GENERATORS OF FUNCTION FIELDS OF THE MODULAR CURVES $X_1(5)$ AND $X_1(6)$

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Abstract. We show that the modular functions $j_{1,5}$ and $j_{1,6}$ generate function fields of the modular curves $X_1(N)$ ($N = 5, 6$ respectively) and find some number theoretic properties of these modular functions.

1. Introduction

Let $\mathfrak{H}$ be the complex upper half plane and let $\Gamma_1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N$ ($N = 1, 2, 3, \dots$). Since the group $\Gamma_1(N)$ acts on $\mathfrak{H}$ by linear fractional transformations, we get the modular curve $X_1(N) = \Gamma_1(N)\backslash\mathfrak{H}^*$, as the projective closure of smooth affine curve $\Gamma_1(N)\backslash\mathfrak{H}$, with genus $g_{1,N}$.

Let $r \in \mathbb{Z}$ and $r \not\equiv 0 \mod N$. For $z \in \mathfrak{H}$, Ishii ([7]) found a family of modular functions $X_r(z)$ defined by

$$X_r(z) = \exp\left(2\pi i \frac{-(r-1)(N-1)}{4N}\right) \prod_{s=0}^{N-1} \frac{K_{r,s}(z)}{K_{1,s}(z)},$$

where $K_{u,v}(z)$ are Klein forms of level N. For the Klein forms we refer to Kubert and Lang [14]. For $\zeta_N = e^{2\pi i/N}$, let $\mathfrak{F}_N$ be the field of modular functions for the principal congruence group $\Gamma(N)$ with $\mathbb{Q}(\zeta_N)$-rational Fourier coefficients at the cusp $i\infty$. Then $X_r(z) \in \mathfrak{F}_N$ (resp. $X_r(z)^{\varepsilon_N} \in \mathfrak{F}_N$) if r is odd (resp. if r is even), where ε_N is 1 or 2 according as N is odd or even. When $N \geq 7$, by utilizing such modular functions, Ishida and Ishii showed in [8] that $X_2(z)^{\varepsilon_N N}, X_3(z)^N$ are generators of function fields of the modular curves $X_1(N)$. As for the cases $N = 1, 2, 3$ we know that the elliptic modular function $j(z)$ ($N = 1$), and the Thompson series of type $2B$ ($N = 2$, Table 3 in [2]) and the Thompson series of type $3B$ ($N = 3$, Table 3 in [2]) are generators, respectively because $\overline{\Gamma}_1(2)$=$\overline{\Gamma}_0(2)$ and $\overline{\Gamma}_1(3)$=$\overline{\Gamma}_0(3)$. In the case $N = 4$, we refer to [10]. Thus, in order to find the rest two cases $N = 5, 6$ we use the following general

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fact. Since $g_{1,N} = 0$ only for the eleven cases $1 \leq N \leq 10$ and $N = 12$ ([9]), the function field $\mathbb{C}(X_1(N))$ of the curve $X_1(N)$ is a rational function field over $\mathbb{C}$ for such N.

In this article we shall find the field generators $j_{1,5}$ and $j_{1,6}$ as uniformizers of the modular curves $X_1(N)$ when $N = 5$ and 6, respectively. In §3 $j_{1,5}$ is constructed by making use of the Dedekind eta functions and Eisenstein series of weight 2. And in §4 we build up $j_{1,6}$ from the Eisenstein series of weight 2. In §5 we estimate the normalized generators (or hauptmodulus) $N(j_{1,5})$ and $N(j_{1,6})$. And, when $z \in \mathfrak{H} \cap \mathbb{Q}(\sqrt{-d})$ for a square free positive integer d, we show that $N(j_{1,N})(z)$ $(N = 5, 6)$ becomes an algebraic integer. In §6 we show that the hauptmodulus $N(j_{1,5})$ has integral Fourier coefficients. Lastly, in §7 we find certain connection between hauptmodulus $N(j_{1,N})$ and the parameter t emerging from the moduli problem of elliptic curves.

Throughout the article we adopt the following notations:

$\mathfrak{H}^*$ the extended complex upper half plane

Γ a congruence subgroup of $SL_2(\mathbb{Z})$

$\Gamma(N) = \{\gamma \in SL_2(\mathbb{Z}) |\ \gamma \equiv I \mod N\}$

$\Gamma_0(N)$ the Hecke subgroup $\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) |\ c \equiv 0 \mod N\}$

$X(\Gamma) = \Gamma \backslash \mathfrak{H}^*$

$X(N) = \Gamma(N) \backslash \mathfrak{H}^*$

$X_0(N) = \Gamma_0(N) \backslash \mathfrak{H}^*$

$\mathbb{C}(X(\Gamma))$ function field of the curve $X(\Gamma)$

$\overline{\Gamma}$ the inhomogeneous group of $\Gamma (= \Gamma / \pm I)$

$\sigma_1(n) = \sum_{\substack{d|n \\ d>0}} d$ the sum of positive divisors of n

$q_h = e^{2\pi i z/h},\ z \in \mathfrak{H}$

$f|_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z)$

$f|_{[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)]_k} = (ad - bc)^{\frac{k}{2}} \cdot f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z) \cdot (cz + d)^{-k}$

$M_k(\Gamma)$ the space of modular forms of weight k with respect to the group Γ

$M_k(\Gamma_0(N),\chi) = \{f \in M_{\frac{k}{2}}(\Gamma_0(N)) \mid f(\gamma z) = \chi(d)(cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\}$

$a \sim b$ means that a is equivalent to b

$z \to i\infty$ denotes that z goes to $i\infty$.

$\nu_0(F)$ the sum of orders of zeros of a modular form (or function) F

$\nu_\infty(F)$ the sum of orders of poles of a modular form (or function) F

$\sigma_\infty(\Gamma)$ the number of Γ-inequivalent cusps of Γ

We shall always take the branch of the square root having argument in $(-\frac{\pi}{2}, \frac{\pi}{2}]$. Thus, $\sqrt{z}$ is a holomorphic function on the complex plane with the negative real axis $(-\infty, 0]$ removed. For any integer k, we define $z^{\frac{k}{2}}$ to mean $(\sqrt{z})^k$.

2. Fundamental region of $X_1(N)$

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$.

Definition. An *(open) fundamental region* R for Γ is an open subset of $\mathfrak{H}^*$ with the properties:

1. there do not exist $\gamma \in \Gamma$ and $w, z \in R$ for which $w \neq z$ and $w = \gamma z$;
2. for any $z \in \mathfrak{H}^*$, there is $\gamma \in \Gamma$ such that $\gamma z \in \overline{R}$ the closure of R.

We will examine some necessary results about fundamental regions, which will give us useful geometric informations for the modular curve $X_1(N)$. Let $\Gamma^1(N)$ be a congruence subgroup of $SL_2(\mathbb{Z})$ whose elements are congruent to $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ mod N ($N = 1, 2, 3, \cdots$). We note that the two groups $\Gamma_1(N)$ and $\Gamma^1(N)$ are conjugate:

(1)
$$\Gamma^1(N) = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(N) \begin{pmatrix} 1/N & 0 \\ 0 & 1 \end{pmatrix}.$$

It turns out that the Γ^1 groups are more convenient than their Γ_1 counterparts for drawing pictures and making geometric computations. Now we will draw fundamental regions by using Ferenbaugh's idea ([4], §3). Suppose $c, r \in \mathbb{R}$ with $r > 0$. Then we define the sets

$$\mathrm{arc}(c,r) = \{z \in \mathfrak{H}^* |\ |z-c| = r\}$$

$$\mathrm{inside}(c,r) = \{z \in \mathfrak{H}^* |\ |z-c| < r\}$$

$$\mathrm{outside}(c,r) = \{z \in \mathfrak{H}^* |\ |z-c| > r\}.$$

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of Γ, and assume $c \neq 0$. Then we define

$$\mathrm{arc}(\gamma) = \mathrm{arc}(a/c, 1/|c|),$$

$$\mathrm{inside}(\gamma) = \mathrm{inside}(a/c, 1/|c|) \quad \text{and}$$

$$\mathrm{outside}(\gamma) = \mathrm{outside}(a/c, 1/|c|).$$

If $c = 0$, γ is of the form $z \mapsto z + n$ for some integer n. We shall assume γ is not the identity, so $n \neq 0$. We then adopt the following conventions: for $n > 0$, we define

$$\mathrm{arc}(\gamma) = \left\{z \in \mathfrak{H}^* |\ \mathrm{Re}(z) = \frac{n}{2}\right\}$$
$$\mathrm{inside}(\gamma) = \left\{z \in \mathfrak{H}^* |\ \mathrm{Re}(z) > \frac{n}{2}\right\}$$
$$\mathrm{outside}(\gamma) = \left\{z \in \mathfrak{H}^* |\ \mathrm{Re}(z) < \frac{n}{2}\right\}.$$

As for the case $n < 0$, we define "arc" in the same way and reverse the inequalities in the definitions of "inside" and "outside". Then we have

Proposition 1. *The element $\gamma \in \Gamma - \{I\}$ sends arc(γ^{-1}) to arc(γ), inside(γ^{-1}) to outside(γ) and outside(γ^{-1}) to inside(γ).*

Proof. [4], Proposition 3.1. □

Theorem 2. *With notations as in the above, a fundamental region R for Γ is given by*

$$R = \bigcap_{\gamma \in \Gamma - \{I\}} \mathit{outside}(\gamma).$$

Proof. [4], Theorem 3.3. □

Now the following theorem enables us to get the generators of the group $\overline{\Gamma}$.

Theorem 3. *Let* $\overline{\Gamma}$ *be a congruence subgroup of* $\overline{\Gamma}(1)$ *of finite index and* R *be a fundamental region for* $\overline{\Gamma}$. *Then the sides of* R *can be grouped into pairs* λ_i, λ'_i $(i = 1, 2, \cdots, s)$ *in such a way that* $\lambda_i \subseteq \overline{R}$ *and* $\lambda'_i = \gamma_i \lambda_i$ *where* $\gamma_i \in \overline{\Gamma}$ $(i = 1, 2, \cdots, s)$. γ_i*'s are called boundary substitutions of* R. *Furthermore,* $\overline{\Gamma}$ *is generated by the boundary substitutions* $\gamma_1, \cdots, \gamma_s$.

Proof. [19], Theorem 2.4.4 (or [10], Theorem 1). □

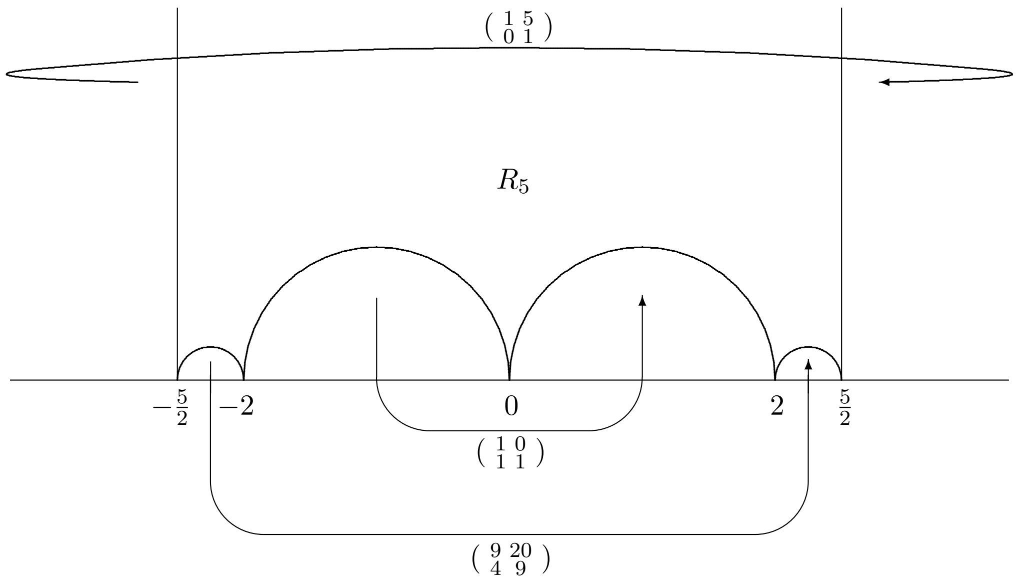
3. Modular function $j_{1,5}$

Let us take $\Gamma = \Gamma^1(5)$ and put $\gamma_1 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\gamma_3 = \begin{pmatrix} 9 & 20 \\ 4 & 9 \end{pmatrix}$. If R_5 is a fundamental region of $\Gamma^1(5)$, then by Theorem 2 it is given by

$$R_5 = \bigcap_{i=1}^{3} outside(\gamma_i^{\pm 1})$$

and is drawn as follows.

Fundamental domain of $\Gamma^1(5)$



We denote by S_Γ the set of inequivalent cusps of Γ. Then we see from the above figure that $S_{\Gamma^1(5)} = \{\infty, 0, 2, \frac{5}{2}\}$. Furthermore it follows from Theorem 3 that $\overline{\Gamma}^1(5)$ is generated by γ_1, γ_2 and γ_3. Thus we obtain the following theorem by (1).

Theorem 4. *(i)* $S_{\Gamma_1(5)} = \{\infty, 0, \frac{2}{5}, \frac{1}{2}\}$. *All cusps of* $\Gamma_1(5)$ *are regular* ([16], [22]).
(ii) $\overline{\Gamma}_1(5)$ *is generated by* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 9 & 4 \\ 20 & 9 \end{pmatrix}$.

For later use we are in need of calculating the widths of the cusps of $\Gamma_1(5)$.

Lemma 5. *Let* $a/c \in \mathbb{P}^1(\mathbb{Q})$ *be a cusp with* $(a, c) = 1$. *Then the width of* a/c *in* $X_1(N)$ *is given by* $N/(c, N)$ *if* $N \neq 4$.

Proof. [11], Lemma 3. □

Therefore, we have the following table of inequivalent cusps of $\Gamma_1(5)$:

Table 1. Cusps of $\Gamma_1(5)$

cusp	∞	0	$\frac{2}{5}$	$\frac{1}{2}$
width	1	5	1	5

Let G_2 be the Eisenstein series of weight 2 defined by

$$G_2(z) = 2\zeta(2) - 8\pi^2 \sum_{n\geq 1} \sigma_1(n) q^n, \quad z \in \mathfrak{H}. \tag{2}$$

Then G_2 has the following transformation formula ([20], p.68) for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ and $z \in \mathfrak{H}$:

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - 2\pi i c(cz+d). \tag{3}$$

Lemma 6. *For each prime* p, *let* $G_2^{(p)}(z) = G_2(z) - pG_2(pz)$. *Then* $G_2^{(p)}(z) \in M_2(\Gamma_0(p))$.

Proof. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\Gamma_0(p)$, then

$$\begin{aligned}
G_2^{(p)}(z)|_{[\gamma]_2} &= (cz+d)^{-2}G_2^{(p)}(\gamma z) \\
&= (cz+d)^{-2}(G_2(\gamma z) - pG_2(p\ \gamma z)) \\
&= (cz+d)^{-2}(G_2(\gamma z) - pG_2(\left(\begin{smallmatrix} a & pb \\ c/p & d \end{smallmatrix}\right) \cdot pz) \\
&\qquad \text{using } \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} a & pb \\ c/p & d \end{smallmatrix}\right) \\
&= (cz+d)^{-2}((cz+d)^2G_2(z) - 2\pi ic(cz+d) \\
&\qquad - p((\frac{c}{p}pz+d)^2G_2(pz) - 2\pi i\frac{c}{p}(\frac{c}{p}pz+d))) \quad \text{by (3)} \\
&= G_2^{(p)}(z).
\end{aligned}$$

Recall that there are 2 cusps $\infty,\ 0$ in $X_0(p)$. The q-expansion of G_2 implies the holomorphicity of $G_2^{(p)}$ at ∞. At 0

$$\begin{aligned}
G_2^{(p)}(z)|_{[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2} &= z^{-2}G_2^{(p)}(-1/z) \\
&= z^{-2}(G_2(-1/z) - pG_2(-p/z)) \\
&= z^{-2}(z^2G_2(z) - 2\pi iz - p((z/p)^2G_2(z/p) - 2\pi iz/p)) \quad \text{by (3)} \\
&= G_2(z) - 1/pG_2(z/p),
\end{aligned}$$

hence it is holomorphic there. □

Lemma 7. *For $F \in M_k(\Gamma_0(N), \chi)$, let $W_N(F)$ be the Fricke involution of F, i.e., $W_N(F) = F|_{[\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)]_k}$. Then for a quadratic character χ on $(\mathbb{Z}/N\mathbb{Z})^*$, W_N preserves $M_k(\Gamma_0(N), \chi)$.*

Proof. [13], p.145. □

Let $\eta(z) = e^{\frac{\pi iz}{12}}\prod_{n=1}^{\infty}(1-q^n),\ z \in \mathfrak{H}$ be the Dedekind eta function. It is well-known ([12], p.235) that

$$\eta(z+1) = e^{\frac{\pi i}{12}}\eta(z) \ \text{ and } \ \eta(-1/z) = (-iz)^{\frac{1}{2}}\eta(z). \tag{4}$$

Lemma 8. *(i) $\eta^p(z)/\eta(pz) \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$ for a prime $p > 3$.*
(ii) $W_p(\eta^p(z)/\eta(pz)) =$ constant $\times\, \eta^p(pz)/\eta(z) \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right)$.

Proof. For (i) we refer to [18], p.28.

(ii)

$$\begin{aligned} W_p(\eta^p(z)/\eta(pz)) &= \frac{\eta^p(z)}{\eta(pz)}\Big|_{[\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)]_{\frac{p-1}{2}}} \\ &= p^{\frac{p-1}{4}}(pz)^{-\frac{p-1}{2}}\eta^p\left(-\frac{1}{pz}\right)/\eta\left(p\cdot\left(-\frac{1}{pz}\right)\right) \\ &= p^{-\frac{p-1}{4}}z^{-\frac{p-1}{2}}\frac{(-ipz)^{\frac{p}{2}}\eta^p(pz)}{(-iz)^{\frac{1}{2}}\eta(z)} \quad \text{by (4)} \\ &= \text{ constant } \times \eta^p(pz)/\eta(z). \end{aligned}$$

Hence, this completes the proof by Lemma 7. □

Now, put $x(z) = 4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)$ and $y(z) = \eta^5(5z)/\eta(z)$, where $E_2(z) = G_2(z)/(2\zeta(2))$ is the normalized Eisenstein series of weight 2 and $E_2^{(5)}(z) = E_2(z) - 5E_2(5z)$. From the q-expansions of G_2 and η it follows that

$$x(z) = -44q - 52q^2 - 56q^3 - 228q^4 + \cdots\ ,$$

$$y(z) = q + q^2 + 2q^3 + 3q^4 + 5q^5 + \cdots\ .$$

We set $j_{1,5}(z) = x(z)/y(z)$.

Theorem 9. *(a)* $x, y \in M_2(\Gamma_1(5))$.

(b) $\mathbb{C}(X_1(5))$ *is equal to* $\mathbb{C}(j_{1,5}(z))$.

(c) $j_{1,5}$ *takes the following value at each cusp:* $j_{1,5}(\infty) = -44$, $j_{1,5}(0) = -20\sqrt{5}$, $j_{1,5}(1/2) = 20\sqrt{5}$, *and* $j_{1,5}(2/5) = \infty$ *(a simple pole).*

Proof. (a) follows from Lemma 6 and 8. Next, it is clear by (a) that $j_{1,5}(z) \in \mathbb{C}(X_1(5))$. We see from the construction of x and y that both x and y vanish at ∞. Also, we know from [22], p.39 that $\nu_0(x) = \nu_0(y) = 2$. Let ∞ and z_0 (resp. z_0') be the zeros of x (resp. y). If z_0 is equivalent to z_0' under $\Gamma_1(5)$, then x/y has no poles in $X_1(5)$ so that it would be a constant. However, the q-expansions of x and y show that the quotient x/y cannot be a constant. Thus z_0 is not $\Gamma_1(5)$-equivalent to z_0'. And $\nu_0(j_{1,5}) = \nu_\infty(j_{1,5}) = 1$, which implies that $j_{1,5}$ generates $\mathbb{C}(X_1(5))$ over $\mathbb{C}$. Now we will prove (c). As mentioned in the Table 1, we note that there are 4 inequivalent cusps $\infty, 0, 1/2, 2/5$ in $X_1(5)$.

(i) $s=\infty$:

$$j_{1,5}(\infty) = \lim_{z\to i\infty}\frac{x}{y} = \lim_{q\to 0}\frac{-44q-52q^2-56q^3-228q^4+\cdots}{q+q^2+2q^3+3q^4+5q^5+\cdots}$$
$$= -44.$$

(ii) $s=0$: Since $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ sends ∞ to 0,

$$\begin{aligned}
j_{1,5}(0) &= \lim_{z\to i\infty}\left.\frac{4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)}\right|_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}\\
&= \lim_{z\to i\infty}\frac{4\cdot\eta^5(-1/z)/\eta(-5/z)+E_2^{(5)}(-1/z)}{\eta^5(-5/z)/\eta(-1/z)}\\
&= \lim_{z\to i\infty}\frac{4\cdot(\sqrt{-iz}^5\eta^5(z))/(\sqrt{-iz/5}\;\eta(z/5))+z^2E_2(z)-(z^2/5)E_2(z/5)}{(\sqrt{-iz/5}^5\eta^5(z/5))/(\sqrt{-iz}\;\eta(z))}\\
&\qquad\text{by (3) and (4)}\\
&= -20\sqrt{5}.
\end{aligned}$$

(iii) $s=1/2$: Now that $\left(\begin{smallmatrix}3&1\\5&2\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ sends ∞ to 1/2,

$$\begin{aligned}
j_{1,5}(1/2) &= \lim_{z\to i\infty}\left.\frac{4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)}\right|_{\left(\begin{smallmatrix}3&1\\5&2\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}\\
&= \lim_{z\to i\infty}\left.\frac{-4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)}{-\eta^5(5z)/\eta(z)}\right|_{\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)}\quad\text{by Lemma 6 and 8}\\
&= 20\sqrt{5}\quad\text{similarly to (ii).}
\end{aligned}$$

(iv) $s=2/5$: $\left(\begin{smallmatrix}2&1\\5&3\end{smallmatrix}\right)\infty=2/5$.

$$\begin{aligned}
j_{1,5}(2/5) &= \lim_{z\to i\infty}\left.\frac{4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)}{\eta^5(5z)/\eta(z)}\right|_{\left(\begin{smallmatrix}2&1\\5&3\end{smallmatrix}\right)}\\
&= \lim_{z\to i\infty}\frac{-4\cdot\eta^5(z)/\eta(5z)+E_2^{(5)}(z)}{-\eta^5(5z)/\eta(z)}\quad\text{by Lemma 6 and 8}\\
&= \infty\ \text{(a simple pole).}
\end{aligned}$$

□

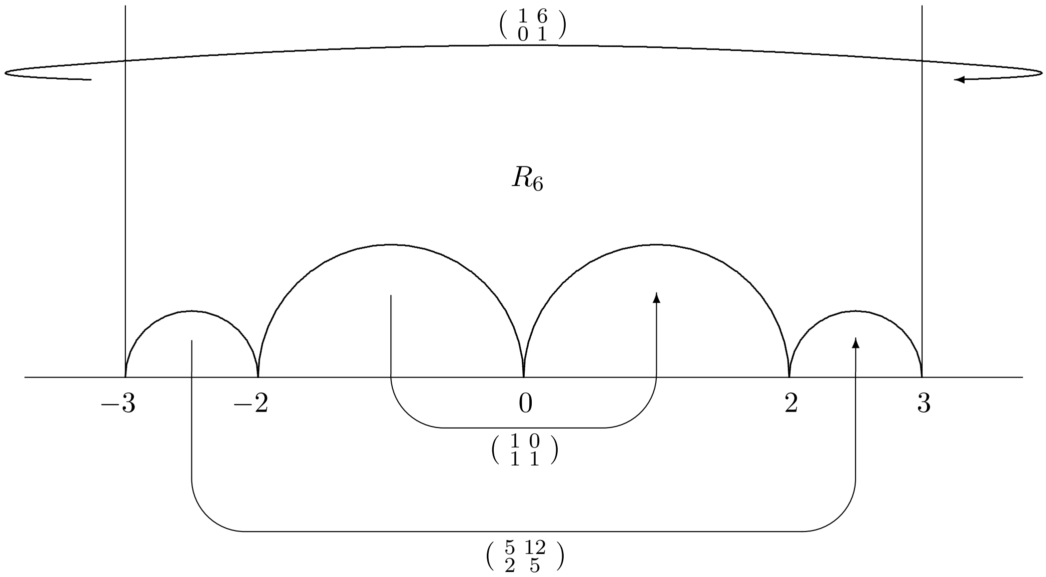
4. Modular function $j_{1,6}$

Let us take $\Gamma = \Gamma^1(6)$ and set $\gamma_1 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\gamma_3 = \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}$. If R_6 is a fundamental region of $\Gamma^1(6)$, then R_6 is described as

$$R_6 = \bigcap_{i=1}^{3} outside(\gamma_i^{\pm 1}).$$

Hence we have the following picture for R_6.

Fundamental domain of $\Gamma^1(6)$



Then as we see in the above figure $S_{\Gamma^1(6)} = \{\infty, 0, 2, 3\}$. Furthermore, it follows from Theorem 3 that $\overline{\Gamma}^1(6)$ is generated by γ_1, γ_2 and γ_3. Therefore we obtain the following theorem by (1).

Theorem 10. *(i)* $S_{\Gamma_1(6)} = \{\infty, 0, \frac{1}{3}, \frac{1}{2}\}$. *All cusps of* $\Gamma_1(6)$ *are regular* ([16], [22]).
(ii) $\overline{\Gamma}_1(6)$ *is generated by* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}$.

We then have the following table of inequivalent cusps of $\Gamma_1(6)$ in virtue of Lemma 5:

Table 2. Cusps of $\Gamma_1(6)$

cusp	∞	0	$\frac{1}{3}$	$\frac{1}{2}$
width	1	6	2	3

Let $G_2^{(p)}(z)$ be the series as in Lemma 6. Put $X(z) = G_2^{(2)}(z) - G_2^{(2)}(3z) = G_2(z) - 2G_2(2z) - G_2(3z) + 2G_2(6z)$ and $Y(z) = 2G_2^{(2)}(z) - G_2^{(3)}(z) = G_2(z) - 4G_2(2z) + 3G_2(3z)$. We set $j_{1,6}(z) = X(z)/Y(z)$.

Theorem 11. *(a)* $X, Y \in M_2(\Gamma_1(6))$.
(b) $\mathbb{C}(X_1(6))$ *is equal to* $\mathbb{C}(j_{1,6}(z))$.
(c) $j_{1,6}$ *takes the following value at each cusp:* $j_{1,6}(\infty) = 1$, $j_{1,6}(0) = 4/3$, $j_{1,6}(1/3) = 0$, *and* $j_{1,6}(1/2) = 1/3$.

Proof. By Lemma 6, $G_2^{(p)}(z) \in M_2(\Gamma_0(p))$ for a prime p. Meanwhile, the identity

$$\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \Gamma_0(p) \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right) \cap \Gamma_0(p) = \Gamma_0(pq)$$

allows us to have $G_2^{(p)}(qz) \in M_2(\Gamma_0(pq))$. Therefore we easily get (a), from which $j_{1,6} = X/Y \in \mathbb{C}(X_1(6))$. By the q-expansion of G_2 as in (2) we derive that

$$X(z) = -8\pi^2 \cdot (q + q^2 + 3q^3 + q^4 + 6q^5 + \cdots), \tag{5}$$

$$Y(z) = -8\pi^2 \cdot (q - q^2 + 7q^3 - 5q^4 + 6q^5 + \cdots). \tag{6}$$

Thus both X and Y vanish at ∞. And, the zero formula ([22], p.39) yields $\nu_0(X) = \nu_0(Y) = 2$. If ∞ and w_0 (resp. w_0') are the zeros of X (resp. Y), then w_0 is not $\Gamma_1(6)$-equivalent to w_0'. Therefore $\nu_0(j_{1,6}) = \nu_\infty(j_{1,6}) = 1$, which means that $j_{1,6}$ generates $\mathbb{C}(X_1(6))$ over $\mathbb{C}$. Next, as for the statement (c), we first recall that there are four $\Gamma_1(6)$-inequivalent cusps ∞, 0, 1/3 and 1/2. Put $f_1(z) = G_2^{(2)}(z)$, $f_2(z) = f_1(3z)$ and $f_3(z) = G_2^{(3)}(z)$. Then

$$X(z) = f_1(z) - f_2(z) \quad \text{and} \quad Y(z) = 2f_1(z) - f_3(z). \tag{7}$$

We shall then evaluate the values of f_i $(i = 1, 2, 3)$ at each cusp. First we note that

$$G_2^{(p)}(\infty) = \lim_{z \to i\infty} G_2^{(p)}(z) = 2\zeta(2)(1 - p) \quad \text{by (2)} \tag{8}$$

$$G_2^{(p)}(0) = \lim_{z \to i\infty} G_2^{(p)}(-1/z) = 2\zeta(2)(1 - 1/p) \quad \text{by (2) and (3).} \tag{9}$$

(i) Cusp values of f_1:

$$f_1(\infty) = G_2^{(2)}(\infty) = -2\zeta(2) \quad \text{by (8)},$$
$$f_1(0) = G_2^{(2)}(0) = \zeta(2) \quad \text{by (9)},$$
$$f_1(1/3) = f_1(0) = \zeta(2) \quad \text{since } f_1 \in M_2(\Gamma_0(2)) \text{ and } 1/3 \sim 0 \text{ under } \Gamma_0(2),$$
$$f_1(1/2) = f_1(\infty) = -2\zeta(2) \quad \text{since } 1/2 \sim \infty \text{ under } \Gamma_0(2).$$

(ii) Cusp values of f_2: Observe that $f_2(z) = f_1(3z) = \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_2}$.

$$\begin{aligned}
f_2(\infty) &= \lim_{z\to i\infty} f_2(z) = \lim_{z\to i\infty} f_1(3z) = f_1(\infty) = -2\zeta(2),\\
f_2(0) &= \lim_{z\to i\infty} f_2|_{[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2} = \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2}\\
&= \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right)]_2} = \frac{1}{3} f_1(0)\cdot 3\cdot \frac{1}{9} = \frac{1}{9}\zeta(2),\\
f_2(1/3) &= \lim_{z\to i\infty} f_2|_{[\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)]_2} = \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)]_2}\\
&= \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_2} = \frac{1}{3} f_1(1)\cdot 3 = f_1(0) = \zeta(2),\\
f_2(1/2) &= \lim_{z\to i\infty} f_2|_{[\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)]_2} = \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)]_2}\\
&= \lim_{z\to i\infty} \frac{1}{3} f_1|_{[\left(\begin{smallmatrix} 3 & 1 \\ 2 & 1 \end{smallmatrix}\right)]_2 [\left(\begin{smallmatrix} 1 & -1 \\ 0 & 3 \end{smallmatrix}\right)]_2} = \frac{1}{3} f_1(3/2)\cdot 3\cdot \frac{1}{9} = \frac{1}{9} f_1(1/2) = -\frac{2}{9}\zeta(2).
\end{aligned}$$

(iii) Cusp values of f_3:

$$f_3(\infty) = G_2^{(3)}(\infty) = -4\zeta(2) \quad \text{by (8)},$$
$$f_3(0) = G_2^{(3)}(0) = \frac{4}{3}\zeta(2) \quad \text{by (9)},$$
$$f_3(1/3) = f_3(\infty) = -4\zeta(2) \quad \text{since } f_3 \in M_2(\Gamma_0(3)) \text{ and } 1/3 \sim \infty \text{ under } \Gamma_0(3),$$
$$f_3(1/2) = f_3(0) = \frac{4}{3}\zeta(2) \quad \text{since } 1/2 \sim 0 \text{ under } \Gamma_0(3).$$

By (i), (ii), (iii) and (7) we conclude that

$X(\infty) = 0$, $Y(\infty) = 0$, $j_{1,6}(\infty) = 1$, (see (5) and (6))

$X(0) = \frac{8}{9}\zeta(2)$, $Y(0) = \frac{2}{3}\zeta(2)$, $j_{1,6}(0) = 4/3$,

$X(1/3) = 0$, $Y(1/3) = 6\zeta(2)$, $j_{1,6}(1/3) = 0$,

$X(1/2) = -\frac{16}{9}\zeta(2)$, $Y(1/2) = -\frac{16}{3}\zeta(2)$, $j_{1,6}(1/2) = 1/3$. □

5. Normalized generators

For a modular function f, we call f *normalized* if its q-series is

$$\frac{1}{q} + 0 + a_1 q + a_2 q^2 + \cdots .$$

Lemma 12. *The normalized generator of a genus zero function field is unique.*

Proof. [10], Lemma 8. □

We will construct the normalized generator (or the hauptmodulus) of the function field $\mathbb{C}(X_1(N))$ $(N = 5, 6)$ from the modular function $j_{1,N}$ $(N = 5, 6)$ described in Theorem 9 and Theorem 11. First, we note that

$$\begin{aligned}\frac{-8}{j_{1,5}(z) + 44} &= \frac{-8y}{x + 44y} \\ &= \frac{1}{q} + 5 + 10q + 5q^2 - 15q^3 - 24q^4 + 15q^5 + \cdots ,\end{aligned}$$

which is in $q^{-1}\mathbb{Z}[[q]]$. This will be justified later in §6. Thus let $N(j_{1,5}) = \frac{-8}{j_{1,5}+44} - 5$. As for the modular function $j_{1,6}$, we observe that

$$\begin{aligned}\frac{2}{j_{1,6} - 1} &= \frac{2Y}{X - Y} = \frac{2(G_2(z) - 4G_2(2z) + 3G_2(3z))}{2G_2(2z) - 4G_2(3z) + 2G_2(6z)} = \frac{G_2(z) - 4G_2(2z) + 3G_2(3z)}{G_2(2z) - 2G_2(3z) + G_2(6z)} \\ &= \frac{-8\pi^2 \cdot (q - q^2 + 7q^3 - 5q^4 + \cdots)}{-8\pi^2 \cdot (q^2 - 2q^3 + 3q^4 + \cdots)} \\ &= \frac{1}{q} + 1 + 6q + 4q^2 - 3q^3 - 12q^4 - 8q^5 + \cdots ,\end{aligned}$$

which is also in $q^{-1}\mathbb{Z}[[q]]$ because the q-series of $\frac{1}{-8\pi^2} \cdot (G_2(z) - 4G_2(2z) + 3G_2(3z))$ and $\frac{1}{-8\pi^2} \cdot (G_2(2z) - 2G_2(3z) + G_2(6z))$ belong to $\mathbb{Z}[[q]]$, and the leading coefficient of the latter series is 1. Define $N(j_{1,6}) = \frac{2}{j_{1,6}-1} - 1$. Then the above computation shows that $N(j_{1,5})$ and $N(j_{1,6})$ are the normalized generators of $\mathbb{C}(X_1(5))$ and $\mathbb{C}(X_1(6))$, respectively. By Theorem 9-(c) and 11-(c) we have the following tables:

Table 3. Cusp values of $j_{1,5}$ and $N(j_{1,5})$

s	∞	0	$1/2$	$2/5$
$j_{1,5}(s)$	-44	$-20\sqrt{5}$	$20\sqrt{5}$	∞
$N(j_{1,5})(s)$	∞	$\frac{1+5\sqrt{5}}{2}$	$\frac{1-5\sqrt{5}}{2}$	-5

Table 4. Cusp values of $j_{1,6}$ and $N(j_{1,6})$

s	∞	0	$1/3$	$1/2$
$j_{1,6}(s)$	1	$4/3$	0	$1/3$
$N(j_{1,6})(s)$	∞	5	-3	-4

Lemma 13. *Let N be a positive integer such that the modular curve $X_1(N)$ is of genus 0. Let t be an element of $\mathbb{C}(X_1(N))$ for which (i) $\mathbb{C}(X_1(N)) = \mathbb{C}(t)$ and (ii) t has no poles except for a simple pole at one cusp s. Let $f \in \mathbb{C}(X_1(N))$. If f has a pole of order n only at s, then f can be written as a polynomial in t of degree n.*

Proof. Take $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma\infty = s$. Let h be the width of s. Then we have

$$t|_\gamma = \frac{1}{c}\,\frac{1}{q_h} + \cdots$$

and

$$f|_\gamma = b_n\,\frac{1}{q_h^n} + \cdots$$

for some $c \neq 0$ and $b_n \neq 0$. Thus

$$(f - b_n(ct)^n)|_\gamma = \lambda_{n-1}\,\frac{1}{q_h^{n-1}} + \cdots$$

for some λ_{n-1}. And

$$(f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1})|_\gamma = \lambda_{n-2}\,\frac{1}{q_h^{n-2}} + \cdots$$

for some λ_{n-2}. In this way we can choose $\lambda_i \in \mathbb{C}$ such that

$$(f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1} - \cdots - \lambda_1(ct))|_\gamma \in \mathbb{C}[[q_h]].$$

Let $g = f - b_n(ct)^n - \lambda_{n-1}(ct)^{n-1} - \cdots - \lambda_1(ct)$. Then g has no poles in $\mathfrak{H}^*$, and so g must be a constant, say λ_0. Therefore we end up with $f = b_n c^n t^n + \lambda_{n-1}c^{n-1}t^{n-1} + \cdots + \lambda_1 ct + \lambda_0$, as desired. $\square$

Theorem 14. *Let d be a square free positive integer and t be the hauptmodulus $N(j_{1,N})$, $(N = 5, 6)$. For $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathfrak{H}$, $t(z)$ is an algebraic integer.*

Proof. Let $j(z) = \frac{1}{q} + 744 + 196884q + \cdots$ be an elliptic modular function. It is well-known that $j(z)$ is an algebraic integer for $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathfrak{H}$ ([15], [22]). For algebraic proofs, see [3],

[17], [21] and [23]. Now, we view j as a function on the modular curve $X_1(N)$. Let s be a cusp of $\Gamma_1(N)$ other than ∞, whose width is h_s. Then j has a pole of order h_s at the cusp s. On the other hand, $t(z)-t(s)$ has a simple zero at s. Thus

$$j \times \prod_{s\,\in\, S_{\Gamma_1(N)}\setminus\{\infty\}} (t(z)-t(s))^{h_s}$$

has a pole only at ∞ whose degree is 12 if $N=5$ or 6. And so by Lemma 13, it is a monic polynomial in t of degree 12, which we denote by $f(t)$. With the aid of datum from Tables 1,2,3 and 4, we can compute the product part in the above more explicitly, that is,

$$\prod_{s\,\in\, S_{\Gamma_1(N)}\setminus\{\infty\}} (t(z)-t(s))^{h_s} = \begin{cases} (t^2-t-31)^5(t+5), & \text{if } N=5 \\ (t-5)^6(t+3)^2(t+4)^3, & \text{if } N=6. \end{cases}$$

Since j and t have integer coefficients in the q-expansions, $f(t)$ is a monic polynomial in $\mathbb{Z}[t]$ of degree 12. This claims that $t(z)$ is integral over $\mathbb{Z}[j(z)]$. Therefore $t(z)$ is integral over $\mathbb{Z}$ for $z \in \mathbb{Q}(\sqrt{-d}) \cap \mathfrak{H}$. $\square$

6. Integrality of Fourier coefficients of $N(j_{1,5})$

We recall that $N(j_{1,5}) = \frac{-8}{j_{1,5}+44} - 5 = \frac{-8y}{x+44y} - 5$ where $x(z) = 4\cdot\eta^5(z)/\eta(5z) + E_2^{(5)}(z)$ and $y(z) = \eta^5(5z)/\eta(z)$. Since the q-series of $-8y$ and $x+44y$ start with $-8(q+q^2+\cdots)$ ($\in -8q\mathbb{Z}[[q]]$) and $-8q^2+32q^3+\cdots$ ($\in q^2\mathbb{Z}[[q]]$) respectively, the q-series of $N(j_{1,5})$ is in $q^{-1}\mathbb{Z}[[q]]$ if all the Fourier coefficients of $x+44y$ is divisible by 8, in which case we simply write $8 \mid x+44y$. Then

$$\begin{aligned} 8 \mid x+44y \Leftrightarrow 8 \mid x+4y &\Leftrightarrow 8 \mid 4\cdot\eta^5(z)/\eta(5z) + 4\cdot\eta^5(5z)/\eta(z) + E_2^{(5)}(z) \\ &\Leftrightarrow 2 \mid \eta^5(z)/\eta(5z) + \eta^5(5z)/\eta(z) \quad \text{except the constant term} \end{aligned}$$

because $24 \mid E_2^{(5)}(z)$ except the constant term. Hence it suffices to show that $2 \mid \eta^5(z)/\eta(5z) + \eta^5(5z)/\eta(z)$ except the constant term.

Let Δ^n be the set of 2×2 integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $a \in 1+N\mathbb{Z}$, $c \in N\mathbb{Z}$, and $ad-bc=n$. For $f \in M_k(\Gamma_1(N))$ we define the Hecke operator T_n by

$$f|_{T_n} = n^{(k/2)-1}\sum f|_{[\alpha_j]_k} \tag{10}$$

where $\Gamma_1(N)\alpha_j$ runs through the right cosets of $\Gamma_1(N)$ in Δ^n. Then T_n preserves the space $M_k(\Gamma_0(N),\chi)$ for a Dirichlet character χ ([13], §5). Let $W_N(f) = f|_{\left[\left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)\right]_k}$ be the action of Fricke involution on f.

Lemma 15. *Let n be a positive integer prime to N and $f \in M_k(\Gamma_0(N),\chi)$ for a Dirichlet character χ. Then we have $W_N \circ T_n(f) = \chi(n)T_n \circ W_N(f)$.*

Proof. Δ^n has the following right coset decomposition: (See [13], [16], [22])

$$\Delta^n = \bigcup_{\substack{a|n \\ (a,N)=1}} \bigcup_{i=0}^{\frac{n}{a}-1} \Gamma_1(N)\sigma_a \begin{pmatrix} a & i \\ 0 & \frac{n}{a} \end{pmatrix} \tag{11}$$

where $\sigma_a \in SL_2(\mathbb{Z})$ such that $\sigma_a \equiv \left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right) \mod N$. By (10) and (11),

$$T_n \circ W_N(f) = n^{(k/2)-1} \sum_{a,b} f|_{\left[\alpha_N\sigma_a\left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right)\right]_k},$$

where $\alpha_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. Let $\alpha_{a,b} = \sigma_n\alpha_N\sigma_a \left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right) \alpha_N^{-1} \in \Delta^n$. Then it is easy to show that $\alpha_{a,b}$ are in distinct cosets of $\Gamma_1(N)$ in Δ^n, and hence form a set of representatives; so by (10),

$$\begin{aligned} T_n \circ W_N(f) &= n^{(k/2)-1} \sum_{a,b} f|_{[\alpha_{a,b}\alpha_N]_k} = n^{(k/2)-1} \sum_{a,b} f|_{\left[\sigma_n\alpha_N\sigma_a\left(\begin{smallmatrix} a & b \\ 0 & n/a \end{smallmatrix}\right)\right]_k} \\ &= \chi(n)T_n(W_N(f)) \quad \text{since } f|_{[\sigma_n]_k} = \chi(n)f. \end{aligned}$$

This completes the proof. □

Next, we observe that

$$M_2(\Gamma_1(5)) = \bigoplus_{\chi \in \widehat{(\mathbb{Z}/5\mathbb{Z})^\times}} M_2(\Gamma_0(5),\chi).$$

Since $(\mathbb{Z}/5\mathbb{Z})^\times$ is generated by $\bar{2}(= 2 \mod 5\mathbb{Z})$, any $\chi \in \widehat{(\mathbb{Z}/5\mathbb{Z})^\times}$ is determined by the value at $\bar{2}$. Let χ_1 be the character such that $\chi_1(\bar{2}) = i$. Then $\widehat{(\mathbb{Z}/5\mathbb{Z})^\times}$ is generated by χ_1 so that ${\chi_1}^4 = \chi_{triv}$ and ${\chi_1}^2 = \left(\frac{\cdot}{5}\right)$. Note that if χ is an odd character, then $M_2(\Gamma_0(5),\chi) = \{0\}$. Thus

$$M_2(\Gamma_1(5)) = M_2(\Gamma_0(5)) \bigoplus M_2(\Gamma_0(5), \left(\frac{\cdot}{5}\right)). \tag{12}$$

Now that the dimension of the space $M_2(\Gamma)$ is equal to $\sigma_\infty(\Gamma) - 1$, it follows from (12) that $M_2\left(\Gamma_0(5), \left(\frac{\cdot}{5}\right)\right)$ is two dimensional. In fact it is generated by $\eta^5(z)/\eta(5z)$ and $\eta^5(5z)/\eta(z)$.

It then follows from the proof of Lemma 8-(ii) that

$$W_5(\eta^5(z)/\eta(5z)) = -5\sqrt{5}\cdot\eta^5(5z)/\eta(z). \tag{13}$$

The fact that W_5 is an involution and (13) imply that

$$W_5(\eta^5(5z)/\eta(z)) = (-5\sqrt{5})^{-1}\cdot\eta^5(z)/\eta(5z).$$

Since T_m preserves $M_k(\Gamma_0(N),\chi)$, we may set

$$T_m(\eta^5(z)/\eta(5z)) = p_m\cdot\eta^5(z)/\eta(5z) + q_m\cdot\eta^5(5z)/\eta(z) \tag{14}$$

and

$$T_m(\eta^5(5z)/\eta(z)) = r_m\cdot\eta^5(z)/\eta(5z) + s_m\cdot\eta^5(5z)/\eta(z) \tag{15}$$

for $p_m, q_m, r_m, s_m \in \mathbb{C}$. Here, we recall from [13], p.163 that if $f(z) = \sum a_n q^n$ and $T_m(f(z)) = \sum b_n q^n$,

$$b_n = \sum_{\substack{d|(m,n)\\ d>0}} \chi(d) d^{k-1} a_{mn/d^2}.$$

If we compare the constant terms in (15), we get $r_m = 0$. In like manner, from (14) we have

$$p_m = \sum_{\substack{d|m\\ d>0}} \left(\frac{d}{5}\right) d^{k-1}\cdot 1. \tag{16}$$

When $(m,5)=1$, by Lemma 15 we obtain

$$T_m\circ W_5\left(\frac{\eta^5(z)}{\eta(5z)}\right) = \left(\frac{m}{5}\right) W_5\circ T_m\left(\frac{\eta^5(z)}{\eta(5z)}\right).$$

Then, by (13) the LHS of the above is equal to $-5\sqrt{5}\cdot T_m\left(\frac{\eta^5(5z)}{\eta(z)}\right) = -5\sqrt{5}\left(s_m\cdot\frac{\eta^5(5z)}{\eta(z)}\right)$. On the other hand the RHS is equal to

$$\begin{aligned}\text{RHS} &= \left(\frac{m}{5}\right) W_5\left(p_m\cdot\frac{\eta^5(z)}{\eta(5z)} + q_m\cdot\frac{\eta^5(5z)}{\eta(z)}\right)\\ &= \left(\frac{m}{5}\right)\left[-5\sqrt{5}\cdot p_m\cdot\frac{\eta^5(5z)}{\eta(z)} + (-5\sqrt{5})^{-1} q_m\cdot\frac{\eta^5(z)}{\eta(5z)}\right].\end{aligned}$$

Hence, by equating both sides we deduce that $q_m = 0$ and $s_m = \left(\frac{m}{5}\right)p_m = \left(\frac{m}{5}\right)\cdot\sum_{\substack{d|m\\ d>0}}\left(\frac{d}{5}\right)d^{k-1}$ by (16). Therefore for each positive integer m prime to 5, it holds that

$$T_m\left(\frac{\eta^5(z)}{\eta(5z)}\right) = p_m\cdot\frac{\eta^5(z)}{\eta(5z)} \tag{17}$$

and

$$T_m\left(\frac{\eta^5(5z)}{\eta(z)}\right)=\left(\frac{m}{5}\right)p_m\cdot\frac{\eta^5(5z)}{\eta(z)}. \tag{18}$$

Let $\frac{\eta^5(z)}{\eta(5z)}=\sum c_m q^m$ and $\frac{\eta^5(5z)}{\eta(z)}=\sum d_m q^m$. If we compare the q^1-coefficients in (17) and (18), then we get

$$c_m=-5\cdot p_m,\ \ d_m=\left(\frac{m}{5}\right)p_m \quad \text{for } (m,5)=1. \tag{19}$$

Now, let $m=5$. It then follows from (16) that $p_5=1$. Moreover in (17) and (18) by comparing the q^1-coefficients, we have $q_5=0$ and $s_5=5$. More generally, we take $m=5^l\cdot m_0$ with $l\geq 0$ and $5\nmid m_0$. Then

(20)

$$\begin{aligned}T_{5^l\cdot m_0}\left(\frac{\eta^5(z)}{\eta(5z)}\right)&=T_{5^l}\circ T_{m_0}\left(\frac{\eta^5(z)}{\eta(5z)}\right)=T_{5^l}\left(p_{m_0}\cdot\frac{\eta^5(z)}{\eta(5z)}\right) \quad \text{by (19)}\\ &=(T_5)^l\left(p_{m_0}\cdot\frac{\eta^5(z)}{\eta(5z)}\right)=p_{m_0}\cdot {p_5}^l\cdot\frac{\eta^5(z)}{\eta(5z)}=p_{m_0}\cdot\frac{\eta^5(z)}{\eta(5z)} \quad \text{since } p_5=1.\end{aligned}$$

Similarly,

$$T_{5^l\cdot m_0}\left(\frac{\eta^5(5z)}{\eta(z)}\right)=\left(\frac{m_0}{5}\right)\cdot p_{m_0}\cdot 5^l\cdot\frac{\eta^5(5z)}{\eta(z)}. \tag{21}$$

In the equations (20) and (21), if we compare the q^1-coefficients, we obtain

$$c_{_{5^l\cdot m_0}}=-5\cdot p_{m_0} \quad \text{and} \quad d_{_{5^l\cdot m_0}}=5^l\cdot\left(\frac{m_0}{5}\right)\cdot p_{m_0}$$

with $p_{m_0}=\sum_{\substack{d|m_0\\ d>0}}\left(\frac{d}{5}\right)d^{k-1}$. And, it is clear that 2 divides $c_{_{5^l\cdot m_0}}+d_{_{5^l\cdot m_0}}$, hence we conclude that

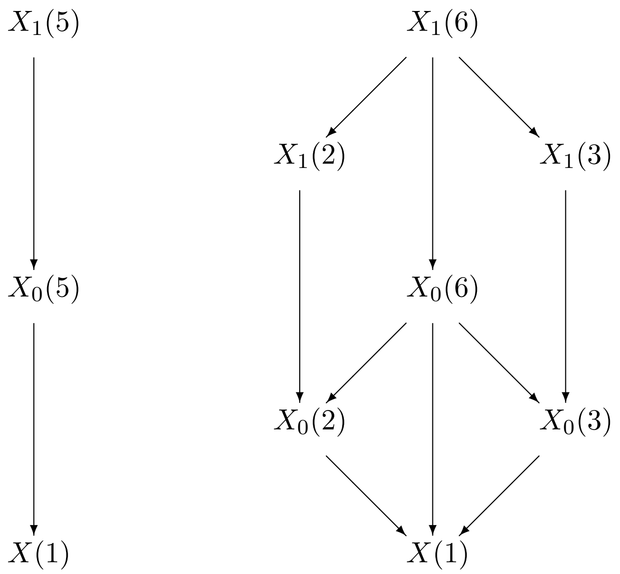
$$2\mid\frac{\eta^5(z)}{\eta(5z)}+\frac{\eta^5(5z)}{\eta(z)}$$

except the constant term.

7. Relationship with moduli of elliptic curves

When k is a field of characteristic prime to N, the k-rational points on the curve $X_0(N)$ ($X_1(N)$, respectively) parametrize pairs (E,C) (pairs (E,P), respectively) - modulo equivalence over an algebraic closure k^{alg} - of elliptic curves E with a k-rational cyclic subgroup C (k-rational point P, respectively) of order N. There are "forgetful" maps $X_1(N)$ to $X_0(N)$

which send $(E, P) \to (E, C)$ in terms of the subgroup $C = \{P, [2]P, \ldots, [N]P\}$. There are two diagrams of interest coming from these "forgetful" maps:



All of these curves have genus zero, but some of theses modular curves are easier to describe than others. For example, there is a canonical bijection $\mathbb{P}^1 \to X(1)$ of the "j-line" which sends $j \mapsto (E_j, O_j)$ in terms of the normal form

$$E_j : y^2 + xy = x^3 - \frac{36}{j - 1728} x - \frac{1}{j - 1728}$$

with a specified base point $O_j = (0 : 1 : 0)$. Clearly the function field of $X(1)$ is $k(j)$.

Similarly, there are canonical bijections $\mathbb{P}^1 \to X_1(N)$ which send $t \mapsto (E_t, P_t)$ in terms of the Tate normal forms

$$E_t : \begin{cases} y^2 = x^3 + 2x^2 + tx, & \text{if } N = 2; \\ y^2 + 3xy + ty = x^3, & \text{if } N = 3; \\ y^2 + (1+t)xy + ty = x^3 + tx^2, & \text{if } N = 5; \\ y^2 + (1+t)xy + (t - t^2)y = x^3 + (t - t^2)x^2, & \text{if } N = 6; \end{cases} \tag{22}$$

each with a specified point $P_t = (0 : 0 : 1)$ of order N. Such formulas can be found in [6, pp.94-95]. Using the "forgetful" maps $X_1(N)$ to $X(1)$, one has the expressions

$$j = \begin{cases} 64(4 - 3t)^3/(t^2(1 - t)), & \text{if } N = 2; \\ 27(9 - 8t)^3/(t^3(1 - t)), & \text{if } N = 3; \\ (1 - 12t + 14t^2 + 12t^3 + t^4)^3/(t^5(1 - 11t - t^2)), & \text{if } N = 5; \\ ((1 - 3t)(1 - 9t + 3t^2 - 3t^3))^3/(t^6(1 - t)^3(1 - 9t)), & \text{if } N = 6. \end{cases}$$

Clearly the function field of $X_1(N)$ is $k(t)$ in these cases; it may be thought of as an algebraic extension of $k(j)$. When the parameter t is interpreted as a modular function $t(z)$, we can find the following identities between our modular function $N(j_{1,N})(z)$ and $t(z)$.

Theorem 16. *(i)* $N(j_{1,5})(z) + 5 = \frac{\varepsilon^5 t(z)+1}{-t(z)+\varepsilon^5}$.
(ii) $N(j_{1,6})(z) + 1 = 6\frac{1+3t(z)}{1-9t(z)}$.
Here we set $\varepsilon = \zeta_5 + \zeta_5^{-1}$.

Proof. (i) First we note that ε satisfies $\varepsilon^2 + \varepsilon - 1 = 0$. Since $\varepsilon = 2\cos(2\pi/5) > 0$, we have $\varepsilon = \frac{-1+\sqrt{5}}{2}$ and hence $\varepsilon^5 = \frac{-11+5\sqrt{5}}{2}$. Let $f(z) = N(j_{1,5})(z) + 5$. The values of $f(z)$ at the cusps (obtained from Table 3) are:

s	∞	$2/5$	$1/2$	0
$f(s)$	∞	0	$-\varepsilon^5$	ε^{-5}

Since $\Delta(E_t) = -t^5(t^2 + 11t - 1)$ from the equation of E_t in (22), the set of possible values of $t(z)$ at the cusps are $\{\infty, 0, \varepsilon^5, -\varepsilon^{-5}\}$. Since $t(z)$ is a fractional linear transformation of $f(z)$, we come up with

$$[f(\infty), f(2/5), f(1/2), f(0)] = [t_1, t_2, t_3, t_4]$$

$$[\infty, 0, -\varepsilon^5, f(z)] = [t_1, t_2, t_3, t(z)]$$

where $t_1 = t(\infty), t_2 = t(2/5), t_3 = t(1/2), t_4 = t(0)$. Thus we obtain that

$$\frac{(t(z) - t_1)(t_2 - t_3)}{(t(z) - t_3)(t_2 - t_1)} = \frac{\varepsilon^5}{f(z) + \varepsilon^5}. \tag{23}$$

Suppose $t(z)$ has a pole or zero at a cusp s. Let h be the width of the cusp s. Considering the q_h-expansion of $t(z)$ at s we see from the identity

$$j = \frac{(1 - 12t + 14t^2 + 12t^3 + t^4)^3}{t^5(1 - 11t - t^2)}$$

that $\frac{1}{q} + O(1) = \frac{1}{q_h^5} + O(1)$. This yields $h = 5$. It then follows from Table 1 that $s = 1/2$ or $s = 0$. This means that $t_3, t_4 \in \{\infty, 0\}$ and so $t_1, t_2 \in \{\varepsilon^5, -\varepsilon^{-5}\}$. There are four possibilities for the cusp values $t(s)$:

Case (i). $t_1 = \varepsilon^5, t_2 = -\varepsilon^{-5}, t_3 = 0, t_4 = \infty$

Case (ii). $t_1 = \varepsilon^5, t_2 = -\varepsilon^{-5}, t_3 = \infty, t_4 = 0$

Case (iii). $t_1 = -\varepsilon^{-5}, t_2 = \varepsilon^5, t_3 = 0, t_4 = \infty$

Case (iv). $t_1 = -\varepsilon^{-5}, t_2 = \varepsilon^5, t_3 = \infty, t_4 = 0$

We see by routine check that only the second and third case satisfy the identity (23), from which we conclude that $t(z)$ should be either

$$u(z) = \frac{\varepsilon^5 f(z) - 1}{f(z) + \varepsilon^5} \quad \text{or} \quad v(z) = \frac{f(z) + \varepsilon^5}{-\varepsilon^5 f(z) + 1}.$$

Now we consider the elliptic curve $E_1 : y^2 + 2xy + y = x^3 + x^2$. By making appropriate change of variables we achieve the elliptic curve

$$E : y^2 = 4x^3 - \frac{4}{3}x + \frac{19}{27}$$

which is isomorphic to E_1. We note that under this isomorphism the point $P_1 = (0, 0) \in E_1$ is sent to $(2/3, -1) \in E$. The period lattice L of E is given by $L = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with

$$\omega_1 = 6.346046521397767108443973083772736526087\cdots,$$

$$\omega_2 = 3.173023260698883554221986541886368263043 8\cdots$$
$$+ 1.458816616938495229330889612903675257158\cdots i$$

from which we can estimate that

$$g_2(L) = 1.33333\cdots, \quad g_3(L) = -0.703703703\cdots,$$

$$\mathcal{P}(\omega_1/5, L) = 0.66666\cdots, \quad \mathcal{P}'(\omega_1/5, L) = -1.00000\cdots.$$

Here $\mathcal{P}(z, L)$ stands for the Weierstrass $\mathcal{P}$-function attached to the lattice L. Thus it turns out that the point of $X_1(5)$ corresponding to the pair (E_1, P_1) is ω_2/ω_1. Using the Fourier expansion of $f(z)$ we can find $u(\omega_2/\omega_1) = 1.00000\cdots$ and $v(\omega_2/\omega_1) = -1.00000\cdots$. Therefore we are forced to have $t(z) = u(z)$.

(ii) Let $g(z) = N(j_{1,6})(z) + 1$. Then it is immediate from Table 4 that the values of $g(z)$ at the cusps of $X_1(6)$ are as follows:

s	∞	0	1/3	1/2
$g(s)$	∞	6	-2	-3

Since $\Delta(E_t) = (t-1)^3 t^6 (9t-1)$ from the equation of E_t in (22), the set of possible values of $t(z)$ at the cusps are $\{\infty, 1, 0, 1/9\}$. Since $t(z)$ is a fractional linear transformation of $g(z)$, we have the equality

$$[g(\infty), g(0), g(1/3), g(1/2)] = [t_1, t_2, t_3, t_4]$$
$$[\infty, 6, -2, g(z)] = [t_1, t_2, t_3, t(z)]$$

where $t_1 = t(\infty), t_2 = t(0), t_3 = t(1/3), t_4 = t(1/2)$. Thus we establish

(24)
$$\frac{(t(z)-t_1)(t_2-t_3)}{(t(z)-t_3)(t_2-t_1)} = \frac{8}{g(z)+2}.$$

Suppose $t(s) = \infty$ for some cusp s. We let h be the width of the cusp s and consider the q_h-expansion of $t(z)$ at s. We choose an element $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma\infty = s$. It then follows that $t|_\gamma = \frac{c}{q_h} + O(1)$ for some $c \in \mathbb{C}$. Now, from the identity

$$j = \frac{((1-3t)(1-9t+3t^2-3t^3))^3}{t^6(1-t)^3(1-9t)}$$

we see that $\frac{1}{q} + O(1) = \frac{1}{q_h^2} + O(1)$. This yields $h = 2$. It then follows from Table 2 that $s = 1/3$ and hence $t_3 = t(1/3) = \infty$. Similarly if $t(s) = 0$, then we come up with $\frac{1}{q} + O(1) = \frac{1}{q_h^6} + O(1)$. Thus we have $h = 6$ and $s = 0$. And we deduce that $t_2 = t(0) = 0$. Therefore, the identity (24) is simplified to

(25)
$$\frac{t(z)-t_1}{-t_1} = \frac{8}{g(z)+2}.$$

Here we have two choices for the values t_1 and t_4: $t_1 = 1$ and $t_4 = 1/9$, or $t_1 = 1/9$ and $t_4 = 1$. Only the latter case fits the identity (25), from which we get the assertion as desired. □

According to the referee's comment we can have canonical bijections $\mathbb{P}^1 \to X_0(N)$ which send $r \mapsto (E_r, C_r)$ in terms of the normal forms

$$E_r : \begin{cases} y^2 = x^3 + \frac{2(r+64)}{r^2}x^2 + \frac{r+64}{r^3}x, & \text{if } N = 2; \\ y^2 + \frac{3(r+27)}{r}xy + \frac{(r+27)^2}{r^2}y = x^3, & \text{if } N = 3; \\ y^2 + \frac{2(2r+25)}{r}xy + \frac{4(r^2+22r+125)}{r^2}y = x^3 + \frac{r+10}{r}x^2, & \text{if } N = 5; \\ y^2 + \frac{5r+36}{r}xy + \frac{9(r+8)(r+9)}{r^2}y = x^3 + \frac{2(r+9)}{r}x^2, & \text{if } N = 6; \end{cases}$$

and cyclic subgroups $C_r =< (x : y : 1) \mid \psi_r(x) = 0 >$ of order N which are generated by the roots of certain divisors of the division polynomials:

$$\psi_r(x) = \begin{cases} x & \text{if } N = 2; \\ x & \text{if } N = 3; \\ 5x^2 - \frac{4(r^2+22r+125)}{r^2} & \text{if } N = 5; \\ x & \text{if } N = 6. \end{cases}$$

Using the "forgetful" maps $X_1(N) \to X_0(N)$, one has the expressions

$$r = \begin{cases} 64t/(1-t), & \text{if } N = 2; \\ 27t/(1-t), & \text{if } N = 3; \\ 125t/(1-11t-t^2), & \text{if } N = 5; \\ 72t/(1-9t), & \text{if } N = 6. \end{cases}$$

Clearly the function field of $X_0(N)$ is $k(r)$ in these cases; it may be thought of as an algebraic extension of $k(j)$ which is contained in $k(t)$. These curves are chosen on the parameter r. For $z \in \mathfrak{H}^*$, define the hauptmoduli

$$r(z) = \begin{cases} \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} = \frac{1}{q} - 24 + 276q - 2048q^2 + \cdots & \text{if } N = 2; \\ \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} = \frac{1}{q} - 12 + 54q - 76q^2 + \cdots & \text{if } N = 3; \\ \left(\frac{\eta(z)}{\eta(5z)}\right)^{6} = \frac{1}{q} - 6 + 9q + 10q^2 + \cdots & \text{if } N = 5; \\ \frac{\eta(z)^5\eta(3z)}{\eta(2z)\eta(6z)^5} = \frac{1}{q} - 5 + 6q + 4q^2 + \cdots & \text{if } N = 6, \end{cases}$$

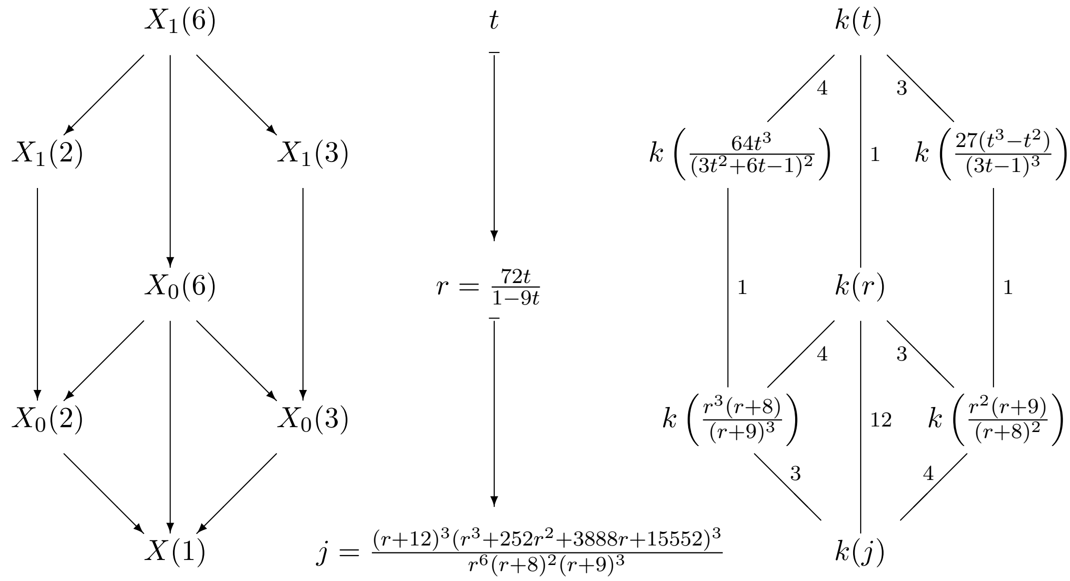
in terms of the Dedekind eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad \text{for } q = e^{2\pi i z}.$$

We may summarize all of this discussion in a lattice diagram of function fields. As for $X_1(5)$, the "forgetful" maps correspond to the following for a field of k of characteristic not dividing 5:

$$\begin{array}{ccccc} X_1(5) & & t & & k(t) \\ \downarrow & & \downarrow & & \Big| \, 2 \\ X_0(5) & & r = \frac{125t}{1-11t-t^2} & & k(r) \\ \downarrow & & \downarrow & & \Big| \, 6 \\ X(1) & & j = \frac{(r^2+250r+3125)^3}{r^5} & & k(j) \end{array}$$

For $X_1(6)$, the "forgetful" maps correspond to the following for a field of k of characteristic not dividing 6:



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