ARTIN L-FUNCTIONS AND MODULAR FORMS ASSOCIATED TO QUASI-CYCLOTOMIC FIELDS

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Abstract. We determine all irreducible representations of primary quasi cyclotomic fields, compute the Artin conductors of the representations and the Artin L-functions for a class of quasi-cyclotomic fields. By Deligne-Serre's theorem, these L-functions give a series of normalized newforms of weight one. We describe these modular forms explicitly.

1. Introduction

A quadratic extension of a cyclotomic field, which is non-abelian Galois over the rational number field $\mathbb{Q}$, is called a quasi-cyclotomic field. All quasi-cyclotomic fields are described explicitly in [9] following the works in [1] and [3]. Actually for any cyclotomic field $\mathbb{Q}(\zeta_n)$ we construct a canonical $\mathbb{Z}/2\mathbb{Z}$-basis of the quotient space of $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q}$ is Galois$\}$ modulo the subspace $\{\alpha \in \mathbb{Q}^*/\mathbb{Q}^{*2} \mid \mathbb{Q}(\zeta_n, \sqrt{\alpha})/\mathbb{Q}$ is Abelin$\}$. The minimal quasi-cyclotomic field containing the square root of a special element of the basis is called a primary quasi-cyclotomic field. L.Yin and C.Zhang [8] have studied the arithmetic of any quasi-cyclotomic field. In this paper we determine all irreducible representations of primary quasi-cyclotomic fields. The methods apply to determine the irreducible representations of an arbitrary quasi-cyclotomic field. We also compute the Artin conductors of the representations and the Artin L-functions for a class of quasi-cyclotomic fields. They correspond to a series of normalized newforms of weight one by Deligne-Serre's theorem [Th.2, 7]. We describe these modular forms explicitly.

First we recall the construction of primary quasi-cyclotomic fields. Let S be the set consisting of -1 and all prime numbers. For $p \in S$, we put $\bar{p} = 4, 8, p$ and set $p^* = -1, 2, (-1)^{\frac{p-1}{2}}p$ if $p = -1, 2$ and an odd prime number, respectively. For prime numbers $p < q$, we define

$$v_{pq} = \prod_{i=0}^{\frac{p-1}{2}} \prod_{j=0}^{\frac{q-1}{2}} \frac{\sin \frac{iq+j}{pq}\pi}{\sin \frac{jp+i}{pq}\pi} \qquad ((i,j) \neq (0,0),\ p > 2)$$

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and

$$v_{2q} = \frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{4q}} \prod_{j=1}^{\frac{q-1}{2}} \frac{\sin\frac{j\pi}{2q} \cdot \sin\frac{2j-1}{4q}\pi}{2\sin\frac{4j+1}{4q}\pi \cdot \sin\frac{j}{q}\pi \cdot \sin\frac{2j-1}{2q}\pi}.$$

For $p < q \in S$, we put

$$u_{pq} = \begin{cases} \sqrt{q^*} & \text{if } p = -1 \\ v_{pq} & \text{if } p = 2 \text{ or } p \equiv q \equiv 1 \bmod 4 \\ \sqrt{p} \cdot v_{pq} & \text{if } p \equiv 1,\ q \equiv 3 \bmod 4 \\ \sqrt{q} \cdot v_{pq} & \text{if } p \equiv 3,\ q \equiv 1 \bmod 4 \\ \sqrt{pq} \cdot v_{pq} & \text{if } p \equiv q \equiv 3 \bmod 4. \end{cases}$$

The canonical $\mathbb{Z}/2\mathbb{Z}$-basis of the quotient space mentioned above consists of a subset of $\{u_{pq} \mid p < q \in S\}$. For $p < q \in S$ let $K = \mathbb{Q}(\zeta_{\bar{p}q})$ be the cyclotomic field of conductor $\bar{p}q$ and let $\tilde{K} = K(\sqrt{u_{pq}})$. Then $\tilde{K}$ is the smallest quasi-cyclotomic fields containing $\sqrt{u_{pq}}$. We call these fields $\tilde{K}$ primary quasi-cyclotomic fields. Let $G = \mathrm{Gal}(K/\mathbb{Q})$ and $\tilde{G} = \mathrm{Gal}(\tilde{K}/\mathbb{Q})$. We always denote by ϵ the unique non-trivial element of $\mathrm{Gal}(\tilde{K}/K)$. If $(p,q) = (-1,2)$, then the group G is generated by two elements σ_{-1} and σ_2, where $\sigma_{-1}(\zeta_8) = \zeta_8^{-1}$ and $\sigma_2(\zeta_8) = \zeta_8^5$. If $p = -1$ and $q \neq 2$, or if $p > 2$, then G is generated by two elements σ_p and σ_q, where $\sigma_p(\zeta_p) = \zeta_p^a, \sigma_p(\zeta_q) = \zeta_q$ and $\sigma_q(\zeta_p) = \zeta_p, \sigma_q(\zeta_q) = \zeta_q^b$, with a, b being generators of $(\mathbb{Z}/\bar{p}\mathbb{Z})^*$ and $(\mathbb{Z}/q\mathbb{Z})^*$ respectively. If $p = 2$, then G is generated by three elements σ_{-1}, σ_2 and σ_q, where σ_{-1}, σ_2 act on ζ_8 as above and on ζ_q trivially, and σ_q acts on ζ_q as above and on ζ_8 trivially.

Next we describe the group $\tilde{G}$ by generators and relations. An element $\sigma \in G$ has two lifts in $\tilde{G}$. By [Sect.3, 7] the action of the two lifts on $\sqrt{u_{pq}}$ has the form $\pm\alpha\sqrt{u_{pq}}$ or $\pm\alpha\sqrt{u_{pq}}/\sqrt{-1}$ with $\alpha > 0$. We fix the lift $\tilde{\sigma}$ of σ to be the one with the positive sign. Then the other lift of σ is $\tilde{\sigma}\epsilon$. The group $\tilde{G}$ is generated by $\epsilon, \tilde{\sigma}_p$ and $\tilde{\sigma}_q$ (and $\tilde{\sigma}_{-1}$ if $p = 2$). Clearly ϵ commutes with the other generators. In addition, we have $\tilde{\sigma}_p\tilde{\sigma}_q = \tilde{\sigma}_q\tilde{\sigma}_p\epsilon$ (and $\tilde{\sigma}_{-1}$ commutes with $\tilde{\sigma}_2$ and $\tilde{\sigma}_q$ if $p = 2$). For an element g of a group, we denote by $|g|$ the order of g in the group. Let $\log_{-1} : \{\pm 1\} \to \mathbb{Z}/2\mathbb{Z}$ be the unique isomorphism. For an odd prime number p and an integer a with $p \nmid a$, let $(\frac{a}{p})$ be the quadratic residue symbol. We also define $(\frac{a}{2}) = (\frac{a}{-1}) = 1$ for any a. Then we have, see [Th.3, 7],

$$|\tilde{\sigma}_p| = \left(1 + \log_{-1}\left(\frac{q^*}{p}\right)\right)|\sigma_p| \quad \text{and} \quad |\tilde{\sigma}_q| = \left(1 + \log_{-1}\left(\frac{p^*}{q}\right)\right)|\sigma_q|,$$

with the exception that $|\tilde{\sigma}_2| = 2|\sigma_2|$ when $(p,q) = (-1,2)$. If $p = 2$, we have furthermore $|\tilde{\sigma}_{-1}| = |\sigma_{-1}|$. Thus we have determined the group $\tilde{G}$ completely by generators and relations.

2. Abelian subgroup of index 2

In this section we construct a special abelian subgroup of $\tilde{G}$ of index 2 and determine its structure. We consider the following three cases separately:

Case A: $|\widetilde{\sigma}_p| = |\sigma_p|$ and $|\widetilde{\sigma}_q| = |\sigma_q|$;
Case B: $|\widetilde{\sigma}_p| = 2|\sigma_p|$, $|\widetilde{\sigma}_q| = |\sigma_q|$ or $|\widetilde{\sigma}_p| = |\sigma_p|$, $|\widetilde{\sigma}_q| = 2|\sigma_q|$;
Case C: $|\widetilde{\sigma}_p| = 2|\sigma_p|$ and $|\widetilde{\sigma}_q| = 2|\sigma_q|$.

All the three cases may happen: Case (A) if and only if $(\frac{p^*}{q}) = (\frac{q^*}{p}) = 1$; Case (B) if and only if $(\frac{p^*}{q}) \neq (\frac{q^*}{p})$ or $(p, q) = (-1, 2)$; Case (C) if and only if $(\frac{p^*}{q}) = (\frac{q^*}{p}) = -1$.

In Case A, we define the subgroup N of $\widetilde{G}$ to be

$$N = \begin{cases} < \widetilde{\sigma}_{-1}, \widetilde{\sigma}_2, \widetilde{\sigma}_q^2, \varepsilon > & \text{if } p = 2 \\ < \widetilde{\sigma}_p, \widetilde{\sigma}_q^2, \varepsilon > & \text{if } p \neq 2 \end{cases} \tag{$A2.1$}$$

It is easy to see that the subgroup N is abelian of index 2 in $\tilde{G}$ and is a direct sum of the cyclic groups generated by the elements. Thus we have

$$N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = -1 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/((q-1)/2)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } p > 2. \end{cases} \tag{$A2.2$}$$

In Case B, we define the subgroup N of $\widetilde{G}$ to be

$$N = \begin{cases} < \widetilde{\sigma}_{-1}, \widetilde{\sigma}_2, \widetilde{\sigma}_q^2 > & \text{if } p = 2 \\ < \widetilde{\sigma}_p, \widetilde{\sigma}_q^2 > & \text{if } p \neq 2 \text{ and } |\widetilde{\sigma}_q| = 2|\sigma_q| \\ < \widetilde{\sigma}_p^2, \widetilde{\sigma}_q > & \text{if } |\widetilde{\sigma}_p| = 2|\sigma_p|. \end{cases} \tag{$B2.1$}$$

Again N is abelian and has index 2 in $\widetilde{G}$. In addition, we have

$$N \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } (p, q) = (-1, 2) \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p = -1,\ q > 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p = 2 \\ \mathbb{Z}/(p-1)\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} & \text{if } p > 2. \end{cases} \tag{$B2.2$}$$

In Case C, p, q both are odd prime numbers. Let $v_2(p-1)$ denote the power of 2 in $p-1$. We define the subgroup N of $\widetilde{G}$ to be

$$N = \begin{cases} < \widetilde{\sigma}_p^2\,, \widetilde{\sigma}_q > & \text{if } v_2(p-1) \leq v_2(q-1) \\ < \widetilde{\sigma}_p\,, \widetilde{\sigma}_q^2 > & \text{if } v_2(p-1) > v_2(q-1). \end{cases} \tag{$C2.1$}$$

Then N is an abelian subgroup of $\widetilde{G}$. When $v_2(p-1) \le v_2(q-1)$, we have

$$|N| = \frac{|\widetilde{\sigma}_p^2| \cdot |\widetilde{\sigma}_q|}{| < \widetilde{\sigma}_p^2 > \cap < \widetilde{\sigma}_q > |} = \frac{(p-1)\cdot 2(q-1)}{2},$$

thus $[\widetilde{G} : N] = 2$ and N is a normal subgroup of $\widetilde{G}$. We have the same result when $v_2(p-1) > v_2(q-1)$. Although the subgroup $< \widetilde{\sigma}_p^2 , \widetilde{\sigma}_q >$ is always an abelian subgroup of $\widetilde{G}$ of index 2, when $v_2(p-1) > v_2(q-1)$ we are not able to get all irreducible representations of $\tilde{G}$ from this subgroup. So we define N in two cases.

Next we determine the structure of the subgroup N in the case C. We consider the case $v_2(p-1) \le v_2(q-1)$ in detail. Let $d = \gcd(\frac{p-1}{2}, q-1)$, $s = (p-1)/2d$ and $t = (q-1)/d$. Choose $u, v \in \mathbb{Z}$ such that $us+vt = 1$. We have the relations

$$(\widetilde{\sigma}_p^2)^{p-1} = 1,\ (\widetilde{\sigma}_p^2)^{\frac{p-1}{2}} = \varepsilon = \widetilde{\sigma}_q^{q-1}.$$

Let M be the free abelian group generated by two words α, β. Let

$$\alpha_1 = (p-1)\alpha \ \ ; \ \ \beta_1 = \frac{p-1}{2}\alpha - (q-1)\beta \ ;$$

and let M_1 be the subgroup of M generated by α_1 , β_1. Then M_1 is the kernel of the homomorphism

$$M \longrightarrow N;\ \alpha \mapsto \widetilde{\sigma}_p^2,\ \beta \mapsto \widetilde{\sigma}_q \ \ .$$

So we have $N \cong M/M_1$. Define the matrix

$$A = \begin{pmatrix} p-1 & \frac{p-1}{2} \\ 0 & 1-q \end{pmatrix} \ .$$

Then $(\alpha_1 , \beta_1) = (\alpha , \beta) \cdot A$. We determine the structure of M_1 by considering the standard form of A. Define

$$P = \begin{pmatrix} u & v \\ -t & s \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ ; \qquad Q = \begin{pmatrix} 1 & 2tv-1 \\ -1 & -2tv+2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then

$$B = PAQ = \begin{pmatrix} d & 0 \\ 0 & -2s(q-1) \end{pmatrix}$$

is the standard form of A. Let

$$(\tau , \mu) = (\alpha , \beta)P^{-1} \quad \text{and} \quad (\tau_1 , \mu_1) = (\alpha_1 , \beta_1)Q.$$

Then $(\tau_1 , \mu_1) = (\tau , \mu)B$, $M = \mathbb{Z}\tau \oplus \mathbb{Z}\mu$ and $M_1 = \mathbb{Z}d\tau \oplus \mathbb{Z}2s(q-1)\mu$. We thus have

$$N \cong M/M_1 \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z} \ .$$

By abuse of notation, we also write

$$(\tau , \mu) = (\widetilde{\sigma}_p^2 , \widetilde{\sigma}_q)P^{-1} = (\widetilde{\sigma}_p^{2s}\widetilde{\sigma}_q^{t} , \widetilde{\sigma}_p^{-2v}\widetilde{\sigma}_q^{u}) \ .$$

Then τ, μ are of order $d, 2s(q-1)$ respectively, and N is a direct sum of $<\tau>$ and $<\mu>$. We have $\widetilde{\sigma}_p^2 = \tau^u\mu^{-t}$ and $\widetilde{\sigma}_q = \tau^v\mu^s$. When $v_2(p-1) > v_2(q-1)$, we get the structure of N in the same way. So in the case (C) we have

$$N \cong \begin{cases} \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z} & \text{if } v_2(p-1) \le v_2(q-1) \\ \mathbb{Z}/d'\mathbb{Z} \oplus \mathbb{Z}/2s'(p-1)\mathbb{Z} & \text{if } v_2(p-1) > v_2(q-1), \end{cases} \tag{C2.2}$$

where $d = \gcd(\frac{p-1}{2}, q-1), s = (p-1)/2d$ and $d' = \gcd(p-1, \frac{q-1}{2}), s' = (q-1)/2d'$.

Now we summarize our results in the following

Proposition 2.1. *The abelian subgroup N of the group $\widetilde{G}$ of index 2 defined in (A2.1), (B2.1) and (C2.1) has the structure described in (A2.2), (B2.2) and (C2.2) in the cases (A), (B) and (C), respectively. In particular, every irreducible representation of $\widetilde{G}$ has dimension 1 or 2.*

3. 2-dimensional representations

We determine all irreducible representations of $\widetilde{G}$ in this section. We will use some basic facts from representation theory freely. For the details, see [6].

It is well-known that the 1-dimensional representations of $\widetilde{G}$ correspond bijectively to those of the maximal abelian quotient G of $\widetilde{G}$, which are Dirichlet characters. So we mainly construct the 2-dimensional irreducible representations of $\widetilde{G}$. From the dimension formula of all irreducible representations, we see that $\widetilde{G}$ has $|G|/4$ irreducible representations of dimension 2, up to isomorphism. Let N be the subgroup of $\widetilde{G}$ defined in last section. Let $\widetilde{G} = N \cup \sigma N$ be the decomposition in cosets. If $\rho : N \to \mathbb{C}^*$ is a representation of N, the induced representation $\widetilde{\rho}$ of ρ is a representation of $\widetilde{G}$ of dimension 2. The space of the representation $\widetilde{\rho}$ is $V = \mathrm{Ind}_N^{\widetilde{G}}(\mathbb{C}) = \mathbb{C}[\widetilde{G}] \otimes_{\mathbb{C}[N]} \mathbb{C}$ with basis $e_1 = 1 \otimes 1$ and $e_2 = \sigma \otimes 1$. The group homomorphism

$$\widetilde{\rho} : \widetilde{G} \longrightarrow \mathrm{GL}(V) \simeq \mathrm{GL}_2(\mathbb{C})$$

is given by

$$\widetilde{\rho}(\widetilde{\sigma}) = \begin{pmatrix} \rho(\widetilde{\sigma}) & \rho(\widetilde{\sigma}\sigma) \\ \rho(\sigma^{-1}\widetilde{\sigma}) & \rho(\sigma^{-1}\widetilde{\sigma}\sigma) \end{pmatrix}, \quad \forall\, \widetilde{\sigma} \in \widetilde{G}, \tag{3.1}$$

where $\rho(\widetilde{\sigma}) = 0$ if $\widetilde{\sigma} \notin N$. The representation $\widetilde{\rho}$ is irreducible if and only if $\rho \not\cong \rho^\tau$ for every $\tau \in \widetilde{G} \setminus N$, where ρ^τ is the conjugate representation of ρ defined by

$$\rho^\tau(x) = \rho(\tau^{-1}x\tau), \quad \forall\, x \in N.$$

Since N is abelian, we only need to check $\rho \not\cong \rho^\sigma$.

Now we begin to construct all 2-dimensional irreducible representations of $\widetilde{G}$. As in last section, we consider the three cases separately. In addition, we consider the case when p and q are odd prime numbers in details, and only state the results in the cases when $p=-1$ or 2.

3.1. Case A. Assume $p>2$. We have in this case $N=\langle\widetilde{\sigma}_p\,,\,\widetilde{\sigma}_q^2\,,\varepsilon\rangle$ and

$$N\cong\mathbb{Z}/(p-1)\mathbb{Z}\oplus\mathbb{Z}/((q-1)/2)\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\,.$$

Every irreducible representation of N can be written as $\rho_{ijk}:N\longrightarrow\mathbb{C}^*$ with

$$\rho_{ijk}(\widetilde{\sigma}_p)=\zeta_{p-1}^i\ ;\ \rho_{ijk}(\widetilde{\sigma}_q^2)=\zeta_{q-1}^{2j}\ ;\ \rho_{ijk}(\varepsilon)=(-1)^k\,.$$

where $0\le i<p-1,\ 0\le j<\frac{q-1}{2}$ and $k=0,1$. Since $\widetilde{G}=N\cup\widetilde{\sigma}_qN$ and $\rho_{ijk}^{\widetilde{\sigma}_q}(\widetilde{\sigma}_p)=\rho_{ijk}(\varepsilon)\rho_{ijk}(\widetilde{\sigma}_p)=(-1)^k\rho_{ijk}(\widetilde{\sigma}_p)$, we have

$$\rho_{ijk}^{\widetilde{\sigma}_q}\ncong\rho_{ijk}\Longleftrightarrow k=1.$$

Write $\rho_{ij}=\rho_{ij1}$. The induced representation $\widetilde{\rho}_{ij}:\widetilde{G}\longrightarrow\mathrm{GL}_2(\mathbb{C})$ of ρ_{ij} is given by

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p)=\begin{pmatrix}\zeta_{p-1}^i&0\\0&-\zeta_{p-1}^i\end{pmatrix},\ \widetilde{\rho}_{ij}(\widetilde{\sigma}_q)=\begin{pmatrix}0&\zeta_{q-1}^{2j}\\1&0\end{pmatrix},\ \widetilde{\rho}_{ij}(\varepsilon)=-I,\ \ (A3.1)$$

where I is the identity matrix of degree 2. Since

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p^2)=\begin{pmatrix}\zeta_{p-1}^{2i}&0\\0&\zeta_{p-1}^{2i}\end{pmatrix}\quad\text{and}\quad\widetilde{\rho}_{ij}(\widetilde{\sigma}_q^2)=\begin{pmatrix}\zeta_{q-1}^{2j}&0\\0&\zeta_{q-1}^{2j}\end{pmatrix},$$

we see that the representations $\widetilde{\rho}_{ij}$ with $0\le i<\frac{p-1}{2},\ 0\le j<\frac{q-1}{2}$ are irreducible and are not isomorphic to each other, by considering the values of the characters of these representations at $\widetilde{\sigma}_p^2$ and $\widetilde{\sigma}_q^2$. The number of these representations is $\frac{p-1}{2}\cdot\frac{q-1}{2}=\frac{|G|}{4}$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, when $p=-1$, all irreducible representations of $\widetilde{G}$ of dimension 2 are $\widetilde{\rho}_j$ with $0\le j<\frac{q-1}{2}$, where

$$\widetilde{\rho}_j(\widetilde{\sigma}_{-1})=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad\widetilde{\rho}_j(\widetilde{\sigma}_q)=\begin{pmatrix}0&\zeta_{q-1}^{2j}\\1&0\end{pmatrix},\quad\widetilde{\rho}(\varepsilon)=-I\qquad(A3.2)$$

and when $p=2$, all irreducible representations of $\widetilde{G}$ of dimension 2 are $\bar{\rho}_{ij}$ with $0\le i\le 1$ and $0\le j<\frac{q-1}{2}$, where $\bar{\rho}_{ij}(\varepsilon)=-I$ and

$$\bar{\rho}_{ij}(\widetilde{\sigma}_{-1})=(-1)^iI,\ \bar{\rho}_{ij}(\widetilde{\sigma}_2)=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \bar{\rho}_{ij}(\widetilde{\sigma}_q)=\begin{pmatrix}0&\zeta_{q-1}^{2j}\\1&0\end{pmatrix}.\qquad(A3.3)$$

3.2 Case B. Assume $p>2$ and $|\widetilde{\sigma}_q|=2|\sigma_q|$. Then $N=\langle\widetilde{\sigma}_p\,,\,\widetilde{\sigma}_q^2\rangle$, and

$$N\cong\mathbb{Z}/(p-1)\mathbb{Z}\oplus\mathbb{Z}/(q-1)\mathbb{Z}.$$

Any irreducible representation of N has the form $\rho_{ij} : N \longrightarrow \mathbb{C}^*$, where

$$\rho_{ij}(\widetilde{\sigma}_p) = \zeta_{p-1}^i \ , \ \rho_{ij}(\widetilde{\sigma}_q^2) = \zeta_{q-1}^j \ , \ \rho_{ij}(\varepsilon) = \rho_{ij}(\widetilde{\sigma}_q^2)^{\frac{q-1}{2}} = (-1)^j \ ,$$

and $0 \le i < p-1, \ 0 \le j < q-1$. It is easy to check that

$$\rho_{ij}^{\widetilde{\sigma}_q} \not\cong \rho_{ij} \Longleftrightarrow j \equiv 1 \pmod 2 \ .$$

The induced representation $\widetilde{\rho}_{ij} : \widetilde{G} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ of ρ_{ij} with odd j is given by

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} \zeta_{p-1}^i & 0 \\ 0 & -\zeta_{p-1}^i \end{pmatrix} , \ \widetilde{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix} . \qquad (B3.1)$$

Since

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p^2) = \begin{pmatrix} \zeta_{p-1}^{2i} & 0 \\ 0 & \zeta_{p-1}^{2i} \end{pmatrix} \quad \text{and} \quad \widetilde{\rho}_{ij}(\widetilde{\sigma}_q^2) = \begin{pmatrix} \zeta_{q-1}^{j} & 0 \\ 0 & \zeta_{q-1}^{j} \end{pmatrix} ,$$

we see that the representations $\widetilde{\rho}_{ij}$ with $0 \le i < \frac{p-1}{2}$ and $0 \le j < q-1, \ 2 \nmid j$ are irreducible and are not isomorphic to each other. The number of these representations is $\frac{|G|}{4}$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, when $(p,q) = (-1,2)$, there is only one irreducible representation $\widetilde{\rho}_0$ of dimension 2 defined by

$$\widetilde{\rho}_0(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \text{and} \quad \widetilde{\rho}_0(\widetilde{\sigma}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \qquad (B3.2)$$

When $p = -1$ and $q > 2$, all irreducible representations of dimension 2 are $\widetilde{\rho}_j$ with $0 \le j < q-1, \ 2 \nmid j$, where $\widetilde{\rho}_j$ is defined by

$$\widetilde{\rho}_j(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \widetilde{\rho}_j(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix} . \qquad (B3.3)$$

When $p = 2$, all irreducible representations of dimension 2 are $\bar{\rho}_{ij}$ with $0 \le i \le 1$ and $0 \le j < q-1, \ 2 \nmid j$, where $\bar{\rho}_{ij}$ is defined by

$$\bar{\rho}_{ij}(\widetilde{\sigma}_{-1}) = (-1)^i I, \ \bar{\rho}_{ij}(\widetilde{\sigma}_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \ \bar{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix} . \qquad (B3.4)$$

When $|\widetilde{\sigma}_p| = 2|\sigma_p|$, all irreducible representations of dimension 2 are $\hat{\rho}_{ij}$ with $0 \le i < p-1, \ 2 \nmid i$ and $0 \le j < \frac{q-1}{2}$, where $\hat{\rho}_{ij}$ is defined by

$$\hat{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} 0 & \zeta_{p-1}^i \\ 1 & 0 \end{pmatrix} , \ \hat{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} \zeta_{q-1}^j & 0 \\ 0 & -\zeta_{q-1}^j \end{pmatrix} . \qquad (B3.5)$$

3.3. Case C. Assume $v_2(p-1) \le v_2(q-1)$. Let

$$d = \gcd(\frac{p-1}{2}, q-1) \ , \ s = \frac{p-1}{2d} , t = \frac{q-1}{d} \ ; \ \ us + vt = 1$$

as before. We must have that t is even and u is odd. Let $\tau = \widetilde{\sigma}_p^{2s} \cdot \widetilde{\sigma}_q^t$ and $\mu = \widetilde{\sigma}_p^{-2v} \cdot \widetilde{\sigma}_q^u$. Then $N = \langle \widetilde{\sigma}_p^2 , \widetilde{\sigma}_q \rangle = \langle \tau , \mu \rangle$ and

$$N \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/2s(q-1)\mathbb{Z}.$$

Any irreducible representation $\rho_{ij} : N \longrightarrow \mathbb{C}^*$ is of the form

$$\rho_{ij}(\tau) = \zeta_d^i = \zeta_{(p-1)(q-1)}^{2s(q-1)i} \quad \text{and} \quad \rho_{ij}(\mu) = \zeta_{2s(q-1)}^j = \zeta_{(p-1)(q-1)}^{dj}.$$

From $\widetilde{\sigma}_p^2 = \tau^u \mu^{-t}$ and $\widetilde{\sigma}_q = \tau^v \mu^s$, we have

$$\rho_{ij}(\widetilde{\sigma}_p^2) = \zeta_{p-1}^{2sui-j} \ ; \ \rho_{ij}(\widetilde{\sigma}_q) = \zeta_{2(q-1)}^{2tvi+j} \ ; \ \rho_{ij}(\varepsilon) = \rho_{ij}(\widetilde{\sigma}_p^2)^{\frac{p-1}{2}} = (-1)^j \ .$$

It is easy to show

$$\rho_{ij}^{\widetilde{\sigma}_p} \ncong \rho_{ij} \Longleftrightarrow j \equiv 1 \pmod 2 \ .$$

The induced representation $\widetilde{\rho}_{ij} : \widetilde{G} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ of ρ_{ij} with odd j is given by

$$\widetilde{\rho}_{ij}(\tau) = \begin{pmatrix} \zeta_d^i & 0 \\ 0 & \zeta_d^i \end{pmatrix} \ ; \quad \widetilde{\rho}_{ij}(\mu) = \begin{pmatrix} \zeta_{2s(q-1)}^j & 0 \\ 0 & -\zeta_{2s(q-1)}^j \end{pmatrix}.$$

Here in the first equality we used the fact that t is even, and in the second equality we used the fact that u is odd. Furthermore we have

$$\widetilde{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} 0 & \zeta_{p-1}^{2sui-j} \\ 1 & 0 \end{pmatrix} \ ; \ \widetilde{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} \zeta_{2(q-1)}^{2tvi+j} & 0 \\ 0 & -\zeta_{2(q-1)}^{2tvi+j} \end{pmatrix}. \qquad (C3.1)$$

By considering the values of the character of $\widetilde{\rho}_{ij}$ at τ and μ^2, we see that all the representations $\widetilde{\rho}_{ij}$ with $0 \le i < d$ and $0 \le j < s(q-1)$, $2 \nmid j$ are irreducible and are not isomorphic to each other. The number of these representations is $d \cdot \frac{s(q-1)}{2} = \frac{|G|}{4}$. So they are all the irreducible representations of $\widetilde{G}$ of dimension 2.

Similarly, if $v_2(p-1) > v_2(q-1)$, we let

$$d' = \gcd(p-1 \, , \frac{q-1}{2}) \ , \ s' = \frac{p-1}{d} \, , t' = \frac{q-1}{2d} \ ; \ \ u's' + v't' = 1.$$

Then all the irreducible representations of $\widetilde{G}$ of dimension 2 are $\hat{\rho}_{ij}$ with $0 \le i < d'$ and $0 \le j < t'(p-1)$, $2 \nmid j$, where $\hat{\rho}_{ij}$ is defined by

$$\hat{\rho}_{ij}(\widetilde{\sigma}_p) = \begin{pmatrix} \zeta_{2(p-1)}^{2s'u'i+j} & 0 \\ 0 & -\zeta_{2(p-1)}^{2s'u'i+j} \end{pmatrix} ; \ \hat{\rho}_{ij}(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^{2t'v'i-j} \\ 1 & 0 \end{pmatrix} \ . \quad (C3.2)$$

Let $\mathrm{R}^2(\widetilde{G})$ be the set of all irreducible representations, up to isomorphism, of $\widetilde{G}$ of dimension 2. As a summary, we have proved the following

Theorem 3.1. *All 2-dimensional irreducible representations of $\widetilde{G}$ are induced from the representations of N. In detail, we have*

In the case (A)

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_j \mid 0 \leq j < \frac{q-1}{2}\} & \text{if } p = -1 \\ \{\bar{\rho}_{ij} \mid i = 0, 1,\ 0 \leq j < \frac{q-1}{2}\} & \text{if } p = 2 \\ \{\widetilde{\rho}_{ij} \mid 0 \leq i < \frac{p-1}{2},\ 0 \leq j < \frac{q-1}{2}\} & \text{if } p > 2, \end{cases}$$

where $\widetilde{\rho}_j$, $\bar{\rho}_{ij}$ and $\widetilde{\rho}_{ij}$ are defined in (A3.2), (A3.3) and (A3.1) respectively.

In the case (B)

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_0\} & \text{if } (p,q) = (-1,2) \\ \{\widetilde{\rho}_j \mid 0 \leq j < q-1,\ 2 \nmid j\} & \text{if } p = -1,\ q > 2 \\ \{\bar{\rho}_{ij} \mid i = 0,1,\ 0 \leq j < q-1,\ 2 \nmid j\} & \text{if } p = 2 \\ \{\hat{\rho}_{ij} \mid 0 \leq i < p-1,\ 2 \nmid i,\ 0 \leq j < \frac{q-1}{2}\} & \text{if } |\widetilde{\sigma}_p| = 2|\sigma_p| \\ \{\widetilde{\rho}_{ij} \mid 0 \leq i < \frac{p-1}{2},\ 0 \leq j < q-1,\ 2 \nmid j\} & \text{otherwise,} \end{cases}$$

where $\widetilde{\rho}_0$, $\widetilde{\rho}_j$, $\bar{\rho}_{ij}$, $\hat{\rho}_{ij}$ and $\widetilde{\rho}_{ij}$ are defined in (B3.2), (B3.3), (B3.4), (B3.5) and (B3.1) respectively.

In the case (C)

$$\mathrm{R}^2(\widetilde{G}) = \begin{cases} \{\widetilde{\rho}_{ij} \mid 0 \leq i < d,\ 0 \leq j < s(q-1),\ 2 \nmid j\} & \text{if } v_2(p-1) \leq v_2(q-1), \\ \{\hat{\rho}_{ij} \mid 0 \leq i < d',\ 0 \leq j < t'(p-1),\ 2 \nmid j\} & \text{otherwise,} \end{cases}$$

where $\widetilde{\rho}_{ij}$ and $\hat{\rho}_{ij}$ are defined in (C3.1) and (C3.2) respectively.

4. The Frobenius maps

This section is a preparation for the next two sections to compute the Artin conductors of the representations and the Artin L-functions of some quasi-cyclotomic fields $\widetilde{K}$. For a prime number ℓ, we say that ℓ is ramified (resp. inert, splitting) in the relative quadratic extension $\tilde{K}/K$ if the prime ideals of K over ℓ are ramified (resp. inert, splitting) in $\tilde{K}$. For a prime number ℓ which is unramified in $\widetilde{K}/K$, let I_ℓ (resp. $\widetilde{I}_\ell$) be the inert group of ℓ in the extension $K/\mathbb{Q}$ (resp. $\widetilde{K}/\mathbb{Q}$). Let Fr_ℓ be the Frobenius automorphism of ℓ in G/I_ℓ and $\widetilde{\mathrm{Fr}}_\ell$ the Frobenius automorphism of ℓ in $\widetilde{G}/\widetilde{I}_\ell$ associated to some prime ideal over ℓ.

To compute the Artin conductors of the representations, we need to construct an uniformizer in the completion of $\tilde{K}$ at a prime ideal, especially at a prime ideal over 2. Generally we are not able to get such an uniformizer, but we can do it in the case $p = -1$. In addition, to calculate the Artin L-functions of the representations, we need to know the $\widetilde{\mathrm{Fr}}_\ell$, especially for $\ell = 2$, and so we need to know the decomposition of 2 in $\tilde{K}$. For odd $p < q \in S$, we calculated some examples by a computer which show that 2 is always unramified in $\tilde{K}$. But we are not able to show it. Furthermore, we do not know when 2 splits in $\tilde{K}/K$ and when 2 is inert in $\tilde{K}/K$. But when $p=-1$, we can solve these questions well (see below). So in this paper we only compute the Artin conductors and Artin L-functions of the representations in the case $p=-1$.

From now on, we always assume that $p=-1$, namely, $K=\mathbb{Q}(\zeta_{4q})$ and $\widetilde{K}=K(\sqrt[4]{q^*})$. In this section we determine $\widetilde{\mathrm{Fr}}_\ell$ by Fr_ℓ for $\ell=2$. In [Sect.5, 7] we have determined the decomposition of some odd prime numbers in $\widetilde{K}/K$. Now we determine the decomposition of 2 in $\widetilde{K}/K$. The result below is a more explicit reformulation of Theorem 2 in [8].

Proposition 4.1. *If $q=2$, then 2 is ramified in $\widetilde{K}/K$. If q is odd, then 2 is unramified in $\widetilde{K}/K$ if and only if $(\frac{2}{q})=1$, and in this case 2 splits in $\widetilde{K}/K$ if $q^*\equiv 1 \mathrm{mod} 16$ and is inert in $\widetilde{K}/K$ if $q^*\not\equiv 1\mathrm{mod}16$.*

Proof. We first consider the case $q=2$. The unique prime ideal of K over 2 is the principal ideal generated by $\pi_2=1-\zeta_8$. Since the ramification degree of 2 in $K/\mathbb{Q}$ is 4 and $\sqrt{2}=\pi_2(\pi_2+2\zeta_8)\zeta_8$, we have that 2 is ramified in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{2}\mathrm{mod}\pi_2^8$ is not solvable in the ring O_K of the integers of K by [Th.2(1), 8], which is equivalent to that $(1+\frac{2}{\pi_2}\zeta_8)\zeta_8$ is not a square modulo π_2^6. Since $2=u\pi_2^4$ for some unit u, we have

$$\left(1+\frac{2}{\pi_2}\zeta_8\right)\zeta_8\equiv\zeta_8\equiv(1-\pi_2)\mathrm{mod}\pi_2^3,$$

namely $(1+\frac{2}{\pi_2}\zeta_8)\zeta_8$ is not a square modulo π_2^3. So 2 is ramified in $\widetilde{K}/K$.

Now we assume that q is odd. Let $\pi_2=1-\zeta_4$. Since the ramification degree of 2 in K is 2, we have 2 is unramified in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{q^*}\mathrm{mod}\pi_2^4$ is solvable in O_K, see [Th.2(1), 8]. Furthermore, 2 splits in $\widetilde{K}/K$ if and only if $x^2\equiv\sqrt{q^*}\mathrm{mod}\pi_2^5$ is solvable in O_K. The explicit computation of Gauss sum gives

$$\sqrt{q^*}=\sum_{a=1}^{q-1}(\frac{a}{q})\zeta_q^a=1+2\sum_{(\frac{a}{q})=1}\zeta_q^a.$$

Let $\alpha=\sum_{(\frac{a}{q})=1}\zeta_q^a$, $\beta=\sum_{(\frac{a}{q})=1}\zeta_{2q}^a$, and $\gamma=\sum_{(\frac{a}{q})=1}\sum_{(\frac{b}{q})=1,\ a<b}\zeta_{2q}^{a+b}$, where in the summations a,b run over $1,2,\cdots,q-1$. Then $\alpha=\beta^2-2\gamma$,

that with the equality $2=\pi_2^2-\pi_2^3$ gives

$$\begin{aligned}\sqrt{q^*}&=1+2\beta^2-4\gamma=1+\pi_2^2\beta^2-\pi_2^3\beta^2-4\gamma\\&\equiv(1+\pi_2\beta)^2-\pi_2^3(\beta+\beta^2)+\pi_2^4(\beta-\gamma)\\&\equiv(1+\pi_2\beta)^2-\pi_2^3(\alpha+\beta)+\pi_2^4(\beta+\gamma)\mathrm{mod}\pi_2^5.\end{aligned}$$

Since $\zeta_{2q}=-\zeta_q^{-\frac{q-1}{2}}=-\zeta_q^t$, where t is the inverse of 2 in $(\mathbb{Z}/q\mathbb{Z})^*$, we see $\beta=\sum_{(\frac{a}{q})=1}(-1)^a\zeta_q^{ta}\equiv\sum_{(\frac{a}{q})=1}\zeta_q^{ta}\mathrm{mod}2$. So if $(\frac{2}{q})=1$ we have $\alpha\equiv\beta\mathrm{mod}2$ and thus 2 is unramified in $\widetilde{K}/K$, and if $(\frac{2}{q})=-1$ we have $\alpha+\beta\equiv\sum_{a=1}^{q-1}\zeta_q^a=-1\mathrm{mod}2$ and thus 2 is ramified in $\widetilde{K}/K$.

Now we assume $(\frac{2}{q})=1$. Then $\sqrt{q^*}\mathrm{mod}\pi_2^5$ is a square if and only if $\pi_2\mid\beta+\gamma$. We consider $2(\beta+\gamma)$. Since $\alpha\equiv\beta\mathrm{mod}2$, we have

$$2(\beta+\gamma)=2\beta+\beta^2-\alpha\equiv\alpha(\alpha+1)\mathrm{mod}4.$$

From $\sqrt{q^*}=1+2\alpha$, we see $\alpha(\alpha+1)=\frac{q^*-1}{4}$. Since $8\mid q^*-1$ under the assumption $(\frac{2}{q})=1$, we have $\beta+\gamma\equiv\frac{q^*-1}{8}\mathrm{mod}2$. So $\pi_2\mid\beta+\gamma$ if and only if $\pi_2\mid\frac{q^*-1}{8}$, namely $2\mid\frac{q^*-1}{8}$. The proof is complete. $\square$

Although the following result determining the ring $O_{\widetilde{K}}$ of the integers of $\widetilde{K}$ will not be used in the paper, we write it down since it is a more explicit reformulation of [Th.2(3), 8] in the case $p=-1$.

Corollary 4.2. *For odd prime q, let $t_q=(q-1)/4$ if $q\equiv1\mathrm{mod}4$, and $t_q=(q-3)/4$ if $q\equiv3\mathrm{mod}4$. Then*

$$O_{\widetilde{K}}=\begin{cases}\mathbb{Z}\left[\zeta_8,\frac{\sqrt[4]{2}}{1-\zeta_8}\right] & \text{if } q=2\\ \mathbb{Z}\left[\zeta_{4q},\frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2(2\sin\frac{\pi}{q})^{t_q}}\right] & \text{if } (\frac{2}{q})=-1\\ \mathbb{Z}\left[\zeta_{4q},\frac{\sqrt[4]{q^*}+1+\pi_2\beta}{2(2\sin\frac{\pi}{q})^{t_q}}\right] & \text{if } (\frac{2}{q})=1.\end{cases}$$

Proof. See [Th.2(3), 8]. $\square$

Now we assume that 2 is unramified in $\widetilde{K}/K$. Let $\mathrm{Fr}_2\in G$ such that $\mathrm{Fr}_2(\zeta_4)=1$ and $\mathrm{Fr}_2(\zeta_q)=\zeta_q^2$. It is a Frobenius element of 2 in G modulo I_2. We have $\mathrm{Fr}_2=\sigma_2^{b_2}$ for some $b_2\in\mathbb{Z}$ with $2\mid b_2$ as $(\frac{2}{q})=1$. Thus $\widetilde{\mathrm{Fr}}_2=\widetilde{\sigma}_2^{b_2}$ or $\widetilde{\mathrm{Fr}}_2=\widetilde{\sigma}_2^{b_2}\varepsilon$. We need to determine $\widetilde{\mathrm{Fr}}_2$ completely. Since $(\frac{2}{q})=1$, we have

$$\sqrt{q^*}\equiv(1+\pi_2\alpha)^2+\pi_2^4(\beta+\gamma)\mathrm{mod}\pi_2^5.$$

Write $u=1+\pi_2\alpha$ for simplicity. Since $\sqrt{q^*}\equiv u^2\mathrm{mod}\pi_2^4$, we see $\frac{\sqrt[4]{q^*}-u}{2}\in O_{\widetilde{K}}$. Let $\wp$ be the prime ideal of $\widetilde{K}$ over 2 associated to $\widetilde{\mathrm{Fr}}_2$. By the definition, we have

$$\widetilde{\mathrm{Fr}}_2\left(\frac{\sqrt[4]{q^*}-u}{2}\right)\equiv\left(\frac{\sqrt[4]{q^*}-u}{2}\right)^2\equiv(\beta+\gamma)+\frac{\sqrt[4]{q^*}-u}{2}\mathrm{mod}\wp.$$

On the other hand, since $\widetilde{\sigma}_q^{b_2}(\sqrt[4]{q^*}) = (-1)^{\frac{b_2}{2}}\sqrt[4]{q^*}$ and $\widetilde{\sigma}_q^{b_2}(u) = u$ as $2 \mid b_2$, we have

$$\widetilde{\sigma}_q^{b_2}\left(\frac{\sqrt[4]{q^*}-u}{2}\right) = \frac{(-1)^{\frac{b_2}{2}}\sqrt[4]{q^*}-u}{2}$$

and

$$\widetilde{\sigma}_q^{b_2}\varepsilon\left(\frac{\sqrt[4]{q^*}-u}{2}\right) = \frac{(-1)^{\frac{b_2}{2}+1}\sqrt[4]{q^*}-u}{2}.$$

So if $2 \mid \frac{b_2}{2}$ we have $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}$ if and only if $\pi_2 \mid \beta+\gamma$ (namely 2 splits in $\widetilde{K}/K$), and if $2 \nmid \frac{b_2}{2}$ we have $\widetilde{\mathrm{Fr}}_2 = \widetilde{\sigma}_2^{b_2}$ if and only if $\pi_2 \nmid \beta+\gamma$ (namely 2 is inertia in $\widetilde{K}/K$). In the case $q \equiv 3 \mathrm{mod} 4$, we can always assume that $2 \nmid \frac{b_2}{2}$, since if $4 \mid b_2$, we may replace b_2 by $b_2+(q-1)$. In the case $q \equiv 1 \mathrm{mod} 4$, we have $2 \mid \frac{b_2}{2}$ iff $2^{\frac{q-1}{4}} \equiv 1 \mathrm{mod} q$ iff q has the form A^2+64B^2 for $A, B \in \mathbb{Z}$, by the Exercise 28 in Chap.5 in [5]. So we get the following result

Proposition 4.3. *Assume that 2 is unramified in $\widetilde{K}/K$. Let $\mathrm{Fr}_2 = \sigma_2^{b_2}$. We have $2 \mid b_2$. If $q \equiv 3 \mathrm{mod} 4$, we always assume $b_2 \equiv 2 \mathrm{mod} 4$. Let P_0 be the set of the prime numbers of the form A^2+64B^2 with $A, B \in \mathbb{Z}$. Then we have*

$$\widetilde{\mathrm{Fr}}_2 = \begin{cases} \widetilde{\sigma}_2^{b_2} & \text{if } q \notin P_0, 16 \nmid q^*-1, \text{ or } q \in P_0, 16 \mid q^*-1 \\ \widetilde{\sigma}_2^{b_2}\varepsilon & \text{if } q \in P_0, 16 \nmid q^*-1, \text{ or } q \notin P_0, 16 \mid q^*-1. \end{cases}$$

The following lemma is useful in the computation of Artin L-functions.

Lemma 4.4. *We have $\varepsilon \in \widetilde{I}_\ell$ if and only if ℓ is ramified in $\widetilde{K}/K$.*

Proof. The canonical projection $\widetilde{G} \longrightarrow G \simeq \widetilde{G}/\langle\varepsilon\rangle$ induces a surjective homomorphism $\widetilde{I}_\ell \longrightarrow I_\ell$ which implies the isomorphism $\widetilde{I}_\ell / < \varepsilon > \cap \widetilde{I}_\ell \cong I_\ell$. Thus ℓ is ramified in $\widetilde{K}/K$ iff $|\widetilde{I}_\ell| = 2|I_\ell|$ iff $|\widetilde{I}_\ell \cap < \varepsilon > | = 2$ iff $\varepsilon \in \widetilde{I}_\ell$. $\square$

5. The conductors of the representations

In this section we compute the Artin conductors of all 2-dimensional irreducible representations of $\tilde{G}$ in the case $p=-1$. First we recall the definition of the Artin conductor. For details, see [Chap.6, 2].

The notations are as before. Let ℓ be a prime number in $\mathbb{Q}$, and choose a prime ideal $\mathfrak{p}$ in $\tilde{K}$ over ℓ. Let $\tilde{G}_\ell = \tilde{G}(\tilde{K}_\mathfrak{p}/\mathbb{Q}_\ell)$ be the corresponding decomposition subgroup. Let v be the normalized valuation in $\tilde{K}_\mathfrak{p}$. For $i \geq 0$, define the ramification groups

$$\tilde{G}_{\ell,i} = \{\sigma \in \tilde{G}_\ell \mid v(\sigma(x)-x) > i \text{ for all } x \in O_{\tilde{K}_\mathfrak{p}}\}.$$

The group $\tilde{G}_{\ell,0}$ is the inertia subgroup of $\tilde{G}_\ell$. Let π be a uniformizer in $\tilde{K}_\mathfrak{p}$. Then for $i > 0$

$$\tilde{G}_{\ell,i} = \{\sigma \in \tilde{G}_\ell \mid v(\sigma(\pi) - \pi) > i\}.$$

For a representation ρ of $\tilde{G}$ with the character χ and the representation space V, let

$$f(\chi, \ell) = f(\rho, \ell) = \sum_{i=0}^{\infty} \frac{|\tilde{G}_{\ell,i}|}{|\tilde{G}_{\ell,0}|}(\chi(1) - \chi(\tilde{G}_{\ell,i})),$$

where $\chi(\tilde{G}_{\ell,i}) = |\tilde{G}_{\ell,i}|^{-1} \sum_{s \in \tilde{G}_{\ell,i}} \chi(s)$. We have $f(\chi, \ell) = 0$ if ρ is unramified over ℓ, i.e. $V = V^{\tilde{G}_{\ell,0}}$. The Artin conductor of the representation ρ is defined as

$$\mathfrak{f}(\chi) = \mathfrak{f}(\rho) = \prod_\ell \ell^{f(\chi,\ell)}.$$

From the result in last section, we know that ℓ is unramified in $\tilde{K}/\mathbb{Q}$ if $\ell \neq 2, q$. Thus to compute the conductor $\mathfrak{f}(\chi)$, we only need to calculate $f(\chi, 2)$ and $f(\chi, q)$. We consider the cases $q = 2$ and q odd separately.

5.1. **Case $q = 2$.** In the case $(p, q) = (-1, 2)$, there is only one 2-dimensional irreducible representation $\tilde{\rho}_0$ of $\tilde{G}$. Let $\tilde{\chi}_0$ be the character of $\tilde{\rho}_0$. Since only 2 is ramified in $\tilde{K}$, we only need calculate $f(\tilde{\chi}_0, 2)$.

As in last section, let $\pi_2 = 1 - \zeta_8$. Let $\wp$ be a prime ideal in $\tilde{K}$ over 2 and let v be the normalized valuation in $\tilde{K}_\wp$. From the proof of Prop.4.1, we see $\frac{\sqrt{2}}{\pi_2^2} \equiv 1 - \pi_2 \bmod \pi_2^3$. Thus

$$v\left(\frac{\sqrt{2}}{\pi_2^2} - 1\right) = v\left(\frac{\sqrt[4]{2}}{\pi_2} - 1\right) + v\left(\frac{\sqrt[4]{2}}{\pi_2} + 1\right) = v(\pi_2) = 2.$$

We have $v(\frac{\sqrt[4]{2}}{\pi_2} - 1) = v(\frac{\sqrt[4]{2}}{\pi_2} + 1) = 1$. So $\pi = \frac{\sqrt[4]{2}}{\pi_2} - 1$ is a uniformizer of $\tilde{K}_\wp$. The group $\tilde{G}$ is generated by $\tilde{\sigma}_{-1}$ and $\tilde{\sigma}_2$, and $\tilde{\sigma}_{-1}(\sqrt[4]{2}) = \sqrt[4]{2}$ and $\tilde{\sigma}_2(\sqrt[4]{2}) = \sqrt[4]{2}/\sqrt{-1}$. Clearly we have $G_{2,0} = \tilde{G}$. Furthermore, we have

$$v(\tilde{\sigma}_{-1}(\pi) - \pi) = v\left(\frac{\sqrt[4]{2}}{1 - \zeta_8^{-1}} - \frac{\sqrt[4]{2}}{1 - \zeta_8}\right) = v\left(-\frac{1 + \zeta_8}{1 - \zeta_8}\sqrt[4]{2}\right) = 2,$$

$$v(\tilde{\sigma}_2(\pi) - \pi) = v\left(\frac{\sqrt[4]{2}/\sqrt{-1}}{1 + \zeta_8} - \frac{\sqrt[4]{2}}{1 - \zeta_8}\right) = v\left(-\frac{1 - \zeta_8^3}{1 - \zeta_8}\sqrt[4]{2}\right) = 2,$$

and

$$v(\epsilon(\pi) - \pi) = v(-2(\pi + 1)) = 8.$$

Thus $G_{2,1} = G_{2,0} = \tilde{G}$ and $G_{2,2} = \cdots = G_{2,7} =< \epsilon >$. By easy computation we get that $\tilde{\chi}_0(G_{2,0}) = \tilde{\chi}_0(G_{2,1}) = \cdots = \tilde{\chi}_0(G_{2,7}) = 0$, and $\tilde{\chi}_0(G_{2,n}) = 2$ for $n \geq 8$. So we obtain

$$f(\tilde{\chi}_0, 2) = 2 + 2 + \frac{1}{4} \times 2 \times 6 = 7. \tag{5.1}$$

5.2. **Case odd q.** To compute $f(\chi, q)$, we consider the cases $(\frac{-1}{q}) = 1$ and $(\frac{-1}{q}) = -1$ separately. Let $\wp$ be a prime ideal in $\tilde{K}$ over q. Let v be the normalized valuation in $\tilde{K}_\wp$.

5.2.1. Assume $(\frac{-1}{q}) = 1$. Then q is unramified in $\tilde{K}/K$ but ramified in $K/\mathbb{Q}$. We see $\pi = 1 - \zeta_q$ is a uniformizer of $\tilde{K}_\wp$. Now all 2-dimensional irreducible representations of $\tilde{G}$ are as in Case A. Let $\tilde{\chi}_j$ be the character of $\tilde{\rho}_j$. It is easy to see that $\tilde{G}_{q,0} =< \tilde{\sigma}_q >$. Notice that $\epsilon \notin \tilde{G}_{q,0}$.

Let $1 \neq \tilde{\sigma} \in \tilde{G}_{q,0}$ and $\tilde{\sigma}(\zeta_q) = \zeta_q^a$, $1 < a \leq q-1$. We have

$$v(\tilde{\sigma}\pi - \pi) = v(\zeta_q - \zeta_q^a) = v(1 - \zeta_q^{a-1}) = 1.$$

Thus $\tilde{G}_{q,n} = \{1\}$ for $n \geq 1$. By easy computation we get that $\tilde{\chi}_j(\tilde{G}_{q,0}) = 0$ and $\tilde{\chi}_j(\tilde{G}_{q,n}) = 2$, for $n \geq 1$. We obtain

$$f(\tilde{\chi}_j, q) = 2. \tag{5.2}$$

5.2.2. Assume $(\frac{-1}{q}) = -1$. Then q is ramified both in $\tilde{K}/K$ and in $K/\mathbb{Q}$, and all 2-dimensional irreducible representations are as in Case B. Let $\tilde{\chi}_j$ be the character of $\tilde{\rho}_j$. Since $v(1-\zeta_q) = 2$ and $v(\sqrt[4]{-q}) = \frac{1}{4}(2(q-1)) = \frac{q-1}{2}$, we see that $\pi = \sqrt[4]{-q}/(1-\zeta_q)^{\frac{q-3}{4}}$ is a uniformizer of q in $\tilde{K}$. It is obvious that $\tilde{G}_{q,0} =< \tilde{\sigma}_q >$. Notice that in this case $\epsilon \in \tilde{G}_{q,0}$.

Let $1 \neq \tilde{\sigma} \in \tilde{G}_{q,0}$ and $\tilde{\sigma}(\zeta_q) = \zeta_q^a$, $1 < a \leq q-1$. We have

$$\begin{aligned} v(\tilde{\sigma}\pi - \pi) + v(\tilde{\sigma}\epsilon\pi - \pi) &= v(\tilde{\sigma}\pi - \pi) + v(-\tilde{\sigma}\pi - \pi) = v(\tilde{\sigma}\pi^2 - \pi^2) \\ &= v\left(\frac{(\frac{a}{q})\sqrt{-q}}{(1-\zeta_q^a)^{\frac{q-3}{2}}} - \frac{\sqrt{-q}}{(1-\zeta_q)^{\frac{q-3}{2}}} \right) \\ &= v(\pi^2) + v\left(\frac{1 - (\frac{a}{q})(\sum_{i=0}^{a-1} \zeta_q^i)^{\frac{q-3}{2}}}{(\sum_{i=0}^{a-1} \zeta_q^i)^{\frac{q-3}{2}}} \right) \\ &= 2 + v\left(1 - (\frac{a}{q}) \left(\sum_{i=0}^{a-1} \zeta_q^i \right)^{\frac{q-3}{2}} \right). \end{aligned}$$

Let $t = v(1 - (\frac{a}{q})(\sum_{i=0}^{a-1}\zeta_q^i)^{\frac{q-3}{2}})$. We claim that $t = 0$. Otherwise $t > 0$. Since

$$\left(\sum_{i=0}^{a-1}\zeta_q^i\right)^{\frac{q-3}{2}} \equiv a^{\frac{q-3}{2}} \equiv \begin{cases} 1 \bmod (1-\zeta_q) & \text{if } (\frac{a}{q}) = 1 \\ -1 \bmod (1-\zeta_q) & \text{if } (\frac{a}{q}) = -1, \end{cases}$$

we always have $a \equiv 1 \bmod q$ and thus $a = 1$, which contradicts to the assumption that $a > 1$. This shows the claim. Thus $v(\tilde{\sigma}\pi - \pi) = v(\tilde{\sigma}\epsilon\pi - \pi) = 1$, as $v(\tilde{\sigma}\pi - \pi + \tilde{\sigma}\epsilon\pi - \pi) = v(2\pi) = 1$. So we get $\tilde{G}_{q,n} = \{1\}$ for $n \geq 1$. By easy computation we get that $\tilde{\chi}_j(\tilde{G}_{q,0}) = 0$ and $\tilde{\chi}_j(\tilde{G}_{q,n}) = 2$ for $n \geq 1$. We obtain

$$f(\tilde{\chi}_j, q) = 2. \tag{5.3}$$

Next we compute $f(\tilde{\chi}_j, 2)$. we consider the cases $(\frac{2}{q}) = 1$ and $(\frac{2}{q}) = -1$ separately. Let $\wp$ be a prime ideal in $\tilde{K}$ over 2. Let v be the normalized valuation in $\tilde{K}_\wp$.

5.2.3. Assume $(\frac{2}{q}) = 1$. Then 2 is unramified in $\tilde{K}/K$ but ramified in $K/\mathbb{Q}$, and $\pi = 1 - \zeta_4$ is a uniformizer in $\tilde{K}_\wp$. It is easy to see that $\tilde{G}_{2,0} =< \tilde{\sigma}_{-1} >$. Notice that in this case $\epsilon \notin \tilde{G}_{2,0}$. We have

$$v(\tilde{\sigma}_{-1}\pi - \pi) = v(\zeta_4 - \zeta_4^{-1}) = v(2) = 2.$$

Thus we have $\tilde{G}_{2,0} = \tilde{G}_{2,1} =< \tilde{\sigma}_{-1} >$ and $\tilde{G}_{2,n} = \{1\}$ for $n > 1$. By easy computation we have $\tilde{\chi}_j(\tilde{G}_{2,0}) = \tilde{\chi}_j(\tilde{G}_{2,1}) = 1$ and $\tilde{\chi}_j(\tilde{G}_{2,n}) = 2$ for $n > 1$. We obtain

$$f(\tilde{\chi}_j, 2) = 1 + 1 = 2 \tag{5.4}$$

5.2.4. Assume $(\frac{2}{q}) = -1$. Now 2 is ramified both in $\tilde{K}/K$ and in $K/\mathbb{Q}$. As in last section, let $\pi_2 = 1 - \zeta_4$, $\alpha = \sum_{(\frac{a}{q})=1}\zeta_q^a$ and $\beta = \sum_{(\frac{a}{q})=1}\zeta_{2q}^a$, where the summations are over $1 \leq a \leq q-1$. From last section we have

$$\sqrt{q^*} \equiv (1 + \pi_2\beta)^2 + \pi_2^3 \bmod \pi_2^4 \tag{5.5}$$

Let $\mu = 1 + \pi_2\beta$. We claim that $\pi = \frac{\sqrt[4]{q^*}+\mu}{\pi_2}$ is a uniformizer in $\tilde{K}_\wp$. In fact, since

$$v(\sqrt[4]{q^*} + \mu) + v(\sqrt[4]{q^*} - \mu) = v(\sqrt{q^*} - \mu^2) = v(\pi_2^3) = 6$$

and $v((\sqrt[4]{q^*} + \mu) + (\sqrt[4]{q^*} - \mu)) = v(2\sqrt[4]{q^*}) = 4$, we must have

$$v(\sqrt[4]{q^*} + \mu) = v(\sqrt[4]{q^*} - \mu) = 3,$$

and thus $v(\frac{\sqrt[4]{q^*}+\mu}{\pi_2}) = 1$.

It is obvious that $\tilde{G}_{2,0} = \{1,\ \epsilon,\ \tilde{\sigma}_{-1},\ \tilde{\sigma}_{-1}\epsilon\}$. Since $\tilde{\sigma}_{-1}(\sqrt[4]{q^*}) = \sqrt[4]{q^*}$ and $\tilde{\sigma}_{-1}\epsilon(\sqrt[4]{q^*}) = -\sqrt[4]{q^*}$, we have

$$\begin{aligned} v(\tilde{\sigma}_{-1}\pi - \pi) &= v\left(\tilde{\sigma}_{-1}\frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2}\right) \\ &= v\left(\tilde{\sigma}_{-1}\frac{\sqrt[4]{q^*}+1}{\pi_2} - \frac{\sqrt[4]{q^*}+1}{\pi_2}\right) \ (\text{since } \tilde{\sigma}_{-1}\beta = \beta) \\ &= v\left(\frac{\sqrt[4]{q^*}+1}{1-\zeta_4^{-1}} - \frac{\sqrt[4]{q^*}+1}{1-\zeta_4}\right) = v(\sqrt[4]{q^*}+1). \end{aligned}$$

To compute it, we first claim that $\pi_2 \nmid \beta$. Otherwise, $2 \mid \beta$ as $\beta \in \mathbb{Q}(\zeta_q)$. From last section, we have $\sqrt{q^*} = 1 + 2\alpha$ and $\alpha + \beta \equiv 1 \bmod 2$, thus $\sqrt{q^*} \equiv -1 + 2\beta \equiv -1 \bmod 4$ and so $q^* \equiv 1 \bmod 8$. This contradicts to the assumption $(\frac{2}{q}) = -1$. We showed the claim. Thus $v(\beta) = 0$. Since $v(\sqrt[4]{q^*}+1+\pi_2\beta) = 3$, we have $v(\sqrt[4]{q^*}+1) = 2$, namely $v(\tilde{\sigma}_{-1}\pi - \pi) = 2$.

We now compute $v(\tilde{\sigma}_{-1}\epsilon\pi - \pi)$. We have

$$\begin{aligned} v(\tilde{\sigma}_{-1}\epsilon\pi - \pi) &= v\left(\tilde{\sigma}_{-1}\epsilon\frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2}\right) \\ &= v\left(\frac{-\sqrt[4]{q^*}+1}{1-\zeta_4^{-1}} - \frac{\sqrt[4]{q^*}+1}{1-\zeta_4}\right) = v(\sqrt[4]{q^*}+\zeta_4). \end{aligned}$$

Observe that $v(\sqrt[4]{q^*}+\zeta_4) + v(\sqrt[4]{q^*}-\zeta_4) = v(\sqrt{q^*}+1) = v(2\frac{\sqrt{q^*}+1}{2}) = 4$, since $\pi_2 \nmid \frac{\sqrt{q^*}+1}{2}$. Furthermore, since

$$v((\sqrt[4]{q^*}+\zeta_4) + (\sqrt[4]{q^*}-\zeta_4)) = v(2\sqrt[4]{q^*}) = 4,$$

we must have $v(\sqrt[4]{q^*}+\zeta_4) = v(\sqrt[4]{q^*}-\zeta_4) = 2$, namely $v(\tilde{\sigma}_{-1}\epsilon\pi - \pi) = 2$. In addition, we have

$$v(\epsilon\pi - \pi) = v\left(\frac{-\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2} - \frac{\sqrt[4]{q^*}+1+\pi_2\beta}{\pi_2}\right) = 2.$$

By the discussion above we have $\tilde{G}_{2,0} = \tilde{G}_{2,1}$ and $\tilde{G}_{2,n} = \{1\}$ for $n > 1$. By easy computation we have $\tilde{\chi}_j(\tilde{G}_{2,0}) = \tilde{\chi}_j(\tilde{G}_{2,1}) = 0$ and $\tilde{\chi}_j(\tilde{G}_{2,n}) = 2$ for $n > 2$. We obtain

$$f(\tilde{\chi}_j, 2) = 2 + 2 = 4. \tag{5.6}$$

5.3. **Global Conductors.** By the equalities (5.1)-(5.6) above, we get the following

Theorem 5.1. *In the case $q = 2$, the conductor of the unique 2-dimensional irreducible representation $\tilde{\rho}_0$ of $\tilde{G}$ is equal to $\mathfrak{f}(\tilde{\rho}_0) = 2^7$. In the case that q is odd, all the 2-dimensional irreducible representations $\tilde{\rho}_j$ of $\tilde{G}$ have the conductor $\mathfrak{f}(\tilde{\rho}_j) = 2^{2(1+\log_{-1}(\frac{2}{q}))}q^2$.*

6. The Artin L-functions

In this section we compute the Artin L-functions of the quasi-cyclotomic fields $\widetilde{K} = \mathbb{Q}(\zeta_{4q}, \sqrt[4]{q^*})$.

The L-functions associated to the 1-dimensional representations of $\widetilde{G}$ are the well-known Dirichlet L-functions. So we mainly compute the L-functions associated to the 2-dimensional irreducible representations of $\widetilde{G}$. Let $\varphi : \widetilde{G} \to \mathrm{GL}(V)$ be a 2-dimensional irreducible representation. The Artin L-function $L(\varphi, s)$ associated to φ is defined as the product of the local factors

$$L(\varphi, s) = \prod_{\ell:\text{prime}} L_\ell(\varphi, s),$$

where the local factors are defined as $L_\ell(\varphi, s) = \det(1 - \varphi(\widetilde{\mathrm{Fr}}_\ell)\ell^{-s} | V^{\widetilde{I}_\ell})^{-1}$. Now we begin to compute them. First we notice that if ℓ is ramified in $\widetilde{K}/K$, then $V^{\widetilde{I}_\ell} = 0$ and $L_\ell(\varphi, s) = 1$, which is due to the facts that $\varepsilon \in \widetilde{I}_\ell$ by Lemma 4.4 and $\varphi(\varepsilon) = -I$ for any irreducible representation φ of $\widetilde{G}$ by Theorem 3.1.

6.1. **Case $q = 2$.** By section 3, there is only one 2-dimensional representation $\widetilde{\rho}_0$ in this case, which is defined as

$$\widetilde{\rho}_0(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \widetilde{\rho}_0(\widetilde{\sigma}_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since 2 is ramified in $\widetilde{K}/K$, we have $L_2(\widetilde{\rho}_0, s) = 1$. Assume that ℓ is an odd prime number.

If $\ell \equiv 7 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_{-1}$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_{-1}$ or $\widetilde{\sigma}_{-1}\varepsilon$. In any case we have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 - \ell^{-2s})^{-1}.$$

If $\ell \equiv 5 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_2$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_2$ or $\widetilde{\sigma}_2\varepsilon$. We have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 + \ell^{-2s})^{-1}.$$

If $\ell \equiv 3 \bmod 8$, then $\mathrm{Fr}_\ell = \sigma_{-1}\sigma_2$ and thus $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_{-1}\widetilde{\sigma}_2$ or $\widetilde{\sigma}_{-1}\widetilde{\sigma}_2\varepsilon$. We have

$$L_\ell(\widetilde{\rho}_0, s) = \det\left(I \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ell^{-s}\right)^{-1} = (1 - \ell^{-2s})^{-1}.$$

If $\ell \equiv 1 \bmod 8$, then $\mathrm{Fr}_\ell = 1$ and thus $\widetilde{\mathrm{Fr}}_\ell = 1$ or ε. In this case we must determine $\widetilde{\mathrm{Fr}}_\ell$ completely. Since $\widetilde{\mathrm{Fr}}_\ell(\sqrt[4]{2}) \equiv (\sqrt[4]{2})^\ell \bmod \wp$ for the prime ideal $\wp$ of $\widetilde{K}$ over ℓ associated to $\widetilde{\mathrm{Fr}}_\ell$, we have $\widetilde{\mathrm{Fr}}_\ell = 1$ if $2^{\frac{\ell-1}{4}} \equiv 1 \bmod \ell$, and $\widetilde{\mathrm{Fr}}_\ell = \varepsilon$ if $2^{\frac{\ell-1}{4}} \equiv -1 \bmod \ell$. As in last section, we have that for $\ell \equiv 1 \bmod 8$, $2^{\frac{\ell-1}{4}} \equiv 1 \bmod \ell$ if and only if $\ell \in P_0$. So we have

$$L_\ell(\widetilde{\rho}_0, s) = \begin{cases} (1-\ell^{-s})^{-2} & \text{if } \ell \in P_0 \\ (1+\ell^{-s})^{-2} & \text{otherwise.} \end{cases}$$

We get the Artin L-function in the case $(p,q) = (-1,2)$ as follows:

$$\begin{aligned} L(\widetilde{\rho}_0, s) = & \prod_{\ell \equiv 3 \text{ or } 7(8)} (1-\ell^{-2s})^{-1} \cdot \prod_{\ell \equiv 5(8)} (1+\ell^{-2s})^{-1} \\ & \times \prod_{\ell \in P_0} (1-\ell^{-s})^{-2} \cdot \prod_{\ell \equiv 1(8),\ \ell \notin P_0} (1+\ell^{-s})^{-2}. \end{aligned} \tag{6.1}$$

6.2. **Case odd q.** In this case, all 2-dimensional irreducible representations of $\widetilde{G}$ are $\widetilde{\rho}_j$ with $0 \le j < q-1, 2 \mid j$ if $q \equiv 1 \bmod 4$, and $0 \le j < q-1, 2 \nmid j$ if $q \equiv 3 \bmod 4$, where $\widetilde{\rho}_j$ is defined by

$$\widetilde{\rho}_j(\widetilde{\sigma}_{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{\rho}_j(\widetilde{\sigma}_q) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\rho}_j(\varepsilon) = -I.$$

We first determine the local factors $L_\ell(\widetilde{\rho}_j, s)$ for $\ell \neq 2, q$. For such ℓ we have $V^{\widetilde{I}_\ell} = V$. Let $\mathrm{Fr}_\ell = \sigma_{-1}^{a_\ell}\sigma_q^{b_\ell}$, which is equivalent to $\ell \equiv (-1)^{a_\ell} \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$, where g is the primitive root $\bmod\, q$ associated to σ_q. It is easy to compute that

$$\widetilde{\rho}_j(\widetilde{\sigma}_q^{b_\ell}) = \begin{pmatrix} 0 & \zeta_{q-1}^j \\ 1 & 0 \end{pmatrix}^{b_\ell} = \begin{cases} \zeta_{2(q-1)}^{jb_\ell} I & \text{if } 2 \mid b_\ell \\ \begin{pmatrix} 0 & \zeta_{2(q-1)}^{j(b_\ell+1)} \\ \zeta_{2(q-1)}^{j(b_\ell-1)} & 0 \end{pmatrix} & \text{if } 2 \nmid b_\ell. \end{cases}$$

Furthermore, we have

$$\det(I - \widetilde{\rho}_j(\widetilde{\sigma}_{-1}^{a_\ell}\widetilde{\sigma}_q^{b_\ell})\ell^{-s}) = \begin{cases} (1-\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^2 & \text{if } a_\ell = 0,\ 2 \mid b_\ell \\ 1-\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell = 0,\ 2 \nmid b_\ell \\ & \text{or } a_\ell = 1,\ 2 \mid b_\ell \\ 1+\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell = 1,\ 2 \nmid b_\ell \end{cases}$$

and

$$\det(I + \widetilde{\rho}_j(\widetilde{\sigma}_{-1}^{a_\ell}\widetilde{\sigma}_q^{b_\ell})\ell^{-s}) = \begin{cases} (1+\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^2 & \text{if } a_\ell = 0,\ 2 \mid b_\ell \\ 1-\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell = 0,\ 2 \nmid b_\ell \\ & \text{or } a_\ell = 1,\ 2 \mid b_\ell \\ 1+\zeta_{q-1}^{jb_\ell}\ell^{-2s} & \text{if } a_\ell = 1,\ 2 \nmid b_\ell. \end{cases}$$

So we get

$$L_\ell(\widetilde{\rho}_j, s) = (1-\zeta_{q-1}^{jb_\ell}\ell^{-2s})^{-1}$$

if $\ell \equiv 1 \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$ with $2 \nmid b_\ell$, or if $\ell \equiv 3 \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$ with $2 \mid b_\ell$, and

$$L_\ell(\widetilde{\rho}_j, s) = (1 + \zeta_{q-1}^{jb_\ell} \ell^{-2s})^{-1}$$

if $\ell \equiv 3 \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$ with $2 \nmid b_\ell$.

To compute the local factors when $\ell \equiv 1 \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$ with $2 \mid b_\ell$ we must determine $\widetilde{\mathrm{Fr}}_\ell$ completely. Since $(\frac{\ell}{q}) = 1$, we have $(\frac{q}{\ell}) = 1$ and $(\frac{q^*}{\ell}) = 1$. Let $\alpha_\ell \in \mathbb{Z}$ such that $\alpha_\ell^2 \equiv q^* \bmod \ell$. From $\widetilde{\sigma}_q^{b_\ell}(\sqrt[4]{q^*}) = (-1)^{\frac{b_\ell}{2}} \sqrt[4]{q^*}$, we see $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_q^{b_\ell}$ if $(\frac{\alpha_\ell}{\ell}) = (-1)^{\frac{b_\ell}{2}}$, and $\widetilde{\mathrm{Fr}}_\ell = \widetilde{\sigma}_q^{b_\ell}\varepsilon$ if $(\frac{\alpha_\ell}{\ell}) = (-1)^{\frac{b_\ell}{2}+1}$. So when $\ell \equiv 1 \bmod 4$ and $\ell \equiv g^{b_\ell} \bmod q$ with $2 \mid b_\ell$, we have

$$L_\ell(\widetilde{\rho}_j, s) = \begin{cases} (1 - \zeta_{2(q-1)}^{jb_\ell} \ell^{-s})^{-2} & \text{if } (\frac{\alpha_\ell}{\ell}) = (-1)^{\frac{b_\ell}{2}} \\ (1 + \zeta_{2(q-1)}^{jb_\ell} \ell^{-s})^{-2} & \text{if } (\frac{\alpha_\ell}{\ell}) = (-1)^{\frac{b_\ell}{2}+1}. \end{cases}$$

Next we compute the local factors $L_2(\widetilde{\rho}_j, s)$ and $L_q(\widetilde{\rho}_j, s)$. When $(\frac{2}{q}) = -1$, we know from last section that 2 is ramified in $\widetilde{K}/K$. So $L_2(\widetilde{\rho}_j, s) = 1$ in this case. Now we assume $(\frac{2}{q}) = 1$. Since $I_2 =< \sigma_{-1} >$ and 2 is unramifed in $\widetilde{K}/K$, we have $\widetilde{I}_2 =< \widetilde{\sigma}_{-1} >$ or $\widetilde{I}_2 =< \widetilde{\sigma}_{-1}\varepsilon >$. The matrices $I + \widetilde{\rho}_j(\widetilde{\sigma}_{-1})$ and $I + \widetilde{\rho}_j(\widetilde{\sigma}_{-1}\varepsilon)$ have rank 1, thus $V^{\widetilde{I}_2}$ has dimension 1. Write $\mathrm{Fr}_2 = \sigma_2^{b_2}$ with $2 \mid b_2$. As in last section, we always assume $b_2 \equiv 2 \bmod 4$ if $q \equiv 3 \bmod 4$. Recall that P_0 be the set of the prime numbers of the form $A^2 + 64B^2$ with $A, B \in \mathbb{Z}$. Since $\widetilde{\rho}_j(\widetilde{\sigma}_2^{b_2}) = \zeta_{2(q-1)}^{jb_2} I$, by Prop.4.3 we have

$$L_2(\widetilde{\rho}_j, s) = \begin{cases} 1 - \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \notin P_0, 16 \nmid q^* - 1, \text{ or } q \in P_0, 16 \mid q^* - 1 \\ 1 + \zeta_{2(q-1)}^{jb_2} 2^{-s} & \text{if } q \in P_0, 16 \nmid q^* - 1, \text{ or } q \notin P_0, 16 \mid q^* - 1. \end{cases}$$

When $q \equiv 3 \bmod 4$, we know that q is ramified in $\widetilde{K}/K$. So $L_q(\widetilde{\rho}_j, s) = 1$ for odd j in this case. Assume $q \equiv 1 \bmod 4$. Since $I_q =< \sigma_q >$ and q is unramifed in $\widetilde{K}/K$, we have $\widetilde{I}_q =< \widetilde{\sigma}_q >$ or $\widetilde{I}_2 =< \widetilde{\sigma}_q\varepsilon >$. Thus $V^{\widetilde{I}_q} = 0$ if $j \neq 0$, and $V^{\widetilde{I}_q}$ has dimension 1 if $j = 0$.

The Frobenius map Fr_q of q in G modulo I_q is the identity map. So $\widetilde{\mathrm{Fr}}_q = 1$ or ε. In [Sect.5, 8] we have showed that q splits in $\widetilde{K}/K$ if $q \equiv 1 \bmod 8$ and is inert if $q \equiv 5 \bmod 8$. So $\widetilde{\mathrm{Fr}}_2 = 1$ if $q \equiv 1 \bmod 8$ and $\widetilde{\mathrm{Fr}}_2 = \varepsilon$ if $q \equiv 5 \bmod 8$. Thus we get

$$L_q(\widetilde{\rho}_j, s) = \begin{cases} 1 & \text{if } j \neq 0 \\ 1 - q^{-s} & \text{if } j = 0,\ q \equiv 1 \bmod 8 \\ 1 + q^{-s} & \text{if } j = 0,\ q \equiv 5 \bmod 8. \end{cases}$$

We have computed all the local factors, obtaining that

$$
\begin{aligned}
L(\widetilde{\rho}_j, s) =&(1 - u_q\zeta_{2(q-1)}^{jb_2}2^{-s})^{-1}(1-(-1)^{\frac{q-1}{4}}q^{-s})^{-\delta_{0j}} \\
&\times \prod_{\ell\equiv 1,\ 2\nmid b_\ell \text{ or } \ell\equiv 3,\ 2|b_\ell} (1-\zeta_{q-1}^{jb_\ell}\ell^{-2s})^{-1} \\
&\times \prod_{\ell\equiv 3,\ 2\nmid b_\ell}(1+\zeta_{q-1}^{jb_\ell}\ell^{-2s})^{-1}\prod_{\ell\equiv 1,\ 2|b_\ell}(1-u_\ell\zeta_{2(q-1)}^{jb_\ell}\ell^{-s})^{-2},
\end{aligned} \tag{6.2}
$$

where $u_q = 1$ if $q \notin P_0, 16 \nmid q^* - 1$, or $q \in P_0, 16 \mid q^* - 1$ and $u_q = -1$ otherwise; $\delta_{0j} = 0$ if $j \neq 0$ and $\delta_{00} = 1$; and $u_\ell = (\frac{\alpha_\ell}{\ell})(-1)^{\frac{b_\ell}{2}}$. Here in the products, " $\equiv$ " denotes the congruence modulo 4.

Theorem 6.1. *Except for the Dirichlet L-functions, all Artin L-functions of the Galois extension $\widetilde{K}/\mathbb{Q}$ are explicitly given by (5.1) in the case $q = 2$ and by (5.2) in the case q is odd, where in (5.2) $0 \le j < q-1,\ 2 \mid j$ if $q \equiv 1 \bmod 4$ and $0 \le j < q-1,\ 2 \nmid j$ if $q \equiv 3 \bmod 4$.*

6.3. **A formula.** Let $\zeta_{\widetilde{K}}(s)$ and $\zeta_K(s)$ be the Dedekind zeta functions of $\widetilde{K}$ and K respectively. By Artin's formula of the decomposition of Dedekind zeta functions, we have

$$
\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} = \prod_{\widetilde{\rho}_j}\prod_{\ell:\ \text{prime}} L_\ell(\widetilde{\rho}_j, s)^2,
$$

where $\widetilde{\rho}_j$ runs over all 2-dimensional irreducible representations of $\widetilde{G}$. When $q = 2$, there is only one 2-dimensional irreducible representation of $\widetilde{G}$. So the square of (5.1) gives the formula. When q is odd, by computing $\prod_{\widetilde{\rho}_j} L_\ell(\widetilde{\rho}_j, s)$, we get the following

Corollary 6.2. *For a prime number $\ell \neq q$, let $f_\ell = \frac{q-1}{\gcd(b_\ell, q-1)}$ be the order of $\ell \bmod q$ and let $g_\ell = \gcd(b_\ell, q-1) = \frac{q-1}{f_\ell}$. If $q \equiv 1 \bmod 4$, we have*

$$
\begin{aligned}
\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} =&(1-u_q^{f_2}2^{-f_2 s})^{-g_2}(1-(-1)^{\frac{q-1}{4}}q^{-s})^{-2}\prod_{\ell\equiv 1,\ 2\nmid b_\ell \text{ or } \ell\equiv 3}(1-\ell^{-f_\ell s})^{-2g_\ell} \\
&\times \prod_{\ell\equiv 1,\ 2|b_\ell}(1-u_\ell^{f_\ell}\ell^{-f_\ell s})^{-2g_\ell},
\end{aligned}
$$

and if $q \equiv 3 \bmod 4$, we have

$$
\begin{aligned}
\frac{\zeta_{\widetilde{K}}(s)}{\zeta_K(s)} =&(1+u_q^{f_2}2^{-f_2 s})^{-g_2}\prod_{\ell\equiv 1,\ 2\nmid b_\ell}(1+\ell^{-f_\ell s})^{-2g_\ell}\prod_{\ell\equiv 3}(1-\ell^{-2f_\ell s})^{-g_\ell} \\
&\times \prod_{\ell\equiv 1,\ 2|b_\ell}(1+u_\ell^{f_\ell}\ell^{-f_\ell s})^{-2g_\ell},
\end{aligned}
$$

where u_q and u_ℓ are as above.

6.4. **The corresponding modular forms.** All the 2-dimensional irreducible representations of $\tilde{G}$ in the case $p=-1$ are monomial. It is easy to see that they are odd. By Deligne-Serre's theorem [Th.2, 7], these Artin L-functions above are equal to the L-functions of some normalized newforms of weight one, which allows one to determine a newforms of weight one by a 2-dimensional irreducible odd representations of $\tilde{G}$. More precisely, the irreducible representation $\tilde{\rho}_j$ of conductor N corresponds to a normalized newform $f_j(z)$ of weight one on $\Gamma_0(N)$ with nebentype $\phi_j=\det(\tilde{\rho}_j)$, which has a Fourier expansion at infinity

$$f_j(z)=\sum_{n=1}^{\infty} a_n^{(j)} q^n, \qquad q=e^{2\pi i z},$$

where $a_1^{(j)}=1$ and the other coefficients a_n are equal to those of the L-function $L(\phi_j,s)=\sum_{n=1}^{\infty} a_n n^{-s}$. In this subsection we describe these modular forms explicitly. Since these newforms are eigenfunctions of Hecke operators, to determine all $a_n^{(j)}$ it is enough to determine $a_\ell^{(j)}$ for all primes ℓ.

When $q=2$, we get one normalized newform $f_0(z)$ of weight 1 on $\Gamma_0(2^7)$ with nebentype $\phi_0:(\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{C}^*$, where $\phi_0(\sigma_{-1})=-1$ and $\phi_0(\sigma_2)=1$. By the formula (6.1), we directly have that for primes ℓ the coefficients $a_\ell^{(0)}$ of the newform are given by

$$a_\ell^{(0)}=\begin{cases} 0 & \text{if } \ell=2 \text{ or } \ell\equiv 3,5,7 \bmod 8 \\ 2 & \text{if } \ell\in P_0 \\ -2 & \text{if } \ell\equiv 1 \bmod 8 \text{ but } \ell\notin P_0 \end{cases}$$

When q is odd, we get $\frac{q-1}{2}$ normalized newforms $f_j(z)$ of weight 1 on $\Gamma_0(4^{1+\log_{-1}(\frac{2}{q})}q^2)$ with neputye $\phi_j:(\mathbb{Z}/4q\mathbb{Z})^*\to\mathbb{C}^*$, where $\phi_j(\sigma_{-1})=-1$ and $\phi_j(\sigma_q)=-\zeta_{q-1}^j$. By the formula (6.2) we directly have that for primes $\ell\neq q$ the coefficients of the newforms are given by

$$a_\ell^{(j)}=\begin{cases} u_q\zeta_{2(q-1)}^{-jb_2} & \text{if } \ell=2 \\ 2u_\ell\zeta_{2(q-1)}^{-jb_\ell} & \text{if } \ell\equiv 1 \bmod 4 \text{ and } 2\mid b_\ell \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a_q^{(j)}=\begin{cases} 0 & \text{if } j\neq 0 \\ (-1)^{\frac{q-1}{4}} & \text{if } j=0, \end{cases}$$

where $0\le j<q-1,\ 2\mid j$ if $q\equiv 1 \bmod 4$ and $0\le j<q-1,\ 2\nmid j$ if $q\equiv 3 \bmod 4$; b_ℓ is defined by $\ell\equiv g^{b_\ell} \bmod q$ for a primitive root g modulo q; $u_\ell=(\frac{\alpha_\ell}{\ell})(-1)^{\frac{b_\ell}{2}}$ for an integral number α_ℓ such that $\alpha_\ell^2\equiv q^* \bmod \ell$;

and

$$u_q = \begin{cases} 1 & \text{if } q \notin P_0, 16 \nmid q^* - 1 \text{ or } q \in P_0, 16 \mid q^* - 1 \\ -1 & \text{if } q \notin P_0, 16 \mid q^* - 1 \text{ or } q \in P_0, 16 \nmid q^* - 1. \end{cases}$$

Here P_0 is the set of all prime numbers of the form $A^2 + 64B^2$ with $A, B \in \mathbb{Z}$.

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