mathbf{sst}}]_{\mathbb{A}^1}.$$

In particular, the semi-topological K-theory is representable in $\mathcal{H}_\bullet(\mathbb{C})$.

Theorem 7.4.3. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$ and $p \in \mathbb{Z}$, there is a natural isomorphism*

$$K^{\mathbf{sst}}_p(X) \xrightarrow{\simeq} \left[\Sigma^\infty_T \Sigma^\infty_s X_+[p], (\mathcal{K}^{\mathrm{alg}})^{\mathbf{host}}\right]_{\mathbb{A}^1}. \tag{7.4.1}$$

In particular, the semi-topological K-theory is representable in $\mathcal{SH}(\mathbb{C})$.

Proof. It follows from Theorems 6.2.1, 6.2.3 and Proposition 7.3.3 that the we can replace $(\mathcal{K}^{\mathrm{alg}})^{\mathbf{host}}$ by $(\mathcal{K}^{\mathrm{alg}})^{\mathbf{sst}}$ in (7.4.1).

Let $f : (\mathcal{K}^{\mathrm{alg}})^{\mathbf{sst}} \to F$ be a stable motivic fibrant replacement of $(\mathcal{K}^{\mathrm{alg}})^{\mathbf{sst}}$. Then $F = (F_0, F_1, \cdots)$ is a T-level-wise motivic fibrant motivic Ω_T-bispectrum. We get the isomorphisms

$$\begin{aligned}\left[\Sigma^\infty_T \Sigma^\infty_s X_+[p], (\mathcal{K}^{\mathrm{alg}})^{\mathbf{sst}}\right]_{\mathbb{A}^1} \xrightarrow{\simeq} [\Sigma^\infty_T \Sigma^\infty_s X_+[p], F]_{\mathbb{A}^1} &\xrightarrow{\simeq} [\Sigma^\infty_s X_+[p], \Omega^\infty_T F]_{\mathbb{A}^1} \\ &= [\Sigma^\infty_s X_+[p], F_0]_{\mathbb{A}^1}.\end{aligned} \tag{7.4.2}$$

On the other hand, it follows from Theorems 3.4.3, 6.2.1, and Proposition 7.3.3 that f is a T-level-wise object-wise weak-equivalence. In particular, the map $(\mathcal{K}^{\mathrm{alg}}_{*,0})^{\mathbf{sst}} = (\mathcal{K}^{\mathrm{alg}})^{\mathbf{sst}}_{*,0} \to F_0$ is an object-wise weak-equivalence. Applying this to (7.4.2), we get

$$[\Sigma^{\infty}_s X_+[p], F_0]_{\mathbb{A}^1} \simeq \left[\Sigma^{\infty}_s X_+[p], (\mathcal{K}^{\mathrm{alg}}_{*,0})^{\mathbf{sst}}\right]_{\mathbb{A}^1} \simeq \left[\Sigma^{\infty}_s X_+[p], (\mathcal{K}^{\mathrm{cf}})^{\mathbf{sst}}\right]_{\mathbb{A}^1}. \tag{7.4.3}$$

It follows from Theorems 3.1.5, 5.2.2, and 7.2.1 that the maps $\mathcal{K}^{\mathbf{sst}} \to (\mathcal{K}^{\mathrm{fib}})^{\mathbf{sst}} \leftarrow (\mathcal{K}^{\mathrm{cf}})^{\mathbf{sst}}$ are object-wise weak-equivalences. In particular, the last term in (7.4.3) is isomorphic to $\left[\Sigma^{\infty}_s X_+[p], \mathcal{K}^{\mathbf{sst}}\right]_{\mathbb{A}^1}$. The theorem now follows from Proposition 7.4.1. $\square$

8. Representing morphic cohomology in $\mathcal{SH}(\mathbb{C})$

The morphic cohomology for smooth quasi-projective schemes over $\mathbb{C}$ was introduced by Friedlander and Lawson [5]. This is an ordinary cohomology theory that lies between the motivic cohomology and the singular (topological) cohomology on such schemes. It was originally defined in terms of the homotopy groups of a certain function space (the space of algebraic cocycles) but was later identified by Friedlander and Walker in [12] as the homotopy group of the semi-topologization of the motivic complex of Friedlander and Suslin (see below). This is a bigraded cohomology theory, denoted by $L^pH^q(X)$ for a smooth quasi-projective complex scheme X.

It was shown by Chu [3] that there is an object in Voevodsky's triangulated category of motives $\mathcal{DM}(\mathbb{C})$ that represents the morphic cohomology. Our aim is to show that the morphic cohomology is representable in $\mathcal{SH}(\mathbb{C})$ by simply (homotopy) semi-topologizing the motivic Eilenberg-MacLane spectrum of Voevodsky. The result of Chu can be deduced from this via the work of Röndigs and Østvær [41].

8.1. **Motivic Eilenberg-MacLane spectrum.**

8.1.1. *Friedlander-Suslin complex.* Given a map $f : Z \to U$ of schemes over $\mathbb{C}$ and a point $s \in U$, we shall denote the scheme-theoretic fiber of f over s by Z_s.

Definition 8.1.1 ([8, p. 141])**.** Let $r \geq 0$ be an integer and let $f : Z \to U$ be a map of schemes such that every irreducible component of Z dominates a component of U. We say that *Z is equidimensional of relative dimension r over U* if for every $s \in U$, we have either $Z_s = \emptyset$ or Z_s is equidimensional of dimension r.

For $X \in \mathbf{Sch}_{\mathbb{C}}$ and $U \in \mathbf{Sm}_{\mathbb{C}}$, let $z_{\mathrm{equi}}(X, r)(U)$ be the free abelian group on the closed integral subschemes Z of $X \times U$ that are dominant and equidimensional of relative dimension r over a component of U.

It is known (see *loc. cit.*) that $z_{\mathrm{equi}}(X, r)$ is a presheaf on $\mathbf{Sm}_{\mathbb{C}}$, which is in fact an étale sheaf. Let F be a presheaf of abelian groups on $\mathbf{Sm}_{\mathbb{C}}$. Let $\Delta^{\bullet}$ be the cosimplicial scheme given by the algebraic simplices $\Delta^n = \mathrm{Spec}\,(\mathbb{C}[t_0, \cdots, t_n])/(\sum_{i=0}^{n} t_i - 1)$, where ∂^n_i $(0 \leq i \leq n)$ are the cofaces. For any $U \in \mathbf{Sm}_{\mathbb{C}}$, consider the simplicial abelian group $F(\Delta^{\bullet} \times U)$. Let $\underline{C}_* F(U)$ be the chain complex (known as the Moore complex) of abelian groups associated to it, namely, $\underline{C}_n F(U) = F(\Delta^n \times U)$ with differentials given by the alternating sum of $F(\partial^n_i \times \mathrm{Id}_U)$ over $0 \leq i \leq n$. Then $\underline{C}_* F$ is a presheaf of chain complexes of abelian groups on $\mathbf{Sm}_{\mathbb{C}}$.

Definition 8.1.2 ([30, p. 126])**.** The *Friedlander-Suslin motivic complex* on $\mathbf{Sm}_{\mathbb{C}}$ is defined to be $\mathbb{Z}^{FS}(n) = \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}^n, 0)$ for $n \geq 0$.

It is a theorem of Voevodsky [48] that the hypercohomology of $\mathbb{Z}^{FS}(n)$ are isomorphic to the motivic cohomology $H^*(-, \mathbb{Z}(*))$ and Bloch's higher Chow groups $\mathrm{CH}^*(-, *)$ as functors on $\mathbf{Sm}_{\mathbb{C}}$. In what follows, we shall identify the presheaf of chain complexes of abelian groups $\mathbb{Z}^{FS}(n)$ with the associated presheaf of pointed simplicial abelian groups (hence an object of $\mathbf{Spc}_\bullet(\mathbb{C})$) via the Dold-Kan correspondence.

Remark 8.1.3. The above definition of the Friedlander-Suslin motivic complex differs from the original definition in [30], where $\mathbb{Z}^{FS}(n)$ is defined as $\underline{C}_* z_{\rm equi}(\mathbb{A}^n, 0)[-2n]$.

8.1.2. *Motivic Eilenberg-MacLane spectrum.* Recall ([47, §6.1]) that the motivic Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$ consists of a sequence of pointed simplicial presheaves, whose n-th level is $K(\mathbb{Z}(n), 2n) = \underline{C}_* L(T^n)$ for some functor L, together with the adjoining map $K(\mathbb{Z}(n), 2n) \to \Omega_T K(\mathbb{Z}(n+1), 2n+2)$, that is a motivic weak-equivalence. This functor L is defined for smooth schemes as follows: for $X \in \mathbf{Sm}_\mathbb{C}$, $L(X)$ is the presheaf of abelian groups such that for each $U \in \mathbf{Sm}_\mathbb{C}$, $L(X)(U)$ is the free abelian group on the closed irreducible subschemes of $U \times X$ that are finite over U and surjective over a connected component of U. This L extends to all pointed simplicial presheaves on $\mathbf{Sm}_\mathbb{C}$.

Using the isomorphisms $T^n \simeq \mathbb{P}^n/\mathbb{P}^{n-1}$ and $\underline{C}_* L(A/B) \simeq \underline{C}_* L(A)/\underline{C}_* L(B)$, we see that $K(\mathbb{Z}(n), 2n) \simeq \frac{\underline{C}_* L(\mathbb{P}^n)}{\underline{C}_* L(\mathbb{P}^{n-1})}$. On the other hand, $\frac{\underline{C}_* L(\mathbb{P}^n)}{\underline{C}_* L(\mathbb{P}^{n-1})}$ is isomorphic (via localization) to the presheaf of simplicial abelian groups that corresponds - via Dold-Kan - to the presheaves of chain complexes of abelian groups $\underline{C}_* z_{\rm equi}(\mathbb{A}^n, 0) = \mathbb{Z}^{FS}(n)$. We conclude that $\mathbf{H}\mathbb{Z}$ can be regarded as the motivic T-spectrum $(\mathbb{Z}^{FS}(0), \mathbb{Z}^{FS}(1), \cdots)$.

8.1.3. *$\mathbb{A}^1$-B.G. property of motivic Eilenberg-MacLane spectrum.* In order to apply our previous results on semi-topologization to represent the morphic cohomology in the stable motivic homotopy category, we need to know that the presheaves of chain complexes $\mathbb{Z}^{FS}(n)$ satisfy the $\mathbb{A}^1$-B.G. property. But this is a known fact. The $\mathbb{A}^1$-weak-invariance part is true by [30, Corollary 2.19]. The B.G. property (which is the Mayer-Vietoris property for the Nisnevich topology) follows from [44, Proposition 4.3.9] combined with the proof of the Mayer-Vietoris property for the Zariski topology in [8, Theorem 5.11]. We state this result in the form we need as follows.

Proposition 8.1.4. *The motivic complexes $\mathbb{Z}^{FS}(n) = \underline{C}_* z_{\rm equi}(\mathbb{A}^n, 0)$ satisfy the $\mathbb{A}^1$-B.G. property on $\mathbf{Sm}_\mathbb{C}$ for $n \geq 0$.*

Recall from § 2.3.2 that for a T-spectrum E, the associated (s,p)-bispectrum E is given by $\Sigma_s^\infty E = (\Sigma_s^\infty E_0, \Sigma_s^\infty E_1, \cdots)$. As a consequence of Proposition 8.1.4, we can show the following.

Proposition 8.1.5. *The (s,p)-bispectrum $\Sigma_s^\infty \mathbf{H}\mathbb{Z}$ satisfies the following properties.*

1. *It is a T-level-wise object-wise Ω_s-spectrum, i.e., $\Sigma_s^\infty \mathbb{Z}^{FS}(n)$ is an object-wise Ω_s-spectrum for every $n \geq 0$.*
2. *It is a S^1-level-wise motivic Ω_T-spectrum, i.e., $\Sigma_s^n \mathbf{H}\mathbb{Z}$ is a motivic Ω_T-spectrum for every $n \geq 0$.*
3. *It satisfies the $\mathbb{A}^1$-B.G. property.*
4. *The properties (1) − (3) also hold for $(\Sigma_s^\infty \mathbf{H}\mathbb{Z})^{\mathbf{sst}}$.*

Proof. Let $\mathbb{Z}(S)$ denote the free abelian group on a set S. Recall that given a simplicial abelian group A and a simplicial set K, there is a standard way of constructing the simplicial abelian group $K \otimes A$, which is the tensor product $\mathbb{Z}(K_n) \otimes_\mathbb{Z} A_n$ at level n. The pointed motivic space $S^1 \wedge \mathbb{Z}^{FS}(n)$ corresponds to the presheaf of simplicial abelian groups $S^1 \otimes \mathbb{Z}^{FS}(n)$ under the Dold-Kan correspondence. It follows from [14, Lemma 4.53] that $\Sigma_s^\infty \mathbb{Z}^{FS}(n)$ is an object-wise Ω_s-spectrum. This proves (1).

The assertion (2) follows from [47, Theorem 6.2] together with the facts that the functor Σ_s preserves motivic weak-equivalences and the map $\Sigma_s(\Omega_T E) \xrightarrow{\sim} \Omega_T(\Sigma_s E)$ is an object-wise weak-equivalence for any pointed motivic space E.

To show (3), it is equivalent to showing that $\Sigma_s^\infty \mathbb{Z}^{FS}(n)$ is $\mathbb{A}^1$-B.G. presheaf of spectra for each $n \geq 0$. But this follows immediately from Proposition 8.1.4, the assertion (1), Corollary 3.3.5 and Theorem 3.1.5.

For (4), the $\mathbb{A}^1$-B.G. property of $(\Sigma_s^\infty \mathbf{H}\mathbb{Z})^{\mathbf{sst}}$ follows from the assertion (3) and Theorem 6.1.5. In fact, it follows from Proposition 8.1.4 and Theorem 6.1.5 that each $\left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}$ is an $\mathbb{A}^1$-B.G. presheaf of simplicial abelian groups. We deduce from [14, Lemma 4.53] that $\Sigma_s^\infty \left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}$ is an object-wise Ω_s-spectrum. The isomorphism $(\Sigma_s(-))^{\mathbf{sst}} \simeq \Sigma_s(-)^{\mathbf{sst}}$ now implies that $(\Sigma_s^\infty \mathbf{H}\mathbb{Z})^{\mathbf{sst}}$ is a T-level-wise object-wise Ω_s-spectrum. The assertion that it is an Ω_T-bispectrum follows from the property (2) of $\Sigma_s^\infty \mathbf{H}\mathbb{Z}$ and Theorem 6.2.1(2). $\square$

Recall that for an (s,p)-bispectrum $E = (E_0, E_1, \cdots)$, where each E_n is a motivic spectrum, the (s,p)-bispectrum $E\{m\}$ is defined by $(E_m, E_{m+1}, \cdots)$. It follows from [16, Lemma 3.8, Theorem 3.9] that $E \mapsto E\{1\} := s_- E$ is a right Quillen endo-functor on $\mathbf{Spt}_{(s,p)}(\mathbb{C})$ and there are isomorphisms of endo-functors $\Sigma_T \simeq \mathbf{L}\Sigma_T \simeq \mathbf{R}s_-$ on $\mathcal{SH}(\mathbb{C})$. Also recall from (2.3.3) that there are adjoint functors $\Sigma_T^\infty : \mathcal{SH}_{S^1}(\mathbb{C}) \leftrightarrow \mathcal{SH}(\mathbb{C}) : \mathbf{R}\Omega_T^\infty$. As consequences of Proposition 8.1.5, we obtain the following results.

Corollary 8.1.6. *For every $n \geq 0$, there is an isomorphism in $\mathcal{SH}_{S^1}(\mathbb{C})$:*

$$\Sigma_s^\infty \left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}} \simeq \mathbf{R}\Omega_T^\infty \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}.$$

Proof. Let $f : (\Sigma_s^\infty \mathbf{H}\mathbb{Z})^{\mathbf{sst}} \to F$ be a stable motivic fibrant replacement of $(\Sigma_s^\infty \mathbf{H}\mathbb{Z})^{\mathbf{sst}}$. It follows from Proposition 8.1.5 and Theorem 6.2.1 that f is T-level-wise object-wise weak-equivalence of $\mathbb{A}^1$-B.G. (s,p)-bispectra. This implies that

(8.1.1)
$$\Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} \simeq \mathbf{R}^n s_- \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} \simeq F\{n\} \simeq \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{sst}}\{n\} \simeq \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\{n\}\right)^{\mathbf{sst}}.$$

Applying Proposition 8.1.5 and Theorem 6.2.1 once again, we get

$$\mathbf{R}\Omega_T^\infty \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} \simeq \Omega_T^\infty((\Sigma_s^\infty \mathbf{H}\mathbb{Z}\{n\})^{\mathbf{sst}}) \simeq Ev_0((\Sigma_s^\infty \mathbf{H}\mathbb{Z}\{n\})^{\mathbf{sst}}) \simeq (\Sigma_s^\infty \mathbb{Z}^{FS}(n))^{\mathbf{sst}},$$

where the second isomorphism follows from Proposition 8.1.5. Since $(\Sigma_s(-))^{\mathbf{sst}} \simeq \Sigma_s(-)^{\mathbf{sst}}$, the corollary follows. $\square$

Corollary 8.1.7. *For every $n \geq 0$, there is a canonical isomorphism in $\mathcal{SH}(\mathbb{C})$:*

$$\left(\Sigma_T^n \Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} \simeq \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}.$$

Proof. Following the notations and proof of Corollary 8.1.6, we get

$$\begin{aligned}
\left(\Sigma_T^n \Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} &\simeq \left(\mathbf{R}^n s_- \Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}} \simeq^1 \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\{n\}\right)^{\mathbf{host}} \\
&\simeq^2 \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\{n\}\right)^{\mathbf{sst}} \simeq^3 \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}.
\end{aligned} \tag{8.1.2}$$

The isomorphism $\simeq^1$ follows from Proposition 8.1.5 and Theorem 3.4.3, $\simeq^2$ follows from Proposition 8.1.5 and $\simeq^3$ is shown in (8.1.1). $\square$

Theorem 8.1.8. *For any smooth quasi-projective scheme X over $\mathbb{C}$, $n \geq 0$ and $p \in \mathbb{Z}$, there is an isomorphism*

$$L^n H^{2n-p}(X) \simeq \left[\Sigma_T^\infty \Sigma_s^\infty X_+[p], \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}\right]_{\mathbb{A}^1}.$$

In particular, the morphic cohomology of smooth quasi-projective schemes is representable in $\mathcal{SH}(\mathbb{C})$.

Proof. It follows from (2.3.3) and Corollary 8.1.6 that $\left[\Sigma_T^{\infty}\Sigma_s^{\infty}X_+[p], \Sigma_T^n\left(\Sigma_s^{\infty}\mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}\right]_{\mathbb{A}^1}$

$$\simeq \left[\Sigma_s^{\infty}X_+[p], \mathbf{R}\Omega_T^{\infty}\Sigma_T^n\left(\Sigma_s^{\infty}\mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}\right]_{\mathbb{A}^1} \simeq \left[\Sigma_s^{\infty}X_+[p], \Sigma_s^{\infty}\left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}\right]_{\mathbb{A}^1}. \tag{8.1.3}$$

On the other hand, we have $\left[\Sigma_s^{\infty}X_+[p], \Sigma_s^{\infty}\left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}\right]_{\mathbb{A}^1}$

$$\simeq^1 \pi_p\left(\Sigma_s^{\infty}\left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}(X)\right) \simeq^2 \pi_p\left(\left(\mathbb{Z}^{FS}(n)\right)^{\mathbf{sst}}(X)\right) \simeq^3 L^nH^{2n-p}(X), \tag{8.1.4}$$

where the isomorphism $\simeq^1$ follows from Proposition 8.1.5 and Corollary 3.2.1, $\simeq^2$ follows from Proposition 8.1.5 and $\simeq^3$ follows from [12, Corollary 3.5]. We conclude the proof of the theorem by combining (8.1.3) and (8.1.4). □

8.2. **Excision and Localization for morphic cohomology.** As a consequence of our representability of the semi-topological K-theory and the morphic cohomology in $\mathcal{SH}(\mathbb{C})$, we can prove the following Mayer-Vietoris and localization theorems for these cohomology theories. For the semi-topological K-theory, this recovers a result of Friedlander-Walker [11].

Theorem 8.2.1. *The morphic cohomology of smooth schemes over $\mathbb{C}$ satisfies Nisnevich descent and localization:*

(1) *Given a Nisnevich square of smooth schemes as in* (3.0.5)*, there is a long exact sequence*

$$\cdots \to L^{p-1}H^q(W) \xrightarrow{\partial} L^pH^q(X) \to L^pH^q(U) \oplus L^pH^q(V) \to L^pH^q(W) \to \cdots.$$

(2) *Given a closed immersion of smooth quasi-projective schemes $i : Z \hookrightarrow X$ over $\mathbb{C}$ of codimension d with complement U, there is a long exact sequence*

$$\cdots \to L^{p-1}H^q(U) \xrightarrow{\partial} L^{p-2d}H^{q-d}(Z) \to L^pH^q(X) \to L^pH^q(U) \to \cdots. \tag{8.2.1}$$

Similar results hold for the semi-topological K-theory.

Proof. We give the proof of the theorem for the morphic cohomology as the other case is similar using Theorem 7.4.3. Given a Nisnevich square as in (3.0.5), it follows from [40, Corollary 2.20] that there is a distinguished triangle in $\mathcal{SH}(\mathbb{C})$ of the form

$$\Sigma_T^{\infty}\Sigma_s^{\infty}W_+ \to \Sigma_T^{\infty}\Sigma_s^{\infty}U_+ \vee \Sigma_T^{\infty}\Sigma_s^{\infty}V_+ \to \Sigma_T^{\infty}\Sigma_s^{\infty}X_+ \to \Sigma_T^{\infty}\Sigma_s^{\infty}W_+[1].$$

The proof of (1) follows by applying $[-, (\Sigma_s^{\infty}\mathbf{H}\mathbb{Z})^{\mathbf{host}}]_{\mathbb{A}^1}$ to this distinguished triangle and using Theorem 8.1.8.

To prove (2), we first note that given the open immersion $j : U \hookrightarrow X$, there is a distinguished triangle in $\mathcal{SH}(\mathbb{C})$ (*cf.* [40, Lemma 2.16]) of the form

$$\Sigma_T^{\infty}\Sigma_s^{\infty}U_+ \to \Sigma_T^{\infty}\Sigma_s^{\infty}X_+ \to \Sigma_T^{\infty}\Sigma_s^{\infty}X/U \to \Sigma_T^{\infty}\Sigma_s^{\infty}U_+[1].$$

Furthermore, it follows from [40, Theorem 2.26] that we can replace X/U by the Thom space $\mathrm{Th}(N_{Z/X})$ of the normal bundle $N_{Z/X}$ of Z in X. Applying $[-, (\Sigma_s^{\infty}\mathbf{H}\mathbb{Z})^{\mathbf{host}}]_{\mathbb{A}^1}$ to this distinguished triangle and using Theorem 8.1.8, we get a long exact sequence as in (8.2.1) except that we have $\mathrm{Th}(N_{Z/X})$ in place of Z. Thus we are only left with showing the *Thom isomorphism* $L^*H^*(Z) \xrightarrow{\simeq} L^*H^*(\mathrm{Th}(N_{Z/X}))$ (up to a shift in bidegree).

We recall from [12] that $L^*H^*(-)$ is a ring cohomology theory for smooth quasi-projective schemes over $\mathbb{C}$. The projective bundle theorem for the morphic cohomology (*cf.* [6]), the natural transformation of cohomology theories $H^*_{\mathcal{M}}(-, \mathbb{Z}(*)) \to L^*H^*(-)$, and the existence of a Chern class theory on the motivic cohomology together imply that there is a theory of Chern classes in $L^*H^*(-)$ in the sense of [37, § 3.6] and this implies the above Thom isomorphism by [37, Theorem 3.35]. □

8.3. **Morphic cohomology of smooth schemes.** The definitions of morphic cohomology in [5] and [12] assume quasi-projectivity of the underlying scheme. We can redefine this cohomology theory for any smooth scheme X over $\mathbb{C}$, using the isomorphism proven in Theorem 8.1.8 via our homotopy semi-topologization functor **host** on $\mathcal{SH}(\mathbb{C})$.

Definition 8.3.1. Given $X \in \mathbf{Sm}_{\mathbb{C}}$, $n \geq 0$ and $p \in \mathbb{Z}$, we define

$$L^nH^{2n-p}(X) := \left[\Sigma_T^\infty \Sigma_s^\infty X_+[p], \Sigma_T^n \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}\right]_{\mathbb{A}^1}. \tag{8.3.1}$$

By Theorem 8.1.8, this definition coincides with the original one of the morphic cohomology for smooth quasi-projective schemes. Moreover, the natural transformation $\mathrm{Id} \to \mathbf{host}$ of endo-functors on $\mathcal{SH}(\mathbb{C})$ and the known isomorphism

$$H^{2n-p}_{\mathcal{M}}(X, \mathbb{Z}(n)) \simeq [\Sigma_T^\infty \Sigma_s^\infty X_+[p], \Sigma_T^n \Sigma_s^\infty \mathbf{H}\mathbb{Z}]_{\mathbb{A}^1}$$

show that there is a natural transformation of bigraded cohomology theories on $\mathbf{Sm}_{\mathbb{C}}$:

$$H^{2n-p}_{\mathcal{M}}(-, \mathbb{Z}(n)) \to L^nH^{2n-p}(-).$$

Using the topological realization functor $\mathbf{R}_{\mathbb{C}} : \mathcal{SH}(\mathbb{C}) \to \mathcal{SH}$ (*cf.* [38, § A.7]), we get a commutative diagram

$$\begin{tikzcd} H^{2n-p}_{\mathcal{M}}(X, \mathbb{Z}(n)) \arrow[r] \arrow[d] & L^nH^{2n-p}(X) \arrow[d] \\ {[\Sigma_s^\infty X_+[p], \mathbf{R}_{\mathbb{C}}\left(\Sigma_T^n \mathbf{H}\mathbb{Z}\right)]_{\mathcal{SH}}} \arrow[r] & {[\Sigma_s^\infty X_+[p], \mathbf{R}_{\mathbb{C}}\left(\Sigma_T^n (\mathbf{H}\mathbb{Z})^{\mathbf{sst}}\right)]_{\mathcal{SH}}}. \end{tikzcd} \tag{8.3.2}$$

The isomorphisms $\mathbf{R}_{\mathbb{C}}(T) \simeq S^2$ and $\mathbf{R}_{\mathbb{C}}\left(\mathbf{H}\mathbb{Z}\right) \simeq \mathbf{R}_{\mathbb{C}}\left((\mathbf{H}\mathbb{Z})^{\mathbf{sst}}\right) \simeq \mathbf{H}\mathbb{Z}_{\mathrm{top}}$ show that there are natural transformations of the cohomology theories on $\mathbf{Sm}_{\mathbb{C}}$:

$$H^{2n-p}_{\mathcal{M}}(-, \mathbb{Z}(n)) \to L^nH^{2n-p}(-) \to H^{2n-p}_{\mathrm{sing}}(-, \mathbb{Z}). \tag{8.3.3}$$

8.4. **Equivariant morphic cohomology.** As another application of the representability of the morphic cohomology in $\mathcal{SH}(\mathbb{C})$, we can construct a theory of Borel style equivariant morphic cohomology of smooth schemes with group action as follows.

Let X be a smooth scheme over $\mathbb{C}$ with action of a complex linear algebraic group G. We define the *equivariant morphic cohomology* of X as

$$L^pH^q_G(X) := \left[\Sigma_T^\infty \Sigma_s^\infty (X_G)_+[2p-q], \Sigma_T^p \left(\Sigma_s^\infty \mathbf{H}\mathbb{Z}\right)^{\mathbf{host}}\right]_{\mathbb{A}^1}, \tag{8.4.1}$$

where X_G is the motivic space defined in [22, § 2.2]. It turns out that there is a natural map from the equivariant higher Chow groups to the above equivariant morphic cohomology which in turn maps to the Borel equivariant singular cohomology. This new cohomology theory and its applications will appear in more detail in [23].

9. Semi-topological cobordism

Recall from [47, § 6.3] that the algebraic cobordism MGL is a T-spectrum of the form $\mathrm{MGL} = (\mathrm{MGL}_0, \mathrm{MGL}_1, \cdots)$, where MGL_n is the motivic *Thom space* of the universal rank n vector bundle E_n on the Grassmann ind-scheme $Gr(n, \infty)$. The T-spectrum MGL is also called the *motivic Thom spectrum.* The associated cohomology theory on $\mathbf{Sm}_{\mathbb{C}}$ given by the assignment $X \mapsto \mathrm{MGL}^{p,q}(X)$ (see (2.3.4)) is called the (Voevodsky) *algebraic cobordism* theory. It is known that this is an oriented bigraded cohomology theory and is universal among all oriented cohomology theories on $\mathbf{Sm}_{\mathbb{C}}$.

9.1. **Definition.** Using our homotopy semi-topologization **host** on $\mathcal{SH}(\mathbb{C})$, we define the semi-topological version of the algebraic cobordism as follows.

Definition 9.1.1. The semi-topological Thom spectrum $\mathrm{MGL}_{\mathbf{sst}}$ is defined to be the homotopy semi-topologization $\mathrm{MGL}^{\mathbf{host}}$ in $\mathcal{SH}(\mathbb{C})$. The associated bigraded cohomology theory $\mathrm{MGL}^{p,q}_{\mathbf{sst}}(-)$ on $\mathbf{Sm}_{\mathbb{C}}$ is called the *semi-topological cobordism.*

The natural map $\mathrm{MGL} \to \mathrm{MGL}_{\mathbf{sst}}$ in $\mathcal{SH}(\mathbb{C})$ defines a natural transformation of bigraded cohomology theories $\mathrm{MGL}^{p,q}(-) \to \mathrm{MGL}^{p,q}_{\mathbf{sst}}(-)$ on $\mathbf{Sm}_{\mathbb{C}}$ and it follows from Theorem 8.1.8 that there is a commutative diagram of cohomology theories

$$
\begin{array}{ccc}
\mathrm{MGL}^{p,q}(-) & \longrightarrow & \mathrm{MGL}^{p,q}_{\mathbf{sst}}(-) \\
\big\downarrow & & \big\downarrow \\
H^p_{\mathcal{M}}(-,\mathbb{Z}(q)) & \longrightarrow & L^pH^q(-).
\end{array}
\tag{9.1.1}
$$

A result of Hopkins and Morel shows that for any $X \in \mathbf{Sm}_{\mathbb{C}}$ and $n \geq 0$, there is a spectral sequence

$$
E^{p,q}(n) = H^{p-q}_{\mathcal{M}}(X,\mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}(X), \tag{9.1.2}
$$

where $\mathbb{L} = \oplus_{q\leq 0}\mathbb{L}^q$ is the Lazard ring. Some also call this spectral sequence, the motivic Atiyah-Hirzebruch spectral sequence. The result of Hopkins and Morel is still in an unpublished form to the best of our knowledges. However, based on the lectures of Hopkins [25], a proof of an essential part of this has now appeared in [17]. Our goal in this section is to use the ideas of [17], [42], [43] and [49] to produce an analogous spectral sequence for the semi-topological cobordism.

We recall here that an Atiyah-Hirzebruch type spectral sequence, which relates the motivic cohomology with the algebraic K-theory of smooth schemes, was earlier constructed by Bloch-Lichtenbaum [1] and Friedlander-Suslin [7]. The corresponding spectral sequence for the semi-topological K-theory was obtained by Friedlander-Haesemeyer-Walker [4].

9.2. **The semi-topological spectral sequence.** Recall from [43, §3] that there is a motivic analogue of the Postnikov tower for each $E \in \mathcal{SH}(\mathbb{C})$ constructed as follows. Let $\mathcal{SH}(\mathbb{C})^{\mathrm{eff}} \subset \mathcal{SH}(\mathbb{C})$ be the full localizing triangulated subcategory generated by $\Sigma^i_s\Sigma^j_t\Sigma^\infty_T X_+$ for $i,j \in \mathbb{Z}$, $j \geq 0$ and $X \in \mathbf{Sm}_{\mathbb{C}}$. For $p \in \mathbb{Z}$, let $\iota_p : \Sigma^p_T\mathcal{SH}(\mathbb{C})^{\mathrm{eff}} \to \mathcal{SH}(\mathbb{C})$ be the inclusion. It is known that there is a right adjoint r_p of ι_p such that $r_p \circ \iota_p \simeq \mathrm{Id}$ (*cf.* [40, § 4]). Set $f_p := \iota_p \circ r_p$. There is a natural transformation $\rho_{p+1} : f_{p+1} \to f_p$. Thus, we have a sequence of maps

$$
\cdots \to f_{p+1}E \xrightarrow{\rho_{p+1}} f_pE \xrightarrow{\rho_p} \cdots \xrightarrow{\rho_2} f_1E \xrightarrow{\rho_1} f_0E \xrightarrow{\rho_{-1}} f_{-1}E \to \cdots \to E. \tag{9.2.1}
$$

The object s_pE is defined to be the cofiber of the map ρ_{p+1} so that there is for each $p \in \mathbb{Z}$, a distinguished triangle in $\mathcal{SH}(\mathbb{C})$:

$$
f_{p+1}E \to f_pE \to s_pE \to (f_{p+1}E)[1]. \tag{9.2.2}
$$

The objects s_pE are called the *slices* of E. We say that the spectrum E is *effective* if the map $f_pE \to E$ is an isomorphism for $i \leq 0$.

Lemma 9.2.1. *The natural map* $\mathrm{MGL} \to \mathbf{H}\mathbb{Z}$ *induces an isomorphism* $s_0\,\mathrm{MGL} \xrightarrow{\simeq} \mathbf{H}\mathbb{Z}$.

Proof. The maps of spectra $\mathbf{1} \to \mathrm{MGL} \to \mathbf{H}\mathbb{Z}$ induce the maps of slices $s_0\mathbf{1} \to s_0\,\mathrm{MGL} \to s_0\mathbf{H}\mathbb{Z}$. The first map is an isomorphism by [42, Corollary 3.3] and the isomorphism of the composite map was shown by Voevodsky [49]. We conclude that the second map is an isomorphism. The lemma now follows because Voevodsky also showed that $s_0\mathbf{H}\mathbb{Z} \simeq \mathbf{H}\mathbb{Z}$. $\square$

The following result follows from [40, Remark 4.2] and [42, Corollary 3.2].

Lemma 9.2.2. *We have* $f_p\,\mathrm{MGL} \xrightarrow{\simeq} \mathrm{MGL}$ *for all* $p \le 0$ *and* $s_p\,\mathrm{MGL} = 0$ *for all* $p < 0$. *In particular,* MGL *is effective.*

Recall that there is a morphism of ring spectra $\mathbb{L} \to \mathrm{MGL}$. On the other hand, the natural map $\mathrm{MGL} \to s_0\,\mathrm{MGL} = \mathbf{H}\mathbb{Z}$ (*cf.* Lemma 9.2.1) factors as $\mathrm{MGL} \to \mathrm{MGL} \otimes_{\mathbb{L}} (\mathbb{L}/\mathbb{L}^{<0}) = \mathrm{MGL} \otimes_{\mathbb{L}} \mathbb{Z} \to \mathbf{H}\mathbb{Z}$ and the last map is an isomorphism in $\mathcal{SH}(\mathbb{C})$ by Hoyois [17]. It was proven by Spitzweck [42] that this implies the following.

Proposition 9.2.3 ([42, Theorem 4.7])**.** *There is an isomorphism* $s_p\,\mathrm{MGL} \xrightarrow{\simeq} \Sigma^p_T \mathbf{H}\mathbb{L}^p$.

Let's fix $X \in \mathbf{Sm}_{\mathbb{C}}$ and $n \ge 0$. We write $\Sigma^\infty_T X_+$ as just X and the hom sets $[-,-]_{\mathbb{A}^1}$ in $\mathcal{SH}(\mathbb{C})$ as just $[-,-]$, for simplicity. Applying **host** to (9.2.1), (9.2.2) and applying Lemma 9.2.2, we get the sequences of maps

$$(9.2.3)\qquad \cdots \to (f_{p+1}\,\mathrm{MGL})^{\mathbf{host}} \to (f_p E)^{\mathbf{host}} \to \cdots \to (f_1 E)^{\mathbf{host}} \to (f_0 E)^{\mathbf{host}} = \mathrm{MGL}_{\mathbf{sst}},$$

$$(9.2.4)\qquad (f_{p+1}\,\mathrm{MGL})^{\mathbf{host}} \to (f_p\,\mathrm{MGL})^{\mathbf{host}} \to (s_p\,\mathrm{MGL})^{\mathbf{host}} \to (f_{p+1}\,\mathrm{MGL})^{\mathbf{host}}[1],$$

where (9.2.4) is a distinguished triangle in $\mathcal{SH}(\mathbb{C})$ by Theorem 6.2.3. Applying $[X,-]$ to (9.2.4), we obtain an exact sequence
(9.2.5)

$$[X,(f_{p+1}\,\mathrm{MGL})^{\mathbf{host}}] \to [X,(f_p\,\mathrm{MGL})^{\mathbf{host}}] \to [X,(s_p\,\mathrm{MGL})^{\mathbf{host}}] \to [X,(f_{p+1}\,\mathrm{MGL})^{\mathbf{host}}[1]].$$

We now construct the exact couples needed to produce our semi-topological spectral sequence (see [31, § 2, Theorem 2.8] for basic formalisms of exact couples and associated spectral sequences). For $p, q \in \mathbb{Z}$ and $n \ge 0$, define

$$A^{p,q}(X,n) := [X, \Sigma_s^{p+q-n}\Sigma_t^n (f_p\,\mathrm{MGL})^{\mathbf{host}}].$$

Note that the map $\rho_p^{\mathbf{host}} : (f_p\,\mathrm{MGL})^{\mathbf{host}} \to (f_{p-1}\,\mathrm{MGL})^{\mathbf{host}}$ induces a natural map $\rho_{p-1,q+1} : A^{p,q}(X,n) \to A^{p-1,q+1}(X,n)$. For the slices, we define

$$E^{p,q}(X,n) := [X, \Sigma_s^{p+q-n}\Sigma_t^n (s_p\,\mathrm{MGL})^{\mathbf{host}}].$$

So, the exact sequence (9.2.5) gives an exact sequence

$$A^{p,q}(X,n) \xrightarrow{\rho_{p-1,q+1}} A^{p-1,q+1}(X,n) \xrightarrow{\gamma_{p-1,q+1}} E^{p-1,q+1}(X,n) \xrightarrow{\delta_{p-1,q+1}} A^{p+1,q}(X,n).$$

Set $D_1(X,n) := \oplus_{p,q} A^{p,q}(X,n)$ and $E_1(X,n) := \oplus_{p,q} E^{p,q}(X,n)$. Write $a_1 := \oplus\delta_{p-1,q+1}$, $b_1 := \oplus\rho_{p-1,q+1}$ and $c_1 := \oplus\gamma_{p-1,q+1}$. This gives an exact couple

$$\begin{tikzcd} D_1 \arrow[rr, "b_1"] & & D_1 \arrow[dl, "c_1"] \\ & E_1. \arrow[ul, "a_1"] & \end{tikzcd}$$

Following the usual formalism of exact couples, we let $d_1 := c_1 \circ a_1 : E_1 \to E_1$. By construction, that (9.2.5) is an exact sequence implies that $d_1^2 = 0$. In particular, (E_1, d_1) is a complex. Thus, by repeatedly taking the homology functors, we obtain a spectral sequence.

For the target of the spectral sequence, let $A^m(X,n) := \mathrm{colim}_{q\to\infty} A^{m-q,q}(X,n)$. Since X is a compact object of $\mathcal{SH}(\mathbb{C})$ (*cf.* [47, Proposition 5.5]), the colimit enters into $[-,-]$

so that $A^m(X,n) = [X, \Sigma_s^{m-n}\Sigma_t^n \mathrm{MGL}^{\mathbf{host}}] = \mathrm{MGL}^{m,n}_{\mathbf{sst}}(X)$ by (9.2.3). The formalism of exact couples then yields a spectral sequence

$$E_1^{p,q}(X,n) = E^{p,q}(X,n) \Rightarrow A^{p+q}(X,n). \tag{9.2.6}$$

We identify the E_1-terms as

$$\begin{array}{rcl} E_1^{p,q}(X,n) & \simeq & [X, \Sigma_s^{p+q-n}\Sigma_t^n (s_p \mathrm{MGL})^{\mathbf{host}}] \simeq^1 [X, \Sigma_s^{p+q-n}\Sigma_t^n \left(\Sigma_T^p \mathbf{H}\mathbb{L}^p\right)^{\mathbf{host}}] \\ & \simeq^2 & [X, \Sigma_s^{p+q-n}\Sigma_t^n \Sigma_T^p (\mathbf{H}\mathbb{L}^p)^{\mathbf{host}}] \simeq [X, \Sigma_s^{p+q-2n}\Sigma_T^{p+n} (\mathbf{H}\mathbb{L}^p)^{\mathbf{host}}] \\ & \simeq & [\Sigma_s^{2n-p-q} X, \Sigma_T^{p+n} (\mathbf{H}\mathbb{L}^p)^{\mathbf{host}}] \\ & \simeq^3 & L^{p+n}H^{2(p+n)-(2n-p-q)}(X) \otimes_{\mathbb{Z}} \mathbb{L}^p = L^{p+n}H^{3p+q}(X) \otimes_{\mathbb{Z}} \mathbb{L}^p. \end{array} \tag{9.2.7}$$

The isomorphism $\simeq^1$ in (9.2.7) follows from Proposition 9.2.3, $\simeq^2$ follows from Corollary 8.1.7 and (2.3.2), and $\simeq^3$ follows from Theorem 8.1.8 and (2.3.4).

This E_1-spectral sequence is actually identical to an E_2-spectral sequence after reindexing. Indeed, let $\tilde{E}_2^{p',q'}(X,n) = L^{n-q'}H^{p'-q'}(X) \otimes_{\mathbb{Z}} \mathbb{L}^{q'}$. We show that this is identical to the E_1-spectral sequence of (9.2.6).

For $r \geq 1$, the change $(p,q) \mapsto (p+r, q-r+1)$ transforms $(p+n, 3p+q, p)$ into $(p+r+n, 3p+q+2r+1, p+r)$. Similarly, for $r' = r+1$, the change $(p',q') \mapsto (p'+r', q'-r'+1)$ transforms $(n-q', p'-q', -q')$ into $(n-q'+r'-1, p'-q'+2r'-1, -q'+r'-1) = (n-q'+r, p'-q'+2r+1, -q'+r)$. Hence, the equality $E_r^{p,q} = \tilde{E}_{r'}^{p',q'}$ gives the equalities $p+n = n-q', 3p+q = p'-q', p = -q'$. This implies that

$$\begin{array}{rcl} E_r^{p+r,q-r+1} & = & L^{p+r+n}H^{3p+q+2r+1}(X) \otimes_{\mathbb{Z}} \mathbb{L}^{-p-r} \\ & = & L^{n-q'+r}H^{p'-q'+2r+1}(X) \otimes_{\mathbb{Z}} \mathbb{L}^{q'-r} = \tilde{E}_{r'}^{p',q'}, \end{array}$$

so indeed the E_1-sequence is now identified to an E_2-sequence.

Summarizing what we have shown above and using the diagram (9.1.1), we get the following semi-topological analogue of the Hopkins-Morel spectral sequence.

Theorem 9.2.4. *For $X \in \mathbf{Sm}_{\mathbb{C}}$ and $n \geq 0$, there is a spectral sequence*

$$E_2^{p,q}(n) = L^{n-q}H^{p-q}(X) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}_{\mathbf{sst}}(X).$$

Moreover, there is morphism of spectral sequences natural in X:

$$\begin{array}{ccc} H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q & \Rightarrow & \mathrm{MGL}^{p+q,n}(X) \\ \downarrow & & \downarrow \\ L^{n-q}H^{p-q}(X) \otimes_{\mathbb{Z}} \mathbb{L}^q & \Rightarrow & \mathrm{MGL}^{p+q,n}_{\mathbf{sst}}(X). \end{array} \tag{9.2.8}$$

Remark 9.2.5. Using the argument in the construction of the spectral sequence in Theorem 9.2.4 and the isomorphism $s_p\mathbf{KGL} \simeq \Sigma_T^p \mathbf{H}\mathbb{Z}$, where $\mathbf{KGL}$ is the algebraic K-theory motivic T-spectrum, one also obtains for any $X \in \mathbf{Sm}_{\mathbb{C}}$, a spectral sequence

$$E_2^{p,q} = L^{-q}H^{p-q}(X) \Rightarrow K^{\mathbf{sst}}_{-p-q}(X), \tag{9.2.9}$$

which is the spectral-sequence of [4, Theorem 2.10].

If we repeat the arguments in Section 9.2 for MGL smashed with mod l-Moore spectrum, and use the fact that the left vertical arrow in (9.2.8) is an isomorphism with finite coefficients (*cf.* [13, Theorem 30]), we deduce the following corollary.

Corollary 9.2.6. *For $X \in \mathbf{Sm}_{\mathbb{C}}$, $l \geq 1$ and $p,q \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{MGL}^{p,q}(X, \mathbb{Z}/l) \to \mathrm{MGL}^{p,q}_{\mathbf{sst}}(X, \mathbb{Z}/l).$$

Using the diagram (9.1.1) and applying [36, Corollary 10.6], we obtain the following.

Corollary 9.2.7. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$, there is an isomorphism of graded $\mathbb{L}_{\mathbb{Q}}$-modules*

$$\mathrm{MGL}^{*,*}_{\mathbf{sst}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq L^* H^*_{\mathbb{Q}}(X) \otimes_{\mathbb{Z}} \mathbb{L}.$$

In particular, the spectral sequence of Theorem 9.2.4 degenerates with the rational coefficients.

Another application of the spectral sequence of Theorem 9.2.4 is the following computation in terms of the algebraic cobordism Ω^*_{alg} modulo algebraic equivalence defined in [24]. Using the universal property of $\Omega^*_{\mathrm{alg}}(-)$ discussed in *ibid.*, one easily sees that for each $X \in \mathbf{Sm}_{\mathbb{C}}$, there is a natural map $\Omega^*_{\mathrm{alg}}(X) \to \mathrm{MGL}^{2*,*}_{\mathbf{sst}}(X)$.

Corollary 9.2.8. *The maps $\mathbb{L} \to \Omega^*_{\mathrm{alg}}(pt) \to \mathrm{MGL}^{2*,*}_{\mathbf{sst}}(pt)$ are isomorphisms.*

Proof. The isomorphism of the first map is already shown in [24]. The spectral sequence above shows that the natural map $\Omega^*_{\mathrm{alg}}(pt) \to \mathrm{MGL}^{2*,*}_{\mathbf{sst}}(pt)$ is surjective. On the other hand, composition with the natural map $\mathrm{MGL}^{2*,*}_{\mathbf{sst}}(pt) \to \mathrm{MU}^{2*}(pt)$ gives an isomorphism $\Omega^*_{\mathrm{alg}}(pt) \simeq \mathrm{MU}^{2*}(pt) \simeq \mathbb{L}$ by [24]. In particular, the map $\Omega^*_{\mathrm{alg}}(pt) \to \mathrm{MGL}^{2*,*}_{\mathbf{sst}}(pt)$ is injective. This completes the proof. $\square$

Remark 9.2.9. It was shown by Levine [27] that the map $\Omega^*(X) \to \mathrm{MGL}^{2*,*}(X)$ is an isomorphism for any smooth scheme X, where $\Omega^*(-)$ is the algebraic cobordism theory of Levine and Morel [28]. Using Theorem 9.2.4 and Corollary 9.2.8, probably it is possible to prove, using the arguments of [27], that the map $\Omega^*_{\mathrm{alg}}(X) \to \mathrm{MGL}^{2*,*}_{\mathbf{sst}}(X)$ is an isomorphism for all $X \in \mathbf{Sm}_{\mathbb{C}}$. But we do not attempt this in this paper.

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