The number of exceptional orbits of a pseudo-free circle actions on S^5

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Abstract

Let M be a closed oriented Riemannian manifold of dimension 5 which admits a Riemannian metric of positive sectional curvature. In this short paper, we point out that under certain lower bound of the orders of isotropy subgroups, every pseudo-free and isometric S^1-action on M cannot have more than five exceptional circle orbits. As a consequence, we conclude that such a pseudo-free S^1-action on S^5 cannot have more than five exceptional circle orbits. This gives a new result related to the Montgomery-Yang problem posed in [9].

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1 Introduction and Main Results

Throughout this paper, every odd dimensional sphere S^{2k-1} $(k \geq 2)$ is assumed to be an unit sphere embedded in the Euclidean space $\mathbf{R}^{2k}$ unless stated otherwise.

The main goal of this paper is to address the problem posed by Montgomery and Yang in [9] about the number of exceptional orbits of the unit 5-sphere S^5 equipped with a pseudo-free S^1-action. Recall that a *pseudo-free* S^1*-action* on a sphere S^{2k-1} $(k \geq 2)$ is, in general, defined to be a smooth S^1-action which is free everywhere except for finitely many exceptional circle orbits whose isotropy subgroups are $\mathbf{Z}_{r_1}, \ldots, \mathbf{Z}_{r_n}$, where $r_1, \ldots, r_n$ are pairwise relatively prime. According to a theorem of Smith, the condition that $r_1, \ldots, r_n$ are pairwise relatively prime is actually superfluous for pseudo-free circle actions on any closed $(2n+1)$-dimensional manifold which has the integral homology of a sphere (see, e.g., (2.2) in [9] or p. 43 in [1]). For pseudo-free circle actions on a general closed $(2n+1)$-dimensional manifold, however, it is not automatic. Thus, in this paper we will not require that $r_1, \ldots, r_n$ be pairwise relatively prime for pseudo-free circle actions on a general closed manifold. As a consequence, the number of exceptional circle orbits obtained in Theorem 1.1 would be more optimal (cf. Conjecture 2 in [8]).

It is well-known by a work [11] of Seifert that every pseudo-free S^1-action on S^3 is linear and so has at most two exceptional orbits. On the other hand, it has been shown by Montgomery and Yang in [9] that given any natural number n, there is a pseudo-free S^1-action on a homotopy 7-sphere whose number of exceptional orbits is exactly equal to n. In [10], Petrie proved similar results in all higher odd dimensions. So there exists a drastic difference between odd dimensions 3 and 7 or higher.

However, it has still remained as an interesting open question whether or not a pseudo-free S^1-action on S^5 (or homotopy 5-sphere) has at most three (or more) exceptional orbits. This is usually called the *Montgomery-Yang problem* in the literature. As far as we know, there are almost no known results regarding this problem, despite some attempts. Recently, this problem has been re-casted by J. Kollár as an algebraic Montgomery-Yang problem in [8], and some important partial results to the algebraic version have been obtained by J. Keum and D. Hwang in [5, 6].

In this paper, we show some new result about the original Montgomery and Yang's problem under the lower bound of the orders of isotropy subgroups. In fact, it turns out to be a direct consequence of the following more general theorem.

Theorem 1.1. *Let M be a closed oriented Riemannian manifold of dimension 5 which admits a Riemannian metric of positive sectional curvature, and let S^1 act pseudo-freely and isometrically on M with isotropy subgroups $\mathbf{Z}_{r_1}, \ldots, \mathbf{Z}_{r_n}$. If all the r_i's are greater than or equal to 61, then n is less than or equal to 5, i.e., the pseudo-free S^1-action on M cannot have more than five exceptional circle orbits.*

It is crucial that the dimension of the Riemannian manifold M is 5. In other words, the proof of Theorem 1.1 does not apply to a positively curved manifold of dimension ≥ 7 with a pseudo-free and isometric circle action. We also remark that it is not true in general and seems to be well-known that every effective S^1-action on a closed oriented non-negatively (or positively) curved Riemannian manifold can be made isometric.

Since the sphere S^5 is assumed to be the unit sphere embedded in R^6, it follows that it can be equipped with a Riemannian metric of positive sectional curvature such as the canonical metric on S^5. Hence we have the following corollary which gives some result related to the Montgomery-Yang problem.

Corollary 1.2. *Let S^1 act pseudo-freely and isometrically on S^5 with isotropy subgroups $\mathbf{Z}_{r_1}, \ldots, \mathbf{Z}_{r_n}$. If all the r_i's are greater than or equal to 61, then n is less than or equal to 5, i.e., the pseudo-free S^1-action on S^5 cannot have more than five exceptional circle orbits.*

As mentioned above, the proof of Theorem 1.1 does not apply to an odd dimensional sphere S^{2k-1} ($k \geq 4$), and so this fits well with the result of Montgomery and Yang in [9]. We remark that there are some cases for which the lower bound 61 can be improved (see, e.g., Corollary 8 in [12]), but we do not pursue it in his paper.

Our method of the proof of Theorem 1.1 is just an application of some well-known techniques in [4] developed by Hsiang and Kleiner when they prove a theorem that every closed oriented positively curved Riemannian manifold of dimension 4 with an isometric S^1-action is homeomorphic to S^4 or $\mathbf{CP}^2$ (see [7] for a recent further development).

We organize this paper as follows. In Section 2, we collect some basic definitions and facts for later use. In Section 3, we give a proof of Theorem 1.1 by essentially applying the techniques of Hsiang-Kleiner and a result of Yang in [12].

2 Preliminaries

In order to give a proof of Theorem 1.1 in Section 3, in this section we collect some basic facts and definitions for the sake of reader's convenience (see, e.g., [3] for more details).

We first consider the geometry of the orbit space $X = M/S^1$. To do so, we need to recall some basic notions of Alexandrov geometry. A finite dimensional length space (X, dist) is called an *Alexandrov space* if it has curvature bounded from below. It is true that when M is a complete connected Riemannian manifold and G is a compact Lie group acting effectively on M by isometry, the orbit space X can be equipped with the orbital distance metric induced from M. Hence the distance between two points $\bar{p}$ and $\bar{q}$ in X is given by the distance between two orbits $G \cdot p$ and $G \cdot q$ as subsets of M. Furthermore, if M has sectional curvature bounded from below, sec-curv$(M) \geq k$ (resp. $> k$), then the orbit space X is an Alexandrov space with curvature bounded from below such that sec-curv$(X) \geq k$ (resp. $> k$).

We will also need the notion of the *space of directions of a Alexandrov space X at a point $\bar{p} \in X$*. In general, it is defined to be the completion of the space of geodesic directions at $\bar{p} \in X$. As a particular case, when X is given by an orbit space M/G, the space of directions at a point $\bar{p} \in X$, denoted $S_{\bar{p}}X$, consists of geodesic directions and is isometric to $S_p^{\perp}/G_p$, where $S_p^{\perp}$ denotes the normal sphere to the orbit $G \cdot p$ at $p \in M$. Here we assume that p maps to $\bar{p}$ in X, and G_p denotes the stabilizer subgroup of G at p.

Next, let $\{p_i\}$ be the set of distinct points in M (or X). For a pair of i and j, define Γ_{ij} be the set of minimizing normal geodesics from p_i to p_j. Then for three distinct i, j, k, the angle between p_j and p_k at p_i is defined to be

$$\angle_{p_i}(p_j, p_k) = \min\{\angle(\gamma_j'(0), \gamma_k'(0)) \mid \gamma_j \in \Gamma_{ij}, \gamma_k \in \Gamma_{ik}\}.$$

We also need the notion of n-extent of a compact metric space. To be precise, the n-extent $xt_n(X)$ for $n \geq 2$ of a compact metric space (X, d) is defined to be the maximum average distance between n points in X, i.e.,

$$xt_n(X) = \frac{2}{n(n-1)} \max \left\{ \sum_{1 \leq i < j \leq n} d(x_i, x_j) \mid \{x_i\}_{i=1}^n \subset X \right\}. \tag{2.1}$$

Given a positive integer r and integers k, l coprime to r, let $L(r; k, l)$ be the 3-dimensional lens space which is the quotient of a free isometric $\mathbf{Z}_{r_i}$-action on the unit sphere S^3 defined by

$$\psi_{k,l} : \mathbf{Z}_r \times S^3 \to S^3, \quad (g, (z_1, z_2)) \mapsto (g^k z_1, g^l z_2),$$

where g is a generator of $\mathbf{Z}_r$ and $(z_1, z_2) \in S^3 \subset \mathbf{C}^2$. Then D. Yang showed the following estimate of n-extent of lens space $L(r; k, l)$ in [12].

Lemma 2.1. *Let $L(r; k, l)$ be a 3-dimensional lens space of constant sectional curvature 1. Then we have*

$$\begin{aligned} &xt_q(L(r;k,l)) \\ &\leq \arccos\left(\cos(\alpha_q)\cos\left(\frac{\pi}{\sqrt{r}}\right) - \frac{1}{2}\left(\left(\cos(\frac{\pi}{\sqrt{r}}) - \cos(\frac{\pi}{r})\right)^2 \right.\right. \\ &\quad \left.\left. + \sin^2(\alpha_q)\left(\sqrt{r}\sin(\frac{\pi}{r}) - \sin(\frac{\pi}{\sqrt{r}})\right)^2\right)^{\frac{1}{2}}\right), \end{aligned} \tag{2.2}$$

where $\alpha_q = \frac{\pi}{2(2-[\frac{q+1}{2}]^{-1})}$ and $[x]$ denotes the greatest integer less than or equal to x.

In particular, if $r \geq 61$, then $xt_5(L(r; k, l))$ is less than $\frac{\pi}{3}$.

3 Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1. To do so, from now on we assume that the circle S^1 on M always acts pseudo-freely and isometrically.

To begin the proof of Theorem 1.1, we also assume that there are six exceptional circle orbits $\{E_i\}_{i=1}^6$ and then we will derive a contradiction.

For each i $(1 \le i \le 6)$, let p_i be a point in E_i. Since every exceptional circle orbit is isolated, the six circle orbits $\{E_i\}_{i=1}^6$ map to six distinct points $\bar{p}_i$ in the orbit space X. Recall that X is an Alexandrov space with sec-curv > 0. Hence it follows from Toponogov's theorem for Alexandrov spaces that the sum of the angles of a geodesic triangle in X is greater than or equal to π. Since there are twenty triangles obtained by connecting each pair of distinct points in $\{\bar{p}_i\}_{i=1}^6$ by minimal geodesics, the total sum of the angles in the twenty triangles is greater than or equal to 20π.

Note also that for each point of $\{p_i\}_{i=1}^6$, the tangent space $T_{p_i}M$ decomposes as $T_{p_i}E_i \oplus (T_{p_i}E_i)^{\perp}$, and the normal space $(T_{p_i}E_i)^{\perp}$ is invariant under the free action of the isotropy subgroup $\mathbf{Z}_{r_i}$ of p_i. Thus the quotient of the unit normal sphere $S^3 \subset (T_{p_i}E_i)^{\perp}$ is the lens space $L(r_i; k_i, l_i)$ $(1 \le k_i, l_i < r_i)$ and so $S_{p_i}X = L(r_i; k_i, l_i)$.

By assumption, since all the r_i's are greater than or equal to 61, it follows from Lemma 2.1 that the 5-extent xt_5 of the lens space satisfies

$$xt_5(L(r_i; k_i, l_i)) < \frac{\pi}{3}$$

This implies from (2.1) that for any pairs of distinct points $\{x_j\}_{j=1}^5$ in $L(r_i; k_i, l_i)$, we have

$$\sum_{1 \le j < k \le 5} d(x_j, x_k) < \frac{\pi}{3} \cdot 10. \tag{3.1}$$

Note that $d(x_j, x_k)$ is the same as the angle between two geodesics from $\bar{p}_i$ to points x_j and x_k which are considered to lie in X, respectively. Thus, by summing over all the pairs formed by each point in $\{\bar{p}_i\}_{i=1}^6$, it follows from (3.1) that the total sum of the angles in the twenty triangles obtained by connecting each pair of distinct points in $\{\bar{p}_i\}_{i=1}^6$ by minimal geodesics should be, this time, less than $\frac{\pi}{3} \cdot 10 \cdot 6 = 20\pi$. But this is clearly a contradiction. Therefore, every pseudo-free and isometric circle action on a closed oriented positively curved Riemannian manifold as in Theorem 1.1 can have at most five exceptional circle orbits. This completes the proof of Theorem 1.1.

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