The contact sphere theorem in dimension three or higher

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Abstract

Given a closed contact three-manifold with a compatible Riemannian metric g, in this paper we show that if the Ricci curvature $Ric(g)$ of g is positive, then the contact structure is universally tight. This result can be thought of as a three-dimensional contact version of the well-known sphere theorem in Riemannian geometry, and affirmatively answers an important question posed by Etnyre, Komendarczyk, and Massot. The basic idea of the proof of main result is to make use of the one-parameter family of Riemannian metrics obtained by the Hamilton's Ricci flows and their corresponding family of contact one-forms.

Keywords: contact structure, universally tight, compatible Riemannian metric, Ricci curvature, the contact sphere theorem

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1 Introduction and Main Results

One of the fundamental problems in Riemannian geometry is to study the topology of Riemannian manifolds with certain curvature conditions. In particular, it is still an intriguing open problem to classify the Riemannian manifolds with positive sectional curvature in full generality. However, it have been shown recently in [11] via the works of Fintushel, Pao, and Perelman that every positively curved Riemannian four-manifold with an isometric circle action is diffeomorphic to S^4, $\mathbb{RP}^4$, or $\mathbb{CP}^2$. It is also worthwhile to mention that it is generally believed that every positively curved four-manifold always admits an isometric circle action.

In a similar context, it will be one of the interesting problems in contact geometry and topology to study how the existence of certain Riemannian metric can affect the topology of contact manifolds. According to the authors of the paper [7], one interesting reference result in Riemannian geometry from the point of view of contact geometry is the sphere theorem. The reason is that the sphere theorem is regarded as one of the fundamental results in Riemannian geometry which clearly shows how geometry can control the topology of the underlying manifold.

We say that a manifold M has *pointwise* $\frac{1}{4}$*-pinched sectional curvature* K if M has positive sectional curvature and for every point $p \in M$ the ratio of the maximum to the minimum sectional curvature at that point is less than 4. In other words, for every pair of two-planes π_1 and π_2 of T_pM we have

$$0 < \frac{1}{4}K(\pi_1) < K < K(\pi_2).$$

The sphere theorem recently proved by Brendle and Schoen in [3] says that every compact Riemannian manifold of dimension $n \geq 4$ with pointwise $\frac{1}{4}$-pinched sectional curvature admits a metric of constant curvature and therefore is diffeomorphic to a spherical space form (see also [4] and [2]). In case of dimension there, Hamilton showed a much stronger result (refer to [9]). To be more precise, using the techniques of Ricci flows, he showed that any Riemannian metric on a three-manifold with positive Ricci curvature converges to a Riemannian metric with positive constant curvature. As a consequence, every closed Riemannian manifold of dimension 3 with positive sectional curvature has the universal cover diffeomorphic to S^3.

A contact three-manifold (M, ξ, g) is a *contact metric manifold* if there is a Riemannian metric g that is compatible with the contact structure ξ in the sense that a contact form α of ξ satisfies

$$||\alpha|| = 1 \text{ and } *_g\, d\alpha = \theta'\alpha,$$

where $||\cdot||$ means the pointwise norm, θ' is a positive constant, and $*_g$ is the Hodge star operator associated to g (refer to the paper [5] of Chern and Hamilton for the case of $\theta' = 2$). Note that the positivity of θ' is equivalent to ξ being a positive contact structure (see Section 2 of the paper [7]). The geometric meaning of the constant θ' is the instantaneous rotation speed of ξ with respect to g, and θ' is called the *instantaneous rotation* or just *rotation* of ξ. This definition of the contact metric three-manifold is equivalent to saying that the instantaneous rotation θ' of ξ with respect to g is constant and the Reeb vecto field R_α is of unit length and orthogonal to ξ (see, e.g., the first paragraph of [7], page 634).

Recall that a contact structure on a three-manifold is classified as overtwisted or tight, depending on whether or not it contains an embedded disk whose boundary is tangent to the contact planes (see [6]). In the paper [7], Etnyre, Komendarczyk, and Massot investigated how the sectional curvature bound of a contact metric manifold (M, ξ, g) implies the tightness of ξ. To be precise, their result that is called $\frac{4}{9}$*-pinched contact sphere theorem* is

Theorem 1.1. *Let (M, ξ) be a closed contact manifold of dimension three and g a Riemannian metric compatible with ξ. If there is a constant K_M such that the sectional curvature $K(g)$ of g satisfies*

$$0 < \frac{4}{9}K_M < K(g) \leq K_M,$$

then the universal cover of M is diffeomorphic to the standard 3-sphere S^3 by a diffeomorphism taking the lift of ξ to the standard contact structure on S^3.

This theorem says that every contact structure compatible with a $\frac{4}{9}$-pinched positively curved Riemannian metric on a contact metric three-manifold is universally tight. Some essential ingredients of the proof of Theorem 1.1 are certain nice comparison of Riemannian and almost-complex convexity and pseudo-holomorphic curves arguments of Gromov and Hofer in [10]. It is unfortunate that the proof of Theorem 1.1 given in [7] does not tell us very much information about whether or not the pinching constant $\frac{4}{9}$ can be improved further. However, in view of the results of Hamilton's Ricci flows, it seems to be natural to ask if the positivity of the Ricci curvature is sufficient to obtain the same conclusion of Theorem 1.1 (refer to Question 6.4 in [7]).

Our main result of this present paper is that asking only the Ricci curvature to be positive is indeed enough to prove the various versions of contact sphere theorem. To be precise, we have the following

Theorem 1.2. *Let (M, ξ) be a closed contact manifold of dimension three and g a Riemannian metric compatible with ξ. If the Ricci curvature $Ric(g)$ of g is positive, then the contact structure ξ is universally tight.*

In Section 2, we give a proof of Theorem 1.2.

The basic strategy of the proof of Theorem 1.2 given in Section 2 is to make use of the one-parameter family $\{g_t\}$ $(0 \leq t \leq 1)$ of Riemannian metrics obtained by the Hamilton's Ricci flows and their corresponding family $\{\alpha_t\}$ of contact one-forms. It turns out that for each $0 \leq t \leq 1$ the Riemannian metric g_t is compatible with the contact structure $\xi_t = \ker \alpha_t$ (see

Section 2 for the precise definition). As a consequence of the result of Hamilton's Ricci flows, we may assume without loss of generality that under the positivity of the Ricci curvature of the initial metric g_0 the terminal Riemannian metric g_1 has positive and constant sectional curvature equal to 1. So the universal cover of the manifold should be the standard three-sphere. It then follows from Theorem 1.1 of Etnyre, Komendarczyk, and Massot that the contact structure ξ_1 associated to the terminal Riemannian metric g_1 is universally tight. On the other hand, by essentially using the techniques of the proof of the Gray's theorem it can be shown in Lemma 2.3 that, given the one-parameter family $\{\alpha_t\}$ of contact one-forms, there is a one-parameter family $\{\varphi_t\}$ of diffeomophisms such that $\varphi_t^*(\alpha_t) = \alpha_0$. So we can conclude that the initial contact structure ξ_0 should be also universally tight.

For the purposes of this paper, we do not need any deep facts about the Ricci flows, except for the unnormalized Ricci flow equation (2.6) and Hamilton's results in [9]. So we do not review any general properties on the Ricci flows, in order to make this paper concise as much as possible. We refer any interested reader to the excellent papers and books such as [9], [15], [14], [12], and [1] for more complete details on Ricci flows.

Finally we remark that, as mentioned above, the sphere theorem holds also for higher dimensional Riemannian manifolds with certain curvature conditions. So it is possible to extend and generalize Theorem 1.2 to higher dimensions by using the notions such as PS-overtwisted or PS-tight contact structures given in the paper [8] analogous to overtwisted or tight contact structures on a three-manifold. Actually, this can be easily achieved without much modification by using Theorem 1.1 in [8] along the same lines of this paper. But we will not pursue this direction further in this paper, since, we think, the three dimensional case is most interesting until now.

2 Proof of Theorem 1.2

The aim of this section is to provide a proof of Theorem 1.2 in detail.

To do so, let $\xi_0 := \xi$ be a contact structure on a closed oriented three-manifold M, and let α_0 be a contact one-form on M whose kernel is exactly equal to ξ_0. Let g_0 be a Riemannian metric compatible with ξ_0, i.e.,

$$||\alpha_0|| = 1 \text{ and } *_{g_0} d\alpha_0 = \theta_0' \alpha_0,$$

where θ_0' is a positive constant.

For the rest of this paper, we shall also assume that the Riemannian metric g_0 has a positive Ricci curvature, unless stated otherwise.

Next we apply the Hamilton's theorem to the Riemannian metric g_0 as an initial one so that g_0 can be deformed along the Ricci flows to a Riemannian metric g_1 whose sectional curvature is positive and constant, say $+1$. As a consequence, we may assume without loss of generality that M is simply connected and M is just the standard three-sphere. For the sake of notational convenience, as before let $\{g_t\}$ $(0 \leq t \leq 1)$ be the family of Riemannian metrics starting from g_0 and ending to g_1 obtained by the techniques of Hamilton's Ricci flows.

Let R_β denote the Reeb vector field for a contact one-form β on M. Then the following lemma holds (see Proposition 2.1 in [7]).

Lemma 2.1. *Let β be a positive contact one-form on a three-manifold M and g a Riemannian metric on M. The following statements are equivalent:*

(a) *The Reeb vector field R_β is orthogonal to ξ with respect to g.*

(b) *There is some positive function θ' such that*

$$*_g d\beta = \theta' \beta,$$

where $*_g$ is the Hodge star operator associated to g.

Let $\mathbf{n}$ be the unit normal vector field to the contact structure ξ_0, and let $\mathbf{n}_t$ be the vector field on M given by

$$\mathbf{n}_t = \frac{\mathbf{n}}{\sqrt{g_t(\mathbf{n}, \mathbf{n})}}.$$

Then $\mathbf{n}_0 = \mathbf{n}$ and $\mathbf{n}_t$ is the unit vector field on M with respect to the Riemannian metric g_t.

Next for each $0 \leq t \leq 1$ we define a new one-form α_t on M by

$$\alpha_t(\cdot) = g_t(\mathbf{n}_t, \cdot).$$

Then α_t is a nowhere vanishing one form on M, since g_t is a Riemannian metric and $\mathbf{n}_t$ is a nowhere vanishing (actually, unit) vector field on M. Let $\xi_t = \ker \alpha_t$. Then clearly $\mathbf{n}_t$ is always orthogonal to ξ_t with respect to g_t for all $t \in [0, 1]$. Since $\alpha_t(\mathbf{n}_t)$ is always equal to one for each $0 \leq t \leq 1$, we can define a smooth function $\tilde{\theta}_t : M \to \mathbb{R}$ such that

$$*_{g_t} d\alpha_t = \tilde{\theta}_t \alpha_t, \quad 0 \leq t \leq 1.$$

Then $\tilde{\theta}_t$ is at least continuous as a function of t. Moreover, it is easy to show that

$$\alpha_t \wedge d\alpha_t = \tilde{\theta}_t \mathrm{vol}_{g_t}. \tag{2.1}$$

The following lemma plays a crucial role in the proof of Theorem 1.2.

Lemma 2.2. *Let*

$$t^* = \sup\{t \in [0,1] \,|\, \alpha_t \text{ is a positive contact one-form on } M\}.$$

Then t^* is equal to 1 and α_1 is a positive contact one-form on M.

Proof. It is clear by the choice of ξ_0 and its contact one-form α_0 that t^* is greater than 0 and that α_t is a positive contact one-form for at least t with $0 \leq t < t^*$.

We then show that α_{t^*} is actually a positive contact form on M. To see it, note first that for $0 \leq t < t^*$ the Reeb vector field R_{α_t} is exactly same as the unit normal vector field $\mathbf{n}_t$ to the contact structure ξ_t. Indeed, since $g_t(\mathbf{n}_t, u) = 0$ for any $u \in \xi_t = \ker \alpha_t$, by using the flow lines of $\mathbf{n}_t$ we have

$$0 = g_t([\mathbf{n}_t, \mathbf{n}_t], u) + g_t(\mathbf{n}_t, [\mathbf{n}_t, u]) = g_t(\mathbf{n}_t, [\mathbf{n}_t, u]).$$

This implies that

$$\iota_{\mathbf{n}_t} d\alpha_t = -\alpha_t([\mathbf{n}_t, \cdot]) = -g_t(\mathbf{n}_t, [\mathbf{n}_t, \cdot]) = 0.$$

Since $\alpha_t(\mathbf{n}_t) = 1$, it follows that $\mathbf{n}_t$ is actually the Reeb vector field for α_t, as desired.

Since $R_{\alpha_t} = \mathbf{n}_t$ is now shown to be orthogonal to $\xi_t = \ker \alpha_t$ with respect to the Riemannian metric g_t, it follows from Lemma 2.2 that there is some positive function θ'_t such that

$$*_{g_t} d\alpha_t = \theta'_t \alpha_t, \quad 0 \leq t < t^*.$$

Since $\alpha_t(\mathbf{n}_t) = 1$ for each $0 \leq t \leq 1$, it is also easy to see that the function θ'_t coincides with $\tilde{\theta}_t$ for $0 \leq t < t^*$. Since $\theta'_t = \tilde{\theta}_t$ is positive for $0 \leq t < t^*$ and $\tilde{\theta}_t$ is continuous as a function of $t \in [0,1]$, it follows from the continuity that $\tilde{\theta}_{t^*}$ is non-negative at $t = t^*$. In fact, we can show further that $\tilde{\theta}_{t^*}$ is actually positive, as follows.

Lemma 2.3. *The non-negative function $\tilde{\theta}_{t^*}$ of M does not vanish at $t = t^*$, and, in fact, is positive.*

Proof. To see it, note first that, if $d\alpha_{t^*}|_{\xi_{t^*}} \neq 0$, then it follows from (2.1) and $\alpha_t(\mathbf{n}_t) = 1$ that we have $\tilde{\theta}_{t^*} \neq 0$. Thus we are done.

Next, suppose that $d\alpha_{t^*}|_{\xi_{t^*}} = 0$ at a fixed point p of M, and for the rest of the proof we will do all the computations at the same point p of M, unless stated otherwise.

Let

$$f(t) = (\alpha_t \wedge d\alpha_t)(\mathbf{n}_t, \mathbf{u}_t, \mathbf{v}_t), \quad 0 \leq t \leq 1,$$

where $\mathbf{u}_t$ and $\mathbf{v}_t$ lie in the kernel $\xi_t = \ker(\alpha_t)$ of α_t, and $\{\mathbf{n}_t, \mathbf{u}_t, \mathbf{v}_t\}$ forms an oriented orthogonal frame of the tangent space T_pM of M with respect to the Riemannian metric g_t such that

$$[\mathbf{u}_t, \mathbf{v}_t] \neq 0 \tag{2.2}$$

at the point p. We remark that the extra condition (2.2) can be easily achieved: for instance, if $[\mathbf{u}_t, \mathbf{v}_t]$ happens to be zero at the point p, then it suffices to take $\tilde{\mathbf{u}}_t$ and $\tilde{\mathbf{v}}_t$ instead of $\mathbf{u}_t$ and $\mathbf{v}_t$ such that

$$\tilde{\mathbf{u}}_t := k\mathbf{u}_t, \quad \tilde{\mathbf{v}}_t := l\mathbf{v}_t,$$

where k and l are smooth non-zero functions on M satisfying $k\mathbf{u}_t(l) \neq 0$ or $l\mathbf{v}_t(k) \neq 0$ at p. Then, by construction, $\tilde{\mathbf{u}}_t$ and $\tilde{\mathbf{v}}_t$ which lie in the kernel ξ_t of α_t are still orthogonal to each other, but we now have

$$[\tilde{\mathbf{u}}_t, \tilde{\mathbf{v}}_t] = kl[\mathbf{u}_t, \mathbf{v}_t] + k\mathbf{u}_t(l)\mathbf{v}_t - l\mathbf{v}_t(k)\mathbf{u}_t = k\mathbf{u}_t(l)\mathbf{v}_t - l\mathbf{v}_t(k)\mathbf{u}_t \neq 0$$

at the point p, as desired.

Since $\alpha_t(\mathbf{n}_t) = 1$, we then have

$$f(t) = d\alpha_t(\mathbf{u}_t, \mathbf{v}_t) = -\alpha_t([\mathbf{u}_t, \mathbf{v}_t]) = -g_t(\mathbf{n}_t, [\mathbf{u}_t, \mathbf{v}_t]).$$

Recall that by assumption $f(t^*) = 0$. Thus, it is also true that at $t = t^*$ we have

$$g_t\left(\frac{\partial \mathbf{n}_t}{\partial t}, [\mathbf{u}_t, \mathbf{v}_t]\right) = 0. \tag{2.3}$$

Indeed, since $g_t(\mathbf{u}_t, \mathbf{v}_t) = 0$, we have

$$g_t(\mathbf{u}_t, [\mathbf{u}_t, \mathbf{v}_t]) = g_t([\mathbf{v}_t, \mathbf{u}_t], \mathbf{v}_t) = 0. \tag{2.4}$$

Thus, if we write

$$\frac{\partial \mathbf{n}_t}{\partial t} = a_t\mathbf{n}_t + b_t\mathbf{u}_t + c_t\mathbf{v}_t,$$

then it is easy to see from $f(t^*) = 0$ together with (2.4) that (2.3) holds at $t = t^*$.

By differentiating $f(t)$ with respect to t, it follows from (2.3) that, at $t = t^*$, we can obtain

$$\begin{aligned} f'(t) &= -\frac{\partial g_t}{\partial t}(\mathbf{n}_t, [\mathbf{u}_t, \mathbf{v}_t]) - g_t\left(\frac{\partial \mathbf{n}_t}{\partial t}, [\mathbf{u}_t, \mathbf{v}_t]\right) - g_t\left(\mathbf{n}_t, \frac{\partial [\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right) \\ &= -\frac{\partial g_t}{\partial t}(\mathbf{n}_t, [\mathbf{u}_t, \mathbf{v}_t]) - g_t\left(\mathbf{n}_t, \frac{\partial [\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right) \\ &= 2Ric(\mathbf{n}_t, [\mathbf{u}_t, \mathbf{v}_t]) - g_t\left(\mathbf{n}_t, \frac{\partial [\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right), \end{aligned} \tag{2.5}$$

where in the last equality we used the assumption that g_t satisfies the Ricci flow equation

$$\frac{\partial g_t}{\partial t} = -2Ric. \tag{2.6}$$

Now, we claim that we can take $\mathbf{u}_t$ and $\mathbf{v}_t$ in such a way that they satisfy one more extra condition at $t = t^*$;

$$\left|\left|\frac{\partial [\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right|\right|_{g_t} < 2C_t||[\mathbf{u}_t, \mathbf{v}_t]||_{g_t}, \tag{2.7}$$

where C_t is a positive lower bound for the positive Ricci curvature Ric. To see it, let us first write $\mathbf{u}_t$ and $\mathbf{v}_t$ in ξ_t, as follows: near $t = t^*$,

$$\begin{aligned} \mathbf{u}_t &= \mathbf{u}_{t^*} + (t - t^*)\mathbf{w}_1 + O((t - t^*)^2), \\ \mathbf{v}_t &= \mathbf{v}_{t^*} + (t - t^*)\mathbf{w}_2 + O((t - t^*)^2). \end{aligned}$$

Then, we take

$$\begin{aligned} \tilde{\mathbf{u}}_t &= \mathbf{u}_{t^*} + (t - t^*)\epsilon\mathbf{w}_1 + O((t - t^*)^2), \\ \tilde{\mathbf{v}}_t &= \mathbf{v}_{t^*} + (t - t^*)\epsilon\mathbf{w}_2 + O((t - t^*)^2), \end{aligned}$$

where ϵ is a sufficiently small positive number. It is easy to see that at $t = t^*$ both of $\tilde{\mathbf{u}}_t$ and $\tilde{\mathbf{v}}_t$ still lie in the kernel ξ_t of α_t. Furthermore, they satisfy the required condition (2.7) for a sufficiently small $\epsilon > 0$ and a contact one-form α_{t^*}, since at $t = t^*$ we have

$$[\tilde{\mathbf{u}}_t, \tilde{\mathbf{v}}_t] = [\mathbf{u}_{t^*}, \mathbf{v}_{t^*}], \quad \frac{\partial [\tilde{\mathbf{u}}_t, \tilde{\mathbf{v}}_t]}{\partial t} = \epsilon([\mathbf{u}_{t^*}, \mathbf{w}_2] + [\mathbf{w}_2, \mathbf{v}_{t^*}]).$$

Finally, it should be clear that, by the continuity of α_t near $t = t^*$ and the openness of the condition (2.7), we can achieve the required condition (2.7) for a general one-form α_t, not just α_{t^*}, as claimed.

As a final step of the proof of Lemma 2.3, note that, by (2.5) and (2.7), at $t = t^*$ we have

$$\begin{aligned} f'(t) &= 2Ric(\mathbf{n}_t, [\mathbf{u}_t, \mathbf{v}_t]) - g_t\left(\mathbf{n}_t, \frac{\partial[\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right) \\ &\geq 2C_t||[\mathbf{u}_t, \mathbf{v}_t]||_{g_t} - \left|\left|\frac{\partial[\mathbf{u}_t, \mathbf{v}_t]}{\partial t}\right|\right|_{g_t} > 0. \end{aligned} \tag{2.8}$$

On the other hand, since $f(t)$ is strictly positive for $0 \leq t < t^*$ and $f(t^*)$ is assumed to be zero, $f'(t^*)$ should be non-positive. This observation contradicts the inequality (2.8), and so the case of $d\alpha_{t^*}|_{\xi_{t^*}} = 0$ actually does not occur.

As a consequence, we can conclude that $\tilde{\theta}_{t^*}$ is indeed non-zero and so positive, which completes the proof of Lemma 2.3. □

In particular, Lemma 2.3 implies that from the equation (2.1) we have

$$\alpha_{t^*} \wedge d\alpha_{t^*} = \theta'_{t^*} \mathrm{vol}_{g_{t^*}}$$

with a positive function θ'_{t^*} that is equal to $\tilde{\theta}_{t^*}$. Therefore we see that α_{t^*} is a positive contact one-form on M so that the kernel ξ_{t^*} of α_{t^*} is now a positive contact structure, as claimed.

Finally, if t^* is equal to 1, then we are done. Otherwise, by the standard continuity argument we can show that there is some t greater than t^* where α_t is a positive contact one-form on M. Clearly this contradicts the choice of t^*. This completes the proof of Lemma 2.2. □

As a consequence of the proof of Lemma 2.2, the Reeb vector field R_{α_t} $(0 \leq t \leq 1)$ has unit length, so that the flow lines of R_{α_t} are actually geodesics (refer to, e.g., Proposition 2.5 in [8]).

The proof of the following lemma uses the well-known techniques of Gray's theorem in contact geometry which is analogous to the Moser's techniques in symplectic geometry.

Lemma 2.4. *There is a family* $\{\varphi_t\}$ $(0 \leq t \leq 1)$ *of diffeomorphisms of* M *such that*

$$\varphi_t^*(\alpha_t) = \alpha_0.$$

Proof. As mentioned above, we shall apply the Gray's arguments to prove the lemma (refer to [13], p. 112). To do so, note first that $\frac{d}{dt}\alpha_t$ vanishes on the Reeb vector field $R_{\alpha_t} = \mathbf{n}_t$. Indeed, starting from the identity $\alpha_t(\mathbf{n}_t) = 1$, along the flow of X_t tangent to $\xi_t = \ker \alpha_t$ we have

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\alpha_t\right)(\mathbf{n}_t) + \alpha_t(\mathcal{L}_{X_t}\mathbf{n}_t) \\ &= \left(\frac{d}{dt}\alpha_t\right)(\mathbf{n}_t) + g_t(\mathbf{n}_t, [X_t, \mathbf{n}_t]), \ \alpha_t(\cdot) = g_t(\mathbf{n}_t, \cdot) \text{ and } \mathcal{L}_{X_t}\mathbf{n}_t = [X_t, \mathbf{n}_t], \\ &= \left(\frac{d}{dt}\alpha_t\right)(\mathbf{n}_t), \quad g_t(\mathbf{n}_t, [X_t, \mathbf{n}_t]) = 0 \text{ by (2.2)}, \end{aligned}$$

as desired.

On the other hand, since $d\alpha_t$ is symplectic on ξ_t and $\frac{d}{dt}\alpha_t$ vanishes on $\mathbf{n}_t$, there exists a unique smooth vector field $X_t \in \xi_t$ such that

$$-\iota_{X_t} d\alpha_t = \frac{d}{dt}\alpha_t. \tag{2.9}$$

Let $\{\varphi_t\}$ $(0 \le t \le 1)$ be a family of diffeomorphisms as the flow of a family $\{X_t\}$ of vector fields on M such that

$$\frac{d}{dt}\varphi_t = X_t \circ \varphi_t, \quad \varphi_0 = \mathrm{id}.$$

Then from the identity (2.9) we have

$$\varphi_t^*\left(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t\right) = 0.$$

This implies that $\frac{d}{dt}(\varphi_t^*\alpha_t) = 0$ for all $0 \le t \le 1$, so that $\varphi_t^*\alpha_t = \alpha_0$. This completes the proof of Lemma 2.4. □

The following lemma will be also used in the proof of Theorem 1.2.

Lemma 2.5. *The following statements are true:*

(a) $(\varphi_t)_*(\mathbf{n}_t) = \mathbf{n}_0$.

(b) $\varphi_t^*(g_t) = g_0$.

Proof. For the proof of (a), it suffices to note that every Reeb vector field is preserved by the contact isomorphism φ_t. Since $R_{\alpha_t} = \mathbf{n}_t$ (resp. $R_{\alpha_0} = \mathbf{n}_0$) is the Reeb vector field for α_t (resp. α_0), we are done.

For (b), its proof is also immediate by the definition of $\alpha_t = g_t(\mathbf{n}_t, \cdot)$, since the contact isomorphism φ_t preserves the contact structures as well as the Reeb vector fields. □

Now we are in a position to give a proof of Theorem 1.2.

Proof of Theorem 1.2. To prove the theorem, recall first that M is assumed to be simply connected and so is the standard three-sphere S^3. Note also that by Lemma 2.2 the one-form α_1 is a positive contact one-form whose kernel is a positive contact structure. Moreover, since $\{\alpha_t\}$ $(0 \leq t \leq 1)$ is now a family of (positive) contact one-forms on M, it follows from a version of Gray's theorem (Lemma 2.4) that there is a contact isomorphism φ_t on M such that $\varphi_t^*(\alpha_t) = \alpha_0$.

We can then claim that the Riemannian metric g_t is compatible with ξ_t. To prove it, it suffices to show that θ_t' is constant as a function of M. Note that θ_t' is well-defined for all $0 \leq t \leq 1$ by Lemma 2.2. Indeed, we have

$$\begin{aligned}\varphi_t^*(\theta_t')\alpha_0 &= \varphi_t^*(\theta_t')\varphi_t^*(\alpha_t) = \varphi_t^*(\theta_t'\alpha_t) = \varphi_t^*(*_{g_t} d\alpha_t)\\ &= *_{\varphi_t^*(g_t)} d\varphi_t^*(\alpha_t) = *_{g_0} d\alpha_0\\ &= \theta_0'\alpha_0,\end{aligned}$$

where in the second-to-last equality we used the fact that $\varphi_t^*(g_t) = g_0$ by Lemma 2.5 (b). Thus we have $\varphi_t^*(\theta_t') = \theta_0'$. Since θ_0' is constant, it follows that for each $0 \leq t \leq 1$ the function θ_t' is also constant as a function of M, as claimed.

On the other hand, since M is the standard three-sphere S^3 and g_1 has positive and constant sectional curvature equal to 1, it follows from Theorem 1.1 in [7] (or Theorem 1.1) that ξ_1 is tight on M. Hence ξ_0 is also tight, since ξ_0 and ξ_1 are contactomorphic to each other. This completes the proof of Theorem 1.2. $\square$

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