ALGEBRAIC COBORDISM THEORY ATTACHED TO ALGEBRAIC EQUIVALENCE

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Abstract. After the algebraic cobordism theory of Levine-Morel, we develop a theory of algebraic cobordism modulo algebraic equivalence.

We prove that this theory can reproduce Chow groups modulo algebraic equivalence and the zero-th semi-topological K-groups. We also show that with finite coefficients, this theory agrees with the algebraic cobordism theory.

We compute our cobordism theory for some low dimensional or special types of varieties. The results on infinite generation of some Griffiths groups by Clemens and on smash-nilpotence by Voevodsky and Voisin are also lifted and reinterpreted in terms of this cobordism theory.

1. Introduction

The goal of this paper is to develop a cobordism theory on schemes, that is associated to algebraic equivalence, from the algebraic cobordism theory Ω_* of Levine and Morel [12]. The algebraic cobordism theory is defined from cobordism cycles as the Chow group is defined from algebraic cycles. Here we regard Ω_* as the cobordism theory associated to rational equivalence, which is the finest adequate equivalence on algebraic cycles in the sense of [1, Définition 3.1.1.1.] or [8].

The algebraic cobordism theory is a universal oriented cohomology theory on smooth varieties modeled on the complex cobordism MU^* in [15]. Unlike the cases of ordinary cohomology theories on algebraic varieties or topological spaces, the Chern classes of the algebraic cobordism and the complex cobordism satisfy a relation of the form $c_1(L \otimes M) = F(c_1(L), c_1(M))$ for a formal power series in two variables $F(u, v) = u + v +$ (higher order terms). They have the structures of graded modules over a graded ring called the Lazard ring. The algebraic cobordism theory contains enough data to reproduce Chow groups and Grothendieck groups. Levine and Pandharipande [13] defined the double-point cobordism theory ω_* that is isomorphic to the Levine-Morel algebraic cobordism theory Ω_*.

There are at least three ways to approach the problem of defining cobordism theories associated to algebraic equivalence. The first method is to modify the Levine-Morel algebraic cobordism by adding additional relations coming from algebraic equivalence. The resulting theory is Ω_*^{alg}. The second method is to modify the Levine-Pandharipande double-point relations, allowing curves of all genus $g \geq 0$. We obtain ω_*^{alg} in this method, which was originally mentioned in [13, §11.2] without further studies. The third method is motivated by the relation between the Grothendieck K_0 and the semi-topological K_0 discussed in Friedlander-Walker [4]:

Definition 1.1. Let X be a separated scheme of finite type over a field k. For a smooth projective curve C over k and k-rational points $i_j \colon \{t_j\} \hookrightarrow C$, $j = 1, 2$, let $\omega_*^{\mathrm{FW}}(X)$ be

2010 *Mathematics Subject Classification.* Primary 14F43; Secondary 55N22.

Key words and phrases. cobordism, Chow group, K-theory, algebraic cycle, Griffiths group.

the quotient of $\omega_*(X)$ by the subgroup generated by the images of $i_0^* - i_1^* \colon \omega_*(X \times C) \to \omega_*(X)$. The existence i_j^* follows from [12, 6.5.4] and [13, Theorem 1].

Some of the central results of this paper are summarized as follows:

Theorem 1.2. *Let $\mathbf{Sch}_k$ be the category of separated schemes of finite type over a field k of characteristic 0, and let $\mathbf{Sm}_k$ be the subcategory of smooth quasi-projective reduced k-schemes. Then, on the category $\mathbf{Sch}_k$, we can define three algebraic cobordism theories attached to algebraic equivalence : Ω_*^{alg} in Definition 3.6, ω_*^{alg} in Definition 4.3, ω_*^{FW} in Definition 1.1, all isomorphic to each other, satisfying the following properties:*

(1) *The functor Ω_*^{alg} defines an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ that respects algebraic equivalence, in the sense of* [12, Definitions 5.1.3] *and Definition 7.8. The restriction Ω^*_{alg} on the subcategory $\mathbf{Sm}_k$, with the cohomological indexing in Definition 3.6, defines an oriented cohomology theory that respects algebraic equivalence in the sense of* [12, Definition 1.1.2] *and Definition 7.8.*

(2) *In particular, Ω_*^{alg} satisfies the localization property, the $\mathbb{A}^1$-homotopy invariance and the projective bundle formula.*

(3) *Ω^*_{alg} is universal among all oriented cohomology theories on $\mathbf{Sm}_k$ that respect algebraic equivalence. Similarly, Ω_*^{alg} is universal among all oriented Borel-Moore homology theories on $\mathbf{Sch}_k$ that respect algebraic equivalence.*

(4) *Let $X \in \mathbf{Sch}_k$. Then, $\Omega_*^{\mathrm{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \simeq \mathrm{CH}_*^{\mathrm{alg}}(X)$ and $\Omega_*^{\mathrm{alg}}(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \simeq G_0^{\mathrm{semi}}(X)[\beta, \beta^{-1}]$, where $\mathrm{CH}_*^{\mathrm{alg}}$ is the Chow group modulo algebraic equivalence, G_0^{semi} is the semi-topological Grothendieck group of coherent sheaves, and β is a formal symbol of degree -1. Here, $\mathbb{L}_*$ is the Lazard ring in* [12, p. 4] *(or, see Remark 3.5).*

(5) *Let $X \in \mathbf{Sch}_k$. Then, $\Omega_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \xrightarrow{\simeq} \Omega_*^{\mathrm{alg}}(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m$.*

A good part of the paper, from Sections 2 to 8, is devoted to proving Theorem 1.2. Section 2 recalls the definition of cobordism cycles from [12], and that of algebraic equivalence. In Section 3, we define Ω_*^{alg} in terms of the cobordism cycles of Levine-Morel modulo various relations, one of which reflects algebraic equivalence of line bundles. We prove a universal property of Ω_*^{alg}. Section 4 recalls the rational and algebraic double-point cobordism theories ω_* and ω_*^{alg} from [13].

In Section 5, we prove as Theorem 5.1 a basic exact sequence that relates $\omega_*(X)$ and $\omega_*^{\mathrm{alg}}(X)$ for any $X \in \mathbf{Sch}_k$. This yields $\omega_*^{\mathrm{alg}} \simeq \omega_*^{\mathrm{FW}}$. This sequence proves various results in the paper. Section 6 proves a comparison theorem $\Omega_*^{\mathrm{alg}} \simeq \omega_*^{\mathrm{alg}}$ and Theorem 1.2(2). Section 7 finishes the proof of Theorem 1.2(1). In Section 8, we answer Theorem 1.2(4)(5). This concludes Theorem 1.2. From (3) and (4) we deduce a novel observation that the Chow groups modulo algebraic equivalence have a characterization by a universal property. See Remark 8.2.

In Section 9, we compute Ω_*^{alg} from various angles:

Theorem 1.3. *Let $X \in \mathbf{Sch}_k$ and let $\mathbb{L}^*$ be the Lazard ring with the cohomological indexing (see Remark 3.5).*

(1) *For X smooth over $\mathbb{C}$, there is a cycle class map $\Omega^*_{\mathrm{alg}}(X) \to \mathrm{MU}^{2*}(X(\mathbb{C}))$.*

(2) *$\mathbb{L}^* \simeq \Omega^*(k) \simeq \Omega^*_{\mathrm{alg}}(k)$. Furthermore, Ω_*^{alg} is generically constant in the sense of* [12, Definition 4.1.1]*. (Or, see the paragraph above Proposition 9.4.)*

(3) *If X is a cellular variety, then $\Omega_*(X) \xrightarrow{\simeq} \Omega_*^{\mathrm{alg}}(X)$ as free $\mathbb{L}_*$-modules.*

(4) *If X is smooth, the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is finitely generated if and only if the group $\mathrm{CH}^*_{\mathrm{alg}}(X)$ is finitely generated. If X is smooth projective over $\mathbb{C}$, then the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is finitely generated if and only if the Griffiths group $\mathrm{Griff}^*(X)$ is finitely*

generated. If $\dim X \leq 2$, the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is finitely generated, and if $\dim X \geq 3$, there are non-finitely generated examples.

(5) *If X is a smooth affine curve, then $\mathbb{L}^* \simeq \Omega^*_{\mathrm{alg}}(X)$. If X is a smooth curve over $\mathbb{C}$, then $\Omega^*_{\mathrm{alg}}(X) \overset{\simeq}{\to} \mathrm{MU}^{2*}(X(\mathbb{C}))$. An analogue of Quillen-Lichtenbaum conjecture holds for smooth curves over $\mathbb{C}$:*

$$\Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \overset{\simeq}{\to} \mathrm{MU}^{2*}(X(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{Z}/m.$$

In Section 10, we discuss a cobordism analogue of a theorem of Voevodsky [18] and Voisin [19] about smash-nilpotence of algebraically trivial algebraic cycles:

Theorem 1.4. *Let X be a smooth projective variety and α be a cobordism cycle over X. If α vanishes in $\Omega^*_{\mathrm{alg}}(X)_{\mathbb{Q}}$, then its smash-product $\alpha^{\boxtimes N}$ (see Definition 10.1) vanishes in $\Omega^*(X^N)_{\mathbb{Q}}$ for some integer $N > 0$.*

Some details related to Gysin pull-backs from [12] are placed in the Appendix (Section 11), and there a new lemma related to Ω_*^{alg} is proven.

This paper builds on two grand works [12] and [13] on algebraic cobordism. Whenever necessary, we take the freedom of using the definitions and results of these references. In doing so, we will provide full reference details.

Convension: Throughout the paper, k is a field of characteristic zero. When no confusion arises, we use $\sim$ to mean algebraic equivalence.

2. Cobordism cycles and algebraic equivalence

This section recalls the basic definitions on cobordism cycles in the theory of algebraic cobordism of Levine and Morel [12]. We also recall the notion of algebraic equivalence of vector bundles and algebraic cycles. This will be used in the construction of our cobordism theory in Section 3.

2.1. **Cobordism cycles.** We recall the definition of the *cobordism cycles* of Levine and Morel from [12, Definition 2.1.6]. The cobordism cycles in algebraic cobordism theories play the role of the algebraic cycles in Chow groups.

Definition 2.1. Let $X \in \mathbf{Sch}_k$ be of dimension $n \geq 0$. An *integral cobordism cycle over* X is a family $(f\colon Y \to X, L_1, \cdots, L_r)$, where Y is integral and in $\mathbf{Sm}_k$, consisting of a projective morphism f, and a finite sequence $(L_1, \cdots, L_r)$ of r line bundles on Y, with $r = 0$ case also allowed. Its dimension is defined to be $\dim(Y) - r \in \mathbb{Z}$. An *isomorphism* Φ *of two cobordism cycles* $(Y \to X, L_1, \cdots, L_r) \simeq (Y' \to X, L'_1, \cdots, L'_{r'})$ is a triple $\Phi = (\phi\colon Y \to Y', \sigma, (\psi_1, \cdots, \psi_r))$ consisting of an isomorphism $\phi\colon Y \to Y'$ of X-schemes, a bijection $\sigma\colon \{1, \cdots, r\} \simeq \{1, \cdots, r'\}$, and isomorphisms $\psi_i\colon L_i \simeq \phi^* L'_{\sigma(i)}$ of lines bundles over Y for all i. When Y has multiple connected components, then $(Y \to X, L_1, \cdots, L_r)$ is defined as the sum of the obvious integral cobordism cycles for the components.

Let $\mathcal{Z}_*(X)$ be the free abelian group on the set of isomorphism classes of integral cobordism cycles over X. We let $\mathcal{Z}_d(X)$ be the subgroup generated by the dimension d cobordism cycles. An element of this group is called a cobordism d-cycle. The image of the integral cobordism cycle $(Y \to X, L_1, \cdots, L_r)$ is denoted by $[Y \to X, L_1, \cdots, L_r] \in \mathcal{Z}_*(X)$. When X is smooth equidimensional, the class $[\mathrm{Id}_X\colon X \to X] \in \mathcal{Z}_d(X)$ is denoted by 1_X. A cobordism cycle of the form $[\mathrm{Id}_X\colon X \to X, L_1, \cdots, L_r]$ is often written as $[X \to X, L_1, \cdots, L_r]$ when no confusion arises. Recall the following definitions from [12, 2.1.2, 2.1.3]:

Definition 2.2. Consider the category $\mathbf{Sch}_k$.

(1) For a projective morphism $g\colon X \to X'$ in $\mathbf{Sch}_k$, the *push-forward along* g is the graded group homomorphism $g_*\colon \mathcal{Z}_*(X) \to \mathcal{Z}_*(X')$ given by the composition with g, *i.e.*, $[f\colon Y \to X, L_1, \cdots, L_r] \mapsto [g \circ f\colon Y \to X', L_1, \cdots, L_r]$.

(2) For a smooth equidimensional morphism $g\colon X \to X'$ of relative dimension d, the *pull-back along* g is the homomorphism $g^*\colon \mathcal{Z}_*(X') \to \mathcal{Z}_{*+d}(X)$ given by sending $[f\colon Y \to X', L_1, \cdots, L_r]$ to $[pr_2\colon Y \times_{X'} X \to X, pr_1^*(L_1), \cdots, pr_1^*(L_r)]$.

(3) Let $X \in \mathbf{Sch}_k$, and let L be a line bundle on X. The *first Chern class operator of* L is defined to be the homomorphism $\tilde{c}_1(L)\colon \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X)$ given by $[f\colon Y \to X, L_1, \cdots, L_r] \mapsto [f\colon Y \to X, L_1, \cdots, L_r, f^*(L)]$. If X is smooth, we define the *first Chern class* $c_1(L)$ *of* L to be the cobordism cycle $c_1(L)\colon = [\mathrm{Id}_X\colon X \to X, L]$.

(4) For $X, Y \in \mathbf{Sch}_k$, we define the *external product*

$$\times\colon \mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \to \mathcal{Z}_*(X \times Y)$$

by sending the pair $[f\colon X' \to X, L_1, \cdots, L_r] \times [g\colon Y' \to Y, M_1, \cdots, M_s]$ to

$$[f \times g\colon X' \times Y' \to X \times Y, pr_1^*(L_1), \cdots, pr_1^*(L_r), pr_2^*(M_1), \cdots, pr_2^*(M_s)].$$

It is known that $\mathcal{Z}_*(-)$ defines the universal oriented Borel-Moore functor on $\mathbf{Sch}_k$ with products in the sense of [12, Definition 2.1.10]. This universality is based on the observation in [*ibid.*, Remark 2.1.8] that in $\mathcal{Z}_*(X)$, we have the identity $[f\colon Y \to X, L_1, \cdots, L_r] = f_* \circ \tilde{c}_1(L_r) \circ \cdots \circ \tilde{c}_r(L_1) \circ \pi_Y^*(1)$, where $\pi_Y\colon Y \to \mathrm{Spec}(k)$ is the structure map and $1\colon = 1_{\mathrm{Spec}(k)} \in \mathcal{Z}_0(k)$.

2.2. **Algebraic equivalence on vector bundles.** For algebraic cycles on varieties, the notion of algebraic equivalence was defined first in [16]. For $X \in \mathbf{Sch}_k$, we say that two algebraic cycles Z_1 and Z_2 on X are *algebraically equivalent* if there exists a smooth projective curve C and k-rational points t_1, t_2 on C with a cycle Z on $X \times C$ such that $Z|_{X \times \{t_j\}} = Z_j$ for $j = 1, 2$. We refer to [6, Chapter 10] for basic facts on algebraic equivalence of algebraic cycles. For vector bundles, we have a related notion. Let $X \in \mathbf{Sch}_k$, and let E_1, E_2 be two vector bundles of finite rank on X. We say that E_1 and E_2 are *algebraically equivalent* if there is a smooth projective curve C, k-rational points t_1, t_2 on C and a vector bundle V on $X \times C$ such that $E_i \simeq V|_{X \times \{t_j\}}$ for $j = 1, 2$. We use $\sim_{\mathrm{alg}}$ to mean both of the above notions on cycles and vector bundles.

We say that a vector bundle E of rank m on X is *algebraically trivial* if it is algebraically equivalent to the trivial bundle $O_X^{\oplus m}$. The following facts about algebraic equivalence of vector bundles and algebraic cycles will be useful in the sequel. Note that we immediately have:

Lemma 2.3. *Two vector bundles E_1 and E_2 on a scheme X are algebraically equivalent if and only if $E_1 \otimes L$ and $E_2 \otimes L$ are algebraically equivalent for every $L \in \mathrm{Pic}(X)$.*

Lemma 2.4. *Let X be a smooth variety and let D_1 and D_2 be two Weil divisors on X. Then $D_1 \sim_{\mathrm{alg}} D_2$ as cycles if and only if $O_X(D_1) \sim_{\mathrm{alg}} O_X(D_2)$ as line bundles.*

Proof. If D_1 and D_2 are algebraically equivalent, then there is a smooth connected scheme T of dimension > 0, k-rational points t_1, t_2 on T, and a Weil divisor D on $X \times T$ such that $D_1 - D_2 = D_{t_1} - D_{t_2}$. We can assume that T is projective.

By [9, Theorem 1] (see also [6, Example 10.3.2] if k is algebraically closed), we can replace T by a smooth projective curve C passing through t_1, t_2. Thus, we have a Weil divisor D on $X \times C$ such that $D_1 - D_2 = D_{t_1} - D_{t_2}$. We can modify D by $D - (D_{t_2} \times C) + (D_2 \times C)$ so that $D_{t_i} = D_i$ for $i = 1, 2$. Letting $\mathcal{L} = O_{X \times C}(D)$, we see that $\mathcal{L}|_{X \times \{t_i\}} \simeq O_X(D_i)$ for $i = 1, 2$.

Conversely, suppose there is a line bundle $\mathcal{L}$ on $X \times C$ such that $\mathcal{L}|_{X\times\{t_i\}} \simeq O_X(D_i)$ for $i = 1, 2$. Since $X \times C$ is smooth, there is a Weil divisor D on $X \times C$ whose associated line bundle is $\mathcal{L}$. This implies in particular that $D_{t_i} \sim_{\text{rat}} D_i$ for $i = 1, 2$. In other words, we have $D_1 \sim_{\text{rat}} D_{t_1} \sim_{\text{alg}} D_{t_2} \sim_{\text{rat}} D_2$, which implies that $D_1 \sim_{\text{alg}} D_2$. □

Remark 2.5. Note that if the curve C in the above definition is (a nonempty open subset of) $\mathbb{P}^1$, then we can say that E_1 and E_2 are rationally equivalent. However, when X is a semi-normal variety and E_1, E_2 are line bundles, this is equivalent to saying that E_1 and E_2 are isomorphic.

3. Definition of the algebraic cobordism Ω_*^{alg} modulo algebraic equivalence

In this section, we define an alegbraic cobordism theory of a scheme $X \in \mathbf{Sch}_k$ associated to algebraic equivalence, modifying the definition of algebraic cobordism $\Omega_*(X)$ of Levine and Morel [12]. The basic motivation is the simple observation that the algebraic cobordism is associated to the rational equivalence of line bundles in that, two line bundles on a smooth variety are isomorphic if and only if they are rationally equivalent (see Remark 2.5).

Levine and Morel in *ibid.* constructed $\Omega_*(X)$ from the cobordism cycles $\mathcal{Z}_*(X)$ of Definition 2.1. We will define the cobordism theory $\Omega_*^{\text{alg}}(X)$ that is similar to that of Levine-Morel, with one additional set of relations that identifies two integral cobordism cycles when their line bundles are suitably related by algebraic equivalence. We use $\sim$ to mean algebraic equivalence.

Definition 3.1 (*cf.* [12, Definition 2.4.5])**.** For $X \in \mathbf{Sch}_k$, the $\sim$*-pre-cobordism* $\underline{\Omega}_*^{\text{alg}}(X)$ is the quotient of $\mathcal{Z}_*(X)$ by the following three relations:

(1) (Dim) If there is a smooth quasi-projective morphism $\pi\colon Y \to Z$ with line bundles $M_1, \cdots, M_{s>\dim Z}$ on Z with $L_i \simeq \pi^* M_i$ for $i = 1, \cdots, s \le r$, then $[f\colon Y \to X, L_1, \cdots, L_r] = 0$.

(2) (Sect) For a section $s\colon Y \to L$ of a line bundle L on Y with its smooth associated divisor $i\colon D \to Y$, we impose

$$[f\colon Y \to X, L_1, \cdots, L_r, L] = [f \circ i\colon D \to X, i^*L_1, \cdots, i^*L_r].$$

(3) (Equiv) $[Y \to X, L_1, \cdots, L_r]$ and $[Y' \to X, L'_1, \cdots, L'_r]$ are identified if there exists an isomorphism $\phi\colon Y \to Y'$ over X, a permutation σ of $\{1, \cdots, r\}$ and algebraic equivalences of the line bundles $L_i \sim \phi^*(L'_{\sigma(i)})$.

Remark 3.2. If we take the quotient of $\mathcal{Z}_*(X)$ by only the relations (Dim) and (Sect), then the resulting quotient group is the pre-cobordism $\underline{\Omega}_*(X)$ of Levine-Morel in *ibid.* The cobordism cycles of the form $[Y \to X, L] - [Y \to X, L']$ are zero in $\underline{\Omega}_*(X)$ if $L \simeq L'$. If $\sim$ in (Equiv) is replaced by the rational equivalence $\sim_{\text{rat}}$ of line bundles, then by Remark 2.5, the modified relation $(\text{Equiv})_{\text{rat}}$ plays no role because Y is smooth, thus semi-normal.

Remark 3.3. Note that by definition, we have a natural surjection $\underline{\Phi}_X\colon \underline{\Omega}_*(X) \to \underline{\Omega}_*^{\text{alg}}(X)$.

Lemma 3.4. *All four operations (projective push-forward, smooth pull-back, external product and the first Chern class operation) in Definition 2.2 descend onto the $\sim$-pre-cobordism $\underline{\Omega}_*^{\text{alg}}$.*

Proof. By [12, Remarks 2.1.11, 2.1.14, Lemmas 2.4.2, 2.4.7], $\underline{\Omega}_*$ is an oriented Borel-Moore functor on $\mathbf{Sch}_k$ with product in the sense of [12, Definition 2.1.10], which implies (Dim) and (Sect) are respected by the four operations. For (Equiv), it follows from the fact that the pull-back operations on line bundles via any morphisms respect algebraic equivalence. □

Remark 3.5. To impose the universal formal group law into our cobordism theory as in [12, p. 4, §2.4.4], first recall from *ibids.* that there is a graded polynomial ring $\mathbb{Z}[a_{i,j}|i,j \geq 0]$, where $a_{i,j}$ are variables of degree $i+j-1$ subject to some relations. This is called the Lazard ring written $\mathbb{L}_*$. There is a power series $F_{\mathbb{L}_*}(u,v) \colon = \sum_{i,j} a_{i,j} u^i v^j \in \mathbb{L}_*[[u,v]]$ such that the pair $(\mathbb{L}_*, F_{\mathbb{L}_*})$ is the universal commutative formal group law of rank one. One also uses the cohomological indexing $\mathbb{L}^*$ by letting $\mathbb{L}^n = \mathbb{L}_{-n}$. We have $\mathbb{L}_0 \simeq \mathbb{Z}$ and $\mathbb{L}^{-n} = \mathbb{L}_n = 0$ if $n < 0$. Now we define the main object of study in this article.

Definition 3.6 (*cf.* [12, Definition 2.4.10])**.** For $X \in \mathbf{Sch}_k$, the graded group $\Omega_*^{\mathrm{alg}}(X)$ is defined to be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*^{\mathrm{alg}}(X)$ by the relations (FGL) of the form $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))([f \colon Y \to X, L_1, \cdots, L_r]) = \tilde{c}_1(L \otimes M)([f \colon Y \to X, L_1, \cdots, L_r])$ for lines bundles L and M on X. By the relation (Dim) in Definition 3.1-(1), the expression $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))$ is a finite sum so that the operator is well-defined. This graded abelian group $\Omega_*^{\mathrm{alg}}(X)$ is called the *algebraic cobordism of X modulo algebraic equivalence.*

When X is smooth equidimensional of dimension n, the codimension of a cobordism d-cycle is defined to be $n-d$. By codimension, we let $\Omega^{n-d}_{\mathrm{alg}}(X) \colon = \Omega_d^{\mathrm{alg}}(X)$, and $\Omega^*_{\mathrm{alg}}(X)$ is the direct sum of the groups over the all codimensions.

Remark 3.7. If we omit in the above process the relation (Equiv), then we obtain the algebraic cobordism theory $\Omega_*(X)$ of [12, Definition 2.4.10]. In particular, we have a natural surjection $\Phi_X \colon \Omega_*(X) \to \Omega_*^{\mathrm{alg}}(X)$.

We immediately see the following:

Proposition 3.8. *All four operations (projective push-forward, smooth pull-back, exterior product, and the first Chern class operation) in Definition 2.2 descend onto the cobordism Ω_*^{alg}.*

Remark 3.9. By definition, we have a natural ring homomorphism

$$\Phi^{\mathrm{alg}} \colon \mathbb{L}_* \to \Omega_*^{\mathrm{alg}}(k) \tag{3.1}$$

induced from the quotient map $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*^{\mathrm{alg}}(k) \to \Omega_*^{\mathrm{alg}}(k)$, which factors through the known map $\Phi \colon \mathbb{L}_* \to \Omega_*(k)$ in [12]. We will see later in Proposition 9.2 that this is an isomorphism.

We have a natural map $q^{\mathrm{alg}} \colon \underline{\Omega}_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$. It was proven that the corresponding map $q \colon \underline{\Omega}_*(X) \to \Omega_*(X)$ is surjective by [12, Lemma 2.5.9]. We have a similar result:

Lemma 3.10. *Let $X \in \mathbf{Sch}_k$. Then, the abelian group $\Omega_*^{\mathrm{alg}}(X)$ is generated by the images of the integral cobordism cycles $[Y \to X, L_1, \cdots, L_r]$. In other words, the natural map $\mathcal{Z}_*(X) \to \Omega_*^{\mathrm{alg}}(X)$ is surjective.*

Proof. It suffices to show that the map $q^{\mathrm{alg}} \colon \underline{\Omega}_*^{\mathrm{alg}} \to \Omega_*^{\mathrm{alg}}(X)$ is surjective. But, this follows from the observation that in the commutative diagram

$$\begin{tikzcd} \underline{\Omega}_*(X) \arrow[r, "\Phi_X"] \arrow[d, "q"] & \underline{\Omega}_*^{\mathrm{alg}}(X) \arrow[d, "q^{\mathrm{alg}}"] \\ \Omega_*(X) \arrow[r, "\Phi_X"] & \Omega_*^{\mathrm{alg}}(X), \end{tikzcd} \tag{3.2}$$

the map Φ_X is surjective and so is q by [12, Lemma 2.5.9]. □

Our discussion so far summarizes as follows (*cf.* [12, Theorem 2.4.13] for Ω_*):

Proposition 3.11. *The theory Ω_*^{alg} is an oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathbf{Sch}_k$ of geometric type in the sense of* [12, Definitions 2.1.2, 2.1.12, 2.2.1].

We now discuss a universal property of Ω_*^{alg}.

Definition 3.12. Let A_* be an oriented Borel-Moore $\mathbb{L}_*$-functor on $\mathbf{Sch}_k$ of geometric type. We say that A_* *respects algebraic equivalence* if for any $X \in \mathbf{Sch}_k$ and for any pair of algebraically equivalent line bundles L and M over X, we have $\tilde{c}_1(L) = \tilde{c}_1(M)$ as operators $A_*(X) \to A_{*-1}(X)$.

Certainly, by (Equiv) of Definition 3.1, the theory Ω_*^{alg} satisfies Definition 3.12.

Proposition 3.13. *The theory Ω_*^{alg} is universal among all oriented Borel-Moore $\mathbb{L}_*$-functors on $\mathbf{Sch}_k$ of geometric type that respect algebraic equivalence. In other words, for any theory A_* satisfying Definition 3.12, there exists a unique morphism $\theta_A \colon \Omega_*^{\mathrm{alg}} \to A_*$ of oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type on $\mathbf{Sch}_k$.*

Proof. By construction, the algebraic cobordism Ω_* of Levine-Morel is a universal oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type (*cf.* [13, §0.4]). So, there is a morphism $\theta \colon \Omega_* \to A_*$ of oriented Borel-Moore $\mathbb{L}_*$-functor of geometric type on $\mathbf{Sch}_k$.

To show that it induces $\theta_A \colon \Omega_*^{\mathrm{alg}} \to A_*$, assuming Proposition 3.16, that will be proven below, it is enough to show that $\theta(\eta) = 0$ in $A_*(X)$ for $\eta \colon = [f \colon Y \to X, L] - [f \colon Y \to X, M]$, where $L \sim M$. This is equivalent to $f_*\left(\left(\tilde{c}_1(L) - \tilde{c}_1(M)\right)(1_Y)\right) = 0 \in A_*(X)$. But this holds by the assumption that $\tilde{c}_1(L) = \tilde{c}_1(M)$ on $A_*(Y)$. Hence, we have the induced morphism $\theta_A \colon \Omega_*^{\mathrm{alg}} \to A_*$ as desired. □

We still need to prove Proposition 3.16 to complete the above. We need the following results that provide useful information on the relationship between Ω_* and Ω_*^{alg}.

Lemma 3.14. *The kernel of the map $\underline{\Phi}_X \colon \underline{\Omega}_*(X) \to \underline{\Omega}_*^{\mathrm{alg}}(X)$ is a subgroup generated by elements of the form $[f \colon Y \to X, L] - [f \colon Y \to X, M]$ with $L \sim M$.*

Proof. Let $\theta_X \colon \underline{\Omega}_*(X) \twoheadrightarrow \overline{\underline{\Omega}}_*^{\mathrm{alg}}(X)$ be the quotient of $\underline{\Omega}_*(X)$ by the subgroup generated by elements given in the lemma. It follows from the definition and the surjection $\mathcal{Z}_*(X) \twoheadrightarrow \underline{\Omega}_*(X)$ that $\ker(\underline{\Phi}_X)$ is generated by elements of the form $\eta = [f \colon Y \to X, L_1, \cdots, L_r] - [f' \colon Y' \to X, L_1', \cdots, L_r']$, where $\phi \colon Y \to Y'$ is an isomorphism over X and σ is a permutation of $\{1, \cdots, r\}$ such that $L_i \sim \phi^*(L'_{\sigma(i)})$. It suffices to show that such elements vanish in $\overline{\underline{\Omega}}_*^{\mathrm{alg}}(X)$. We can modify η so that $\eta = [f \colon Y \to X, L_1, \cdots, L_r] - [f \colon Y \to X, L_1', \cdots, L_r']$, where $L_i \sim L_i'$ for $1 \le i \le r$ by virtue of the relations in $\underline{\Omega}_*(X)$ as described in Definition 2.1. Since $\theta_X(\eta) = f_* \circ \theta_Y\{\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y) - \tilde{c}_1(L_1') \circ \cdots \circ \tilde{c}_1(L_r')(1_Y)\}$, it is enough to consider the case when $X = Y$ and $f = \mathrm{Id}_Y$. Then, the lemma follows by repeated applications of the Chern class operators, *i.e.*, $\tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)(1_Y) = \tilde{c}_1(L_1') \circ \cdots \circ \tilde{c}_1(L_r')(1_Y)$ in $\overline{\underline{\Omega}}_*^{\mathrm{alg}}(Y)$. □

For $X \in \mathbf{Sch}_k$, let $\tilde{\mathcal{R}}_*^{\mathrm{alg}}(X)$ denote (*cf.* [12, Definition 2.5.13]) the graded subgroup of $\underline{\Omega}_*^{\mathrm{alg}}(X)$ generated by elements of the form

$$f_* \circ \tilde{c}_1(L_1) \circ \cdots \circ \tilde{c}_1(L_r)\{F\left(\tilde{c}_1(L), \tilde{c}_1(M)\right)(\eta) - \tilde{c}_1(L \otimes M)(\eta)\}, \tag{3.3}$$

where $[f \colon Y \to X, L_1, \cdots, L_r]$ is a standard cobordism cycle, $L, M \in \mathrm{Pic}(Y)$ and $\eta \in \underline{\Omega}_*^{\mathrm{alg}}(Y)$. Since we have a natural surjection $\mathcal{Z}_*(k) \twoheadrightarrow \Omega_*(k)$ and the isomorphism $\Phi \colon \mathbb{L}_* \xrightarrow{\simeq} \Omega_*(k)$ (*cf.* [12, Lemma 2.5.9]), each element $a_{i,j} \in \mathbb{L}_*$ has a lift in $\mathcal{Z}_*(k)$. In particular, the elements of the form $F\left(\tilde{c}_1(L), \tilde{c}_1(M)\right)(\eta)$ are well-defined in $\underline{\Omega}_*(Y)$, thus well-defined in $\underline{\Omega}_*^{\mathrm{alg}}(Y)$. Set $\tilde{\Omega}_*^{\mathrm{alg}}(X) \colon = \underline{\Omega}_*^{\mathrm{alg}}(X)/\tilde{\mathcal{R}}_*^{\mathrm{alg}}(X)$. The following result is a refinement of Lemma 3.10.

Proposition 3.15. *For any $X \in \mathbf{Sch}_k$, there is a natural map $\psi_X^{\mathrm{alg}} \colon \tilde{\Omega}_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$ which is an isomorphism.*

Proof. It follows from Definition 3.6 that the map $\underline{\Omega}_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$ kills $\tilde{\mathcal{R}}_*^{\mathrm{alg}}(X)$. This induces the natural map $\psi_X^{\mathrm{alg}} \colon \tilde{\Omega}_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$. We have already shown in Lemma 3.10 that this map ψ_X^{alg} is surjective. We define an inverse $\phi_X^{\mathrm{alg}} \colon \Omega_*^{\mathrm{alg}}(X) \to \tilde{\Omega}_*^{\mathrm{alg}}(X)$ of ψ_X^{alg} to complete the proof of the proposition.

To do this, we consider the commutative diagram

$$\begin{tikzcd}
\underline{\Omega}_*(X) \arrow[rr, two heads] \arrow[dr] \arrow[dd, two heads] & & \underline{\Omega}_*^{\mathrm{alg}}(X) \arrow[dr] \arrow[dd, two heads] & \\
& \mathbb{L}_* \otimes \underline{\Omega}_*(X) \arrow[rr, two heads, "\overline{\beta}"] \arrow[dd, two heads, "\alpha"'] \arrow[dr, dashed, "\gamma"] & & \mathbb{L}_* \otimes \underline{\Omega}_*^{\mathrm{alg}}(X) \arrow[dl, dashed, "\gamma^{\mathrm{alg}}"'] \arrow[dd, two heads, "\alpha^{\mathrm{alg}}"] \\
\tilde{\Omega}_*(X) \arrow[rr, two heads] \arrow[dr, shift left, "\psi_X"] & & \tilde{\Omega}_*^{\mathrm{alg}}(X) \arrow[dr, shift left, "\psi_X^{\mathrm{alg}}"] & \\
& \Omega_*(X) \arrow[rr, two heads, "\beta"'] \arrow[ul, shift left, "\phi_X"] & & \Omega_*^{\mathrm{alg}}(X), \arrow[ul, dashed, shift left, "\phi_X^{\mathrm{alg}}"]
\end{tikzcd} \tag{3.4}$$

where $\tilde{\Omega}_*(X)$ is defined in [12, Definition 2.5.13]. All the squares in the above diagram commute and the maps ψ_X and ϕ_X are inverse to each other by [12, Proposition 2.5.15]. By Lemma 3.14, the kernel of the map $\overline{\beta}$ is generated by elements of the form $a \otimes ([Y \to X, L] - [Y \to X, M])$, where $L \sim M$ and $a \in \mathbb{L}_*$. On the other hand, such an element maps to $\Phi(a)\left([Y \to X, L] - [Y \to X, M]\right)$ in $\tilde{\Omega}_*(X)$ under $\phi_X \circ \alpha$ (*cf.* (3.1)). In particular, these elements are killed in $\tilde{\Omega}_*^{\mathrm{alg}}(X)$ under the composite map $\gamma \colon \mathbb{L}_* \otimes \underline{\Omega}_*(X) \to \Omega_*(X) \to \tilde{\Omega}_*(X) \to \tilde{\Omega}_*^{\mathrm{alg}}(X)$. Thus it descends to the quotient $\gamma^{\mathrm{alg}} \colon \mathbb{L}_* \otimes \underline{\Omega}_*^{\mathrm{alg}}(X) \to \tilde{\Omega}_*^{\mathrm{alg}}(X)$.

Next, we see from Definition 3.6 that the kernel of α^{alg} is generated by elements the form $F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))([f \colon Y \to X, L_1, \cdots, L_r]) - \tilde{c}_1(L \otimes M)([f \colon Y \to X, L_1, \cdots, L_r])$ for line bundles L_i on Y, and line bundles L and M on X. But these elements also lie in the kernel of the map α. In particular, they die in $\tilde{\Omega}_*^{\mathrm{alg}}(X)$ via γ so that we conclude that $\ker(\alpha^{\mathrm{alg}}) \subseteq \ker(\gamma^{\mathrm{alg}})$. Hence, the map γ^{alg} descends to $\phi_X^{\mathrm{alg}} \colon \Omega_*^{\mathrm{alg}}(X) \to \tilde{\Omega}_*^{\mathrm{alg}}(X)$ which makes all the squares commute. Now, from the construction, $\phi_X^{\mathrm{alg}} \circ \psi_X^{\mathrm{alg}}$ is the identity map. In particular, ψ_X^{alg} is injective, thus an isomorphism. $\square$

Proposition 3.16. *For $X \in \mathbf{Sch}_k$, the kernel of the natural surjection $\Phi_X \colon \Omega_*(X) \to \Omega_*^{\mathrm{alg}}(X)$ is the graded subgroup generated by the cobordism cycles of the form $[f \colon Y \to X, L] - [f \colon Y \to X, M]$, where L and M are algebraically equivalent.*

Proof. In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\mathcal{R}}_*(X) \arrow[r] \arrow[d] & \underline{\Omega}_*(X) \arrow[r] \arrow[d, "\underline{\Phi}_X"] & \Omega_*(X) \arrow[r] \arrow[d, "\Phi_X"] & 0 \\
0 \arrow[r] & \tilde{\mathcal{R}}_*^{\mathrm{alg}}(X) \arrow[r] & \underline{\Omega}_*^{\mathrm{alg}}(X) \arrow[r] & \Omega_*^{\mathrm{alg}}(X) \arrow[r] & 0,
\end{tikzcd} \tag{3.5}$$

the top row is exact by [12, Proposition 2.5.15] and the bottom row is exact by Proposition 3.15. The left vertical arrow in this diagram is surjective by the definition of $\tilde{\mathcal{R}}_*^{\mathrm{alg}}(X)$ above and that of $\tilde{\mathcal{R}}_*(X)$ in [12, Lemma 2.5.14]. Hence, the map $\ker(\underline{\Phi}_X) \to \ker(\Phi_X)$ is surjective by the snake lemma. On the other hand, by Lemmas 2.3 and 3.14, the group $\ker(\underline{\Phi}_X)$ is generated by cobordism cycles of the form $[f\colon Y \to X, L] - [f\colon Y \to X, M]$ where $L \sim M$. This proves the proposition. □

Some fundamental properties of Ω_*^{alg} will be studied in Section 6.2 and Section 7. We shall also show that Ω_*^{alg} gives an oriented cohomology theory on $\mathbf{Sm}_k$ and an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ (see [12, Definitions 1.1.2, 5.1.3]), equipped with a similar universal property.

4. Algebraic double-point cobordism ω_*^{alg}

The purpose of this section is to recall the cobordism theories ω_* and ω_*^{alg} based on the double-point relations of Levine and Pandharipande [13].

4.1. **Double-point cobordism after Levine-Pandharipande.** We recall the double-point cobordism theory ω_* of Levine and Pandharipande in [13, §0.3], and its *algebraic equivalence* analogue ω_*^{alg} in [13, §11.2]. This description is simpler than that of Levine-Morel in [12] in that, first, the cobordism cycles are simpler, *i.e.*, without the attached line bundles and forcing of the formal group law as in Definitions 2.1 and 3.6, and second, the relations are given from a single sort of morphisms called *double-point degenerations*.

For the cobordism cycles in the sense of Levine-Pandharipande in [13], we will still call them cobordism cycles whenever no confusion arises.

Definition 4.1 ([13, §0.2])**.** Let $X \in \mathbf{Sch}_k$. An *integral cobordism cycle* on X is the isomorphism class over X of a projective morphism $f\colon Y \to X$, where Y is integral and in $\mathbf{Sm}_k$. This will be denoted by $[f\colon Y \to X]$. Its dimension is by definition $\dim Y$. A *cobordism cycle* is a finite sum of integral cobordism cycles. If $Y = \coprod Y_i \in \mathbf{Sm}_k$ where each Y_i is integral, then given a projective morphism $f\colon Y \to X$, the cobordism cycle $[f\colon Y \to X]$ is defined to be the sum of $[f|_{Y_i}\colon Y_i \to X]$. Let $\mathcal{M}_*(X)^+$ be the free abelian group of all cobordism cycles over X, and let $\mathcal{M}_d(X)^+$ be the subgroup generated by the cobordism cycles of dimension d.

Now we recall the notion of double-point degenerations and the associated relations in [13, §0.2, §0.3, §11.2].

Definition 4.2. Let $Y \in \mathbf{Sm}_k$ be of pure dimension. Let (C, p) be a pair consisting of a smooth projective curve C and a k-rational point $p \in C$.

(1) A morphism $\pi\colon Y \to C$ of k-varieties is a *double-point degeneration* over $p \in C$ if $\pi^{-1}(p)$ can be written as $\pi^{-1}(p) = A \cup B$, where A and B are smooth closed subschemes of Y of codimension 1 intersecting transversally. The intersection $D = A \cap B$ is called the *double-point locus* of π over $p \in C$. We allow A, B, or D to be empty. Let $N_{A/D}$ and $N_{B/D}$ denote the normal bundles of D in A and B, respectively. As in [13, §0.2], the projective bundles $\mathbb{P}(O_D \oplus N_{A/D}) \to D$ and $\mathbb{P}(O_D \oplus N_{B/D}) \to D$ are isomorphic over D. Either of these is denoted by $\mathbb{P}(\pi) \to D$ of the double-point degeneration π.

(2) Let $X \in \mathbf{Sch}_k$, and let pr_1, pr_2 be the projections from $X \times C$ to X and C, respectively. Let $Y \in \mathbf{Sm}_k$ be of pure dimension, and let $g\colon Y \to X \times C$ be a projective morphism such that $\pi = pr_2 \circ g\colon Y \to C$ is a double-point degeneration over $p \in C$. For each regular value $\zeta \in C(k)$ of π, the triple (g, p, ζ) is called a *double-point cobordism* with

the degenerate fibre over $p \in C$ and the smooth fibre over ζ. The associated *double-point relation* over X of the double-point cobordism (g, p, ζ) is the cobordism cycle

$$\partial_C(g, p, \zeta) \colon = [Y_\zeta \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X] \in \mathcal{M}_*(X)^+,$$

where $Y_\zeta \colon = \pi^{-1}(\zeta)$.

(3) Let $\mathcal{R}_*^{\mathrm{rat}}(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by all double-point relations over X over the pair $(C, p) = (\mathbb{P}^1, 0)$. This is the group of *rational double-point relations.* This group was denoted by $\mathcal{R}_*(X)$ in [13].

(4) Let $\mathcal{R}_*^{\mathrm{alg}}(X) \subset \mathcal{M}_*(X)^+$ be the subgroup generated by all double-point relations over X over all pairs (C, p) of smooth projective curve C and a point $p \in C(k)$. This is the group of *algebraic double-point relations.*

Now we define the associated cobordism theories:

Definition 4.3 ([13])**.** Let $X \in \mathbf{Sch}_k$.

(1) The *(rational) double-point cobordism theory* $\omega_*(X)$ is the quotient

$$\omega_*(X) = \mathcal{M}_*(X)^+/\mathcal{R}_*^{\mathrm{rat}}(X). \qquad ([\textit{ibid.}, \text{Definition } 0.2])$$

(2) The *algebraic double-point cobordism theory* $\omega_*^{\mathrm{alg}}(X)$ is the quotient

$$\omega_*^{\mathrm{alg}}(X) = \mathcal{M}_*(X)^+/\mathcal{R}_*^{\mathrm{alg}}(X). \qquad ([\textit{ibid.}, \S 11.2])$$

4.2. **Basic structures.** There are some basic properties of ω_*^{alg} that follow essentially from the definition and some analogous constructions in [13, §3.1]:

Proposition 4.4. *For $X \in \mathbf{Sch}_k$, the assignment $X \mapsto \omega_*^{\mathrm{alg}}(X)$ has the following structures. Let $X, X' \in \mathbf{Sch}_k$.*

(1) projective push-forward: *For a projective $g \colon X \to X'$, we have $g_* \colon \omega_*^{\mathrm{alg}}(X) \to \omega_*^{\mathrm{alg}}(X')$ given by $g_*([f \colon Y \to X]) = [g \circ f \colon Y \to X']$. This satisfies $(g_1 \circ g_2)_* = g_{1*} \circ g_{2*}$ when g_1, g_2 are both projective.*

(2) smooth pull-back: *For a smooth quasi-projective $g \colon X' \to X$ of pure relative dimension d, we have $g^* \colon \omega_*^{\mathrm{alg}}(X) \to \omega_{*+d}^{\mathrm{alg}}(X')$ given by $g^*([f \colon Y \to X]) = [pr_2 \colon Y \times_X X' \to X']$. This satisfies $(g_1 \circ g_2)^* = g_2^* \circ g_1^*$.*

(3) external product: *We have $\times \colon \omega_*^{\mathrm{alg}}(X) \times \omega_*^{\mathrm{alg}}(X') \to \omega_*^{\mathrm{alg}}(X \times X')$ given by $[f \colon Y \to X] \times [f' \colon Y' \to X'] = [f \times f' \colon Y \times Y' \to X \times X']$.*

(4) unit: *The class $1_{\mathrm{Spec}(k)} \in \omega_0^{\mathrm{alg}}(k)$ is the unit for the external product on ω_*^{alg}.*

(5) Chern classes: *For every line bundle L on X, there is a Chern class operation $\tilde{c}_1(L) \colon \omega_*^{\mathrm{alg}}(X) \to \omega_{*-1}^{\mathrm{alg}}(X)$ which is compatible with smooth pull-back and projective push-forward.*

Proof. (1) Given projective $g \colon X \to X'$, we already have $g_* \colon \mathcal{M}_*(X)^+ \to \mathcal{M}_*(X')^+$. It remains to show that g_* sends the algebraic double-point relations $\mathcal{R}_*^{\mathrm{alg}}(X)$ into $\mathcal{R}_*^{\mathrm{alg}}(X')$. Indeed, given an algebraic double-point cobordism (h, p, ζ) over X, where $h \colon Y \to X \times C$ with a smooth projective curve C, we get an algebraic double-point cobordism $((g \times \mathrm{Id}_C) \circ h, p, \zeta)$ over X', where $(g \times \mathrm{Id}_C) \circ h \colon Y \to X' \times C$. We immediately note that $g_*(\partial_C(h, p, \zeta)) = \partial_C((g \times \mathrm{Id}_C) \circ h, p, \zeta)$. This proves (1).

(2) Given smooth quasi-projective $g \colon X' \to X$, we have $g^* \colon \mathcal{M}_*(X)^+ \to \mathcal{M}_*(X')^+$. It remains to show that g^* sends $\mathcal{R}_*^{\mathrm{alg}}(X)$ into $\mathcal{R}_*^{\mathrm{alg}}(X')$. This follows by observing that given an algebraic double-point cobordism (h, p, ζ) over X, the pull-back $(g^* h, p, \zeta)$, given by the second projection of the fibre product $Y' \colon = Y \times_{X \times C} (X' \times C) \to X' \times C$, is an algebraic double-point cobordism over X'.

(3) The map $\times\colon \mathcal{M}_*(X)^+ \times \mathcal{M}_*(X')^+ \to \mathcal{M}_*(X \times X')^+$ is defined on the level of cobordism cycles. For an algebraic double-point cobordism (h, p, ζ) over X as before, for each $[f\colon Y' \to X'] \in \mathcal{M}_*(X')^+$, we get an induced algebraic double-point cobordism $(h \times f, p, \zeta)$, where $h \times f\colon Y \times Y' \to X \times X' \times C$. Similarly, interchanging the role of X and X', we see that $\times$ descends onto the level of $\omega_*^{\mathrm{alg}}(-)$. This proves (3). Part (4) is immediate.

(5) The construction of the first Chern class operation $\tilde{c}_1(L)$ on $\omega_*^{\mathrm{alg}}(X)$ follows the same arguments as for $\omega_*(X)$ by first assuming that L is globally generated and then deducing the general case, as in [13, §4] and [*ibid.*, §9], respectively. We omit the details. $\square$

5. The basic exact sequence

Let $X \in \mathbf{Sch}_k$. By [13, Theorem 1], the natural map $\omega_*(X) \to \Omega_*(X)$ is an isomorphism. We often identify these implicitly in the following. Let (C, t_1, t_2) denote a smooth projective curve C with distinct $t_1, t_2 \in C(k)$ with the inclusions $i_j\colon X \times \{t_j\} \to X \times C$. By the existence of the l.c.i. pull-backs on Ω_* in [12, §6.5], we have maps $i_1^*, i_2^*\colon \omega_*(X \times C) \to \omega_*(X)$. By definition, we also have a natural surjection $\Psi_X\colon \omega_*(X) \to \omega_*^{\mathrm{alg}}(X)$. The main theorem of the section is:

Theorem 5.1. *Let $X \in \mathbf{Sch}_k$. The sequence*

$$\bigoplus_{(C,t_1,t_2)} \omega_*(X \times C) \xrightarrow{i_1^* - i_2^*} \omega_*(X) \xrightarrow{\Psi_X} \omega_*^{\mathrm{alg}}(X) \to 0,$$

where (C, t_1, t_2) runs over the equivalence classes of triples consisting of a smooth projective curve C and two distinct points $t_1, t_2 \in C(k)$, is exact.

We begin with some remarks on cobordism cycles associated to strict normal crossing divisors on smooth varieties.

5.1. **Remarks on divisor classes.** Recall from [12, §3.1] that given a strict normal crossing divisor E on $Y \in \mathbf{Sm}_k$ with the support $\iota\colon |E| \to Y$, there is a class $[E \to |E|] \in \Omega_*(|E|)$ that satisfies $\iota_*([E \to |E|]) = [Y \to Y, O_Y(E)] = \tilde{c}_1(O_Y(E))(1_Y)$. Since we have a natural surjection $\Omega_* \to \Omega_*^{\mathrm{alg}}$, the class $[E \to |E|]$ makes sense also in $\Omega_*^{\mathrm{alg}}(|E|)$.

The construction $[E \to |E|] \in \Omega_*(|E|)$ uses the formal group law F for $\Omega_*(k)$. We look at only the following case from [12, §3.1]. The special case we need is when $E = E_1 + E_2$, where E_1, E_2 are transversal smooth divisors on $Y \in \mathbf{Sm}_k$. Let $\iota_D\colon D = E_1 \cap E_2 \to Y$ be the inclusion. We let $O_D(E_i)\colon = \iota_D^*(O_Y(E_i))$. Then, the class $[E \to Y] \in \Omega_*(Y)$ is defined as

$$[E \to Y]\colon = [E_1 \to Y] + [E_2 \to Y] + \iota_{D*}\left(F^{1,1}(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2)))(1_D)\right),$$

where $F^{1,1}(u, v) = \sum_{i,j \geq 1} a_{i,j} u^{i-1} v^{j-1} \in \Omega_*(k)[[u, v]]$ and $a_{i,j} \in \Omega_{i+j-1}(k)$ are the coefficients of the formal group law.

In addition, suppose that $O_D(E)\colon = \iota_D^*(O_Y(E))$ is trivial, *i.e.*, $O_D(E_1) \simeq O_D(E_2)^{-1}$ on D. Let $\mathbb{P}_D \to D$ be the $\mathbb{P}^1$-bundle $\mathbb{P}(O_D \oplus O_D(E_1))$. Then, by [13, Lemma 3.3], we have

$$F^{1,1}(\tilde{c}_1(O_D(E_1)), \tilde{c}_1(O_D(E_2)))(1_D) = -[\mathbb{P}_D \to D] \in \Omega_*(D). \tag{5.1}$$

Hence, we have the following equation in $\Omega_*(Y)$, thus in $\Omega_*^{\mathrm{alg}}(Y)$, too:

$$[E \to Y] - [E_1 \to Y] - [E_2 \to Y] + [\mathbb{P}_D \to Y] = 0. \tag{5.2}$$

5.2. **Proof of Theorem 5.1.** Consider the commutative diagram with the top exact row:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{R}_*^{\mathrm{alg}}(X) \arrow[r] \arrow[dr, "\theta"] & \mathcal{M}_*(X)^+ \arrow[r] \arrow[d, two heads] & \omega_*^{\mathrm{alg}}(X) \arrow[r] \arrow[d, equal] & 0 \\
& \bigoplus_{(C,t_1,t_2)} \omega_*(X \times C) \arrow[r, "\theta'"'] & \omega_*(X) \arrow[r, "\Psi_X"'] & \omega_*^{\mathrm{alg}}(X) \arrow[r] & 0,
\end{tikzcd}
\tag{5.3}
$$

where θ is the composition of the two arrows, θ' is the sum of the maps $i_1^* - i_2^*$. We want to prove that the bottom row is exact. It is apparent that $\ker(\Psi_X) = \mathrm{Im}(\theta)$, thus it suffices to prove that $\mathrm{Im}(\theta) = \mathrm{Im}(\theta')$.

We prove $\mathrm{Im}(\theta) \subseteq \mathrm{Im}(\theta')$ first. Let (g, p, ζ) be a double-point cobordism as in Definition 4.2, *i.e.*, a projective $g\colon Y \to X \times C$, two points $p, \zeta \in C(k)$ such that for $\pi = pr_2 \circ g$ we have $\pi^{-1}(p) = A \cup B$. Let $\gamma\colon = [g\colon Y \to X \times C] \in \omega_*(X \times C)$.

Let $i_p\colon X \times \{p\} \to X \times C$ be the inclusion. Let $X_p\colon = X \times \{p\}$. Since the divisor $E\colon = g^*(X_p) = A + B$ is strict normal crossing, we have $\gamma \in \Omega_*(X \times C)_{X_p}$ (see Definitions 11.1, 11.2, 11.3), and by Theorem 11.4, Definition 11.5 and [12, Lemma 6.5.6], we have $i_p^*(\gamma) = g'_*([E \to |E|]) \in \omega_*(X_p)$, where $g' = g|_{|E|}\colon |E| \to X_p$. Consider the following commutative diagram:

$$
\begin{tikzcd}
{|E|} \arrow[r, "\iota_E"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \arrow[dr, "\pi'"] & \\
X_p \arrow[r, "i_p"] & X \times C \arrow[r, "pr_1"] & X.
\end{tikzcd}
$$

Note that $pr_1 \circ i_p = \mathrm{Id}_X$ via $X \simeq X_p$ and π' is projective. Thus, $i_p^*(\gamma) = g'_*([E \to |E|]) = pr_{1*} i_{p*} g'_*([E \to |E|]) = \pi'_* \iota_{E*}([E \to |E|]) = \pi'_*([E \to Y]) =^\dagger [A \to X] + [B \to X] - [\mathbb{P}(\pi) \to X]$ in $\omega_*(X)$, where $\dagger$ follows from (5.2). Since Y_ζ is smooth, $i_\zeta^*(\alpha) = [Y_\zeta \to X]$. Hence, we get $\theta(\partial_C(g, p, \zeta)) = [Y_\zeta \to X] - [A \to X] - [B \to X] + [\mathbb{P}(\pi) \to X] = -(i_p^* - i_\zeta^*)(\gamma)$. That is, $\mathrm{Im}(\theta) \subseteq \mathrm{Im}(\theta')$.

To prove the reverse inclusion $\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta')$, we consider two cases.

Case 1: First assume that X is smooth. For (C, t_1, t_2) as before, let $\gamma\colon = [g\colon Y \to X \times C]$ be a cobordism cycle. Since X is smooth, by the transversality [12, Proposition 3.3.1], we may assume that g is transverse to i_j, $j = 1, 2$. The composition $Y \to X \times C \to C$ now has smooth fibres over t_1, t_2 so that we have $-(i_1^* - i_2^*)(\gamma) = \theta\left(\partial_C(g, t_1, t_2)\right)$. So, if X is smooth, then $\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta')$ holds.

Case 2: Suppose X is any scheme in $\mathbf{Sch}_k$. We prove by induction on $\dim X$. Note that every cobordism cycle is a formal sum of integral cobordism cycles of the form $[f\colon Y \to X]$ where Y is smooth and irreducible, and such f factors uniquely through an irreducible component of X_{red}. Thus, we may reduce to the case when X is integral.

If $\dim X = 0$, then X is smooth so that the statement holds by *Case 1.* Suppose $\dim X > 0$, and assume the statement holds for all lower dimensional schemes in $\mathbf{Sch}_k$.

Let $\iota\colon Z \hookrightarrow X$ be the singular locus, and let $U\colon = X \backslash Z$ be the open complement. Using Hironaka's resolution of singularities, we can find a projective morphism $\pi\colon \widetilde{X} \to X$ that is an isomorphism over U such that the inverse image of Z is a strict normal crossing divisor. Let $[g\colon Y \to X \times C] \in \omega_*(X \times C)$, and let $t_1, t_2 \in C(k)$ be two distinct points. Consider the diagram:

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \widetilde{Y} \arrow[r, "\widetilde{g}"] \arrow[d, "\mu"] & \widetilde{X} \times C \arrow[d, "\pi_C"] & U \times C \arrow[l, hook'] \arrow[d, equal] \\
W \arrow[r, hook, "j"] \arrow[dr, equal] & Y \arrow[r, "g"] \arrow[ur, dashed, "f"] & X \times C & U \times C \arrow[l, hook'] \\
& W \arrow[u, hook, "j"] \arrow[r, "g'"] & Z \times C, \arrow[u, hook, "\iota_C"] &
\end{tikzcd}$$

where $W\colon = g^{-1}(Z \times C)$, g' is the restriction of g on W, f is the rational map $\pi_C^{-1} \circ g$, and μ is a sequence of blow-ups of the indeterminacy of f, which is an isomorphism on the complement of W such that the exceptional divisor E is a strict normal crossing divisor, and such that there is a morphism $\widetilde{g}$ making the diagram commute. Moreover, the upper-right and the lower squares are Cartesian.

Let $\alpha\colon = [g\colon Y \to X \times C] \in \omega_*(X \times C), \widetilde{\alpha}\colon = [\widetilde{g}\colon \widetilde{Y} \to \widetilde{X} \times C] \in \omega_*(\widetilde{X} \times C)$, and $\beta\colon = [\mu\colon \widetilde{Y} \to Y] \in \omega_*(Y)$. Recall that for $V \in \mathbf{Sm}_k$, we write $1_V = [\mathrm{Id}\colon V \to V] \in \omega_*(V)$. Then as cobordism cycle classes, we have

$$\alpha = g_*(1_Y), \quad \widetilde{\alpha} = \widetilde{g}_*(1_{\widetilde{Y}}), \quad \pi_{C*}(\widetilde{\alpha}) = g_*\mu_*(1_{\widetilde{Y}}) = g_*(\beta). \tag{5.4}$$

Thus, $\alpha - \pi_{C*}(\widetilde{\alpha}) = g_*(1_Y - \beta)$. But, by a blow-up formula [12, Proposition 3.2.4], there is a cobordism cycle $\eta \in \omega_*(W)$ such that $1_Y - \beta = j_*(\eta)$. We thus have

$$\alpha - \pi_{C*}(\widetilde{\alpha}) = g_*(1_Y - \beta) = g_*j_*(\eta) = \iota_{C*}g'_*(\eta). \tag{5.5}$$

In particular, we have $i_j^*(\alpha) - i_j^*(\pi_{C*}(\widetilde{\alpha})) = i_j^*(\iota_{C*}g'_*(\eta))$ for $j = 1, 2$ so that

$$\theta'(\alpha) - \theta'(\pi_{C*}(\widetilde{\alpha})) = \theta'(\iota_{C*}g'_*(\eta)). \tag{5.6}$$

On the other hand, in the Cartesian diagrams below whose rows are regular embeddings,

$$\begin{tikzcd}
\widetilde{X} \times \{t_j\} \arrow[r] \arrow[d, "\pi"] & \widetilde{X} \times C \arrow[d, "\pi_C"] \\
X \times \{t_j\} \arrow[r] & X \times C,
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \times \{t_j\} \arrow[r] \arrow[d, "\iota"] & Z \times C \arrow[d, "\iota_C"] \\
X \times \{t_j\} \arrow[r] & X \times C,
\end{tikzcd}$$

we can use [12, Proposition 6.5.4] and Lemma 5.3 (to be proven below) to deduce that $\theta'(\pi_{C*}(\widetilde{\alpha})) = \pi_*\left(\theta'(\widetilde{\alpha})\right)$ and $\theta'(\iota_{C*}g'_*(\eta)) = \iota_*\left(\theta'(g'_*(\eta))\right)$. (N.B. The Tor-independence assumption in [12, Proposition 6.5.4] is only to guarantee that pull-backs of regular embeddings are regular embeddings. In our case, the rows are regular embeddings, thus, the conclusion of the proposition applies here without Lemma 5.3.) Applying this to (5.6), we conclude that

$$\theta'(\alpha) = \pi_*\left(\theta'(\widetilde{\alpha})\right) + \iota_*\left(\theta'(g'_*(\eta))\right). \tag{5.7}$$

By the *Case 1* applied to $\widetilde{X}$, we have $\theta'(\widetilde{\alpha}) \in \theta(\mathcal{R}_*^{\mathrm{alg}}(\widetilde{X}))$ so that $\pi_*(\theta'(\widetilde{\alpha})) \in \pi_*(\theta(\mathcal{R}_*^{\mathrm{alg}}(\widetilde{X}))) \subset \theta(\mathcal{R}_*^{\mathrm{alg}}(X))$ by Proposition 4.4. Thus, to show $\theta'(\alpha) \in \theta(\mathcal{R}_*^{\mathrm{alg}}(X))$, it is enough to prove that $\theta'(g'_*(\eta)) \in \theta(\mathcal{R}_*^{\mathrm{alg}}(Z))$. But this holds by the induction hypothesis since $\dim Z < \dim X$. Hence, we have shown that $\mathrm{Im}(\theta) \supseteq \mathrm{Im}(\theta')$ for X. This finishes the proof of the theorem. $\square$

Corollary 5.2. *Let $X \in \mathbf{Sch}_k$. Then, we have an isomorphism*

$$\omega_*^{\mathrm{FW}}(X) \simeq \omega_*^{\mathrm{alg}}(X).$$

We used the following lemma in the proof of Theorem 5.1, while it was not absolutely necessary for the theorem. But it is used a few times at other locations in the paper:

Lemma 5.3. *Let T be a smooth scheme over k and let $W \subset T$ be a smooth closed subscheme. Then for any morphism $f\colon V' \to V$ in $\mathbf{Sch}_k$, the schemes $V' \times T$ and $V \times W$ are Tor-independent over $V \times T$.*

Proof. Since the assertion is local on V and T, by shrinking them to small enough affine open subschemes if necessary, we may assume that both are affine such that $W \subseteq V$ is a complete intersection subscheme. In particular, there is a finite resolution $\mathcal{F}_\bullet \to \mathcal{O}_W$ by free $\mathcal{O}_T$-modules of finite rank. This in turn shows that $\mathcal{F}_\bullet \otimes_k \mathcal{O}_V \to \mathcal{O}_{V\times W}$ is a finite free resolution of $\mathcal{O}_{V\times W}$ as $\mathcal{O}_{V\times T}$-module.

Since $(\mathcal{F}_\bullet \otimes_k \mathcal{O}_V) \otimes_{\mathcal{O}_{V\times T}} \mathcal{O}_{V'\times T} \simeq \mathcal{F}_\bullet \otimes_k \mathcal{O}_{V'}$, we have

$$\mathrm{Tor}_i^{\mathcal{O}_{V\times T}}(\mathcal{O}_{V\times W}, \mathcal{O}_{V'\times T}) = \mathcal{H}_i(\mathcal{F}_\bullet \otimes_k \mathcal{O}_{V'}) = 0$$

for $i > 0$. This proves the lemma. □

6. Equivalence of Ω_*^{alg} and ω_*^{alg} and consequences

The purpose of this section is to prove Theorem 6.2, and to establish some fundamental properties of our cobordism theory.

6.1. **The comparison theorem.** First, we state an analogue of [13, Lemma 3.2] for algebraic equivalence:

Lemma 6.1. *Let $Y \in \mathbf{Sm}_k$, and let E, F be strict normal crossing divisors on Y that are algebraically equivalent. Then, we have $[E \to Y] = [F \to Y]$ in $\Omega_*^{\mathrm{alg}}(Y)$.*

Proof. By [12, Proposition 3.1.9], we have $[E \to Y] = [Y \to Y, O_Y(E)]$ and $[F \to Y] = [Y \to Y, O_Y(F)]$ in $\Omega_*(Y)$. Via the natural map $\Omega_*(Y) \to \Omega_*^{\mathrm{alg}}(Y)$, these equalities still hold in $\Omega_*^{\mathrm{alg}}(Y)$. It follows from the relation $(\sim)$ of Definition 3.1 and Lemma 2.4 that $[Y \to Y, O_Y(E)] = [Y \to Y, O_Y(F)]$ in $\Omega_*^{\mathrm{alg}}(Y)$. Hence $[E \to Y] = [F \to Y]$ in $\Omega_*^{\mathrm{alg}}(Y)$. □

Theorem 6.2. *For $X \in \mathbf{Sch}_k$, there is a canonical isomorphism $\Omega_*^{\mathrm{alg}}(X) \simeq \omega_*^{\mathrm{alg}}(X)$.*

Proof. We first define a natural map $\vartheta_X^{\mathrm{alg}}\colon \omega_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$. We let $\vartheta_X^{\mathrm{alg}}\colon \mathcal{M}_*(X)^+ \to \Omega_*^{\mathrm{alg}}(X)$ be given by $\vartheta_X^{\mathrm{alg}}([f\colon Y \to X]_{\omega^{\mathrm{alg}}})\colon = [f\colon Y \to X]_{\Omega^{\mathrm{alg}}}$. We need to show that $\vartheta_X^{\mathrm{alg}}$ kills the algebraic double-point relations.

So let (g, p, ζ) be an algebraic double-point cobordism given by a projective $g\colon Y \to X \times C$, where C is a smooth projective curve. It is enough to show that $\partial_C(g, p, \zeta)$ vanishes in $\Omega_*^{\mathrm{alg}}(X)$. Let $f\colon = pr_1 \circ g$ and $\pi\colon = pr_2 \circ g$. Since

$$\partial_C(g, p, \zeta) = f_*([Y_\zeta \to Y] - [A \to Y] - [B \to Y] + [\mathbb{P}(\pi) \to Y]),$$

it suffices to show that we have in $\Omega_*^{\mathrm{alg}}(Y)$,

$$[Y_\zeta \to Y] - [A \to Y] - [B \to Y] + [\mathbb{P}(\pi) \to Y] = 0. \tag{6.1}$$

We apply the equation (5.2) to the divisor $E\colon = A + B$ on Y to obtain

$$[E \to Y] - [A \to Y] - [B \to Y] + [\mathbb{P}(\pi) \to Y] = 0 \in \Omega_*^{\mathrm{alg}}(Y). \tag{6.2}$$

On the other hand, the divisor E is algebraically equivalent to the divisor Y_ζ and hence by Lemma 6.1, we also have the equality $[E \to Y] = [Y_\zeta \to Y] \in \Omega_*^{\mathrm{alg}}(Y)$. Combining this with (6.2), we obtain (6.1). This shows that the natural map $\vartheta_X^{\mathrm{alg}}\colon \mathcal{M}_*(X)^+ \to \Omega_*^{\mathrm{alg}}(X)$ descends to give $\vartheta_X^{\mathrm{alg}}\colon \omega_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(X)$.

To define the inverse $\varrho_X^{\mathrm{alg}}\colon \Omega_*^{\mathrm{alg}}(X) \to \omega_*^{\mathrm{alg}}(X)$ of $\vartheta_X^{\mathrm{alg}}$, we consider the diagram

$$
\begin{tikzcd}
 & \Omega_*(X) \arrow[r, "\Phi_X"] \arrow[d, "\varrho_X"'] & \Omega_*^{\mathrm{alg}}(X) \arrow[r] \arrow[d, dashed] & 0 \\
\bigoplus\limits_{(C,t_1,t_2)} \omega_*(X \times C) \arrow[r, "i_1^*-i_2^*"'] & \omega_*(X) \arrow[r, "\Psi_X"'] & \omega_*^{\mathrm{alg}}(X) \arrow[r] & 0,
\end{tikzcd}
\tag{6.3}
$$

where the bottom row is exact by Theorem 5.1 and the isomorphism ϱ_X is from [13, Theorem 1]. Let θ' be the sum of the maps $i_1^*-i_2^*$. We need to show $\varrho_X(\ker(\Phi_X)) \subseteq \mathrm{Im}(\theta')$ in order to define ϱ_X^{alg}. By Proposition 3.16, it is enough to show it for a cobordism cycle α of the form $[f\colon Y \to X, L_1] - [f\colon Y \to X, L_2]$ such that $L_1 \sim L_2$. We can write $\alpha = f_*\left(\tilde{c}_1(L_1)(1_Y) - \tilde{c}_1(L_2)(1_Y)\right)$. By Lemma 5.3 and [12, Theorem 6.5.12], we replace X by Y. So, suppose $X = Y$, replace f by Id_X, and assume that X is smooth.

Since $L_1 \sim L_2$, there exists a smooth projective curve C, two distinct points $t_1, t_2 \in C(k)$ and a line bundle $\mathcal{L}$ on $X \times C$ such that $\mathcal{L}|_{X\times\{t_j\}} \simeq L_i$ for $j = 1, 2$. We can then write

$$[\mathrm{Id}_X\colon X \to X, L_j] = \tilde{c}_1(L_j)(1_X) = \left(\tilde{c}_1\left(i_j^*(\mathcal{L})\right) \circ i_j^*\right)(1_{X\times C}) = \left(i_j^* \circ \tilde{c}_1(\mathcal{L})\right)(1_{X\times C}),$$

where the last equality follows from [12, Lemma 7.4.1 (2)].

In particular, we see that $\alpha = i_1^*(\widetilde{\alpha}) - i_2^*(\widetilde{\alpha})$, where $\widetilde{\alpha} = [\mathrm{Id}\colon X \times C \to X \times C, \mathcal{L}]$. That is, $\varrho_X(\alpha) = \theta'(\widetilde{\alpha})$. This shows that $\varrho_X(\ker(\Phi_X)) \subseteq \mathrm{Im}(\theta')$ and it defines ϱ_X^{alg} such that the above diagram commutes. Both $\omega_*^{\mathrm{alg}}(X)$ and $\Omega_*^{\mathrm{alg}}(X)$ are generated by cobordism cycles of the form $[f\colon Y \to X]$ and for those cycles ϱ^{alg} and $\vartheta_X^{\mathrm{alg}}$ are inverse to each other. This proves the theorem. $\square$

As an immediate consequence of Theorems 5.1 and 6.2, we obtain:

Theorem 6.3. *Let $X \in \mathbf{Sch}_k$. The sequence*

$$\bigoplus_{(C,t_1,t_2)} \Omega_*(X \times C) \xrightarrow{i_1^*-i_2^*} \Omega_*(X) \xrightarrow{\Phi_X} \Omega_*^{\mathrm{alg}}(X) \to 0,$$

where (C, t_1, t_2) runs over the equivalence classes of triples consisting of a smooth projective curve C and two distinct points $t_1, t_2 \in C(k)$, is exact.

6.2. **Fundamental properties of Ω_*^{alg}.** We prove some important properties of our cobordism theory.

Theorem 6.4 (Localization sequence)**.** *The cobordism Ω_*^{alg} satisfies the localization exact sequence $\Omega_*^{\mathrm{alg}}(Y) \to \Omega_*^{\mathrm{alg}}(X) \to \Omega_*^{\mathrm{alg}}(U) \to 0$, where Y is closed in $X \in \mathbf{Sch}_k$, and $U = X \backslash Y$.*

Proof. Let $\iota\colon Z \to X$ and $j\colon U \to X$ be the inclusions. Consider the diagram

$$\begin{tikzcd}
\bigoplus_{(C,t_1,t_2)} \Omega_*(Z \times C) \arrow[r, "i_1^*-i_2^*"] \arrow[d, "\iota_*"] & \Omega_*(Z) \arrow[r, "\Phi_Z"] \arrow[d, "\iota_*"] & \Omega_*^{\mathrm{alg}}(Z) \arrow[r] \arrow[d, "\iota_*"] & 0 \\
\bigoplus_{(C,t_1,t_2)} \Omega_*(X \times C) \arrow[r, "i_1^*-i_2^*"] \arrow[d, "j^*"] & \Omega_*(X) \arrow[r, "\Phi_X"] \arrow[d, "j^*"] & \Omega_*^{\mathrm{alg}}(X) \arrow[r] \arrow[d, "j^*"] & 0 \\
\bigoplus_{(C,t_1,t_2)} \Omega_*(U \times C) \arrow[r, "i_1^*-i_2^*"] \arrow[d] & \Omega_*(U) \arrow[r, "\Phi_U"] \arrow[d] & \Omega_*^{\mathrm{alg}}(U) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0, &
\end{tikzcd}$$

where the rows are exact by Theorem 6.3 and the first two columns are exact by [12, Theorem 3.2.7]. The top square on the left commutes by Lemma 5.3 and [12, Theorem 6.5.12], and the bottom square on the left commutes by the composition law of the pull-back map. The top and the bottom squares on the right are easily seen to commute by the naturality of the quotient map Φ_X. In other words, the diagram above is commutative. A simple diagram chase now shows that the third column is also exact which proves the theorem. $\square$

Theorem 6.5 ($\mathbb{A}^1$-homotopy Invariance)**.** *Let $X \in \mathbf{Sch}_k$ and let $p\colon V \to X$ be an affine-space bundle over X of rank n. Then, the map $p^*\colon \Omega_*^{\mathrm{alg}}(X) \to \Omega_{*+n}^{\mathrm{alg}}(V)$ is an isomorphism.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\bigoplus_{(C,t_1,t_2)} \Omega_*(X \times C) \arrow[r, "i_1^*-i_2^*"] \arrow[d, "p^*"] & \Omega_*(X) \arrow[r, "\Phi_X"] \arrow[d, "p^*"] & \Omega_*^{\mathrm{alg}}(X) \arrow[r] \arrow[d, "p^*"] & 0 \\
\bigoplus_{(C,t_1,t_2)} \Omega_{*+n}(V \times C) \arrow[r, "i_1^*-i_2^*"] & \Omega_{*+n}(V) \arrow[r, "\Phi_V"] & \Omega_{*+n}^{\mathrm{alg}}(V) \arrow[r] & 0,
\end{tikzcd}$$

where the rows are exact by Theorem 6.3. The first two vertical arrows are isomorphisms by [12, Theorem 3.6.3] and hence the third vertical arrow must also be an isomorphism. $\square$

Using the projective bundle formula [12, Theorem 3.5.4] for Ω_*, the argument of the proof of Theorem 6.5 can be repeated in verbatim with V replaced by $\mathbb{P}(V)$ to prove the following projective bundle formula for our cobordism theory.

Theorem 6.6 (Projective bundle formula)**.** *Let $X \in \mathbf{Sch}_k$ and let E be a rank $n+1$ vector bundle on X. Then, we have $\bigoplus_{j=0}^{n} \Omega_{*-n+j}^{\mathrm{alg}}(X) \xrightarrow{\simeq} \Omega_*^{\mathrm{alg}}(\mathbb{P}(E))$.*

7. Ω^*_{alg} as an oriented cohomology theory

Recall from [12, Definition 1.1.2] that an oriented cohomology theory A^* on $\mathbf{Sm}_k$ is an additive contravariant functor to the category of commutative graded rings with unit, such that A^* has push-forward maps for projective morphisms and it satisfies the homotopy invariance and projective bundle formula. Moreover, the push-forward and the pull-back maps commute in a Cartesian diagram of transverse morphisms.

On the bigger category $\mathbf{Sch}_k$, from [12, Definition 5.1.3], we have the notion of an oriented Borel-Moore homology theory. This requires some similar axioms, but a nontrivial one is the existence of pull-backs for locally complete intersection (l.c.i.) morphisms. This ensures that an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ restricted onto $\mathbf{Sm}_k$ gives an oriented cohomology theory.

Our goal in this section is to conclude that $\Omega^*_{\rm alg}$ is an oriented cohomology theory on $\mathbf{Sm}_k$ and $\Omega^{\rm alg}_*$ is an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$.

7.1. **Pull-back via l.c.i. morphisms.** By Definition 2.2, one can pull-back cobordism cycles via *smooth* quasi-projective morphisms. One further step needed to turn $\Omega^{\rm alg}_*$ into an oriented Borel-Moore homology is to show that one can pull-back also via l.c.i. morphisms $f\colon X \to Y$ for $X, Y \in \mathbf{Sch}_k$. Recall that $f\colon X \to Y$ is an l.c.i. morphism if it factors as the composition $f = q \circ i\colon X \to P \to Y$, where i is a regular embedding and q is a smooth quasi-projective morphism. Since we have q^* already, defining i^* is the first technical issue to resolve. We shall demonstrate the existence of such pull-backs on $\Omega^{\rm alg}_*$ using Proposition 3.16 and the analogous construction for the algebraic cobordism in [12, §5, 6].

Recall from [6, Definition 2.2.1] that a *pseudo-divisor* D on a scheme X is a triple $D = (Z, \mathcal{L}, s)$, where $Z \subset X$ is a closed subset, $\mathcal{L}$ is an invertible sheaf on X, and s is a section of $\mathcal{L}$ on X such that the support of the zero scheme of s is contained in Z. We call Z *the support of* D and write it as $|D|$. We call the zero scheme $\{s = 0\}$ *the divisor of* D and write it as $\mathrm{Div}(D)$.

Given $X \in \mathbf{Sch}_k$ and a pseudo-divisor D on X, Levine and Morel defined in [12, §6.1.2] a graded group $\Omega_*(X)_D$ with a natural map $\theta_X\colon \Omega_*(X)_D \to \Omega_*(X)$, which is an isomorphism by [12, Theorem 6.4.12]. Roughly speaking, this is the group on which the "intersection product" by the divisor D is well-defined so that we have a map $D(-)\colon \Omega_*(X)_D \to \Omega_{*-1}(|D|)$. (See Section 11 for the definitions of $\Omega_*(X)_D$ and $D(-)$.) This yields

$$i^*_D\colon \Omega_*(X) \underset{\simeq}{\xrightarrow{\theta_X^{-1}}} \Omega_*(X)_D \xrightarrow{D(-)} \Omega_{*-1}(|D|). \tag{7.1}$$

It follows from Proposition 3.16 and Lemma 11.6 that i^*_D descends to

$$i^*_D\colon \Omega^{\rm alg}_*(X) \to \Omega^{\rm alg}_{*-1}(|D|). \tag{7.2}$$

7.1.1. *Gysin map for regular embedding.* Let $\iota_Z\colon Z \to X$ be a closed regular subscheme of codimension d in $\mathbf{Sch}_k$. We use (7.2) and the technique of the deformation to the normal bundle to define the pull-back map $\iota^*_Z\colon \Omega^{\rm alg}_*(X) \to \Omega^{\rm alg}_{*-d}(Z)$, that we call *the Gysin map* for the cobordism classes. Without going into the full construction of the deformation to the normal bundle, we recall here only the necessary summary from [12, §6.5.2 (6.10)]:

Proposition 7.1. *Let $\iota_Z\colon Z \to X$ be a closed regular embedding in $\mathbf{Sch}_k$. Then, there exist a scheme $U \in \mathbf{Sch}_k$, a closed immersion $i_N\colon N \to U$ of codimension 1, a surjective morphism $\mu\colon U \to X \times \mathbb{P}^1$, and its restriction $\mu_N\colon N \to Z \times 0$, that form the following* *commutative diagram*

$$\begin{tikzcd} N \arrow[rr, "i_N"] \arrow[d, "\mu_N"'] & & U \arrow[d, "\mu"] \\ Z \times 0 \arrow[r, "\mathrm{Id}\times 0"] & Z \times \mathbb{P}^1 \arrow[r, "\iota_Z \times \mathrm{Id}"] & X \times \mathbb{P}^1, \end{tikzcd}$$

such that

(1) *N is isomorphic to the normal vector bundle $N_{Z/X}$ of Z in X over Z, under the identification $Z = Z \times 0$, and*

(2) *the restriction $\mu \colon U \backslash N \to X \times \left(\mathbb{P}^1 \backslash \{0\}\right)$ is an isomorphism of k-schemes.*

We have the following analogue of [12, Lemma 6.5.2]:

Lemma 7.2. *The composition $i_N^* \circ i_{N*} \colon \Omega^{\mathrm{alg}}_{*+1}(N) \to \Omega^{\mathrm{alg}}_{*+1}(U) \to \Omega^{\mathrm{alg}}_{*}(N)$ is zero, where i_{N*} is the push-forward via the closed immersion i_N, and i_N^* is the pull-back by the divisor N defined in* (7.2).

Proof. Consider the commutative diagram

$$\begin{tikzcd} \Omega_{*+1}(N) \arrow[r, "i_N^* \circ i_{N*}"] \arrow[d] & \Omega_*(N) \arrow[d] \\ \Omega^{\mathrm{alg}}_{*+1}(N) \arrow[r, "i_N^* \circ i_{N*}"] & \Omega^{\mathrm{alg}}_{*}(N), \end{tikzcd}$$

where the vertical maps are the natural surjections. Since the top map on the algebraic cobordism is zero by [12, Lemma 6.5.2], the bottom map is also zero. $\square$

By Theorem 6.4, we have the localization exact sequence $\Omega^{\mathrm{alg}}_{*+1}(N) \overset{i_{N*}}{\to} \Omega^{\mathrm{alg}}_{*+1}(U) \overset{j^*}{\to} \Omega^{\mathrm{alg}}_{*+1}(U \backslash N) \to 0$, that gives an isomorphism

$$(j^*)^{-1} \colon \Omega^{\mathrm{alg}}_{*+1}(U \backslash N) \to \frac{\Omega^{\mathrm{alg}}_{*+1}(U)}{i_{N*}(\Omega^{\mathrm{alg}}_{*+1}(N))}. \tag{7.3}$$

Combining (7.3) with Lemma 7.2, we see that the composition

$$\alpha \colon \Omega^{\mathrm{alg}}_{*+1}(U \backslash N) \overset{(j^*)^{-1}}{\to} \frac{\Omega^{\mathrm{alg}}_{*+1}(U)}{i_{N*}(\Omega^{\mathrm{alg}}_{*+1}(N))} \overset{i_N^*}{\to} \Omega^{\mathrm{alg}}_{*}(N) \tag{7.4}$$

is well-defined.

Definition 7.3. For a regular embedding $\iota_Z \colon Z \to X$ of codimension d in $\mathbf{Sch}_k$, the *Gysin morphism* $\iota_Z^* \colon \Omega^{\mathrm{alg}}_*(X) \to \Omega^{\mathrm{alg}}_{*-d}(Z)$ is defined to be the composition

$$\Omega^{\mathrm{alg}}_*(X) \overset{pr_1^*}{\to} \Omega^{\mathrm{alg}}_{*+1}(X \times (\mathbb{P}^1 \backslash \{0\})) \underset{\simeq}{\overset{\mu^*}{\to}} \Omega^{\mathrm{alg}}_{*+1}(U \backslash N) \overset{\alpha}{\to} \Omega^{\mathrm{alg}}_*(N) \underset{\simeq}{\overset{(\mu_N^*)^{-1}}{\to}} \Omega^{\mathrm{alg}}_{*-d}(Z),$$

where pr_1 is the projection, μ is the isomorphism of Proposition 7.1(2), α is the map in (7.4), and $\mu_N \colon N \to Z$ is the normal vector bundle of Proposition 7.1(1) so that μ_N^* is an isomorphism by the $\mathbb{A}^1$-homotopy invariance, Theorem 6.5.

We have the following basic properties for the Gysin maps on Ω^{alg}_* that can be easily deduced from [12, Lemmas 6.5.6, 6.5.7, Theorem 6.5.8]:

Proposition 7.4. *The Gysin maps on Ω_*^{alg} satisfy the following:*

(1) *Let $\iota\colon Z \to X$ be a regular embedding of codimension 1. Then, as operators $\Omega_*^{\mathrm{alg}}(X) \to \Omega_{*-1}^{\mathrm{alg}}(Z)$, the pull-back $Z(-)$ by the divisor Z is identical to the Gysin pull-back ι^*.*

(2) *Let $\iota\colon Z \to X$ be a regular embedding, let $p\colon Y \to X$ be a smooth quasi-projective morphism, and let $s\colon Z \to Y$ be a section of Y over Z. Then, $s^* \circ p^* = \iota^*$.*

(3) *Let $\iota\colon Z \to Z'$ and $\iota'\colon Z' \to X$ be regular embeddings. Then, $(\iota' \circ \iota)^* = \iota^* \circ \iota'^*$.*

Proof. All of the above statements follow simply from [12, Lemmas 6.5.6, 6.5.7, Theorem 6.5.8] and the surjectivity of $\Phi_X\colon \Omega_*(X) \to \Omega_*^{\mathrm{alg}}(X)$, as in the proof of Lemma 7.2. We skip the details. □

7.1.2. *Pull-back for l.c.i. morphisms.* Let $f\colon X \to Y$ be an l.c.i. morphism in $\mathbf{Sch}_k$ with a factorization $f = p \circ i\colon X \to P \to Y$, where p is smooth quasi-projective and i is a regular embedding. We now have p^* by Definition 2.2, and we have the Gysin pull-back i^* by Definition 7.3. So, one wishes to define f^* by simply taking the composition $i^* \circ p^*$. To show that this definition is meaningful, one needs to know that if $p_1 \circ i_1 = p_2 \circ i_2$ are two such factorizations, then $i_1^* \circ p_1^* = i_2^* \circ p_2^*$. However, this fact follows at once from such an equality on the level of algebraic cobordism, as shown in [12, Lemma 6.5.9], and from the surjection $\Phi_-\colon \Omega_*(-) \to \Omega_*^{\mathrm{alg}}(-)$. Thus we have:

Definition 7.5. Let $f\colon X \to Y$ be an l.c.i. morphism that has a factorization $f = p \circ i\colon X \to P \to Y$, where i is a regular embedding and p is smooth quasi-projective. The pull-back f^* on $\Omega_*^{\mathrm{alg}}(Y)$ is defined to be $i^* \circ p^*$.

One has the following properties of the l.c.i. pull-backs on Ω_*^{alg} as for Ω_* proven in [12, Theorems 6.5.11, 6.5.12, 6.5.13]. The proof follows immediately from *ibid.* and we omit the arguments.

Theorem 7.6. *The pull-backs via l.c.i. morphisms have the following properties:*

(1) *If $f_1\colon X \to Y$, $f_2\colon Y \to Z$ are l.c.i. morphisms in $\mathbf{Sch}_k$, then $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$.*

(2) *Suppose $f\colon X \to Z$ and $g\colon Y \to Z$ are Tor-independent morphisms in $\mathbf{Sch}_k$, where f is l.c.i. and g is projective. Then, for the Cartesian square*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{pr_2} & Y \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z, \end{array}$$

we have $f^* \circ g_* = pr_{1*} \circ pr_2^*$.

(3) *Let $f_i\colon X_i \to Y_i$ for $i = 1, 2$ be two l.c.i. morphisms in $\mathbf{Sch}_k$. Then, for $\eta_i \in \Omega_*^{\mathrm{alg}}(Y_i)$ with $i = 1, 2$, we have $(f_1 \times f_2)^*(\eta_1 \times \eta_2) = f_1^*(\eta_1) \times f_2^*(\eta_2)$.*

Corollary 7.7. *Let $f\colon X \to Y$ be any morphism of smooth varieties. Then, there is a well-defined pull-back $f^*\colon \Omega^*_{\mathrm{alg}}(Y) \to \Omega^*_{\mathrm{alg}}(X)$. If $f\colon X \to Y$, $g\colon Y \to Z$ are any morphisms of smooth varieties, then $(g \circ f)^* = f^* \circ g^*$.*

Proof. Any morphism $f\colon X \to Y$ of smooth varieties is an l.c.i. morphism, with a factorization $f = pr_2 \circ \Gamma_f\colon X \to X \times Y \to Y$. The rest follows immediately. □

7.2. **Summary and the universal property.** Here is an analogue of Definition 3.12

Definition 7.8. We say that an oriented cohomology theory A^* on $\mathbf{Sm}_k$ *respects algebraic equivalence* if for $X \in \mathbf{Sm}_k$ and two algebraically equivalent line bundles L and M over X, we have $\tilde{c}_1(L) = \tilde{c}_1(M)$ as operators $A^*(X) \to A^{*+1}(X)$.

Similarly, we say that an oriented Borel-Moore homology theory A_* on $\mathbf{Sch}_k$ *respects algebraic equivalence* if for $X \in \mathbf{Sch}_k$ and two algebraically equivalent line bundles L and M over X, we have $\tilde{c}_1(L) = \tilde{c}_1(M)$ as operators $A_*(X) \to A_{*-1}(X)$.

The main results discussed in Sections 6.2 and 7.1 can be crystallized into the following result:

Theorem 7.9. *The theory Ω^*_{alg} is an oriented cohomology theory on $\mathbf{Sm}_k$ that respects algebraic equivalence, and it is universal among such theories. In other words, for any oriented cohomology theory A^* that respects algebraic equivalence, there exists a unique morphism of oriented cohomology theories $\theta\colon \Omega^*_{\mathrm{alg}} \to A^*$ on $\mathbf{Sm}_k$.*

Similarly, the theory Ω^{alg}_* is an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ that respects algebraic equivalence, and it is universal among such theories.

8. Connections to algebraic cobordism, Chow groups and K-theory

In this section, we study how our cobordism theory $\Omega^{\mathrm{alg}}_*(X)$ is related with the Chow groups $\mathrm{CH}^{\mathrm{alg}}_*(X)$ modulo algebraic equivalence and the semi-topological K-groups $K^{\mathrm{semi}}_0(X)$ and $G^{\mathrm{semi}}_0(X)$. We also show that with finite coefficients, our cobordism theory agrees with the algebraic cobordism theory.

8.1. Connection with Chow groups and K-theory.

Theorem 8.1. *For $X \in \mathbf{Sch}_k$, there is a natural map $\Omega^{\mathrm{alg}}_*(X) \to \mathrm{CH}^{\mathrm{alg}}_*(X)$ that induces an isomorphism $\Omega^{\mathrm{alg}}_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \xrightarrow{\simeq} \mathrm{CH}^{\mathrm{alg}}_*(X)$.*

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\bigoplus\limits_{(C,t_1,t_2)} \Omega_*(X \times C) \arrow[r, "i_1^* - i_2^*"] \arrow[d] & \Omega_*(X) \arrow[r] \arrow[d] & \Omega^{\mathrm{alg}}_*(X) \arrow[r] \arrow[d, dashed] & 0 \\
\bigoplus\limits_{(C,t_1,t_2)} \mathrm{CH}_*(X \times C) \arrow[r, "i_1^* - i_2^*"] & \mathrm{CH}_*(X) \arrow[r] & \mathrm{CH}^{\mathrm{alg}}_*(X) \arrow[r] & 0,
\end{tikzcd}$$

where the top row is exact by Theorem 6.3. It follows from the definition of algebraic equivalence on algebraic cycles in [6, Definition 10.3] and the proof of Lemma 2.4 that the bottom row is also exact (see [6, Example 10.3.2] when k is algebraically closed). The existence of the first two vertical maps and their commutativity follow from the universal property of Ω_*. This immediately yields a natural map $\Omega^{\mathrm{alg}}_*(X) \to \mathrm{CH}^{\mathrm{alg}}_*(X)$.

Moreover, the top row remains exact after tensoring with $- \otimes_{\mathbb{L}_*} \mathbb{Z}$ and the first two vertical maps after tensoring are isomorphisms by [12, Theorem 4.5.1]. Thus, the last vertical map after tensoring is also an isomorphism. $\square$

Remark 8.2. By Theorems 7.9, 8.1, and [12, Theorem 1.2.2], we see that $\mathrm{CH}^*_{\mathrm{alg}}$ is universal among oriented cohomology theories on $\mathbf{Sm}_k$ whose Chern class operations are additive, $\tilde{c}_1(L_1 \otimes L_2) = \tilde{c}_1(L_1) + \tilde{c}_1(L_2)$, and respect algebraic equivalence.

For $X \in \mathbf{Sch}_k$, let $K_0(X)$ (resp. $G_0(X)$) be the Grothendieck group of coherent locally free sheaves (resp. coherent sheaves) on X. Recall from [4, Definition 1.1] that the semi-topological K-group $K^{\mathrm{semi}}_0(X)$ (resp. $G^{\mathrm{semi}}_0(X)$) is the quotient by the subgroup generated by the images of the l.c.i. pull-back $i_1^* - i_2^*\colon K_0(X \times C) \to K_0(X)$ (resp. $i_1^* - i_2^*\colon G_0(X \times C) \to G_0(X)$) over the equivalence classes of the triples (C, t_1, t_2). When X is smooth, we have $K^{\mathrm{semi}}_0(X) = G^{\mathrm{semi}}_0(X)$. We have the following analogue of [12, Corollary 4.2.12].

Theorem 8.3. *Let $X \in \mathbf{Sch}_k$ and let β be a formal symbol of degree -1. Then, there is a natural map $\Omega^{\mathrm{alg}}_*(X) \to G^{\mathrm{semi}}_0(X)[\beta, \beta^{-1}]$ which induces an isomorphism $\Omega^{\mathrm{alg}}_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}] \xrightarrow{\simeq} G^{\mathrm{semi}}_0(X)[\beta, \beta^{-1}]$.*

Proof. The existence and the isomorphism of the desired maps follow from the definition of $G^{\mathrm{semi}}_0(X)$ above, Theorem 6.3, together with [12, Corollary 4.2.12] (if X is smooth) and [3, Theorem 1.5] (if X is not smooth) by repeating the same kind of arguments as in the proof of Theorem 8.1 in verbatim. We remark that we take $- \otimes_{\mathbb{L}_*} \mathbb{Z}[\beta, \beta^{-1}]$ in the diagram of the proof of Theorem 8.1. □

8.2. **Comparison with algebraic cobordism with finite coefficients.** By [5, Corollary 3.8], we know that with finite coefficients, the algebraic and the semi-topological K-theories of complex projective varieties coincide. We prove its cobordism analogue using Ω_* and Ω^{alg}_*:

Theorem 8.4. *Let $X \in \mathbf{Sch}_k$ and let $m \geq 1$ be an integer. Then, the natural map $\Phi_X \otimes \mathbb{Z}/m \colon \Omega_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \to \Omega^{\mathrm{alg}}_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m$ is an isomorphism.*

Proof. Using [13, Theorem 1] and Theorem 6.2, we can identify $\Omega_*(X)$ and $\Omega^{\mathrm{alg}}_*(X)$ with $\omega_*(X)$ and $\omega^{\mathrm{alg}}_*(X)$, respectively, whenever necessary. In the diagram (5.3), it suffices to show that $\mathrm{Im}(\theta)$ in $\omega_*(X)$ is divisible.

Let (g, p, ζ) be a double-point cobordism with a projective $g \colon Y \to X \times C$, two points $p, \zeta \in C(k)$ and $\pi = pr_2 \circ g$ such that $\pi^{-1}(p) = A \cup B$ as in Definition 4.2.

Let $\alpha \colon = [Y_\zeta \to Y] - [A \to Y] - [B \to Y] + [\mathbb{P}(\pi) \to Y]$ in $\omega_*(Y)$. Let $f \colon = pr_1 \circ g \colon Y \to X$. Since $\partial_C(g, p, \zeta) = f_*(\alpha)$, it suffices to show that α is divisible in $\omega_*(Y)$. An application of (5.2) to the divisor $E \colon = A + B$ shows that $[A \to Y] + [B \to Y] - [\mathbb{P}(\pi) \to Y] = [E \to Y] = \pi^*([\{p\} \to C])$. We also have $[Y_\zeta \to Y] = \pi^*([\{\zeta\} \to C])$. Thus, $\alpha = \pi^*([\{\zeta\} \to C] - [\{p\} \to C])$ and it reduces to prove that the class $\beta \colon = [\{\zeta\} \to C] - [\{p\} \to C]$ is divisible in $\omega_0(C)$.

By [12, Lemma 4.5.3], the natural map $\omega_0(C) \to \mathrm{CH}_0(C)$ is an isomorphism and the image of β in $\mathrm{CH}_0(C)$ is $[\{\zeta\}] - [\{p\}]$, which lies in $\mathrm{Pic}^0(C)$. Since $\mathrm{Pic}^0(C)$ is an abelian variety, the group $\mathrm{Pic}^0(C)(k)$ is divisible. This concludes the proof. □

9. Computations of Ω^{alg}_* and questions on finite generation

It is not in general easy to compute Ω_*. For the point $X = \mathrm{Spec}(k)$, Levine and Morel [12] showed that the natural map $\mathbb{L}_* \to \Omega_*(k)$ is an isomorphism. In this section, we focus on some computational aspects of Ω^{alg}_*.

9.1. **Comparison with the complex cobordism.** We refer to [15] or [17] for the definition and basic properties of the complex cobordism theory MU^* for locally compact Hausdorff topological spaces. We only mention here that $\mathrm{MU}^*(X)$ is generated by $[f \colon Y \to X]$, where f is proper and Y is a weakly complex real manifold under certain "bordism relations".

Proposition 9.1. *If there is an embedding $\sigma \colon k \hookrightarrow \mathbb{C}$, then there is a natural transformation $\theta^{\mathrm{alg}} \colon \Omega^*_{\mathrm{alg}} \to \mathrm{MU}^{2*}$ of oriented cohomology theories on $\mathbf{Sm}_k$ that factors the natural cycle class map $\theta \colon \Omega^* \to \mathrm{MU}^{2*}$.*

Proof. From [12, Example 1.2.10], we have a morphism $\theta \colon \Omega^* \to \mathrm{MU}^{2*}$ of oriented cohomology theories on $\mathbf{Sm}_k$. Hence by Theorem 7.9, it suffices to show that for any $X \in \mathbf{Sm}_k$ and algebraically equivalent line bundles L_1 and L_2 on X, one has $\tilde{c}_1(L_1) = \tilde{c}_1(L_2) \colon \mathrm{MU}^*(X_\sigma) \to \mathrm{MU}^{*+2}(X_\sigma)$. We can assume $k = \mathbb{C}$.

Let $\mathcal{L}$ be a line bundle on $X \times C$ for some compact Riemann surface C such that for some points $t_1, t_2 \in C$, we have $L_j = \mathcal{L}|_{X \times \{t_j\}}$ for $j = 1, 2$. Let $i_j \colon X \times \{t_j\} \to X \times C$ be the inclusions. Take any differentiable path I in C, diffeomorphic to the unit interval $[0, 1]$, whose end points are t_1 and t_2. Let $\alpha \colon X \times I \to X \times C$ and $\iota_j \colon X \times \{t_j\} \to X \times I$ be the inclusions. Note that $\alpha \circ \iota_j = i_j$ for $j = 1, 2$.

Since X is smooth, we have $\tilde{c}_1(L_j)([Y \to X]) = (\tilde{c}_1(L_j)(1_X)) \cdot [Y \to X] = c_1(L_j) \cdot [Y \to X]$, where the first equality comes from [12, (5.2)-5]. On the other hand, we have $c_1(L_j) = i_j^*(c_1(\mathcal{L})) = \iota_j^* \alpha^*(c_1(\mathcal{L}))$. The desired assertion now follows from the fact that $\iota_j^* \colon \mathrm{MU}^*(X \times I) \to \mathrm{MU}^*(X)$ is an isomorphism for $j = 1, 2$ because I is contractible. $\square$

Remark 9.2. One may view the above result as a lifting of the cycle class map of Totaro [17], $\mathrm{CH}^*_{\mathrm{alg}}(X) \to \mathrm{MU}^{2*}(X) \otimes_{\mathbb{L}^*} \mathbb{Z}$ because $\mathrm{CH}^*_{\mathrm{alg}}(X) \simeq \Omega^*_{\mathrm{alg}} \otimes_{\mathbb{L}^*} \mathbb{Z}$ by Theorem 8.1.

9.2. **Points.**

Proposition 9.3. *The map $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(k)$ is an isomorphism.*

Proof. Composing the isomorphism $\mathbb{L}^* \xrightarrow{\simeq} \Omega^*(k)$ with the surjection $\Omega^*(k) \to \Omega^*_{\mathrm{alg}}(k)$, we see that the map $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(k)$ is surjective. We prove injectivity.

We first prove the injectivity of the map $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{Q} \to \Omega^*_{\mathrm{alg}}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the rational coefficients. Applying Proposition 3.16, we see that $\ker(\Omega^*(k) \to \Omega^*_{\mathrm{alg}}(k))$ is generated by the cobordism cycles of the form $\alpha = [Y \to \mathrm{Spec}(k), L] - [Y \to \mathrm{Spec}(k), M]$, where $L \sim M$ on Y. Since we are working with the rational coefficients, we can use [13, Theorem 1, Corollary 3] to assume that Y is a product of projective spaces. But for such spaces, that two lines bundles are algebraically equivalent is nothing but that the line bundles are isomorphic to each other. In particular, α is zero already in $\Omega^*(k) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, the map $\mathbb{L}^* \otimes_{\mathbb{Z}} \mathbb{Q} \to \Omega^*_{\mathrm{alg}}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. The injectivity of $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(k)$ now follows for $\mathbb{L}^*$ has no torsion. $\square$

Recall from [12, Definition 4.4.1] that an oriented Borel-Moore homology theory A_* on $\mathbf{Sch}_k$ is said to be *generically constant* if for each finitely generated field extension $k \subset F$, the canonical morphism $A_*(k) \to A_*(F/k)$ is an isomorphism, where $A_*(F/k)$ is the colimit of $A_{*+\mathrm{tr}_{F/k}}(X)$ over models X for F over k. Here $\mathrm{tr}_{F/k}$ is the transcendence degree of F over k. Recall that a model for F over k is an integral scheme $X \in \mathbf{Sch}_k$ whose function field is isomorphic to F.

Proposition 9.4. *The cobordism theory Ω^{alg}_* is generically constant.*

Proof. Let $\mathcal{C}$ denote the category of models for F over k. Then, we have a commutative diagram

$$\begin{tikzcd}
\Omega_*(k) \arrow[rr, "\simeq"] \arrow[d, "\eta_F"', "\simeq"] & & \Omega^{\mathrm{alg}}_*(k) \arrow[d, "\eta^{\mathrm{alg}}_F"'] \arrow[dr, "\simeq"] & \\
\operatorname*{colim}_{X \in \mathcal{C}} \Omega_{*+\mathrm{tr}_{F/k}}(X) \arrow[rr] & & \operatorname*{colim}_{X \in \mathcal{C}} \Omega^{\mathrm{alg}}_{*+\mathrm{tr}_{F/k}}(X) \arrow[r] & \Omega^{\mathrm{alg}}_*(F).
\end{tikzcd}$$

We have to show that η^{alg}_F is an isomorphism. It follows from [12, Corollary 4.4.3] that η_F is an isomorphism. Applying Proposition 3.16 to the first horizontal arrow on the bottom, we see that η^{alg}_F is surjective. On the other hand, it follows from Proposition 9.3 that the slanted downward arrow is an isomorphism. This in turn implies that η^{alg}_F must also be injective, and hence an isomorphism. $\square$

Recall from [6, Example 1.9.1] that a scheme $X \in \mathbf{Sch}_k$ is called *cellular* if it has a filtration $\emptyset = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_1 \subsetneq X_0 = X$ by closed subschemes such that each $X_i \backslash X_{i+1}$ is a disjoint union of affine spaces, called *cells*. By the Bruhat decomposition, one sees that the schemes of type G/P are cellular, where P is a parabolic subgroup of a split reductive group G. Smooth projective toric varieties are also examples of cellular schemes. Both these classes contain the projective spaces. As a consequence of Proposition 9.3, we have the following computation of our cobordism theory for cellular schemes.

Proposition 9.5. *For a cellular scheme $X \in \mathbf{Sch}_k$, the natural map $\Phi_X \colon \Omega_*(X) \to \Omega_*^{\mathrm{alg}}(X)$ is an isomorphism. Each of these groups is a free $\mathbb{L}_*$-module.*

Proof. We prove it by induction on the length of a filtration $\{X_i\}_{0 \le i \le n+1}$ on X. Let $U_i = X \backslash X_i$. Here, all U_i and $U_{i+1} \backslash U_i$ are also cellular. By Theorem 6.4 and [11, Proposition 4.3], for each $1 \le i \le n$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \to \Omega_*(U_{i+1} \backslash U_i) \arrow[r] \arrow[d] & \Omega_*(U_{i+1}) \arrow[r] \arrow[d] & \Omega_*(U_i) \arrow[r] \arrow[d] & 0 \\
0 \to \Omega_*^{\mathrm{alg}}(U_{i+1} \backslash U_i) \arrow[r] & \Omega_*^{\mathrm{alg}}(U_{i+1}) \arrow[r] & \Omega_*^{\mathrm{alg}}(U_i) \arrow[r] & 0.
\end{tikzcd}$$

Since U_1 is a disjoint union of affine spaces over k, the map $\Omega_*(U_1) \to \Omega_*^{\mathrm{alg}}(U_1)$ is an isomorphism of free $\mathbb{L}_*$-modules by Proposition 9.3 and Theorem 6.5. By the same reason, the left vertical map in the above diagram is an isomorphism for each $1 \le i \le n$.

Assuming the assertion for U_i, from a diagram chase and the induction hypothesis, it follows that the middle vertical map is an isomorphism. Taking $i = n$, we get the desired result for X. $\square$

Remark 9.6. If there is an embedding $\sigma \colon k \hookrightarrow \mathbb{C}$, Proposition 9.5 can be also deduced from Proposition 3.16, Proposition 9.1 and [10, Theorem 6.1].

9.3. **Curves.** We compute the cobordism theory $\Omega^*_{\mathrm{alg}}(X)$ of a smooth curve X. We show that this is a finitely generated $\mathbb{L}^*$-module. This is usually false for the algebraic cobordism $\Omega^*(X)$ unless X is rational. If $k = \mathbb{C}$, we show that $\Omega^*_{\mathrm{alg}}(X)$ is closely related to the complex cobordism $\mathrm{MU}^*(X(\mathbb{C}))$.

Theorem 9.7. *Let X be a smooth curve over a field k. Then,*

(1) *The $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is generated by at most 2 elements.*
(2) *If X is affine, then the map $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(X)$ is an isomorphism.*
(3) *When $k = \mathbb{C}$, there is a split exact sequence*

$$0 \to \Omega^*_{\mathrm{alg}}(X) \to \mathrm{MU}^*(X(\mathbb{C})) \to H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}^* \to 0. \tag{9.1}$$

In particular, the map $\Omega^*_{\mathrm{alg}}(X) \to \mathrm{MU}^{2*}(X(\mathbb{C}))$ is an isomorphism.

Proof. We have shown in Theorem 6.4 and Proposition 9.4 that Ω_*^{alg} has the localization property and is generically constant. Hence, it satisfies the generalized degree formula [12, Theorem 4.4.7]. By the degree formula, the cobordism $\Omega^*_{\mathrm{alg}}(X)$ is generated as an $\mathbb{L}^*$-module by the cobordism cycles $1_X = [X \to X]$ and $[\{p\} \to X] = [X \to X, O_X(p)]$, where p is a closed point of X. Part (1) now follows from the fact that the map $\deg \colon \mathrm{Pic}(X)/\!\sim \to \mathbb{Z}$ is injective.

If X is affine, we choose a smooth compactification $j \colon X \hookrightarrow \overline{X}$ and set $Z \colon = \overline{X} \backslash X$. This yields an exact sequence

$$\mathrm{CH}^0(Z) \to \mathrm{Pic}(\overline{X})/\!\sim \xrightarrow{j^*} \mathrm{Pic}(X)/\!\sim \to 0$$

by [6, Example 10.3.4], and here the first map in this exact sequence is surjective. In particular, the last term is zero. Thus, $\Omega^*_{\mathrm{alg}}(X)$ is generated by 1_X as an $\mathbb{L}^*$-module, *i.e.*, $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(X)$ is surjective. On the other hand, for a closed point $p \in X$, the composition with the pull-back $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(X) \to \Omega^*_{\mathrm{alg}}(k(p))$ is an isomorphism by Proposition 9.3. Hence, $\mathbb{L}^* \to \Omega^*_{\mathrm{alg}}(X)$ is injective, thus this map is an isomorphism. This proves (2).

For (3), we first compute $\mathrm{MU}^*(X(\mathbb{C}))$ using the Atiyah-Hirzebruch spectral sequence $E^2 = H^*(X(\mathbb{C}), \mathbb{L}^*) \Rightarrow \mathrm{MU}^*(X(\mathbb{C}))$ (see Remark 9.8 for an elementary approach). It is known (*cf.* [17, §1]) that the differentials of this spectral sequence are all torsion. On the other hand, the inclusion $H^*(X(\mathbb{C}), \mathbb{Z}) \hookrightarrow H^*(X(\mathbb{C}), \mathbb{C})$ (use the exponential exact sequence) shows that $H^*(X(\mathbb{C}), \mathbb{Z})$ is torsion-free. Thus, the spectral sequence degenerates and induces isomorphisms

$$H^*(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{L}^* \xrightarrow{\simeq} \mathrm{MU}^*(X(\mathbb{C})), \tag{9.2}$$

$$\mathrm{MU}^*(X(\mathbb{C})) = \mathrm{MU}^{2*+1}(X(\mathbb{C})) \oplus \mathrm{MU}^{2*}(X(\mathbb{C})) \simeq H^1(X(\mathbb{C}), \mathbb{L}^*) \oplus H^{\{0,2\}}(X(\mathbb{C}), \mathbb{L}^*).$$

Since $H^{\{0,2\}}(X(\mathbb{C}), \mathbb{Z})$ is generated by the algebraic cycles on X, we see from the proof of (1) and (2) above that the image of the map $\Omega^*_{\mathrm{alg}}(X) \to \mathrm{MU}^*(X(\mathbb{C}))$ contains $\mathrm{MU}^{2*}(X(\mathbb{C}))$, and is exactly equal to $\mathrm{MU}^{2*}(X(\mathbb{C}))$ if X is affine (since $H^2(X(\mathbb{C}), \mathbb{Z})$ is zero in this case). In particular, this proves (3) if X is affine using (2).

If X is not affine (thus $X(\mathbb{C})$ is a compact Riemann surface in this curve case), we let $U = X \setminus \{p\}$ where p is a closed point. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^*_{\mathrm{alg}}(\{p\}) \arrow[r] \arrow[d] & \Omega^*_{\mathrm{alg}}(X) \arrow[r] \arrow[d] & \Omega^*_{\mathrm{alg}}(U) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{MU}^*(\{p\}) \arrow[r] & \mathrm{MU}^*(X(\mathbb{C})) \arrow[r] & \mathrm{MU}^*(U(\mathbb{C})) \arrow[r] & 0, \end{tikzcd} \tag{9.3}$$

where the bottom row is exact from the computations in (9.2) and the isomorphism $H^*(X(\mathbb{C}), \mathbb{Z}) \simeq H^*(\{p\}, \mathbb{Z}) \oplus H^*(U(\mathbb{C}), \mathbb{Z})$. Since the left vertical arrow is an isomorphism, we see from Theorem 6.4 that the top row is also exact. The exact sequence (9.1) for X now follows from a diagram chase and the case of the affine curve U. The isomorphism $\Omega^*_{\mathrm{alg}}(X) \xrightarrow{\simeq} \mathrm{MU}^{2*}(X(\mathbb{C}))$ follows from (9.1) and (9.2). □

Remark 9.8. The authors thank the referee who pointed out that, Theorem 9.7(3) can be proven in an elementary way: if X is affine, then $X(\mathbb{C})$ has the homotopy type of a wedge of finitely many circles, and if X is not affine, then the suspension $\Sigma X(\mathbb{C})$ has the homotopy type of the sum of S^1, $2g$-copies of S^2, and a copy of S^3. Thus, the suspension isomorphism in MU^* and Mayer-Vietoris give the proof.

As an immediate corollary of Theorems 8.4 and 9.7, we obtain the following analogue of Quillen-Lichtenbaum conjecture for the cobordism of smooth curves.

Corollary 9.9. *For a smooth curve X over $\mathbb{C}$ and an integer $m \geq 1$, the natural map $\Omega^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m \to \mathrm{MU}^{2*}(X(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{Z}/m$ is an isomorphism.*

9.4. **Surfaces.** For an algebraic surface X, under the rational equivalence both the 1-cycles and 0-cycles often form infinitely generated Chow groups. Since the algebraic cobordism contains more data than Chow groups as shown in [12, Theorem 4.5.1], the algebraic cobordism of a surface is often infinitely generated as an $\mathbb{L}^*$-module. However, under algebraic equivalence, the 1-cycles form the Néron-Severi group, which is finitely generated, and the 0-cycles form an infinite cyclic group. We prove an analogous result for the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$. We use the following graded Nakayama lemma whose proof is an elementary application of a backward induction argument. It is left as an exercise.

Lemma 9.10. *Let M^* be a $\mathbb{Z}$-graded $\mathbb{L}^*$-module such that for some integer $N \geq 0$, we have $M^n = 0$ for all $n > N$. Suppose that $S = \{\alpha_1, \cdots, \alpha_r\}$ is a set of homogeneous elements in $M^{\geq 0}$ whose images generate $M^* \otimes_{\mathbb{L}^*} \mathbb{Z}$ as an abelian group. Then M^* is generated by S as an $\mathbb{L}^*$-module.*

Theorem 9.11. *Let X be a smooth projective surface. Then, $\Omega^*_{\mathrm{alg}}(X)$ is a finitely generated $\mathbb{L}^*$-module, with at most $\rho + 2$ generators, where ρ is the minimal number of generators of the Néron-Severi group* $\mathrm{NS}(X)$.

Note that if $\mathrm{NS}(X)$ is torsion free, then ρ is the Picard number of X.

Proof. This follows immediately from Theorem 8.1 and Lemma 9.10 using the fact that $\mathrm{CH}^*_{\mathrm{alg}}(X) \simeq \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. □

9.5. **Threefolds and beyond.** We saw that for a smooth projective variety X of dimension ≤ 2, the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is finitely generated. But, this is the highest we can go, due to the following result and some known deep results about algebraic cycles. Recall that for a smooth projective complex variety X, the Griffiths group $\mathrm{Griff}^r(X)$ of X is the group of codimension r homologically trivial cycles modulo algebraic equivalence. In particular, it is a subgroup of $\mathrm{CH}^r_{\mathrm{alg}}(X)$.

Theorem 9.12. *For any smooth variety X, the following two statements are equivalent:*

(1) *The Chow group* $\mathrm{CH}^*_{\mathrm{alg}}(X)$ *modulo algebraic equivalence is finitely generated.*

(2) *The cobordism* $\Omega^*_{\mathrm{alg}}(X)$ *is a finitely generated* $\mathbb{L}^*$*-module.*

If X is a smooth projective complex variety, then the following statement is also equivalent to the above two:

(3) *The Griffiths group* $\mathrm{Griff}^*(X)$ *is finitely generated.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 8.1 and by applying Lemma 9.10 to $M^* = \Omega^*_{\mathrm{alg}}(X)$.

When X is a smooth projective complex variety, let $\mathrm{CH}^*_{\mathrm{hom}}(X)$ denote the group of algebraic cycles on X modulo homological equivalence. The equivalence (1) $\Leftrightarrow$ (3) follows from the exact sequence

$$0 \to \mathrm{Griff}^*(X) \to \mathrm{CH}^*_{\mathrm{alg}}(X) \to \mathrm{CH}^*_{\mathrm{hom}}(X) \to 0 \tag{9.4}$$

and the observation that $\mathrm{CH}^*_{\mathrm{hom}}(X)$ is a subgroup of $H^{2*}(X(\mathbb{C}), \mathbb{Z})$, which is a finitely generated abelian group since X is smooth and projective. □

Remark 9.13. It was shown by Griffiths [7] that the Griffiths groups can be nontrivial. Clemens [2] later showed that $\mathrm{Griff}^2(X)$ is not finitely generated for a general quintic threefold X. These results were generalized by Nori [14] for algebraic cycles of codimension ≥ 2. Thus, it follows from Theorem 9.12 that the $\mathbb{L}^*$-module $\Omega^*_{\mathrm{alg}}(X)$ is in general not finitely generated for a variety of dimension at least three.

It seems that certain questions about algebraic cycles of smooth projective varieties can be lifted to the level of cobordism cycles. As an example, consider the following. We saw in Section 9.1 that for a smooth complex variety X, there are cycle class maps $\theta_X \colon \Omega^*(X) \to \mathrm{MU}^{2*}(X(\mathbb{C}))$ and $\theta^{\mathrm{alg}}_X \colon \Omega^*_{\mathrm{alg}}(X) \to \mathrm{MU}^{2*}(X(\mathbb{C}))$. Let $\Phi_X \colon \Omega^*(X) \to \Omega^*_{\mathrm{alg}}(X)$ be the natural map. We define the *Griffiths groups for the cobordism cycles* to be the graded group

$$\mathrm{Griff}^*_{\Omega}(X) = \ker(\theta_X)/\ker(\Phi_X). \tag{9.5}$$

The subgroup $\ker(\theta_X)$ can be called the group of cobordism cycles *homologically equivalent to zero.* We ask the following:

Question 9.14. *Let X be a smooth projective complex variety of dimension at least three. Is it true that $\mathrm{Griff}^*_\Omega(X)$ is a finitely generated $\mathbb{L}^*$-module if and only if $\mathrm{Griff}^*(X)$ is a finitely generated abelian group? In particular, are there examples where $\mathrm{Griff}^*_\Omega(X)$ is not finitely generated as an $\mathbb{L}^*$-module?*

10. Rational smash-nilpotence for cobordism

It was proven by Voevodsky [18] and Voisin [19] that if an algebraic cycle α on a smooth projective variety X is 0 in $\mathrm{CH}^{\mathrm{alg}}_*(X)_\mathbb{Q}$, then for some integer $N > 0$ its smash-product $\alpha^{\otimes N} \colon = \alpha \times \cdots \times \alpha$ on $X^N \colon = X \times \cdots \times X$ is 0 in $\mathrm{CH}_*(X^N)_\mathbb{Q}$. We use the notation $\alpha^{\otimes N}$ instead of α^N for the latter is the self-intersection of α in $\mathrm{CH}_*(X)_\mathbb{Q}$. This section studies the corresponding question for cobordism cycles.

Definition 10.1. Let $X \in \mathbf{Sch}_k$. Let $\alpha \in \mathcal{Z}_*(X)$. Let $N \geq 1$ be an integer.

(1) The *N-fold smash-product* $\alpha^{\boxtimes N} \in \mathcal{Z}_*(X^N)$ is the N-fold self-external product $\alpha \times \cdots \times \alpha$ using Definition 2.2.

(2) α is *rationally smash-nilpotent* if there is an integer $N > 0$ such that the image of $\alpha^{\boxtimes N}$ in $\Omega_*(X^N)_\mathbb{Q}$ is zero.

Lemma 10.2. *Let $X \in \mathbf{Sch}_k$. Let $\alpha, \beta \in \mathcal{Z}_*(X)$.*

(1) *If α or β is rationally smash-nilpotent, then so is $\alpha \times \beta$.*

(2) *If α and β are rationally smash-nilpotent, then so is $\alpha + \beta$.*

Proof. Note that the external product $\times$ is commutative because in Definition 2.1 we identified all isomorphic cobordism cycles. For (1), if $\alpha^{\boxtimes N} = 0 \in \Omega_*(X^N)_\mathbb{Q}$, then $(\alpha \times \beta)^{\boxtimes N} = \alpha^{\boxtimes N} \times \beta^{\boxtimes N} = 0 \in \Omega_*(X^{2N})_\mathbb{Q}$. The case $\beta^{\boxtimes N} = 0$ in $\Omega_*(X^N)_\mathbb{Q}$ is similar. For (2), use the binomial theorem since $\times$ is commutative. $\square$

We now prove the cobordism analogue of the result [18, Corollary 3.2]:

Theorem 10.3. *Let X be a smooth projective variety. Let $\alpha \in \mathcal{Z}_*(X)$. If the image of α in $\Omega^{\mathrm{alg}}_*(X)_\mathbb{Q}$ is trivial, then it is rationally smash-nilpotent.*

Proof. By [13, Theorem 1] and Theorem 6.2, we may identify $\Omega_*(X)$ and $\Omega^{\mathrm{alg}}_*(X)$ with $\omega_*(X)$ and $\omega^{\mathrm{alg}}_*(X)$, respectively. Consider the exact sequence of Theorem 5.1 and tensor it with $- \otimes_\mathbb{Z} \mathbb{Q}$ to obtain the exact sequence:

$$\bigoplus_{(C,t_1,t_2)} \omega_*(X \times C)_\mathbb{Q} \xrightarrow{\theta'} \omega_*(X)_\mathbb{Q} \xrightarrow{\Psi_X} \omega^{\mathrm{alg}}_*(X)_\mathbb{Q} \to 0,$$

where (C, t_1, t_2) runs over the equivalence classes of triples consisting of a smooth projective curve C and two distinct points $t_1, t_2 \in C(k)$, and θ' is the sum of the maps $i_1^* - i_2^*$, with the inclusions $i_j \colon X \times \{t_j\} \to X \times C$. Consider the image of α in $\omega_*(X)_\mathbb{Q}$, also denoted by α. Since α belongs to $\ker \Psi_X$ by the given assumption, we have $\alpha \in \mathrm{Im}(\theta')$. By Lemma 10.2-(2), it is enough to consider α of the form $(i_1^* - i_2^*)(\beta)$ for $\beta = [g \colon Y \to X \times C] \in \omega_*(X \times C)$. So, we call $\alpha = (i_1^* - i_2^*)(\beta)$.

Since $X \times C$ is smooth, by the transversality [12, Proposition 3.3.1] combined with [13, Theorem 1], we may assume that g is transversal to the closed immersions i_j, $j = 1, 2$. Hence, the fibre product Y_{t_j} of $X \times \{t_j\}$ and Y over $X \times C$ is smooth, and $i_j^*(\beta) = [Y_{t_j} \to X]$. On the other hand, let $\pi \colon = pr_2 \circ g \colon Y \to X \times C \to C$, which is projective since X is projective, then $\pi^*[\{t_j\} \to C] = [Y_{t_j} \to Y]$. Hence, for $f \colon = pr_1 \circ g \colon Y \to X \times C \to X$, which is projective for C is projective, we have $f_* \pi^*[\{t_j\} \to C] = [Y_{t_j} \to X]$ so that $(i_1^* - i_2^*)(\beta) = [Y_{t_1} \to X] - [Y_{t_2} \to X] = f_* \pi^*([\{t_1\} \to C] - [\{t_2\} \to C])$. Let $\gamma \colon = [\{t_1\} \to C] - [\{t_2\} \to C] \in \omega_0(C)_\mathbb{Q}$.

Then we have $\alpha = f_*\pi^*(\gamma)$ with $\gamma \in \omega_0(C)_{\mathbb{Q}}$, such that $\gamma = 0 \in \omega_0^{\mathrm{alg}}(C)_{\mathbb{Q}}$. We claim that γ is rationally smash-nilpotent.

Under the isomorphism $\omega_0(C)_{\mathbb{Q}} \overset{\simeq}{\to} \mathrm{CH}_0(C)_{\mathbb{Q}}$ of [12, Lemma 4.5.10], the image of γ in $\mathrm{CH}_0(C)_{\mathbb{Q}}$ is the 0-cycle $\bar{\gamma} = [\{t_1\}] - [\{t_2\}] \in \mathrm{CH}_0(C)_{\mathbb{Q}}$, whose image in $\mathrm{CH}_0^{\mathrm{alg}}(C)_{\mathbb{Q}}$ is trivial. Hence, by [18, Corollary 3.2], for some integer $N > 0$, we have $\bar{\gamma}^{\otimes N} = 0 \in \mathrm{CH}_0(C^N)_{\mathbb{Q}}$. Since the isomorphism $\omega_0(C^N)_{\mathbb{Q}} \simeq \mathrm{CH}_0(C^N)_{\mathbb{Q}}$ of [12, Lemma 4.5.10] respects the external products, we have $\gamma^{\boxtimes N} = 0 \in \omega_0(C^N)_{\mathbb{Q}}$.

Since γ is rationally smash-nilpotent, we now easily see that $\alpha = f_*\pi^*(\gamma)$ is also rationally smash-nilpotent since the push-forward and the pull-back maps respect external products (*cf.* Theorem 7.6). □

Remark 10.4. We remark that the proof of Theorem 10.3 uses [18] only for smooth projective curves.

11. Appendix

This section gives a summary of the constructions from [12, §6] related to the Gysin maps and the pull-backs via l.c.i. morphisms on the algebraic cobordism, that are used in the paper. The only new result is Lemma 11.6, used in the construction of the map $i_D^*\colon \Omega_*^{\mathrm{alg}}(X) \to \Omega_{*-1}^{\mathrm{alg}}(|D|)$ of (7.2) for a pseudo-divisor D on X.

Definition 11.1 ([12, 6.1.2])**.** Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X.

(1) $\mathcal{Z}_*(X)_D$ is the subgroup of $\mathcal{Z}_*(X)$ generated by the cobordism cycles $[f\colon Y \to X, L_1, \cdots, L_r]$ such that either $f(Y) \subset |D|$, or $f(Y) \not\subset |D|$ and $\mathrm{Div}\, f^*D$ is a strict normal crossing divisor on Y.

(2) Let $\mathcal{R}_*^{\mathrm{Dim}}(X)_D$ be the subgroup of $\mathcal{Z}(X)_D$ generated by the cobordism cycles of the form $[f\colon Y \to X, \pi^*(L_1), \cdots, \pi^*(L_r), M_1, \cdots, M_s]$, where $\pi\colon Y \to Z$ is smooth quasi-projective, $Z \in \mathbf{Sm}_k$, and $L_1, \cdots, L_{r>\dim Z}$ are line bundles on Z. We let $\underline{\mathcal{Z}}_*(X)_D\colon = \mathcal{Z}_*(X)_D/\mathcal{R}_*^{\mathrm{Dim}}(X)_D$.

The projective push-forward and smooth pull-back on $\mathcal{Z}_*(-)_D$ can be defined as for $\mathcal{Z}_*(-)$, and similarly for $\underline{\mathcal{Z}}_*(-)$.

(3) For a line bundle $L \to X$, define the Chern class operation $\tilde{c}_1(L)\colon \mathcal{Z}_*(X)_D \to \mathcal{Z}_{*-1}(X)_D$ as for $\mathcal{Z}_*(X)$. This descends onto $\underline{\mathcal{Z}}_*(X)_D$.

(4) We have the external product

$$\times\colon \mathcal{Z}_*(X)_D \otimes \mathcal{Z}_*(X')_{D'} \to \mathcal{Z}_*(X \times X')_{pr_1^*D + pr_2^*D'}$$

as for $\mathcal{Z}_*(-)$. This descends onto $\underline{\mathcal{Z}}_*(-)_D$-level.

Given $X \in \mathbf{Sch}_k$, a pseudo-divisor D on X, and a projective morphism $f\colon Y \to X$, where Y is a smooth irreducible variety, a strict normal crossing divisor E on Y is said to be in *very good position with* D if either $f(Y) \subset |D|$, or $f(Y) \not\subset |D|$ and $E + \mathrm{Div}\, f^*D$ is a strict normal crossing divisor on Y. By [12, Remark 6.1.4(1)], if E is in very good position with D, then for each face $i_J\colon E_J \hookrightarrow E$ and the composition $f_J\colon = f \circ i_J\colon E_J \to Y \to X$, either $f_J(E_J) \subset |D|$ or $\mathrm{Div}\, f_J^*D$ is a strict normal crossing divisor on E_J.

Definition 11.2 (([12, Definition 6.1.5]))**.** Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X. Let $\mathcal{R}_*^{\mathrm{Sect}}(X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by elements of the form $[f\colon Y \to X, L_1, \cdots, L_r] - [f \circ i\colon Z \to X, i^*(L_1), \cdots, i^*(L_{r-1})]$, with $r > 0$, such that

(1) $[f\colon Y \to X, L_1, \cdots, L_r] \in \mathcal{Z}_*(X)_D$, and

(2) $i\colon Z \to Y$ is the closed immersion of the subscheme given by the vanishing of a transverse section $s\colon Y \to L_r$ such that Z is in very good position with D.

Then, we let $\underline{\Omega}_*(X)_D\colon = \underline{\mathcal{Z}}_*(X)_D/\mathcal{R}_*^{\mathrm{Sect}}(X)_D$.

Definition 11.3 ([12, Definitions 6.1.6])**.** Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X.

(1) Let $\mathcal{R}_*(X)_D$ be the subgroup of $\mathcal{Z}_*(X)_D$ generated by elements of the form $[Y \to X, L_1, \cdots, L_r] - [Y' \to X, L'_1, \cdots, L'_r]$ such that

- (a) $[Y \to X, L_1, \cdots, L_r]$ and $[Y' \to X, L'_1, \cdots, L'_r]$ are in $\mathcal{Z}_*(X)_D$, and
- (b) there exist an isomorphism $\phi \colon Y \to Y'$ over X, a permutation σ of $\{1, \cdots, r\}$ and an isomorphism $L_i \simeq \phi^*(L'_{\sigma(i)})$.

We define $\underline{\Omega}_*(X)_D \colon = \underline{\Omega}_*(X)_D/\mathcal{R}_*(X)_D$.

(2) Let $\Omega_*(X)_D$ be the quotient of $\mathbb{L}_* \otimes_{\mathbb{Z}} \underline{\Omega}_*(X)_D$ by the relations of the form

$$(\mathrm{Id}_{\mathbb{L}_*} \otimes f_*)(F_{\mathbb{L}_*}(\tilde{c}_1(L), \tilde{c}_1(M))(\eta) - \tilde{c}_1(L \otimes M)(\eta))$$

where L, M are line bundles on Y, $\eta \in \underline{\Omega}_*(Y)_D$, and $[f \colon Y \to X]$ is a cobordism cycle such that either $f(Y) \subset |D|$, or $f(Y) \not\subset |D|$ and $\mathrm{Div}\, f^*D$ is a strict normal crossing divisor on Y.

The Chern class operation and the external product are induced onto $\Omega_*(-)_D$.

It is clear from the above definition that there is a natural map $\theta_X : \Omega_*(X)_D \to \Omega_*(X)$. The main content of [12, §6.4.1] is the proof of the following moving lemma.

Theorem 11.4 ([12, Theorem 6.4.12])**.** *For $X \in \mathbf{Sch}_k$, the natural map $\theta_X : \Omega_*(X)_D \to \Omega_*(X)$ is an isomorphism.*

Now we define the intersection by D on $\Omega_*(X)_D$, namely, $D(-) \colon \Omega_*(X)_D \to \Omega_{*-1}(|D|)$. First recall the map $D(-) \colon \mathcal{Z}_*(X)_D \to \Omega_{*-1}(|D|)$.

Definition 11.5 (([12, §6.2.1]))**.** Let $X \in \mathbf{Sch}_k$ and let $D = (|D|, O_X(D), s)$ be a pseudo-divisor on X. Let $\eta \colon = [f \colon Y \to X, L_1, \cdots, L_r] \in \mathcal{Z}_*(X)_D$.

(1) If $f(Y) \subset |D|$, then let $f^D \colon Y \to |D|$ be the induced morphism from f. Note $\tilde{c}_1(f^*O_X(D))([\mathrm{Id}_Y \colon Y \to Y, L_1, \cdots, L_r]) \in \Omega_{*-1}(Y)$. We define

$$D(\eta) \colon = f^D_* \left\{\tilde{c}_1(f^*O_X(D)) \left([\mathrm{Id}_Y \colon Y \to Y, L_1, \cdots, L_r]\right)\right\} \in \Omega_{*-1}(|D|).$$

(2) If $f(Y) \not\subset |D|$, then $\tilde{D} \colon = \mathrm{Div}\, f^*D$ is a strict normal crossing divisor on Y. Let $f^D \colon |\tilde{D}| \to |D|$ be the restriction of f, and L^D_i be the restriction of L_i on $|\tilde{D}|$. We define $D(\eta) \colon = f^D_* \{\tilde{c}_1(L^D_1) \circ \cdots \circ \tilde{c}_1(L^D_r)([\tilde{D} \to |\tilde{D}|])\} \in \Omega_{*-1}(|D|)$, where the cobordism cycle $[\tilde{D} \to |\tilde{D}|] \in \Omega_*(|\tilde{D}|)$ is discussed in Section 5.1, and with more details in [12, §3.1].

This descends to give $D(-) \colon \Omega_*(X)_D \to \Omega_{*-1}(|D|)$ by [12, §6.2]. To show that it induces (7.2), we need the following:

Lemma 11.6. *Let $X \in \mathbf{Sch}_k$ and let D be a pseudo-divisor on X. Let $[f \colon Y \to X, L]$ and $[f \colon Y \to X, M]$ be two cobordism cycles in $\Omega_*(X)$ such that $L \sim M$. Let $\eta \colon = [Y \to X, L] - [Y \to X, M]$. Then $D \circ \phi_X(\eta) \in \ker(\Omega_{*-1}(|D|) \to \Omega^{\mathrm{alg}}_{*-1}(|D|))$, where $\phi_X = \theta_X^{-1}$ of Theorem 11.4.*

Proof. We first assume that $[f \colon Y \to X, L]$ and $[f \colon Y \to X, M]$ lie in $\Omega_*(X)_D$ and show that $D(\eta) \in \ker(\Omega_{*-1}(|D|) \to \Omega^{\mathrm{alg}}_{*-1}(|D|))$.

By the definition of $D(-)$ in Definition 11.5, if $f(Y) \subset |D|$, then there is nothing to prove. So, suppose $f(Y) \not\subset |D|$. Then, we have

$$D([f \colon Y \to X, L]) = f^D_* \{\tilde{c}_1(L|_{\widetilde{D}})([\widetilde{D} \to |D|])\} \in \Omega_{*-1}(|D|) \tag{11.1}$$

and the similar expression holds for $D([f \colon Y \to X, M])$. On the other hand, $L \sim M$ implies that $L|_{\widetilde{D}} \sim M|_{\widetilde{D}}$ and hence $\tilde{c}_1(L|_{\widetilde{D}}) = \tilde{c}_1(L|_{\widetilde{D}})$ as operators on $\Omega^{\mathrm{alg}}_*(|\widetilde{D}|)$. Using

Proposition 3.13 and applying f_*^D, from (11.1) we get $D([f\colon Y \to X, L]) = D([f\colon Y \to X, M])$ in $\Omega^{\mathrm{alg}}_{*-1}(|D|)$. Equivalently, $D(\eta) \in \ker\Big(\Omega_{*-1}(|D|) \to \Omega^{\mathrm{alg}}_{*-1}(|D|)\Big)$.

To complete the proof we note again that the only case to consider is when $f(Y) \not\subset |D|$. In this case, choose a suitable projective birational map $\rho\colon W \to Y \times \mathbb{P}^1$ as in [12, Lemma 6.4.1]. This yields a commutative diagram

$$\begin{array}{ccc} \Omega_*(W)_{D_W} & \xrightarrow{D_W(-)} & \Omega_{*-1}(|D_W|) \\ {\scriptstyle (f\circ pr_1\circ\rho)_*}\downarrow & & \downarrow{\scriptstyle (f\circ pr_1\circ\rho)_*} \\ \Omega_*(X)_D & \xrightarrow{D(-)} & \Omega_{*-1}(|D|) \end{array} \tag{11.2}$$

where $D_W = \rho^* \circ pr_1^* \circ f^*(D)$ such that

$$D \circ \phi_X([Y \to X, L]) = (f \circ pr_1 \circ \rho)_* \circ D_W\left([\rho^*(Y \times \{0\}) \to W, \rho^*(L)]\right).$$

A similar formula holds for $D \circ \phi_X([Y \to X, M])$. From this the lemma follows by applying what we have shown above to the pair (W, D_W) in place of (X, D). $\square$

Acknowledgments. The authors feel very grateful to Marc Levine for useful comments on various parts of this work. JP would like to thank Juya and Damy for the constant supports at home during the work. He also wishes to acknowledge that part of this work was written during his visits to TIFR and the Universität Duisburg-Essen in 2011. He would like to thank all those institutions that made this work possible.

During this work, JP was partially supported by the National Research Foundation of Korea (NRF) grant (No. 2011-0001182) and Korea Institute for Advanced Study (KIAS) grant, both funded by the Korean government (MEST), and the TJ Park Junior Faculty Fellowship funded by POSCO TJ Park Foundation.

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