Periodic Hamiltonian flows with almost minimality conditions

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Abstract

The primary aim of this paper is to investigate some general properties of a Hamiltonian circle action under certain minimality condition. As main applications of our techniques of this paper, we show the existence and also non-existence theorems of a Hamiltonian circle action of pure type on a compact symplectic manifold under the assumption that the fixed-point set has the smallest possible number of components and satisfies a certain non-minimality condition. Those theorems can be regarded as a first step towards the classification of higher dimensional closed symplectic manifolds admitting a Hamiltonian circle action, and provide some constraints to the existence of certain Hamiltonian circle actions. Some new general formulas for the S^1-equivariant Euler class of the negative normal bundle of a fixed-point component which might be of independent interest play a crucial role in obtaining main applications of this paper.

Keywords: symplectic manifold, Hamiltonian circle action, moment map, minimality condition, almost minimality condition, equivariant cohomology, Chern classes, S^1-equivariant Euler class.

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1 Introduction and Main Results

Throughout this paper, the action of a Lie group on a manifold is always assumed to be effective.

Let (M, ω) be a compact symplectic manifold with a non-degenerate closed 2-form ω. A circle action on M is called **symplectic** if it preserves ω. A symplectic circle action on M is **Hamiltonian** if there is a moment map $\phi : M \to \mathbf{R}$ such that

$$\iota_{\xi_M}\omega = -d\phi,$$

where ξ_M denotes the fundamental vector field induced by the circle action. A compact symplectic manifold with a Hamiltonian circle action will be often called a **Hamiltonian S^1-space**, for simplicity.

As is well-known, Audin, Ahara-Hattori, and Karshon have completed the classification of four dimensional closed symplectic manifolds admitting a Hamiltonian circle action (see [1], [2], and [6]). Recently there have been some interesting works of the classifications of higher dimensional Hamiltonian S^1-spaces satisfying certain minimality conditions (see, e.g., [4], [7], [8], [9], and [11]). In a similar vein, the goal of this paper is to show some existence and also non-existence theorems of a Hamiltonian circle action of pure type on a compact symplectic manifold under the assumption that the fixed-point set has the smallest possible number of components and satisfies a certain non-minimality condition. So results of this paper as well as in [11] and [9] can be regarded as a first step towards the classification of higher dimensional closed symplectic manifolds admitting a Hamiltonian circle action, and give some constraints to the existence of certain Hamiltonian circle actions. They are, in fact, some consequences of general formulas for the S^1-equivariant Euler class of the negative normal bundle of a fixed-point component which are of independent interest in their own right. We also remark that our results are essentially related to the fundamental questions about the existence of a Lie group action on a given manifold posed by T. Petrie (see [12] and [13]).

In order to explain our results more precisely, let (M, ω) be a compact Hamiltonian S^1-space with moment map $\phi : M \to \mathbf{R}$. Then ϕ is a perfect Morse-Bott function whose critical point set is exactly the fixed-point set M^{S^1} (refer to [2] and [10]). Since M is a compact symplectic manifold equipped with a non-degenerate closed 2-form ω, it is obvious that for all $0 \leq i \leq \frac{1}{2}\dim M$, $[\omega]^i$ represents a non-zero class in $H^{2i}(M; \mathbf{R})$. Hence, the following inequality always holds:

$$\sum_{F \subset M^{S^1}} (\dim F + 2) \geq \dim M + 2. \tag{1}$$

Recall also that each fixed connected component F is a symplectic submanifold of even codimension in M.

It is quite natural to consider the extreme case of the above inequality (1), as follows.

Definition 1.1. A Hamiltonian S^1-space (M, ω) is said to satisfy the **minimal dimension condition** if

$$\sum_{F \subset M^{S^1}} (\dim F + 2) = \dim M + 2$$

holds.

On the other hand, there is also the notion of a minimality condition in terms of the Betti numbers.

Definition 1.2. A Hamiltonian S^1-space (M, ω) is said to satisfy the **minimal even Betti number condition** if all even Betti numbers $b_{2i}(M) = 1$ for all $0 \leq i \leq \frac{1}{2}\dim M$.

Since the moment map ϕ is a perfect Morse-Bott function, it can be easily shown that if M satisfies the minimal even Betti number condition, then the Hamiltonian circle action satisfies the minimal dimension condition ([9], Lemma 4.1). However, the converse is not true, in general. On the other hand, if M is 6-dimensional, the minimal even Betti number condition is equivalent to the minimal dimension condition (see Remark 4.3 of [9] for more details).

In their paper [9], Li and Tolman have considered the case that the fixed point set has the smallest possible number of components and these components have the smallest possible dimension (see also [3] for some earlier work). Under the smallest possible conditions as above, they proved in [9] that the cohomology ring $H^*(M;\mathbf{Z})$ and the total Chern class $c(M)$ of M are identical to those of one of the complex projective space $\mathbf{CP}^{n+1}$ and the Grassmannian $\tilde{Gr}_2(\mathbf{R}^{n+2})$ of oriented two-planes in $\mathbf{R}^{n+2}$ ([9], Theorem 1). To be precise, their main results can be summarized, as follows.

Theorem 1.3. *Let the circle act on a compact symplectic manifold (M,ω) of dimension $2n$ with moment map $\phi : M \to \mathbf{R}$. Assume that the fixed-point set consists of exactly two connected components X and Y and that*

$$\dim X + \dim Y + 2 = \dim M.$$

Then one of the following holds:

(a) *The action is semifree, $H^*(M;\mathbf{Z}) = \mathbf{Z}[x]/(x^{n+1})$, and the total Chern class $c(M) = (1+x)^{n+1}$.*

(b) *The action is non-semifree, $n \neq 1$ is odd, $H^*(M;\mathbf{Z}) = \mathbf{Z}[x,y]/(x^{\frac{1}{2}(n+1)} - 2y, y^2)$, and the total Chern class $c(M) = \frac{(1+x)^{n+2}}{1+2x}$.*

Here x has degree 2 and y has degree $n+1$.

Note that as a group $H^i(M;\mathbf{Z})$ is isomorphic to $H^i(\mathbf{CP}^n;\mathbf{Z}) = \mathbf{Z}$ for both cases in Theorem 1.3. In addition, it is interesting to note from a recent paper of Li-Olbermann-Stanley ([8], Theorem 1.4) that the fundamental groups of X, Y, and M as in Theorem 1.3 are all trivial.

In the present paper, we are interested in the somehow next simplest case that the fixed point has the smallest possible number of components but satisfies a certain non-minimal dimension condition. Thus this paper can be regarded as a continuation of a systematic investigation of Hamiltonian circle actions satisfying certain minimality conditions as in [9], and will provide some more topological understanding of such compact Hamiltonian S^1-spaces. If the fixed point set of a Hamiltonian torus action on a symplectic manifold M consists of exactly two connected components, then the symplectic manifold M is usually called a **simple Hamiltonian manifold** in the literature (see, e.g., [5]).

For the purposes of this paper, we need the following definition.

Definition 1.4. A Hamiltonian S^1-space (M, ω) is said to satisfy the **almost minimal dimension condition** if

$$\sum_{F \subset M^{S^1}} (\dim F + 2) = \dim M + 4$$

holds.

In particular, if the fixed point set M^{S^1} consists of exactly two connected components X and Y, i.e., if M is a simple Hamiltonian manifold under a Hamiltonian circle action, then the almost minimal dimension condition implies that

$$\dim X + \dim Y = \dim M.$$

More generally, for simple manifolds the following identity always holds:

$$\dim X + \dim Y = \dim M + 2l, \quad l \geq -1.$$

It is fairly easy to show that there does not exist a Hamiltonian circle action on a symplectic manifold of dimension 4 with exactly two fixed point components, X and Y, such that

$$\dim X + \dim Y = \dim M + 2l, \quad l \geq 1,$$

due to the obvious dimensional reason. On the other hand, there exists a Hamiltonian circle action on a compact symplectic manifold of dimension 4 with exactly two fixed connected components satisfying the almost minimality condition.

Example 1.5. Consider a Hamiltonian circle action on $S^2 \times S^2$ which acts on the first factor by the standard rotation and acts trivially on the second factor. Then the fixed point sets of the circle action consist of only two components $\{NP\} \times S^2$ and $\{SP\} \times S^2$, where NP and SP denote the north and south pole of the first factor S^2, respectively. Moreover, the sum of the dimensions of the fixed point sets is equal to 4 that is the dimension of $S^2 \times S^2$.

As mentioned earlier, Karshon gave a classification of four dimensional closed symplectic manifolds admitting a Hamiltonian circle action (see [6]). According to her classification, every compact four dimensional symplectic manifold with a Hamiltonian circle action can be obtained by a sequence of S^1-equivariant symplectic blow-ups from a minimal space. Here the minimal space is either the complex projective space $\mathbf{CP}^2$, or a Hirzebruch surface $\mathbf{H}_k$, or a space with two fixed surfaces and no interior fixed points. It is well known that $\mathbf{H}_k$ is diffeomorphic to $S^2 \times S^2$ or the twisted S^2-bundle over S^2, depending on whether k is even or odd, respectively. Moreover, the spaces with two fixed surfaces and no interior fixed points turn out to be **ruled manifolds**. They are defined to be an S^2-bundle over a closed Riemann surface with a circle action that fixes the base and rotates each fiber, and an invariant symplectic form and a moment map. In fact, they admits a compatible Kähler structure. The north and south pole of each fiber fit together to form two fixed surfaces which are both diffeomorphic to the base Riemann surface, while by definition the action is free at all other points. Thus the circle action on the ruled manifolds is always semifree (see [6], Section 6.3 for more details).

It is interesting to note that there also exists a Hamiltonian circle action on a compact symplectic manifold of dimension $4n \geq 8$ whose fixed point set consists of the complex projective spaces and satisfies the almost minimality condition.

Example 1.6. Consider the Grassmannian $\tilde{Gr}_2(\mathbf{R}^{2n+2})$ of oriented two-planes in $\mathbf{R}^{2n+2}$. Then $SO(2n+2)$ acts on the space $\mathbf{R}^{2n+2}$ in a natural way, and so the circle S^1 acts on $\mathbf{R}^{2n+2}$ as the diagonal of $SO(2n+2)$. This gives rise to a Hamiltonian circle action on $\tilde{Gr}_2(\mathbf{R}^{2n+2})$. Observe that the fixed point set consists of two connected components which correspond to two orientations on the complex projective space $\mathbf{P}(\mathbf{C}^{n+1}) = \mathbf{CP}^n$.

Assume now that X denotes the minimal fixed point component of a Hamiltonian S^1-space M. Let N_X be the normal bundle of X in M. Then the S^1-equivariant Euler class $e^{S^1}(N_X)$ of N_X lies in $H^{\dim M - \dim X}_{S^1}(X; \mathbf{Z})$. Moreover, it follows from [9], Proposition 2.2 (or Lemma 2.2 below) that there is an element $\lambda \in H^*_{S^1}(X; \mathbf{R}) = H^*(X; \mathbf{R}) \otimes \mathbf{R}[t]$ such that

$$\prod_{F \subset (M \setminus X)^{S^1}} ([\omega|_X] + t(\phi(F) - \phi(X)))^{\frac{1}{2}\dim F + 1} = \lambda \cdot e^{S^1}(N_X). \tag{2}$$

Then the following notion will play an important role in the applications of this paper.

Definition 1.7. A compact Hamiltonian S^1-space (M, ω) with moment map ϕ will be said to be **of pure type** if λ appearing in the identity (2) is of pure type in that λ is an element of $H^*(\mathbf{CP}^\infty; \mathbf{R}) \cong \mathbf{R}[t]$.

In Section 2, we will provide certain criteria for a Hamiltonian circle action with the almost minimality condition to be of pure type (see Theorems 2.3 and 2.4). Moreover, if X consists of just one point, then the Hamiltonian circle action is trivially of pure type. It is also worth mentioning that any Hamiltonian circle action on a compact symplectic manifold with exactly two fixed connected components satisfying the minimal dimension condition is automatically of pure type (see, e.g., the proof of Theorem 2.3).

So it seems to be quite a natural question to ask the existence of a Hamiltonian circle action of pure type on a compact symplectic manifold satisfying the almost minimality condition, and this question will constitute our main applications in this paper. In view of the result of Li and Tolman (Theorem 1.3 and its proof) and some existing examples, it also appears to be reasonable to first begin with investigating the case that each connected component of the fixed point set has the same cohomology ring as the complex projective space. So, let X and Y be two connected components of the fixed point set of a Hamiltonian S^1-space (M, ω). Then the following two conditions (**C1**) and (**C2**) will be used throughout this paper without further mentioning:

(C1) $\dim X + \dim Y = \dim M$.

(C2) X (resp. Y) has the same cohomology ring of the complex projective space $\mathbf{CP}^{\frac{1}{2}\dim X}$ (resp. $\mathbf{CP}^{\frac{1}{2}\dim Y}$) over integer coefficients.

Given a manifold M, there is a natural map from $H^*(M, \mathbf{Z})$ to $H^*(M; \mathbf{R})$. The image of the map forms a lattice in $H^*(M; \mathbf{R})$. A class in $H^*(M; \mathbf{R})$ is called **integral** if it lies in the image of the map, and is called **primitive** if, in addition, it is not a positive integer multiple of any other integral class. For the sake of simplicity, in this paper we will also say that a symplectic class $[\omega]$ is a primitive integral class **in the strong sense** if $[\omega]$ is a primitive integral class and its restriction to any fixed connected component of the circle action again induces a primitive integral class.

With these preliminaries in place, our first main application of our investigation which might be also derived with some computations from the results of Karshon in [6] and, we think, illustrates our techniques very well is

Theorem 1.8. *Let the circle act on a compact symplectic manifold (M, ω) of dimension 4 with moment map $\phi : M \to \mathbf{R}$ of pure type, where $[\omega]$ is a primitive integral class in the strong sense. Assume that the fixed point set of the circle action has exactly two connected components X and Y satisfying* (**C1**) *and* (**C2**). *If we let $m = \phi(Y) - \phi(X) > 0$, then the following properties hold:*

(a) *The circle action is always semifree and $m = 1$.*

(b) $H^*(X; \mathbf{Z}) = \mathbf{Z}[u]/u^2$ *and* $H^*(Y; \mathbf{Z}) = \mathbf{Z}[v]/v^2$, *where* $u = [\omega|_X]$ *and* $v = [\omega|_Y]$.

(c) $c(X) = 1 + 2u$ *and* $c(Y) = 1 - 2v$.

(d) $c(N_X) = 1 + 2u$ *and* $c(N_Y) = 1 - 2v$.

Remark 1.9. (a) Two items (b) and (c) of Theorem 1.8 are almost trivial, but they are stated simply in order to provide some notations used in the statement Theorem 1.8 (d).

(b) It follows from the equation (9) at the end of Section 2 that the condition that λ is of pure type in our case is equivalent to saying that

$$C_1 = \frac{\left(\frac{1}{2}\dim Y + 1\right)\Lambda_X}{m}[\omega|_X] \neq 0,$$

where Λ_X denotes the product of the weights with multiplicities on the normal bundle N_X of X and C_1 denotes the coefficient of $t^{\frac{1}{2}\dim Y - 1}$ in $e^{S^1}(N_X)$ as in

(7) (so, we have $C_1 = c_1(N_X)$ in the semifree case). In particular, this implies that Theorem 1.8 does not apply to the case that $c_1(N_X)$ is trivial. Recall that Hirzebruch surfaces $\mathbf{H}_k$ which form a large class of good examples for Theorem 1.8 are diffeomorphic to $S^2 \times S^2$ or the twisted S^2-bundle over S^2, depending on whether k is even or odd, respectively. So a Hamiltonian circle action on $\mathbf{H}_k$ is of pure type only if k is odd, which fits well with Theorem 1.8 (4).

Remark 1.10. Note also that, if we drop the condition that $[\omega]$ is a primitive integral class in the strong sense, i.e., its restriction to any fixed connected component of the circle action induces a primitive integral class, then clearly Theorem 1.8 is no longer true. For a concrete illustration, consider a semifree Hamiltonian circle action on $\mathbf{CP}^1 \times \mathbf{CP}^1$ given by $t \cdot ([z_0, z_1], [w_0, w_1]) = ([tz_0, z_1], [w_0, w_1])$ with an invariant symplectic form whose cohomology class is $n_1\alpha_1 + n_2\alpha_2$, where α_1 and α_2 denote two generators of the integral cohomology ring of $\mathbf{CP}^1 \times \mathbf{CP}^1$. Then the difference m between the levels of moment map at two fixed components $\{[1,0]\} \times \mathbf{CP}^1$ and $\{[0,1]\} \times \mathbf{CP}^1$ is n_1 which is arbitrary. Exactly the same argument applies to any twisted $\mathbf{CP}^1$-bundle over $\mathbf{CP}^1$. On the other hand, in higher dimensions we do not need any extra assumption except that $[\omega]$ is a primitive integral class. This is due to Proposition 3.1 (2).

It can be shown as in Proposition 3.1 that the dimension of a compact symplectic manifold equipped with a Hamiltonian circle action satisfying the conditions **(C1)** and **(C2)** should be equal to zero modulo four. In contrast to Theorem 1.8, for higher dimensions we can show a non-existence theorem about a Hamiltonian circle action of pure type on a compact symplectic manifold with exactly two fixed connected components under the almost minimal dimension condition which can be regarded as another application about Hamiltonian S^1-spaces with almost minimality condition, as follows.

Theorem 1.11. *Let the circle act on a compact symplectic manifold (M, ω) of dimension $4n$ ($n \geq 2$) with moment map $\phi : M \to \mathbf{R}$ of pure type, where $[\omega]$ is a primitive integral class, whose fixed point set consists of exactly two fixed connected components X and Y satisfying the almost minimal dimension condition* **(C1)***. Then there does not exist any such Hamiltonian circle action on M such that both of X and Y have the same cohomology rings of the complex projective spaces over integer coefficients.*

Remark 1.12. A careful analysis of the proof of Theorem 1.8 shows that one may replace the condition (**C2**) in Theorem 1.8 by a slightly weaker condition by assuming that $\dim X \geq \dim Y$ and that only X has the same cohomology ring of the complex projective space over integer coefficients. This remark also applies to Theorem 1.11. Recall that there exists a Hamiltonian circle action on $\tilde{Gr}_2(\mathbf{R}^{2n+2})$ whose fixed point set consists of the complex projective spaces and satisfies the almost minimality condition. As a consequence of Theorem 1.11, we can conclude that the Hamiltonian circle action on $\tilde{Gr}_2(\mathbf{R}^{2n+2})$ is **not** of pure type.

The proof of Theorem 1.11 will be given in Sections 5 and 6. In fact, we divide the proof into two cases: semifree case and non-semifree case. A new general formula for the S^1-equivariant Euler class of the negative normal bundle of a fixed-point component will play a crucial role. Theorem 1.11 as well as Theorems 1.3 and 1.8 gives a classification of Hamiltonian circle actions on a compact symplectic manifold with exactly two fixed point components for certain special and interesting cases.

Moreover, we remark that Theorem 1.11 might be no longer true, if we drop the condition (**C1**) or (**C2**) of Theorem 1.11.

Example 1.13. Indeed, there does exist a Hamiltonian circle action on a compact symplectic manifold M of dimension $2n \geq 6$ with exactly two fixed point components, X and Y satisfying the condition **(C1)** such that one (or both) of X and Y does not have the cohomology ring of the complex projective space. For example, let N be a compact symplectic manifold with a Hamiltonian circle action whose fixed point set consists of two connected components X' and Y' such that $\dim X' + \dim Y' + 2 = \dim N$. Then consider the product symplectic manifold $M = N \times S^2$ equipped with an obvious Hamiltonian circle action which acts trivially on S^2. Then the fixed point set consists of two connected components $X = X' \times S^2$ and $Y = Y' \times S^2$, and $\dim X + \dim Y = \dim M$ holds. Note that from a result of [9] the cohomology ring of X' is isomorphic to $\mathbf{CP}^{\frac{1}{2}\dim X'}$. Thus the cohomology of X is clearly *not* isomorphic to that of $\mathbf{CP}^{\frac{1}{2}\dim X}$ unless $\dim X'$ is equal to zero, but the almost minimal dimension condition still holds.

Further, there exists a Hamiltonian circle action on a compact symplectic manifold M of dimension greater than or equal to 6 with exactly two fixed point components, X and Y, satisfying the dimension condition $\dim X + \dim Y = \dim M + 2l$ for some integer l greater than or equal to -1.

Example 1.14. As a concrete example, take the Grassmannian manifold $Gr_k(\mathbf{C}^r)$ of unoriented complex k-planes in $\mathbf{C}^r$ with a Hamiltonian circle action whose fixed point set consists of two Grassmannian manifolds $Gr_{k-1}(\mathbf{C}^{r-1})$ and $Gr_k(\mathbf{C}^{r-1})$. Then clearly we have

$$\dim Gr_k(\mathbf{C}^r) + 2l = 2k(r-k) + 2l = \dim Gr_{k-1}(\mathbf{C}^{r-1}) + \dim Gr_k(\mathbf{C}^{r-1}),$$

where $l = k(r-k) - r$ (see Example 1.4 of [5] for more details). However, the cohomology ring of $Gr_p(\mathbf{C}^q)$ is not isomorphic to that of $\mathbf{CP}^{p(q-p)}$, unless $p = 1$ or $p = q-1$. So these examples do not contradict Theorem 1.11.

In addition, Theorems 1.8 and 1.11 also partially answer a question (Question in [9], Section 1) raised by Li and Tolman about the existence of an effective Hamiltonian circle action with two fixed point components whose stabilizer is $\mathbf{Z}_k$ for $k > 2$. In fact, the circle acts on the fiber of the normal bundle of X with weights $\{1, 1, \ldots, 1, 1\}$ for the case (1) of Theorem 1.3 and with weights $\{2, 2, \cdots, 2, 1\}$ for the case (2) of Theorem 1.3, assuming that X is the minimum of the moment map ϕ. So there exists a point whose stabilizer is $\mathbf{Z}_k$ with $k = 1$ or 2 for a Hamiltonian circle action with exactly two fixed point components (here, $\mathbf{Z}_1 = \{0\}$). Moreover, by using a combinatorial argument involving with the weights, Li and Tolman proved in the appendix of the paper [9] that for a Hamiltonian circle action with exactly two fixed point components, no point has stabilizer $\mathbf{Z}_k$ for $k > 6$. Then they raised a question of the existence of an example with stabilizer $\mathbf{Z}_k$ for any $k > 2$. As an interesting by-product of the classification results (Theorems 1.8 and 1.11), we can now negatively answer their question (Question in [9], Section 1) in our case, as follows:

Theorem 1.15. *Let the circle act on a compact symplectic manifold (M, ω) of dimension $4n$ $(n \geq 1)$ with moment map of pure type, where $[\omega]$ is a primitive integral class such that the fixed point set consists of exactly two fixed point components X and Y satisfying the conditions* **(C1)** *and* **(C2)**. *Then there is no point with stabilizer $\mathbf{Z}_k$ for any $k \geq 2$.*

This theorem makes more sense for dimension 4, since in higher dimensions there does not exist a Hamiltonian S^1-space of pure type satisfying two conditions **(C1)** and **(C2)**, as in Theorem 1.11.

This paper is organized as follows. In Section 2, we first prove a certain formulas regarding the S^1-equivariant Euler class of the normal bundle of a certain fixed point set of the circle action on a compact symplectic manifold. It will play an important role in classifying the symplectic circle actions on a compact symplectic manifold with moment map. In Section 3, we show some preparatory results which are needed to show main theorems in Sections 5 and 6. Section 4 is devoted to collecting some general facts already proved by Li and Tolman in [9] which can be still applied to our settings.

Finally, we give proofs of our main Theorems 1.8 and 1.11 in Sections 5 and 6. We first deal with a semifree case in Section 5, and then we complete the proof of Theorem 1.11 for a non-semifree case, in Section 6. In particular, in Section 5 we give a classification of certain semifree Hamiltonian circle actions on a compact symplectic manifold M of dimension 4 with exactly two fixed connected components X and Y satisfying the almost minimal dimension condition $\dim X + \dim Y = \dim M$ (Theorem 5.1).

2 S^1-equivariant Euler classes

The goal of this section is to derive some formulas regarding the S^1-equivariant Euler class of the normal bundle of a certain fixed point set of the circle action on a compact symplectic manifold. This formula will play an important role in classifying the symplectic circle actions on a compact symplectic manifold with moment map of pure type in Sections 5 and 6.

When a circle acts on a manifold M, recall that the equivariant cohomology $H^*_{S^1}(M; R)$

is defined to be the ordinary cohomology

$$H^*(M \times_{S^1} ES^1; R), \quad R = \mathbf{Z} \text{ or } \mathbf{R},$$

where ES^1 denotes the total space of the universal S^1-bundle over the classifying space BS^1.

Throughout this section, we assume that the circle acts on a symplectic manifold (M, ω) of dimension $2n \geq 4$ with exactly two fixed point components, X and Y, such that

$$\dim X + \dim Y = \dim M + 2l, \quad l \geq -1$$

unless stated otherwise. For simplicity, let X denote the minimal fixed point component of the circle action and let N_X be the normal bundle of X in M. Then the S^1-equivariant Euler class $e^{S^1}(N_X)$ of N_X lies in $H_{S^1}^{\dim M - \dim X}(X; \mathbf{Z}) = H_{S^1}^{\dim Y}(X; \mathbf{Z})$ for the case of $l = 0$. More generally, if $\dim X + \dim Y = \dim M + 2l$ for $l \geq -1$, then $e^{S^1}(N_X)$ lies in $H_{S^1}^{\dim Y - 2l}(X; \mathbf{Z})$.

As before, let Λ_X be the product of the weights with multiplicities on the normal bundle N_X of X and let t denote a generator of $H^*(\mathbf{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[t]$. We then recall the following lemma ([9], Lemma 2.3).

Lemma 2.1. *Let the circle act on a compact symplectic manifold (M, ω) with moment map $\phi : M \to \mathbf{R}$. Let F be a fixed point component. Then there exists an element $\tilde{u}$ in $H_{S^1}^2(M; \mathbf{R})$ such that*

$$\tilde{u}|_{F'} = [\omega|_{F'}] + t(\phi(F) - \phi(F'))$$

for all fixed components F'. Moreover, if $[\omega]$ is an integral class, then $\tilde{u}$ is also an integral class and $\phi(F) - \phi(F')$ is an integer.

We also need recall the following interesting lemma ([9], Proposition 2.2).

Lemma 2.2. *Let $\beta \in H_{S^1}^*(M; \mathbf{R})$. If $\beta|_{F'} = 0$ for all fixed point components F' with $\phi(F') < \phi(F)$, then $\beta|_F$ is a multiple of $e^{S^1}(N_F^-)$, i.e.,*

$$\beta|_F = \lambda \cdot e^{S^1}(N_F^-)$$

for some $\lambda \in H_{S^1}^*(X; \mathbf{R}) = H^*(X; \mathbf{R}) \otimes \mathbf{R}[t]$. Here, N_F^- denotes the negative normal bundle of a fixed point component F.

Next, we compute the S^1-equivariant Euler class $e^{S^1}(N_X)$ of the normal bundle N_X.

Theorem 2.3. *Let the circle act on a compact symplectic manifold (M, ω) with moment map $\phi : M \to \mathbf{R}$ of pure type. Assume that the fixed point set consist of only two connected components X and Y such that*

$$\dim X + \dim Y = \dim M + 2l, \quad l \geq -1.$$

Then the following identity holds:

$$t^{l+1}e^{S^1}(N_X) = \Lambda_X \left(t + \frac{[\omega|_X]}{\phi(Y) - \phi(X)}\right)^{\frac{1}{2}\dim Y + 1}. \tag{3}$$

Proof. Since the case of $l = -1$ has already been dealt with in the paper [9], Lemma 4.8 of Li and Tolman, we will consider only the case of $l \geq 0$.

By Lemma 2.1, there exists $\tilde{u}$ in $H^2_{S^1}(M; \mathbf{R})$ such that

$$\tilde{u}|_Y = [\omega|_Y] + t(\phi(X) - \phi(Y)).$$

Let

$$\beta = (\tilde{u} + t(\phi(Y) - \phi(X)))^{\frac{1}{2}\dim Y + 1}.$$

Then clearly we have

$$\beta|_Y = (\tilde{u}|_Y + t(\phi(Y) - \phi(X)))^{\frac{1}{2}\dim Y + 1} = [\omega|_Y]^{\frac{1}{2}\dim Y + 1} = 0,$$

due to the dimensional reason. Now, if we apply Lemma 2.2 to β with $-\phi$, there exists $\lambda \in H^*_{S^1}(X; \mathbf{R})$ such that

$$\begin{aligned} \beta|_X &= ([\tilde{u}|_X] + t(\phi(Y) - \phi(X)))^{\frac{1}{2}\dim Y + 1} \\ &= ([\omega|_X] + t(\phi(Y) - \phi(X)))^{\frac{1}{2}\dim Y + 1} \\ &= \lambda \cdot e^{S^1}(N_X). \end{aligned} \tag{4}$$

Since λ lies in $H^*(\mathbf{CP}^\infty; \mathbf{R}) = \mathbf{R}[t]$ by assumption, we may set

$$\lambda = a_0 + a_1 t + \cdots + a_p t^p, \quad a_i \in \mathbf{R}.$$

We now assume that $l = 0$. Then the S^1-equivariant Euler class $e^{S^1}(N_X)$ is of the form

$$e^{S^1}(N_X) = \Lambda_X t^{\frac{1}{2}\dim Y} + C_1 t^{\frac{1}{2}\dim Y - 1} + \cdots + C_{\frac{1}{2}\dim Y - 1} t + C_{\frac{1}{2}\dim Y},$$

where $C_i \in H^{2i}(X; \mathbf{R})$. Hence, we have the following

$$\begin{aligned} \lambda \cdot e^{S^1}(N_X) = {} & (a_0 + a_1 t + \cdots + a_p t^p) \\ & \cdot (\Lambda_X t^{\frac{1}{2}\dim Y} + C_1 t^{\frac{1}{2}\dim Y - 1} + \cdots + C_{\frac{1}{2}\dim Y - 1} t + C_{\frac{1}{2}\dim Y}). \end{aligned} \tag{5}$$

Since the non-zero highest degree of t of the left hand side of the equation (4) is $\frac{1}{2}\dim Y + 1$, Λ_X is non-zero, and C_i are elements of degree $2i$ of $H^{2i}(X; \mathbf{R})$, it follows from (5) that we should have $a_2 = a_3 = \cdots = a_p = 0$ for $p \geq 2$. Thus it follows that $\lambda = a_0 + a_1 t$.

Next, by comparing the coefficients of $t^{\frac{1}{2}\dim Y+1}$ and $t^{\frac{1}{2}\dim Y}$ of both sides of the equation (4), we obtain

$$
\begin{aligned}
a_1\Lambda_X &= (\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y+1}, \\
a_0\Lambda_X + C_1a_1 &= \left(\frac{1}{2}\dim Y+1\right)[\omega|_X](\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y}.
\end{aligned} \tag{6}
$$

By comparing the cohomological degrees of both sides of the second equation of (6), it is easy to see that $a_0 = 0$. Finally, it follows from the first equation of (6) that

$$a_1 = \frac{1}{\Lambda_X}(\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y+1}.$$

Hence, we have

$$te^{S^1}(N_X) = \Lambda_X\left(t + \frac{[\omega|_X]}{\phi(Y)-\phi(X)}\right)^{\frac{1}{2}\dim Y+1},$$

which completes the proof for the case $l = 0$.

Next, we deal with the general case of $l > 0$. This time, we have

$$\mathrm{rank}_{\mathbf{R}} N_X = \dim M - \dim X = \dim Y - 2l.$$

Thus, $e^{S^1}(N_X) \in H^{\dim M - \dim X}_{S^1}(X;\mathbf{R}) = H^{\dim Y - 2l}_{S^1}(X;\mathbf{R})$.

Let λ be the same as before. Then we have

$$
\begin{aligned}
\lambda\cdot e^{S^1}(N_X) = (a_0 + a_1t + \cdots + a_pt^p)\cdot(&\Lambda_X t^{\frac{1}{2}\dim Y - l} \\
&+ C_1t^{\frac{1}{2}\dim Y-l-1} + \cdots + C_{\frac{1}{2}\dim Y-l-1}t + C_{\frac{1}{2}\dim Y-l}).
\end{aligned} \tag{7}
$$

The highest degree of t of the right hand side of the equation (7) is $\frac{1}{2}\dim Y - l + m$, while the non-zero highest degree of t of the left hand side of the equation (4) is $\frac{1}{2}\dim Y + 1$. Thus at least we should have

$$\frac{1}{2}\dim Y - l + p \leq \frac{1}{2}\dim Y + 1, \text{ i.e., } p \leq l+1.$$

This implies that $a_{l+2} = a_{l+3} = \cdots = a_p = 0$ for all $m \geq l+2$. Next, we compare the coefficients of t^k of both sides of the equation (4) for $0 \leq k \leq \frac{1}{2}\dim Y + 1$. To be precise, by comparing the coefficients of $t^{\frac{1}{2}\dim Y+1}$, $t^{\frac{1}{2}\dim Y}$, and $t^{\frac{1}{2}\dim Y-1}$, respectively, of both sides of the equation (4), it is easy to obtain

$$
\begin{aligned}
\Lambda_X a_{l+1} &= (\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y+1}, \\
\Lambda_X a_l + C_1a_{l+1} &= \left(\frac{1}{2}\dim Y+1\right)[\omega|_X](\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y}, \\
\Lambda_X a_{l-1} + C_1a_l + C_2a_{l+1} &= \binom{\frac{1}{2}\dim Y+1}{2}[\omega|_X]^2(\phi(Y)-\phi(X))^{\frac{1}{2}\dim Y-1},
\end{aligned}
$$

respectively. Hence, we obtain

$$a_{l+1} = \frac{1}{\Lambda_X} \left(\phi(Y) - \phi(X)\right)^{\frac{1}{2}\dim Y + 1}, \ a_l = a_{l-1} = 0. \tag{8}$$

Now divide our proof into two subcases: $l \geq \frac{1}{2}\dim Y - 1$ and $l < \frac{1}{2}\dim Y - 1$. So suppose first that $l \geq \frac{1}{2}\dim Y - 1$. Then we have $\dim X + \dim Y = \dim M + 2l \geq \dim M + \dim Y - 2$. Thus $\dim X \geq \dim M - 2$. Since the circle action is assumed to be effective and so $\dim X \leq \dim M - 2$, this implies that $\dim X = \dim M - 2$ and $\dim Y = 2l + 2 \geq 4$. Moreover, since $\dim X = \dim M - 2$, note also that the S^1-equivariant Euler class $e^{S^1}(N_X)$ of N_X becomes $\Lambda_X t + C_1$. By comparing the constant coefficients of (7) which do not contain any t-factor, we have $a_0 C_1 = [\omega|_X]^{\frac{1}{2}\dim Y + 1}$. Since the cohomological degree of C_1 is two and $\dim Y + 2 \geq 6$, we should have $a_0 = 0$. Similarly, from the coefficients of t^2 in the equation (7), we get

$$a_1 \Lambda_X + a_2 C_1 = \binom{\frac{1}{2}\dim Y + 1}{2} [\omega|_X]^{\frac{1}{2}\dim Y - 1} (\phi(Y) - \phi(X))^2.$$

This implies that $a_1 = 0$, $l = \frac{1}{2}\dim Y - 1 = 1$. Thus it follows from (8) that

$$a_2 = \frac{1}{\Lambda_X}(\phi(Y) - \phi(X))^{\frac{1}{2}\dim Y + 1} = \frac{1}{\Lambda_X}(\phi(Y) - \phi(X))^{l+2} = \frac{1}{\Lambda_X}(\phi(Y) - \phi(X))^3,$$

which completes the proof for the case $l \geq \frac{1}{2}\dim Y - 1$.

Finally, we suppose that $l < \frac{1}{2}\dim Y - 1$. By comparing the coefficients of $t^{\frac{1}{2}\dim Y - l}$ of the equation (7), we have

$$a_0 \Lambda_X + a_1 C_1 + \cdots + a_s C_s = \binom{\frac{1}{2}\dim Y + 1}{\frac{1}{2}\dim Y - l} [\omega|_X]^{l+1} (\phi(Y) - \phi(X))^{\frac{1}{2}\dim Y - l}$$

for some $s \leq m$. This implies that $a_j = 0$ for $0 \leq j \leq l$ and $l + 2 \leq j \leq m$, and it follows from (8) that $a_{l+1} = \frac{1}{\Lambda_X}(\phi(Y) - \phi(X))^{\frac{1}{2}\dim Y + 1}$, as required. This completes the proof of Theorem 2.3. □

In fact, the proof of Theorem 2.3 also shows that if the Hamiltonian circle action satisfies the almost minimal dimension condition, then λ appearing in the identity (2) is always of the form $A_0 + A_1 t$ with $A_i \in H^{2i}(X; \mathbf{R})$:

$$\begin{aligned} \lambda = A_0 + A_1 t = {} & \frac{m^{\frac{1}{2}\dim Y + 1}}{\Lambda_X^2} \left(\frac{(\frac{1}{2}\dim Y + 1)\Lambda_X}{m} [\omega|_X] - C_1 \right) \\ & + \frac{m^{\frac{1}{2}\dim Y + 1}}{\Lambda_X} t. \end{aligned}$$

Thus Theorem 2.3 shows that if, in addition, λ is of pure type in the sense that λ lies in $\mathbf{R}[t]$, then we have $A_0 = 0$, i.e.,

$$C_1 = \frac{\left(\frac{1}{2}\dim Y + 1\right)\Lambda_X}{m}[\omega|_X]. \tag{9}$$

Note that if the circle action is semifree, then C_1 is equal to the first Chern class $c_1(N_X)$ of the normal bundle N_X of X in M (refer to [9], Lemma 2.4 and Remark 1.9 (b)). Hence we have the following practical criterion for a semifree Hamiltonian circle action with the almost minimality condition to be of pure type.

Theorem 2.4. *Let the circle act semifreely on a compact symplectic manifold (M, ω) with moment map $\phi : M \to \mathbf{R}$ of pure type. Assume that the fixed point set consist of only two connected components X and Y such that*

$$\dim X + \dim Y = \dim M.$$

Then the first Chern class $c_1(N_X)$ of the normal bundle N_X satisfies

$$c_1(N_X) = \frac{\left(\frac{1}{2}\dim Y + 1\right)\Lambda_X}{m}[\omega|_X].$$

3 Topology of certain Hamiltonian S^1-spaces

The goal of this section is to study topology of certain Hamiltonian S^1-spaces with almost minimality condition, which will be used to show our main theorems in Section 1. To do so, we prove the following proposition.

Proposition 3.1. *Let the circle act on a compact symplectic manifold (M, ω) of dimension $2n$ $(n \geq 2)$ with moment map $\phi : M \to \mathbf{R}$. Assume that the fixed point set M^{S^1} of the circle action has exactly two fixed connected components X and Y satisfying the conditions* **(C1)** *and* **(C2)**. *Then the following properties hold:*

(a) *$H^i(M;\mathbf{R}) \cong H^i(\mathbf{CP}^n;\mathbf{R})$ for all $0 \leq i \leq 2n$ except for $i = n$.*

(b) *$H^n(M;\mathbf{R}) = \mathbf{R}^{\oplus 2}$.*

(c) *The dimension of X is equal to that of Y, and the real dimension of M is zero mod 4. In particular, this implies that as a ring $H^*(X;\mathbf{R})$ is isomorphic to $H^*(Y;\mathbf{R})$.*

(d) *Assume that the condition* **(C2)** *holds for any field, as well. If $[\omega]$ is an integral class and the integer $\phi(X) - \phi(Y)$ is equal to ± 1, then, as a ring, $H^*(X;\mathbf{Z}) = \mathbf{Z}[u]/u^{\frac{1}{2}\dim X + 1}$ and $H^*(Y;\mathbf{Z}) = \mathbf{Z}[v]/v^{\frac{1}{2}\dim Y + 1}$, where u and v maps to $[\omega|_X]$ and $[\omega|_Y]$, respectively.*

Note that the Hamiltonian circle action on $S^2 \times S^2 = \mathbf{CP}^1 \times \mathbf{CP}^1$ which acts on the first factor by the standard rotation and acts trivially on the second factor satisfies all the assumptions of Proposition 3.1. We remark that Proposition 3.1 (d) will not be used in this paper.

Proof. To start the proof of the proposition, we first need the following easy lemma.

Lemma 3.2. *Assume that the hypotheses of Proposition 3.1 hold, and that, in addition, X is the minimal fixed connected component. Then the following properties hold.*

(a) *X is the unique fixed point component such that $H^i(M;\mathbf{R}) = H^i(X;\mathbf{R})$ for all $0 \leq i \leq \dim X - 1$.*

(b) *Y is the unique fixed point component such that*

$$H^j(M;\mathbf{R}) = H^{j-\dim X}(Y;\mathbf{R})$$

for all $\dim X + 1 \leq j \leq \dim M$.

(c) $H^{\dim X}(M;\mathbf{R}) = \mathbf{R}^{\oplus 2} = H^{\dim Y}(M;\mathbf{R})$.

Proof. Since ϕ is a perfect Morse-Bott function, we have

$$\begin{aligned} H^*(M;\mathbf{R}) &= H^*(X;\mathbf{R}) \oplus H^{*-2\lambda_Y}(Y;\mathbf{R}) \\ &= H^*(X;\mathbf{R}) \oplus H^{*-\dim X}(Y;\mathbf{R}), \end{aligned} \tag{10}$$

where λ_Y is half of the Morse-Bott index of Y and so $2\lambda_Y = \dim M - \dim Y = \dim X$.

If $0 \leq i \leq \dim X - 1$, then we have $i - \dim X \leq -1$ and so $H^{i-\dim X}(Y;\mathbf{R}) = 0$. Thus (a) follows immediately from the equation (10).

On the other hand, if $\dim X + 1 \leq j \leq \dim M$, then we have

$$1 \leq j - \dim X \leq \dim M - \dim X = \dim Y,$$

but clearly $H^j(X;\mathbf{R}) = 0$ due to the dimensional reason of X, thus obtaining that we are done with (b).

Finally, it is easy to see from (10) that

$$H^{\dim X}(M;\mathbf{R}) = H^{\dim X}(X;\mathbf{R}) \oplus H^{\dim X - \dim X}(Y;\mathbf{R}) = \mathbf{R}^{\oplus 2}.$$

Next, apply the same argument to $-\phi$ instead of ϕ, we can get

$$H^{\dim Y}(M;\mathbf{R}) = \mathbf{R}^{\oplus 2}.$$

This completes the proof of Lemma 3.2. $\square$

From the above discussion, it is now immediate to deduce the assertion (c) of Proposition 3.1. To see it, note from Lemma 3.2 (a) and (b) that X and Y are the unique fixed point components such that $H^i(M;\mathbf{R}) = H^i(X;\mathbf{R})$ for all $0 \le i \le \dim X - 1$ and such that $H^j(M;\mathbf{R}) = H^{j-\dim X}(Y;\mathbf{R})$ for all j with $\dim X + 1 \le j \le \dim M$. But, since $H^{\dim X}(M;\mathbf{R}) = \mathbf{R}^{\oplus 2} = H^{\dim Y}(M;\mathbf{R})$ by Lemma 3.2 (c), it follows from the Poincaré duality of M that we should have $\dim X = \dim Y$, as required. This completes the proof of statement (c) of Proposition 3.1.

Also, we note that the following lemma ([9], Lemma 4.7) is true in our case.

Lemma 3.3. *Let X be a compact manifold. Assume that $H^{2i+1}(X;\mathbf{Z}_p)$ vanishes for all i and all primes p. Then $H^*(X;\mathbf{Z})$ is torsion-free.*

So, if $u = [\omega]$ is an integral class, then $\tilde{u}$ can be taken as an integral class in $H^2_{S^1}(X;\mathbf{Z})$ such that $\tilde{u}|_X = u$, $\tilde{u}|_y = t(\phi(X) - \phi(Y))$ for all fixed points $y \in Y$. Since $\phi(X) - \phi(Y)$ is equal to ± 1, for any prime p there exists a fixed point y so that $\phi(X) - \phi(y) \neq 0 \bmod p$. By Lemma 4.6 of [9] and Lemma 3.3, this implies that $H^*(X;\mathbf{Z})$ is torsion-free. This completes the proof of Proposition 3.1 (d). $\square$

4 Chern classes of fixed connected componenets

The goal of this section is to collect some general facts which have already been proved by Li and Tolman in [9] but also applies to our settings.

We begin with the following proposition which is similar to Proposition 5.1 in [9].

Proposition 4.1. *Let the circle act on a compact symplectic manifold (M,ω) of dimension $2n$ $(n \ge 2)$ with moment map $\phi : M \to \mathbf{R}$. Assume that the fixed point set M^{S^1} of the circle action has exactly two fixed connected components X and Y satisfying the conditions* **(C1)** *and* **(C2)**. *Given a regular value c of ϕ so that the intersection of the fixed point set with $\phi^{-1}(c,+\infty)$ is equal to Y, for each $0 \le i \le \dim Y$ there exists an isomorphism*

$$\kappa_{Y,c} : H^i_{S^1}(Y;\mathbf{Z})/e^{S^1}(N_Y) \to H^i_{S^1}(\phi^{-1}(c);\mathbf{Z})$$

such that $\kappa_{X,c}(\tilde{\alpha}|_Y) = \tilde{\alpha}|_{\phi^{-1}(c)}$ for all $\tilde{\alpha} \in H^i_{S^1}(M;\mathbf{Z})$.

Proof. Since $\dim M - \dim Y = \dim X = \dim Y$ in our case, Proposition 5.1 in [9] continues to hold over integer coefficients for all $0 \le i \le \dim Y$. This completes the proof. $\square$

As a consequence of Proposition 4.1, we have the following corollary which is similar to Corollary 5.2 in [9].

Corollary 4.2. *Let the circle act on a compact symplectic manifold (M,ω) of dimension $2n$ $(n \ge 2)$ with moment map $\phi : M \to \mathbf{R}$. Assume that the fixed point set of the circle action has exactly two fixed connected components X and Y satisfying* *the conditions* **(C1)** *and* **(C2)***. For each* $0 \le i \le \dim X = \dim Y$*, there exists an isomorphism*

$$f : H^i_{S^1}(X; \mathbf{Z})/e^{S^1}(N_X) \to H^i_{S^1}(Y; \mathbf{Z})/e^{S^1}(N_Y)$$

such that $f(\tilde{\alpha}|_X) = \tilde{\alpha}|_Y$ *for all* $\tilde{\alpha} \in H^i_{S^1}(M; \mathbf{Z})$*.*

Moreover, the following properties hold:

$$\begin{aligned} &f([\omega|_X]) = [\omega|_Y] + t(\phi(Y) - \phi(X)), \\ &sf([\omega|_X]) + (1-s)[\omega|_Y] \neq 0, \quad s \in (0,1). \end{aligned}$$

Proof. In our case, since $\dim X = \dim Y$, the minimum of $\dim M - \dim X$ and $\dim M - \dim Y$ is equal to $\dim X = \dim Y$. Hence Corollary 5.2 in [9] proves Corollary 4.2 with integer coefficients. This completes the proof. $\square$

Finally, we also need the following lemma which is similar to Lemma 5.4 in [9].

Lemma 4.3. *Let the circle act on a compact symplectic manifold* (M, ω) *of dimension* $2n$ $(n \ge 2)$ *with moment map* $\phi : M \to \mathbf{R}$ *of pure type. Assume that the fixed point set of the circle action has exactly two fixed connected components* X *and* Y *satisfying the conditions* **(C1)** *and* **(C2)***. Under the natural isomorphism*

$$H^*(X; \mathbf{R}) \to H^*_{S^1}(X; \mathbf{R})/\left([\omega|_X] + t(\phi(Y) - \phi(X))\right),$$

the total Chern class $c(X)$ *of the tangent bundle of* X *is given by*

$$c(X) = \frac{\prod_\lambda (1 + \lambda t)}{c^{S^1}(N_X)},$$

where $c^{S^1}(N_X)$ *denotes the* S^1*-equivariant total Chern class of* N_X *and the product is taken over all weights* λ *with multiplicity in the normal bundle* N_Y *of* Y*.*

Proof. The proof given below is a slight modification of that of Lemma 5.4 in [9]. Let $m = \phi(Y) - \phi(X)$, $u = [\omega|_X]$, and $v = [\omega|_Y]$. Then, as in the proof of Lemma 5.4 in [9], there is a map

$$g : H^*_{S^1}(X; \mathbf{R})/(u + mt, e^{S^1}(N_X)) \to H^*_{S^1}(\{y\}; \mathbf{R})/t^{\frac{1}{2}\dim X} = \mathbf{R}[t]/t^{\frac{1}{2}\dim X}$$

such that

$$g(u) = -mt \text{ and } g(c^{S^1}(M)|_X) = c^{S^1}(M)|_y = \prod_\lambda (1 + \lambda t),$$

where the product is taken over all weights with multiplicity in the normal bundle N_Y of Y in M.

Now, recall from Theorem 2.3 that

$$te^{S^1}(N_X) = \Lambda_X \left(t + \frac{u}{m}\right)^{\frac{1}{2}\dim X + 1}.$$

Thus, $te^{S^1}(N_X)$ is a multiple of $(mt+u)$. This implies that

$$\begin{aligned}
&H^*_{S^1}(X;\mathbf{R})/((mt+u), e^{S^1}(N_X)) = tH^*_{S^1}(X;\mathbf{R})/(t(mt+u), te^{S^1}(N_X)) \\
&= tH^*_{S^1}(X;\mathbf{R})/(mt+u) = tH^*(X;\mathbf{R}) = t\mathbf{R}[t]/t^{\frac{1}{2}\dim X+1} \\
&= \mathbf{R}[t]/t^{\frac{1}{2}\dim X}.
\end{aligned}$$

Therefore, g is indeed an isomorphism. Moreover, since $c^{S^1}(M)|_X = c(X)c^{S^1}(N_X)$, we obtain

$$c(X) = \frac{\prod_\lambda (1+\lambda t)}{c^{S^1}(N_X)},$$

as required. $\square$

5 Proof of Theorem 1.11: semifree case

The goal of this section is to prove Theorem 1.11 for the semifree case. We will do this by contradiction, assuming that all the assumptions in Theorem 1.11 hold.

To do so, we shall also assume that $\phi(X) < \phi(Y)$ and that $\dim_{\mathbf{R}} X = 2n$. Note that, since the Hamiltonian circle action for dimensions greater than or equal to 6 is assumed to be of pure type, Theorem 2.3 applies to our settings. Then we have $\operatorname{rank}_{\mathbf{C}} N_X = \operatorname{rank}_{\mathbf{C}} N_Y = n$, and

$$te^{S^1}(N_X) = \left(t + \frac{u}{m}\right)^{n+1}. \tag{11}$$

Recall from Lemma 2.1 that $m = \phi(Y) - \phi(X)$ is a positive integer, since $[\omega]$ is assumed to be an integral class. Since $c^{S^1}(N_X) = \prod^n (1+t+\alpha_i)$ with $\alpha_i \in H^{2i}(X;\mathbf{Z})$ for semifree case, we should have

$$t\prod^n (t+\alpha_i) = \left(t + \frac{u}{m}\right)^{n+1}. \tag{12}$$

By comparing the coefficients of t^j of both sides of (12) for $1 \le j \le n$, we obtain

$$\begin{aligned}
&\sigma_1(\alpha_1, \cdots, \alpha_n) := \sum_{i=1}^n \alpha_i = \binom{n+1}{1}\frac{u}{m}, \\
&\sigma_2(\alpha_1, \cdots, \alpha_n) := \sum_{1\le i<j\le n} \alpha_i\alpha_j = \binom{n+1}{2}\left(\frac{u}{m}\right)^2, \\
&\cdots \\
&\sigma_n(\alpha_1, \cdots, \alpha_n) := \alpha_1\alpha_2\cdots\alpha_n = \binom{n+1}{1}\left(\frac{u}{m}\right)^n.
\end{aligned} \tag{13}$$

In case of four dimensions, due to the extra assumption about the primitive integral class $[\omega]$, it generates the cohomology ring of X over integer coefficients. This argument also applies to $v = [\omega|_Y]$. On the other hand, in higher dimensions u and v generate the cohomology rings of X and Y, respectively, without any extra assumption except that $[\omega]$ is a primitive integral class. As mentioned in Section 1, this is due to Proposition 3.1 (2).

Now, if $n = 1$, then we have $ku = \alpha_1 = 2\frac{u}{m}$ for some integer k. Thus $mk = 2$, which implies either $m = 1$ and $k = 2$ or $m = 2$ and $k = 1$. If $m = 1$ and $k = 2$, then it follows from Lemma 4.3 that

$$c(X) = \frac{(1-t)}{c^{S^1}(N_X)} = \frac{(1-t)}{(1+t+2u)} = 1 + 2u,$$

where we put $t = -2u$ in the last equality in order to obtain the last expression. Similarly, if $m = 2$ and $k = 1$, then we obtain $c(X) = 1 + u$, which is impossible under the condition $H^*(X; \mathbf{Z}) = \mathbf{Z}[u]/u^2$. Note also that the action of the circle should be semifree, since by Proposition 3.1 we have $\dim X = \dim Y$ and the action is assumed to be effective. Hence it is easy to see that the following theorem (Theorem 1.8) holds.

Theorem 5.1. *Let the circle act on a compact symplectic manifold (M, ω) of dimension 4 with moment map $\phi : M \to \mathbf{R}$ of pure type, where $[\omega]$ is a primitive integral class in the strong sense. Assume that the fixed point set of the circle action has exactly two fixed connected components X and Y satisfying (**C1**) and (**C2**). If we let $m = \phi(Y) - \phi(X) > 0$, then the following properties hold:*

(a) *The circle action is always semifree and $m = 1$.*

(b) *$H^*(X; \mathbf{Z}) = \mathbf{Z}[u]/u^2$ and $H^*(Y; \mathbf{Z}) = \mathbf{Z}[v]/v^2$, where $u = [\omega|_X]$ and $v = [\omega|_Y]$.*

(c) *$c(X) = 1 + 2u$ and $c(Y) = 1 - 2v$.*

(d) *$c(N_X) = 1 + 2u$ and $c(N_Y) = 1 - 2v$.*

Next, let us take $n = 2$. Then it follows from (13) that we have

$$\alpha_1 + \alpha_2 = 3\frac{u}{m}, \quad \alpha_1\alpha_2 = 3\left(\frac{u}{m}\right)^2. \tag{14}$$

Let $\alpha_1 = k_1 u$ and $\alpha_2 = k_2 u$ for some integers k_1 and k_2. Then it follows from (14) that

$$(k_1 + k_2)m = 3, \quad k_1 k_2 m^2 = 3.$$

Thus we have $m = 1$, $k_1 k_2 = 3$, and $k_1 + k_2 = 3$, which is a contradiction. Therefore, we can conclude that there is no semifree Hamiltonian circle action of pure type on a compact symplectic manifold of dimension 8 with exactly two fixed point components, X and Y, satisfying the conditions **(C1)** and **(C2)**.

Now, we are ready to prove our main theorem of this section which can be regarded as an immediate consequence of (12).

Theorem 5.2. *Let the circle act semifreely on a compact symplectic manifold (M, ω) of dimension $4n$ $(n \geq 2)$ with moment map $\phi : M \to \mathbf{R}$ of pure type, where $[\omega]$ is a primitive integral class, whose fixed point set has exactly two fixed connected components X and Y satisfying the condition* **(C1)**. *Then there does not exist any such Hamiltonian circle action on M such that both of X and Y have the same cohomology rings of the complex projective spaces over integer coefficients.*

Proof. We prove it by contradiction. By using a similar argument as above, it is easy to obtain from the last equation of (13) that

$$k_1 k_2 \cdots k_n m^n = n + 1, \tag{15}$$

where k_i are integers. Since $m^n > (n+1)$ for all $n \geq 2$ and $m \geq 2$, it follows from the equation (15) that $m = 1$ and $k_i \neq 0$. Thus we have the system of equations corresponding to (13), as follows:

$$\begin{aligned}
&\sigma_1(k_1, k_2 \cdots, k_n) = k_1 + k_2 + \cdots + k_n = \binom{n+1}{1}, \\
&\sigma_2(k_1, k_2, \cdots, k_n) = \sum_{1 \leq i < j \leq n} k_i k_j = \binom{n+1}{2}, \\
&\sigma_3(k_1, k_2, \cdots, k_n) = \sum_{1 \leq i < j < p \leq n} k_i k_j k_p = \binom{n+1}{3}, \\
&\cdots, \\
&\sigma_n(k_1, k_2, \cdots, k_n) = \prod_{i=1}^{n} k_i = \binom{n+1}{n}.
\end{aligned} \tag{16}$$

Thus we have $\prod^n (1 + k_i) = 2^{n+1} - 1$, which implies that all the k_i's are even. Since all symmetric functions σ_i of k_1, k_2, $\cdots$, k_n are positive by (16), note that all non-zero integers k_i should be also positive.

If n is even, then $n+1$ is odd, while $\sum_{i=1}^n k_i$ is even. This is a contradiction. So n is odd. If n is of the form $4s+1$, then it follows from the second equation of (16) that the left hand side $\sigma_2(k_1, k_2, \cdots, k_n)$ is zero mod 4, while the right hand side $\binom{n+1}{2}$ is 1 or 3 mod 4. So this case cannot occur, either. Finally, assume that n is of the form $4s+3$. If $s = 0$, i.e., $n = 3$, then by the third symmetric function σ_3 in (16) we have

$$8 \leq \sigma_3(k_1, k_2, k_3) = k_1 k_2 k_3 = 4,$$

since all k_i are greater than or equal to 2, as noted above. This is a contradiction. On the other hand, if $s > 0$, then it follows from σ_3 of (16) that we have

$$\begin{aligned}
&3\sigma_3(k_1, k_2, \cdots, k_n) - (n+1)n(n-1) \\
&\geq 3 \cdot 8 \cdot \binom{4s+3}{3} - (4s+4)(4s+3)(4s+2) \\
&= 4(4s+3)(4s+2)(4s+1) - (4s+4)(4s+3)(4s+2) \\
&= (4s+3)(4s+2)(4(4s+1) - (4s+4)) \\
&= 12s(4s+3)(4s+2) > 0.
\end{aligned}$$

So again we have a contradiction. This completes the proof of Theorem 5.2. $\square$

6 Proof of Theorem 1.11: non-semifree case

The goal of this section is to prove Theorem 1.11 for the non-semifree case. To do so, we begin with this section by noticing all the results in Section 7 of the paper [9] still hold to be true in our case, i.e., under two conditions **(C1)** and **(C2)**. In particular, the following proposition ([9], Proposition 7.9) continues to hold.

Proposition 6.1. *Let the circle act on a compact symplectic manifold* (M, ω) *with moment map* $\phi : M \to \mathbf{R}$. *Assume that the fixed point set* M^{S^1} *consists of exactly two connected components* X *and* Y *and that* $b_{2i}(X) = 1$ *for all* $0 \leq i \leq \frac{1}{2}\dim X$. *Then the following properties hold.*

(a) *There are no points with stabilizer* $\mathbf{Z}_k$ *for* $k > 2$.

(b) *If the circle action is not semifree, then* $\dim M^{\mathbf{Z}_2} - \dim Y = 2$ *or* $\dim M - \dim M^{\mathbf{Z}_2} = 2$. *Here both possibilities can occur.*

As a consequence of Proposition 6.1 (2) and Proposition 3.1 or a direct argument, it is easy to see that there cannot exist a *non-semifree* and effective Hamiltonian circle action on a compact symplectic manifold of dimension 4 with exactly two fixed connected components satisfying the almost minimal dimension condition.

Now, we want to rule out even the case of $\dim M^{\mathbf{Z}_2} - \dim Y = 2$ in Proposition 6.1 (b).

Proposition 6.2. *Let the circle act on a compact symplectic manifold* (M, ω) *of dimension* $4n$ $(n \geq 2)$ *with moment map* $\phi : M \to \mathbf{R}$ *of pure type, where* $[\omega]$ *is a primitive integral class, whose fixed point set has exactly two connected components* X *and* Y *satisfying the condition* **(C1)**. *Then there does not exist any such Hamiltonian circle action on* M *such that both of* X *and* Y *have the same cohomology rings of the complex projective spaces over integer coefficients and such that* $\dim M^{\mathbf{Z}_2} - \dim Y = 2$.

Proof. We prove it by contradiction. For simplicity, let $\dim_{\mathbf{R}} X = \dim_{\mathbf{R}} Y = 2n$. Since $\dim M^{\mathbf{Z}_2} - \dim Y = 2$, note that $\Lambda_X = 2$ and $\dim M - \dim M^{\mathbf{Z}_2} = 2n-2 \geq 2$. Thus it follows from Theorem 2.3 that we have

$$te^{S^1}(N_X) = 2\left(t + \frac{u}{m}\right)^{n+1},$$

where $m = \phi(Y) - \phi(X)$. Since $c_1(N_X^{M^{\mathbf{Z}_2}}) = 0$ by Lemma 7.6 in [9], we have

$$e^{S^1}(N_X^{M^{\mathbf{Z}_2}}) = 2t + c_1(N_X^{M^{\mathbf{Z}_2}}) = 2t.$$

Now, it follows from the relation $e^{S^1}(N_X) = e^{S^1}(N_X^{M^{\mathbf{Z}_2}})e^{S^1}(N_{M^{\mathbf{Z}_2}})|_X$ that

$$\begin{aligned} e^{S^1}(N_{M^{\mathbf{Z}_2}})|_X &= \frac{1}{2t}e^{S^1}(N_X) = \frac{1}{2t^2}te^{S^1}(N_X) \\ &= \frac{1}{t^2}\left(t + \frac{u}{m}\right)^{n+1} \\ &= t^{n-1} + (n+1)\frac{u}{m}t^{n-2} + \cdots + (n+1)\frac{u^n}{m^n}\frac{1}{t}. \end{aligned} \tag{17}$$

On the other hand, since $c^{S^1}(N_{M^{\mathbf{Z}_2}})|_X = \prod_{i=1}^{n-1}(1 + t + \alpha_i)$ for some $\alpha_i \in H^2(X;\mathbf{Z})$, it follows from (17) that we have

$$\begin{aligned} \prod_{i=1}^{n-1}(t + \alpha_i) = e^{S^1}(N_{M^{\mathbf{Z}_2}})|_X &= \frac{1}{t^2}\left(t + \frac{u}{m}\right)^{n+1} \\ &= t^{n-1} + (n+1)\frac{u}{m}t^{n-2} + \cdots + (n+1)\frac{u^n}{m^n}\frac{1}{t}, \quad u^n \neq 0. \end{aligned} \tag{18}$$

However, the lowest non-zero degree of t in the left hand side of (18) is zero, which is not equal to the lowest non-zero degree of t, that is now -1, in the right hand side of (18). This is a contradiction, which completes the proof of Proposition 6.2. $\square$

Now, we are ready to prove Theorem 1.11 for non-semifree case.

Theorem 6.3. *Let the circle act on a compact symplectic manifold (M, ω) of dimension $4n$ ($n \geq 2$) with moment map $\phi : M \to \mathbf{R}$ of pure type, where $[\omega]$ is a primitive integral class, whose fixed point set has exactly two fixed connected components X and Y satisfying the condition* **(C1)**. *Then there does not exist any such Hamiltonian circle action on M such that both of X and Y have the same cohomology rings of the complex projective spaces over integer coefficients.*

Proof. We prove it by contradiction. In view of Propositions 6.1 (b) and 6.2, from now on we assume that the circle action is not semifree, $\dim M^{\mathbf{Z}_2} - \dim Y > 2$, and $\dim M - \dim M^{\mathbf{Z}_2} = 2$. Then the following relations hold:

- $te^{S^1}(N_X) = \Lambda_X \left(t + \frac{u}{m}\right)^{n+1} = 2^{n-1}\left(t + \frac{u}{m}\right)^{n+1}$.

- $e^{S^1}(N_{M^{\mathbf{Z}_2}})|_X = t + 2\frac{u}{m}$.
- $te^{S^1}(N_X^{M^{\mathbf{Z}_2}}) = \frac{2^{n-1}\left(t+\frac{u}{m}\right)^{n+1}}{(t+2\frac{u}{m})}$,

where we used Lemma 7.7 of [9] in order to obtain the second item. From [9], Lemma 2.4, it is easy to deduce that

$$tc^{S^1}(N_X^{M^{\mathbf{Z}_2}}) = \frac{\left(1+2t+2\frac{u}{m}\right)^{n+1}}{4(t+2\frac{u}{m})}.$$

This in turn implies that

$$c^{S^1}(N_X) = \frac{\left(1+2t+2\frac{u}{m}\right)^{n+1}}{4t(t+2\frac{u}{m})}(1+t+2\frac{u}{m}).$$

It is also true from Lemma 4.3 that

$$\begin{aligned} c(X) = \frac{\prod_\lambda (1+\lambda t)}{c^{S^1}(N_X)} &= \frac{(1-t)(1-2t)^{n-1}}{c^{S^1}(N_X)} \\ &= \frac{4t(1-t)(1-2t)^{n-1}(t+2\frac{u}{m})}{\left(1+2t+2\frac{u}{m}\right)^{n+1}(1+t+2\frac{u}{m})} \\ &= -\frac{4u^2}{m^2}\left(1+2\frac{u}{m}\right)^{n-1}, \end{aligned}$$

where we put $t = -\frac{u}{m}$ in the last equality in order to obtain the last expression.

Next, we use the calculation of Euler-Pincaré characteristic of X:

$$\begin{aligned} n+1 &= \sum_{i=0}^{n}(-1)^i \dim H^i(X;\mathbf{R}) \\ &= \int_X c_n(X) = \int_X -\frac{4u^2}{m^2}(n-1)\frac{2^{n-2}u^{n-2}}{m^{n-2}} \\ &= \int_X -\frac{2^n(n-1)u^n}{m^n}. \end{aligned}$$

Thus we have

$$1 = \int_X -\frac{2^n(n-1)u^n}{(n+1)m^n}, \tag{19}$$

which implies that $-\frac{2^n(n-1)}{(n+1)m^n}u^n$ is a primitive integral class in $H^{2n}(X;\mathbf{Z})$. Since we assumed that $[\omega]$ is a primitive integral class, u also becomes a primitive integral class ([9], Lemma 4.10). Note also that, since u^n is a primitive class, $\frac{m^n(n+1)}{2^n(n-1)}$ should be 1.

Furthermore, from $c^{S^1}(N_M\mathbf{z_2})|_X = 1 + t + 2\frac{u}{m}$, we see that $\frac{2}{m}$ should be an integer. So we have either $m = 1$ or $m = 2$.

Finally, if $m = 1$, then $\frac{2^n(n-1)}{n+1}$ would be equal to one. But this does not make sense, since $4(n-1) \leq 2^n(n-1) = n+1$ implies $n \leq \frac{5}{3} < 2$. On the other hand, if $m = 2$, then $\frac{n-1}{n+1}$ would be equal to one, which is also a contradiction. This completes the proof of Theorem 6.3. □

References

[1] K. Ahara and A. Hattori, *4-dimensional symplectic S^1-manifolds admitting moment map*, J. Fac. Sci. Univ. Tokyo Sect. 1A **38** (1991), 251-298.

[2] M. Audin, *The topology of torus actions on symplectic manifolds*, Birkhäuser, 1991.

[3] T. Delzant, *Hamiltoniens périodique et images convexes de lapplication moment*, Bull. Soc. Math. France **116** (1998), 315–339.

[4] L. Godinho and S. Sabatini, *New tools for classifying Hamiltonian circle actions with isolated fixed points*, preprint (2012); arXiv:1206.3195v1.

[5] J.-C. Hausmann and T. Holm, *Simple Hamiltonian manifolds*, preprint (2010); arXiv:1012.4740v1.

[6] Y. Karshon, *Periodic Hamiltonian flows of four dimensional manifolds*, Memoirs of Amer. Math. Soc. **672**, 1999.

[7] H. Li, *Certain circle actions on Kähler manifolds*, to appear in Int. Math. Res. Not.

[8] H. Li, M. Olbermann, and D. Stanley, *One-connectivity and finiteness of Hamiltonian circle actions with minimal fixed sets*, preprint (2010); arXiv:1010.2505v1.

[9] H. Li and S. Tolman, *Hamiltonian circle actions with minimal fixed sets*, Int. J. Math. **23** (2012), 1250071; arXiv:0909.4049v1.

[10] D. McDuff and D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford Science Publ., 1998.

[11] S. Tolman, *On a symplectic generalization of Petrie's conjecture*, Tans. Amer. Math. Soc. **326** (2010), 3963–3996; arXiv:0903.4918v1.

[12] T. Petrie, *Smooth S^1-actions on cohomology projective spaces and related topics*, Bull. Amer. Math. Soc. **78** (1972), 105-153.

[13] T. Petrie, *Torus actions on homotopy projective spaces*, Invent. Math. **20** (1973), 139–146.

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