Finiteness of solvable automorphisms with null entropy on a compact Kähler manifold

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Abstract

Let G be a solvable subgroup of the automorphism group $\mathrm{Aut}(X)$ of a compact Kähler manifold X of complex dimension n, and let $N(G)$ be the normal subgroup of G consisting of elements with null entropy. Let us denote by G^* the image of G under the natural map from $\mathrm{Aut}(X)$ to $GL(V,\mathbf{R})$, where V is the Dolbeault cohomology group $H^{1,1}(X,\mathbf{R})$. Assume that the Zariski closure of G^* in $GL(V_{\mathbf{C}})$ is connected. The main aim of this paper is to show that, when the rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$ and the identity component of $\mathrm{Aut}(X)$ is trivial, the normal subgroup $N(G)$ of G is finite. This affirmatively answers a question in *Invent. Math.* posed by D.-Q. Zhang.

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1 Introduction and Main Results

Let X be a compact Kähler manifold of complex dimension n, and let us denote by $\mathrm{Aut}(X)$ the biholomorphism (or automorphism) group of X. In this paper we mainly study the structure of solvable subgroups of $\mathrm{Aut}(X)$ which are of null entropy. One motivation for this study comes from the paper [5] of Dinh and Sibony that deals with only abelian subgroups of automorphisms of a compact Kähler manifold. It will be also worth pointing out the fact that the class of solvable subgroups of automorphisms of a compact Kähler manifold is not exhausted by abelian subgroups. Moreover, we mention that recently there have been some great interests in this field (see, e.g., [2], [6] and references therein).

In order to describe our results in more detail, we first need to set up some notations and terminology. Let f be an automorphism of X. The spectral radius

$$\rho(f) = \rho(f^*|_{H^2(X,\mathbf{C})})$$

of the action of f on the cohomology ring $H^2(X,\mathbf{C})$ is defined to be the maximum of the absolute values of eigenvalues on the $\mathbf{C}$-linear extension of $f^*|_{H^2(X,\mathbf{R})}$. We call f *of null entropy* (resp. *of positive entropy*) if the spectral radius $\rho(f)$ is equal to 1 (resp. > 1). It is well-known as in [5] (or [10], Theorem 4.1 (2)) that

$$\rho(f^*|_{H^2(X,\mathbf{C})}) = \rho(f^*|_{H^{1,1}(X,\mathbf{C})}).$$

We say that a subgroup G of automorphisms is *of null entropy* (resp. *of positive entropy*) if all non-trivial elements of G are of null entropy (resp. of positive entropy). It is easy to show from the definition that $\rho(f^{\pm 1})$ is always less than or equal to $\rho(f^{\mp})^{n-1}$ (e.g., see [9]). Similarly, a *dynamical entropy* of f can also be defined to be the logarithm of the maximum of the absolute values of eigenvalues on the $\mathbf{C}$-linear extension of $f^*|_{H^*(X,\mathbf{R})}$. It has been shown in the works [13] and [8] by Yomdin and Gromov that the definition of dynamical entropy as above is, in fact, equivalent to that of topological entropy.

Now let G^* denote the image of a subgroup G of $\mathrm{Aut}(X)$ under the natural map from $\mathrm{Aut}(X)$ to $GL(V,\mathbf{R})$, where V is the Dolbeault cohomology group $H^{1,1}(X,\mathbf{R})$. Then clearly G^* is a subgroup of $GL(V,\mathbf{R})$. We call a group *virtually solvable* if it has a solvable subgroup of finite index. Let $V_{\mathbf{C}}$ be the complexification of V so that $V_{\mathbf{C}}$ is now a finite dimensional vector space over $\mathbf{C}$. Then G^* can be regarded as a subgroup of $GL(V_{\mathbf{C}})$ in a natural way, and a solvable subgroup G^* of $GL(V_{\mathbf{C}})$ is called *Zariski-connected*, or simply *Z-connected*, if its Zariski closure $\bar{G}^*$ in $GL(V_{\mathbf{C}})$ is connected. One important point to note here is that G^* itself may not be connected, in general. From now on, for the sake of simplicity, we will say that G is *Zariski-connected* or simply *Z-connected* if G^* is Zariski-connected.

It is easy to see that given a virtually solvable subgroup G^* of $GL(V_{\mathbf{C}})$, one can find a Zariski-connected solvable finite-index subgroup G_1 of G^*. Moreover, any subgroup of a solvable group G^* and any quotient group of G^* are also solvable, and the closure of G^* is solvable as well (see Section 2 of [10]).

Next, let

$$N(G) = \{f \in G \,|\, f \text{ is of null entropy}\}.$$

Then it is easy to see that $N(G)$ is a normal subgroup of G, when G is a Zariski-connected solvable subgroup of the automorphism group $\mathrm{Aut}(X)$ (see, e.g., Section 2 or [10]). Moreover, in this case it has been proved that $G/N(G)$ is a finitely generated abelian group of rank at most $n-1$ (see [14]).

In their paper [5], Dinh and Sibony proved that if, in addition, G is abelian and the rank $r(G)$ is equal to $n-1$, then $N(G)$ is finite ([5], Proposition 4.7). Let $\mathrm{Aut}_0(X)$ denote the identity component of $\mathrm{Aut}(X)$ consisting of automorphisms homotopically equivalent to the identity. In a recent paper [16], Zhang also investigated a question of finiteness of $N(G)$ for Zariski-connected solvable subgroups G of $\mathrm{Aut}(X)$ for a minimal complex projective threefold X. As a consequence, he proved that if $r(G) = n-1 = 2$ and $\mathrm{Aut}_0(X)$ is trivial, then $N(G)$ is finite ([16], Theorem 1.1 (3)).

The primary aim of this paper is to extend the result of Zhang, which holds only for minimal complex projective threefolds, to any compact Kähler manifolds with complex dimension n at least 3. To be precise, our main result of this paper which affirmatively answers Question 2.17 in [14] is

Theorem 1.1. *Let X be a compact Kähler manifold of complex dimension $n \geq 3$, and let G be a Zariski-connected solvable subgroup of the automorphism group* $\mathrm{Aut}(X)$. *Assume that the rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$ and that* $\mathrm{Aut}_0(X)$ *is trivial. Then $N(G)$ is finite.*

It is quite interesting to notice that, if G is further assumed to be abelian in Theorem 1.1, then $N(G)$ is finite even without the assumption of the triviality of $\mathrm{Aut}_0(X)$ (see also [11], Proposition 2.2).

In order to explain that the result of this paper is quite sharp, we then recall Example 4.5 in [5]. To do so, let $\mathbf{T} = \mathbf{C}/(\mathbf{Z} + \sqrt{-1}\mathbf{Z})$ and $X = \mathbf{T}^n$. Then X is an abelian variety of complex dimension n. Let $\mathrm{SL}(n, \mathbf{Z})$ denote the group of $n \times n$-integer matrices whose determinant is equal to 1. Then $\mathrm{SL}(n, \mathbf{Z})$ acts diagonally on X, and it contains an abelian subgroup G_1 of rank $n-1$ whose non-trivial elements are diagonalizable over real numbers and are also of positive entropy. Now let G be the group generated by G_1 and $\mathrm{Aut}_0(X)$. Then G is a Zariski-connected solvable subgroup of $\mathrm{Aut}(X)$ which acts on the abelian variety X of complex dimension n such that $N(G) = \mathrm{Aut}_0(X) \cong X$ and the rank $r(G) = n-1$ (see also [16], Remark 1.3 (1)). So the triviality of $\mathrm{Aut}_0(X)$ cannot be dropped in Theorem 1.1, in general.

This example also shows that our result in this paper is somewhat optimal in several other respects. For example, by simply taking an abelian subgroup of rank $n-2$ of G_1, we can give an example whose $G/N(G)$ has rank $n-2$, but $N(G)$ is not finite. Moreover, note that a recent result of B. Fu and D.-Q. Zhang in [6] shows that the Zariski-connectedness assumption in Theorem 1.1 is really necessary, since otherwise the Kähler manifold X is essentially a complex torus under some mild assumptions and so the question of finiteness of $N(G)$ becomes significantly trivial, as the preceding paragraph clearly shows (see [6] for more details).

A class c in $H^{1,1}(X; \mathbf{R})$ is called a *Kähler class* if it is a class of a Kähler form, and is called *numerically effective*, or simply *nef*, if it lies in the closure of the set of Kähler classes. On the other hand, a class c in $H^{1,1}(X; \mathbf{R})$ is called *nef and big* if it is nef and $\int_X c^n > 0$.

Let $\mathrm{Aut}_c(X)$ denote the automorphism group preserving a class $c \in H^{1,1}(X; \mathbf{R})$. If, in addition, the class c is a Kähler class, then it is well-known that the quotient group $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is finite by a theorem of Lieberman and independently Fujiki ([11], Proposition 2.2 or [7], Theorem 4.8). For a compact complex projective manifold X of dimension n, by using the arguments of Lieberman in [11] Zhang proved in [15], Lemma 2.23 that $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is finite even for a nef and big class c (see also the proof of Theorem 4.6 in the published version of Dinh and Sibony's paper [5] for a certain other case). As far as we know, however, it appears to be still unknown whether $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is finite for a nef and big class c on a general compact Kähler manifold X.

In view of this fact, the following theorem is interesting in its own right, and plays an important role in finishing the proof of our main Theorem 1.1.

Theorem 1.2. *Let X be a compact Kähler manifold of complex dimension $n \geq 3$, and G a Zariski-connected solvable subgroup of* $\mathrm{Aut}(X)$ *whose rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$. For a nef and big, but not necessarily Kähler, class $c \in H^{1,1}(X, \mathbf{R})$, the quotient group* $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ *is finite.*

In Section 3, we give a proof of Theorem 1.2 (see Theorem 3.8).

Finally, we close this section with a refined statement of Theorem 1.1. To do so, note that $N(G)$ can be shown to be a subgroup of $\mathrm{Aut}_c(X)$ for a nef and big class c, as in Section 3, and that $N(G) \cap \mathrm{Aut}_0(X)$ is clearly a normal subgroup of $N(G)$. So we have the following corollary:

Corollary 1.3. *Let X be a compact Kähler manifold of complex dimension $n \geq 3$, and let G be a Zariski-connected solvable subgroup of the automorphism group* $\mathrm{Aut}(X)$. *Assume that the rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$. Then $N(G)/N(G) \cap \mathrm{Aut}_0(X)$ is always a finite group.*

We organize this paper as follows. In Section 2, we first collect some basic facts which are relevant to the proof of Theorem 1.1. In this section, we construct a homomorphism from a solvable subgroup of automorphisms to the abelian group $(\mathbf{R}^m, +)$. Here one of the key technical ingredients is a theorem of Lie-Kolchin type in [10] (Theorem 2.3 of Section 2). In Section 3, we finally give proofs of our main Theorem 1.1 and its Corollary 1.3 as well as Theorem 1.2.

Form now on, we always assume that the complex dimension of the manifold X is at least 3, unless stated otherwise.

2 Theorem of Lie-Kolchin type and its Applications

The goal of this section is to set up some preliminary results necessary for the proof of our main Theorem 1.1. As mentioned earlier, one of the key ingredients for the proof is a theorem of Lie-Kolchin type established in [10].

Let V be a finite dimensional real vector space and let $V_{\mathbf{C}}$ be its complexification. For a solvable group G, let $\psi : G \to GL(V_{\mathbf{C}})$ be a complex linear representation of G. Then we take the Zariski closure, denoted by Z, of $\psi(G)$ in $GL(V_{\mathbf{C}})$. Let Z_0 be the connected component of the identity in Z and let $G_0 = \psi^{-1}(Z_0)$. Since G is a solvable group, the group Z_0 is conjugate to a group of upper triangular matrices whose determinant is non-zero. Let $N(G_0)$ be the subgroup of G_0 whose elements are defined by the statement that f is an element of $N(G_0)$ if and only if the absolute values of all eigenvalues of f on $V_{\mathbf{C}}$ are equal to 1. Now observe that $N(G_0)$ is a normal subgroup of G_0 and that the abelian group $G_0/N(G_0)$ embeds into $(\mathbf{C}^*)^{\dim V_{\mathbf{C}}}$. Hence the rank of an abelian group $G_0/N(G_0)$ should be finite and, moreover, bounded from above by $\dim V_{\mathbf{C}}$. From now on, we shall denote by G such a group G_0.

Then we will need the following lemma whose proof is simple (e.g., see [10], Lemma 2.5).

Lemma 2.1. *Let Z_0 be a Zariski-connected solvable subgroup of $GL(V_{\mathbf{C}})$. Then the eigenvalues of every element of the commutator subgroup $[Z_0, Z_0]$ of Z_0 are all equal to* 1.

Since G is solvable, there exists a derived series of G, as follows.

$$G = G^{(0)} \triangleright G^{(1)} \triangleright G^{(2)} \triangleright \ \cdots \ \triangleright G^{(k)} \triangleright G^{(k+1)} = \{\mathrm{id}\},$$

where $G^{(i+1)}$ is a normal subgroup of $G^{(i)}$ and $G^{(i+1)}$ is the commutator subgroup $[G^{(i)}, G^{(i)}]$ of $G^{(i)}$ $(0 \le i \le k)$. Let $A = G^{(k)}$. Then A is an abelian subgroup of G, and clearly A is a subset of $[G,G]$. Thus, by Lemma 2.1, every element of A has all the eigenvalues equal to 1.

Recall that if C is a subset of a real vector space V, then C is said to be a *strictly convex closed cone* of V if C is closed in V, closed under addition and multiplication by a non-negative scalar, and contains no 1-dimensional linear space.

We also recall the following theorem of Birkhoff-Perron-Frobenius in [1] which plays an important role in the proof of Theorem 2.3.

Theorem 2.2. *Let C be a non-trivial strictly convex closed cone of V with non-empty interior in V. Then any element f of $GL(V)$ such that $f(C) \subset C$ has an eigenvector v_f in C whose eigenvalue is the spectral radius $\rho(f)$ of f in V.*

In fact, if we use the subgroup A of $[G,G]$ and Lemma 2.1 together with Theorem 2.2, we obtain a stronger version for Zariski-connected solvable subgroups of $GL(V)$ as in [10] which is called the *theorem of Lie-Kolchin type for a cone*. For more precise statement of Theorem 2.3, see Theorem 2.1 in [10].

Theorem 2.3. *Let V be a finite dimensional real vector space, and let $C \neq \{0\}$ be a strictly convex closed cone of V. Let G be a Zariski-connected solvable subgroup of $GL(V)$ such that $G(C) \subset C$. Then there exists a nonzero vector in $C \backslash \{0\}$ which spans a one-dimensional subcone of C invariant under G.*

From now on, let X be a compact connected Kähler manifold of complex dimension n as before, and let V denote the Dolbeault cohomology group $H^{1,1}(X, \mathbf{R})$. In this paper, we will apply the above general discussion to a solvable subgroup G of $\mathrm{Aut}(X)$ acting on $V_{\mathbf{C}} = V \otimes \mathbf{C} = H^{1,1}(X, \mathbf{R}) \otimes \mathbf{C}$. Since every element of $[G,G]$ has all the eigenvalues equal to 1 (i.e., every element of $[G,G]$ is *unipotent*) and A is a subset of $[G,G]$, every element of A also has all the eigenvalues equal to 1. In other words, this says that every element of A is of null entropy.

Now let $\mathscr{K}$ denote the Kähler cone in the Dolbeault cohomology group $H^{1,1}(X, \mathbf{R})$. Then $\mathscr{K}$ is the cone of strictly positive smooth $(1,1)$-forms in $H^{1,1}(X, \mathbf{R})$, and it is a strictly convex open cone in $H^{1,1}(X, \mathbf{R})$ whose closure $\bar{\mathscr{K}}$ is also a strictly convex closed cone such that $\bar{\mathscr{K}} \cap -\bar{\mathscr{K}} = \{0\}$.

With these understood, we have the following lemma which is an immediate consequence of Theorem 2.3 and its proof in [10], but will play an important role in the proof of Theorem 1.1.

Lemma 2.4. *Let G be a Zariski-connected solvable group of automorphisms of a compact Kähler manifold, and let f_0 be an element of G. Then the following properties hold:*

(a) *For all $f \in G$, there exist a non-zero class c_{f_0} in $\bar{\mathscr{K}}$ and a positive real number $\chi(f) \leq \rho(f)$ such that*

$$f^*(c_{f_0}) = \chi(f)c_{f_0},$$

(that is, c_{f_0} spans a one-dimensional subcone of $\bar{\mathscr{K}}$ invariant under G) and such that $\chi(f_0)$ is greater than or equal to 1.

(b) *If, in addition, $f \in G$ is of null entropy, then $\chi(f)$ is exactly equal to* 1.

Remark 2.5. This corollary is a generalization of Lemma 4.1 in [5]. That is, if G is abelian and f_0 is of positive entropy, then the statement holds to be true with $\chi(f_0)$ replaced by the spectral radius $\rho(f_0)$ of f_0 greater than 1 and also with $\chi(f)$ replaced by $\rho(f)$.

Proof. To prove (a), we simply take $C := \bar{\mathscr{K}}$ in order to apply Theorem 2.3. Then it follows from Theorem 2.3 of Lie-Kolchin type (or [10], Theorem 2.1 and its proof) that there exists a non-zero eigenvector $c_{f_0} \in C = \bar{\mathscr{K}}$ for G which spans a one-dimensional subcone of C invariant under all of G. That is, we have $f^*(c_{f_0}) = \chi(f)c_{f_0}$ with a positive real number $\chi(f) \leq \rho(f)$. Moreover, the proof of Theorem 2.3 (or [10], Theorem 2.1) actually shows that given an element $f_0 \in G$, $\chi(f_0)$ is always taken to be a positive real number greater than or equal to 1. This completes the proof of Lemma 2.4 (a).

For the proof of (b), note first that if f is of null entropy, then so is f^{-1}. Hence, if $\chi(f)$ is less than 1, it follows from $(f^{-1})^*(c_{f_0}) = \chi(f)^{-1}c_{f_0}$ that f^{-1} cannot be of null entropy. This contradicts the choice of f. Note also that, again by the choice of f, $\chi(f)$ cannot be greater than 1. This completes the proof of Lemma 2.4 (b). □

To give a proof of Theorem 1.1, we need one more notation.

Definition 2.6. Let $\tau = (\tau(f))_{f \in G} \in \mathbf{R}^G$, and let Γ_τ be the cone of classes c in $\bar{\mathscr{K}}$ such that

$$f^*(c) = \exp(\tau(f))c$$

for all $f \in G$.

Then we set

$$F = \{\tau \in \mathbf{R}^G \,|\, \Gamma_\tau \neq \{0\}\}.$$

If $\Gamma_\tau \neq \{0\}$, then $\exp(\tau(f))$ is an eigenvalue of f^* on $V = H^{1,1}(X, \mathbf{R})$. Note that the maps $\tau : G \to \mathbf{R}$ in the set F are continuous with respect to the standard topologies. So, since V is finite dimensional and thus the number of eigenvalues of f^* are finite, F must be also finite.

Now, let $\tau_1, \tau_2, \cdots, \tau_m$ be all the elements of the finite set F. We then define a map $\pi : G \to \mathbf{R}^m$ given by

$$\pi : G \to \mathbf{R}^m, \quad f \mapsto (\tau_1(f), \tau_2(f), \ldots, \tau_m(f)). \tag{2.1}$$

It is not difficult to show that the following lemma holds:

Lemma 2.7. (a) *The integer m satisfies the inequality*

$$m \le h_1 := \dim H^{1,1}(X, \mathbf{R}).$$

(b) *The map π is always a homomorphism into the abelian group $(\mathbf{R}^m, +)$. In particular, the image $\pi(G)$ is also abelian.*

Proof. For the proof of (a), since $\tau_1, \tau_2, \cdots, \tau_m$ are all distinct, there exists an element $f_0 \in G$ such that $\tau_i(f_0) \neq \tau_j(f_0)$ for $1 \le i < j \le m$. Thus f_0^* on V has at most m distinct eigenvalues. Since the number of eigenvalues is clearly less than or equal to the dimension of V and the maps $\tau : G \to \mathbf{R}$ in the set F are continuous, m should be less than or equal to the dimension of V that is equal to h_1 in our case. This completes the proof of (a).

For the proof of (b), it suffices to prove that π is a homomorphism. To do so, for each $c_i \in \Gamma_{\tau_i}$ observe first that

$$(f \circ g)^* c_i = \exp(\tau_i(f) + \tau_i(g)) c_i.$$

Hence we have $\pi(f \circ g) = \pi(f) + \pi(g)$ for all f and g in G, which means that π is a group homomorphism. This completes the proof of Lemma 2.7. □

3 Proofs of Theorems 1.1 and 1.2

The aim of this section is to give proofs of Theorems 1.1 and 1.2 as well as Corollary 1.3.

To do so, we first need the following key technical Lemma 3.1 below (or see [5], Lemma 4.3). Assume that X is a compact Kähler manifold of dimension n, as before.

Lemma 3.1. *Let c, c', and c_i be the non-zero classes in $\bar{\mathscr{K}}$ with $1 \le i \le t \le n-2$ for some integer t, and let $f \in \mathrm{Aut}(X)$. Assume that there exist two distinct positive real constants λ and λ' such that*

$$f^*(c_1 \wedge \cdots \wedge c_t \wedge c) = \lambda c_1 \wedge \cdots \wedge c_t \wedge c,$$
$$f^*(c_1 \wedge \cdots \wedge c_t \wedge c') = \lambda' c_1 \wedge \cdots \wedge c_t \wedge c'.$$

Assume also that $c_1 \wedge \cdots \wedge c_t \wedge c \neq 0$ and $c_1 \wedge \cdots \wedge c_t \wedge c \wedge c' = 0$. Then we have

$$c_1 \wedge \cdots \wedge c_t \wedge c' = 0.$$

Note that Lemma 3.1 will play an essential role in the proofs of Lemma 3.2 and Theorem 1.1. For their proofs, we adapt a variation of some arguments originated from [5] (see also [14]).

Lemma 3.2. *Let $\tilde{r}$ denote the rank of the image of the homomorphism π defined in (2.1). Then there are non-zero classes c_j $(j = 1, 2, \cdots, \tilde{r})$ in $\bar{\mathscr{K}}$ such that*

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{\tilde{r}} \neq 0, \text{ and } f^*(c_j) = \exp(\tau_j(f)) c_j \text{ for all } f \in G,$$

where $\tau_j : G \to \mathbf{R}$ is a homomorphism.

Remark 3.3. By the way of construction, it is obvious that each subcone of V spanned by a non-zero class c_j $(1 \le j \le \tilde{r})$ in $\bar{\mathcal{K}}$ in the statement of Lemma 3.2 is not necessarily invariant under G. However, it follows from Lemma 2.4 (b) that all of $c_1, c_2, \cdots$, and $c_{\tilde{r}}$ are invariant under $N(G)$. That is, all of

$$\tau_1(f), \cdots, \tau_{\tilde{r}}(f)$$

are zero for all $f \in N(G)$. This fact will play a crucial role later. In particular, it enables us to prove Theorem 1.1.

Proof of Lemma 3.2. To prove it, we assume without loss of generality that the first $\tilde{r}$ coordinates of the map π generate the image of the map π. Let us denote by $\tau_1, \cdots, \tau_{\tilde{r}}$ such $\tilde{r}$ coordinates. Let c_i be a non-zero class in Γ_{c_i} $(1 \le i \le \tilde{r})$. Then for any $I = \{i_1, \cdots, i_k\} \subset \{1, 2, \cdots, \tilde{r}\}$, set

$$c_I = c_{i_1} \wedge \cdots \wedge c_{i_{k-1}} \wedge c_{i_k}.$$

For the proof, it suffices to show that $c_I \neq 0$ for any subset I of $\{1, 2, \cdots, \tilde{r}\}$, and we prove it only for the case of $I = \{1, 2, \cdots, k\}$, since other cases are similar. Indeed, if $k = 1$, by construction we have $c_1 \neq 0$ and so there is nothing to prove. For $k \ge 2$, we can prove that $c_1 \wedge c_2 \wedge \cdots \wedge c_k \neq 0$ by contradiction. So suppose that

$$c_1 \wedge \cdots \wedge c_{k-2} \wedge c_{k-1} \neq 0, \quad c_1 \wedge \cdots \wedge c_{k-2} \wedge c_k \neq 0, \text{ and}$$
$$c_1 \wedge \cdots \wedge c_{k-1} \wedge c_k = 0.$$

We then apply Lemma 3.1 for $t = k-2$, $c = c_{k-1}$, and $c' = c_k$. As a consequence, it is easy to see that $\tau_{k-1}(f) = \tau_k(f)$ for all $f \in G$ (or see the proof of Lemma 3.5 below for a similar argument). But, this implies that the image of π lies in the hyperplane $\{x_{k-1} = x_k\}$, which contradicts the choice of τ_i's. This completes the proof of Lemma 3.2. □

Observe that the arguments in the proof of Lemma 3.2 show that the rank $\tilde{r}$ cannot be strictly greater than $n-1$, due to the obvious dimensional reason (see also [14]).

Definition 3.4. Let $\tilde{r}$ denote the rank of the image of the homomorphism π. From now on, we assume without loss of generality that the first $\tilde{r}$ coordinates of the map π generate the image of the map π. So, in case of $\tilde{r} = n-1$, a homomorphism $\pi : G \to \mathbf{R}^{n-1}$ is just defined to be the homomorphism π in (2.1) composed by the canonical projection onto the first $n-1$ factors.

Now, we are ready to give a proof of Theorem 1.1.

Proof of Theorem 1.1. To begin with the proof, by hypothesis assume first that the rank $r(G)$ of $G/N(G)$ is equal to $n-1$. Then the proof consists of a series of lemmas, as follows.

Lemma 3.5. *There exist non-zero classes $c_1, \cdots, c_n$ in the closure $\bar{\mathcal{K}}$ of the Kähler cone $\mathcal{K}$ such that*

- *$f^* c_i = c_i$ for all $1 \le i \le n$ and all $f \in N(G)$,*

- $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \neq 0$.

Proof. Let $c_1, c_2, \cdots, c_{n-2}$, and c_{n-1} be the classes given by Lemma 3.2. Then, by Remark 3.3, we have $f^* c_i = c_i$ for all $1 \leq i \leq n-1$ and all $f \in N(G)$. Since the rank $r(G)$ is assumed to be $n-1$ and the kernel of π is same as $N(G)$ (see, e.g., Claim 2.8 of [16]), the homomorphism $\pi : G \to \mathbf{R}^{n-1}$ induces an isomorphism from $G/N(G)$ to $\mathbf{R}^{n-1}$. This, in particular, implies that its image spans $\mathbf{R}^{n-1}$, so there exists an element $f_0 \in G$ such that all of $\tau_j(f_0)$'s ($1 \leq j \leq n-1$) are negative.

Applying Theorem 2.3 of Lie-Kolchin type for a cone (or Lemma 2.4 (a)) to the closure $\bar{\mathscr{K}}$ of the Kähler cone, we see that there exists another non-zero class $c_n \in \bar{\mathscr{K}}$ such that c_n spans a one-dimensional subcone of $\bar{\mathscr{K}}$ invariant under G and such that

$$f_0^*(c_n) = \exp(\tau_n(f_0))c_n$$

for a non-negative real number $\tau_n(f_0)$. In particular, this implies that $\tau_j(f_0)$ is not equal to $\tau_n(f_0)$ for all $1 \leq j \leq n-1$ and, by Lemma 2.4 (b), $f^*(c_n) = c_n$ for all $f \in N(G)$. Recall from Lemma 3.2 that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}$ is a non-zero class.

Next we claim that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_n$ is also a non-zero class. To see it, suppose first that $c_j \wedge c_n = 0$ for $1 \leq j \leq n-1$. Then, by applying Lemma 3.1 for f_0 with $c = c_j$ and $c' = c_n$ for $1 \leq j \leq n-1$, we would have $0 > \tau_j(f_0) = \tau_n(f_0) \geq 0$, since both c_j and c_n are non-zero classes and satisfy

$$f_0^*(c_j) = \exp(\tau_j(f_0))c_j, \quad f_0^*(c_n) = \exp(\tau_n(f_0))c_n.$$

But this is a contradiction. Thus all of $c_j \wedge c_n$ are non-zero for $1 \leq j \leq n-1$.

Using this fact, we can show further that this time $c_1 \wedge c_2 \wedge c_n$ is nonzero. For this, note first that we have

$$\begin{aligned} f_0^*(c_1 \wedge c_2) &= \exp(\tau_1(f_0))\exp(\tau_2(f_0))c_1 \wedge c_2, \\ f_0^*(c_1 \wedge c_n) &= \exp(\tau_1(f_0))\exp(\tau_n(f_0))c_1 \wedge c_n. \end{aligned} \tag{3.1}$$

Suppose then that $c_1 \wedge c_2 \wedge c_n$ is zero. By applying Lemma 3.1 to the equations in (3.1) for f_0 with $t = 1$, $c = c_2$ and $c' = c_n$ once again, it is easy to obtain

$$\exp(\tau_1(f_0))\exp(\tau_2(f_0)) = \exp(\tau_1(f_0))\exp(\tau_n(f_0)).$$

That is, $\exp(\tau_2(f_0)) = \exp(\tau_n(f_0))$. This implies that $0 > \tau_2(f_0) = \tau_n(f_0) \geq 0$, which is a contradiction. By inductive arguments, it is now clear that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_n$ can be shown to be a non-zero class.

Similarly, we can also show that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \neq 0$. To be precise, suppose, on the contrary, that $c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n = 0$. Then we have

$$\begin{aligned} &f_0^*(c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}) \\ &= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0))\exp(\tau_{n-1}(f_0))c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_{n-1}, \\ &f_0^*(c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_n) \\ &= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0))\exp(\tau_n(f_0))c_1 \wedge c_2 \wedge \cdots \wedge c_{n-2} \wedge c_n. \end{aligned} \tag{3.2}$$

As above, if we apply Lemma 3.1 to the equations in (3.2) for f_0 with $t = n-2$, $c = c_{n-1}$ and $c' = c_n$, then we obtain

$$\exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_{n-1}(f_0))$$
$$= \exp(\tau_1(f_0)) \cdots \exp(\tau_{n-2}(f_0)) \exp(\tau_n(f_0)).$$

Thus we have $0 > \tau_{n-1}(f_0) = \tau_n(f_0) \geq 0$, which is again a contradiction. Therefore we have

$$c_1 \wedge c_2 \wedge \cdots \wedge c_{n-1} \wedge c_n \neq 0,$$

as desired. This completes the proof of Lemma 3.5. □

Lemma 3.6. *Let*

$$c = c_1 + c_2 + \cdots + c_{n-1} + c_n.$$

Then c is actually a nef and big class which is invariant under $N(G)$.

Proof. To see it, note first that clearly $f^*(c) = c$ for all f in $N(G)$, but not in the whole of G. Since $c_1 \wedge c_2 \wedge \cdots \wedge c_n \neq 0$, it is also clear that $c^n \neq 0$. Moreover, it follows from its construction that c lies in $\bar{\mathscr{K}}$. Hence c is a nef and big class, as desired. □

Finally, in order to finish the proof of Theorem 1.1, as before let $\mathrm{Aut}_c(X)$ denote the automorphism group preserving the nef and big class c given in Lemma 3.6. Then, by the way of construction of all c_i's and c, $N(G)$ is a subgroup of $\mathrm{Aut}_c(X)$. If c is a Kähler class and also $\mathrm{Aut}_0(X)$ is trivial, then $N(G)$ is always finite by a theorem of Lieberman and independently Fujiki mentioned in Section 1, as desired.

On the other hand, if c is not a Kähler class, we first need to recall the following theorem of Demailly and Paun ([4], Theorem 0.1 and [3], Theorem 2.2).

Theorem 3.7. *Let X be a compact Kähler manifold, and $c \in H^{1,1}(X, \mathbf{R})$ a nef and big class which is not a Kähler class. Then the following holds:*

(a) *There exists an irreducible analytic subset $Y \subset X$ of positive dimension such that $\int_Y c^{\dim Y} = 0$.*

(b) *The union of all irreducible analytic subsets Y in* (a) *above is a proper Zariski closed subset $Z \subset X$.*

We then prove a version of the result of Liberman and Fujiki for a nef and big class, as follows (see Theorem 1.2). For the sake of reader's convenience, we recall the statement of the theorem.

Theorem 3.8. *Let X be a compact Kähler manifold of complex dimension $n \geq 3$, and G a Zariski-connected solvable subgroup of* $\mathrm{Aut}(X)$ *whose rank $r(G)$ of the quotient group $G/N(G)$ is equal to $n-1$. For a nef and big, but not necessarily Kähler, class $c \in H^{1,1}(X, \mathbf{R})$, the quotient group* $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ *is still finite.*

Proof. To prove it, note first that the union Z of all irreducible analytic subsets Y of maximal dimension as in Theorem 3.7 (b) is invariant under the action of the quotient group $G/N(G)$. Moreover, since X is compact, the number of all irreducible analytic subsets Y of maximal dimension in Z is finite. Thus, by taking a finite index subgroup of $G/N(G)$, if necessary, we may assume without loss of generality that each irreducible analytic subset Y as above is invariant under $G/N(G)$.

Since $G/N(G)$ is a finite generated abelian group of rank $n-1$ isomorphic to $\mathbf{R}^{n-1}$, clearly it contains a lattice Γ of rank $n-1 \geq 2$ acting faithfully on X. Note also that the rank $n-1$ of Γ is exactly equal to the dimension of X minus one and that, by the very choice of Γ, any non-trivial element of Γ as an element of $\mathrm{Aut}(X)$ is never virtually contained in $\mathrm{Aut}_0(X)$. Now, by applying Theorem 4.5 in the paper [3] (or Main Theorem on page 449 of [3]) to the proper Zariski closed subset Z as in Theorem 3.7 (b), it is easy to see that the proper Zariski closed subset Z should be the union of a finite number of disjoint copies of the complex projective space $Z_i = \mathbf{CP}^{n-1}$. This, in particular, implies that there is no irreducible analytic subset of maximal dimension less than $n-1$ in Z. Moreover, the normal bundle of Z_i is just $O_{Z_i}(-k_i)$ for some integer $k_i > 0$, and, by the rigidity theorem of Grauert, the neighborhood of an exceptional divisor is isomorphic to the normal bundle of Y. Thus we can consider a birational morphism $p : X \to X_0$ onto a compact Kähler orbifold X_0 by blowing down all irreducible analytic subsets of Z to points of X_0. Then the image of the nef and and big class c in X_0, denoted by the letter c_0, is now an orbifold Kähler class (or just an ample class). Note that each automorphism f of X induces an automorphism f_0 of X_0, and this correspondence is injective, since all automorphisms of X are holomorphic (so bi-holomorphic) and Z is a proper Zariski closed subset of X by Theorem 3.7 (b).

It is straightforward to check that the result of Liberman and Fujiki saying that the quotient group $\mathrm{Aut}_{c'}(X)/\mathrm{Aut}_0(X)$ is finite for any Kähler class c' on a compact Kähler manifold X still holds to be true for any orbifold Kähler class on a compact orbifold manifold. Hence the quotient group $\mathrm{Aut}_{c_0}(X_0)/\mathrm{Aut}_0(X_0)$ is finite. This implies that $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is also a finite group, as asserted. □

As a consequence of Theorem 3.8, the arguments used to show that $N(G)$ is finite when c is a Kähler class can be applied even for a nef and big class c. Therefore we can conclude that if, in addition, $\mathrm{Aut}_0(X)$ is trivial, $N(G)$ is always finite. This completes the proof of Theorem 1.1. □

We are finally ready to prove Corollary 1.3.

Proof of Corollary 1.3. Recall that $N(G)$ is a subgroup of $\mathrm{Aut}_c(X)$ for the choice of the nef and big class c. So clearly $N(G)/N(G)\cap\mathrm{Aut}_0(X)$ is a subgroup of $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$. Since the quotient group $\mathrm{Aut}_c(X)/\mathrm{Aut}_0(X)$ is finite by Theorem 1.2, the corollary easily follows. □

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