On some first sign change problems of modular forms

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August 26, 2014

Abstract

1 Introduction

The problem of sign change of Hecke eigenvalues (or Fourier coefficients) has been studied recently by many researchers. In [4] it was proved that a nontrivial cusp form $f(z)$ with real Fourier coefficients $a(n)$ has infinitely many sign changes. Further, many quantitative results for the number of sign changes for the Fourier coefficients have been established. The best known result about Hecke eigenforms is that there is $n \ll (k^2N)^{3/8}$ such that $a(n) < 0$. Same questions about Siegel forms and Hilbert forms have been studied.

On the other hand, the sign changes of the subsequence of the Fourier coefficients at prime numbers was first studied by Ram Murty [10]. In this paper we give a bound for the first prime p such that the p-th Hecke eigenvalue $\lambda_f(p)$ is negative when f is a normalized cuspidal newform of level $\Gamma_0(N)$. We also give a bound for the first sign change for general cusp forms, which improves the bound given by Choie and Kohnen [1] in the level aspect in the case of prime level.

2 Preliminaries

In this section we briefly recall the basic tools to be used.

2.1 Some properties of L-functions

Let f be a normalized eigen cuspform of weight $k \in \mathbb{Z}$ for the group $\Gamma_0(N)$, having Fourier expansion $\sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$ $(e(z) := e^{2\pi i z})$. One can associate the Hecke L-function $L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s}$. It is well-known that $|\lambda_f(n)| \leq 2$ for $p \not| N$, thanks to the proof of Ramanujan conjecture given by Deligne [2], and hence $L(f,s)$ converges for $Re(s) > 1$. It has the following Euler product expansion converging in the right half-plane $Re(s) > 1$

$$L(f,s) = \prod_{p \not| N} (1-\alpha_p p^{-s})^{-1}(1-\alpha_p^{-1}p^{-s})^{-1} \prod_{p|N} (1-\beta_p p^{-s})^{-1}$$

where $|\alpha_p| = 1$ and $|\beta_p| \leq 1$.

We recall the definitions of the Rankin-Selberg L-function for two normalized newforms f and g and the symmetric square L-function of f. For given Fourier expansion $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz)$ and $g = \sum_{n=1}^{\infty} \lambda_g(n) n^{\frac{k-1}{2}} e(nz)$ the Rankin-Selberg L-function $L(f \otimes g, s)$ is defined by

$$L(f \otimes g, s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n)\lambda_g(n)}{n^s},$$

and the symmetric square L-function $L(sym^2 f, s)$ is defined by

$$L(sym^2 f, s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s},$$

where $\zeta(s)$ is the Riemann zeta function. It is well known that two L-functions are related by the relation $L(f \times f, s) = \zeta(s)L(sym^2 f, s)$. Recall that $L(sym^2 f, s)$ is entire, and $L(f \otimes g)$ is entire unless $f = g$, when $L(f \otimes f, s)$ is analytic except at $s = 1$, which is a simple pole.

For the later purpose, we recall the convexity bounds for these L-functions(see [3, p.100, 131]):

$$L(f, \frac{1}{2} + it) \ll_\epsilon q(f, \frac{1}{2} + it)^{\frac{1}{4}+\epsilon}, \tag{2.1}$$

$$L(f \otimes f, \frac{1}{2} + it) \ll q(f \otimes f, \frac{1}{2} + it)^{\frac{1}{4}+\epsilon} \ll q(f, \frac{1}{4} + \epsilon)^{1+4\epsilon}, \tag{2.2}$$

where $q(f, s)$ is the analytic conductor defined by $q(f)\prod_{j=1}^{d}(|s + \kappa_j| + 3)$, where $q(f)$ is the conductor of L-function and d is the degree of L coming from the gamma factor of the L-function $\pi^{-ds}\prod_{j=1}^{d}\Gamma(\frac{s+\kappa_j}{2})$ (for more detail, see [3, p.94]). Note that the L-function of a given elliptic cusp form f of weight k and level N has the property $q(f, s) \le (k^2N)(|s| + |k| + 3)^2$ and $q(f \otimes f, s) \le (k^2N)^4(|s| + |k| + 3)^4$. Note that $q(f) \sim k^2N$ when f is an elliptic cusp form of weight k and level N.

2.2 Perron type formula

In this subsection we briefly recall the Perron type formula. We refer as a general reference to [8].

Consider a L-function $L(s) = \sum \frac{a(n)}{n^s}$ analytic on some right half-plane and fix a smooth function w defined on $[0, \infty)$ such that supp $w \subset [0, 2]$ and $w \equiv 1$ on $[0, 1]$ and $0 \le w \le 1$ on $[1, 2]$. Then for any positive x we have the following Perron type formula

$$\sum_{n \ge 1} a(n)w(\frac{n}{x}) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \check{w}(s)x^s L(s)ds, \tag{2.3}$$

where $\check{w} = \int_0^\infty w(x)x^{s-1}dx$ is the Mellin transform of w and σ is any positive number greater than the absissa of convergence σ_0 of $L(s)$. Note that $\check{w}$ is of rapid decay as $Im(s) \to \pm\infty$, since w is compactly supported smooth, and this guarantees the convergence of the integral given on the right.

3 First sign change problem at prime argument

Let f be a normalized Hecke eigennewform of weight k and level N, with Fourier expansion $\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e(nz)$. In this section we give a bound for the first sign change at prime arguments. The main idea is to compare the Euler product expansion of $L(f, s) = \sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}$ and that of another L-series defined by

$$M(f, s) := \sum_{\substack{n=1 \\ n\ square-free}}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 + \lambda_f(p)p^{-s}).$$

Then it can be seen that $\frac{M(f,s)}{L(f,s)} \le \prod_p(1 + 6p^{-2s}) \ll_s 1$ when $Re(s) > \frac{1}{2}$. Then by (2.3) we have

$$\sum_{\substack{n \ge 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x}) = \frac{1}{2\pi i}\int_{\frac{1}{2}+\epsilon-i\infty}^{\frac{1}{2}+\epsilon+i\infty} \check{w}(s)x^s M(f, s)ds$$

where ϵ is an arbitrary positive number. Then $\frac{M(f,s)}{L(f,s)} \ll_s 1$ and the convexity bound for $L(f, s)$ gives an upper bound

$$\sum_{\substack{n \ge 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x}) \ll q(f, s)^{\frac{1}{4}+\epsilon}x^{\frac{1}{2}+\epsilon}.$$

On the other hand, we can apply the same argument to $L(f \otimes f, s)$ to get a lower bound for $\sum_{\substack{n \ge 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x})$. Comparing the Euler product expansions of two L-series $L(f \otimes f, s)$ and $M(f \otimes f, s) := \sum_{\substack{n \ge 1 \\ n\ square-free}} \lambda_f^2(n)n^{-s}$, we get $\frac{M(f\otimes f,s)}{L(f\otimes f,s)} \ll_s 1$ when $Re(s) > \frac{1}{2}$, and obtain

$$\sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f^2(n)w(\frac{n}{x}) = \frac{1}{2\pi i}\int_\sigma M(f\otimes f,s)x^s\check{w}(s)ds,$$

where $\sigma = Re(s) > 1$.

We need to translate the line of integration, but in this case $L(f\otimes f,s)$ has a simple pole at $s=1$, and we get the residue term to get

$$\sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f^2(n)w(\frac{n}{x}) = Res_{s=1}(M(f\otimes f,s))x\check{w}(1) + \frac{1}{2\pi i}\int_{\frac{1}{2}+\epsilon-i\infty}^{\frac{1}{2}+\epsilon+i\infty} M(f\otimes f,s)x^s\check{w}(s)ds.$$

Therefore we conclude

$$\sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f^2(n)w(\frac{n}{x}) = Res_{s=1}(M(f\otimes f,s))\check{w}(1)x + O(q^{1+\epsilon}x^{\frac{1}{2}+\epsilon}).$$

Note that for $x \gg q^{2+\epsilon}$ the first term $Res_{s=1}(M(f\otimes f,s))\check{w}(1)x$ dominates the right hand side of the equation.

On the other hand, suppose that $\lambda_f(n) \geq 0$ for all square-free $n \leq x$. Then the inequality $|\lambda_f(n)| \ll_\eta n^\eta$ (for any $\eta > 0$) coming from Deligne's bound, gives

$$x^\eta \sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x}) \gg_\eta \sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f^2(n)w(\frac{n}{x}),$$

hence consequently we get

$$\sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x}) \gg_\eta Res_{s=1}(M(f\otimes f,s))x^{1-\eta}\check{w}(1) + O(q^{1+\epsilon}x^{\frac{1}{2}+\epsilon-\eta}).$$

Suppose $x \gg q^{2+\epsilon}$. Then we get

$$q(f,s)^{\frac{1}{4}+\epsilon}x^{\frac{1}{2}+\epsilon} \gg_\epsilon \sum_{\substack{n\geq 1 \\ n\ square-free}} \lambda_f(n)w(\frac{n}{x}) \gg_\eta Res_{s=1}(M(f\otimes f,s))x^{1-\eta}\check{w}(1) + O(q^{1+\epsilon}x^{\frac{1}{2}+\epsilon-\eta}),$$

but then we get $x \ll q^{\frac{1}{2}+\epsilon}$, contradicting the assumption $x \gg q^{2+\epsilon}$. Therefore we conclude that there is a square-free $n_0 \ll_\epsilon q^{2+\epsilon}$ such that $\lambda_f(n_0) < 0$ and also there is a prime $p \ll_\epsilon q^{2+\epsilon}$ such that $\lambda_f(p) < 0$.
We summarize the result below.

Theorem 3.1. *For a normalized Hecke eigennewform $f = \sum_{n=1}^\infty \lambda_f(n)n^{\frac{k-1}{2}}e(nz)$ of weight k and level $\Gamma_0(N)$ which is cuspidal, there is a prime $p \ll_\epsilon (k^2N)^{2+\epsilon}$ such that $\lambda_f(p)$ is negative.*

4 First sign change for general cusp forms of prime level

Let p be a prime. From the newform theory it is well-known that the space $S_k(\Gamma_0(p))$ is decomposed into the orthogonal direct sum(with respect to the Petersson inner product $<,>$) of the space of old forms $S_k^{old}(\Gamma_0(p))$ and the space of newforms $S_k^{new}(\Gamma_0(p))$, and we have $S_k^{old}(\Gamma_0(p)) = S_k^{new}(\Gamma_0(1))|V_p$, where $V_p : g(z) \mapsto g(pz)$. Also note that $< f,g >=< f|V_p, g|V_p >$ (see for instance [6, p.294]). Moreover it is known that(see for example the introduction of [5]) there is an orthogonal basis consisting of that of $S_k^{new}(\Gamma_0(p))$ and the basis coming from the orthogonal basis of $S_k^{old}(\Gamma_0(p))$ such that each form is a normalized Hecke eigenform in each space. Fix such a basis $f_1, \cdots, f_{m_1}$ of $S_k^{new}(\Gamma_0(p))$ and $f_{m_1+1}, \cdots, f_{m_1+m_2}$ of $S_k^{old}(\Gamma_0(p))$ and let $g_1, \cdots, g_{m_2}$ be the orthogonal basis of $S_k^{new}(\Gamma_0(1))$ such that $g_i|V_p = f_{m_1+i}$ for $i = 1, \cdots, m_2$. Note that $m_1 + m_2 \sim kp$.

Let f be a nonzero cusp form of weight k for the group $\Gamma_0(p)$ with the Fourier expansion $f(z) = \sum_{n\geq 1}\alpha_n n^{\frac{k-1}{2}}e(nz)$ and $f_j(z) = \sum_{n\geq 1}\lambda_j(n)n^{\frac{k-1}{2}}e(nz)$. Then if we write $f(z) = \sum a_j f_j(z)$ $(a_j \in \mathbb{C})$, it is easily seen that $\alpha_n = \sum a_j\lambda_j(n)$.

Now we would like to get a bound for the first sign change for the general cusp forms. The main argument is to use the Perron type formula (2.3) and the convexity bounds (2.1), (2.2) to each f_j and pairs of f_j. Even though $f_{m_1+1}, \cdots, f_{m_1+m_2}$ are not Hecke eigenforms for the group $\Gamma_0(p)$ in general, we can obtain some bounds coming from the convexity bounds of $g_1, \cdots, g_{m_2}$.

Lemma 4.1. *$L(g_i, s) = p^{s+\frac{k-1}{2}} L(f_{m_1+i}, s)$ for $i = 1, \cdots, m_2$.*

Proof Comparing the Fourier expansions we get the result immediately. □

Now we find lower and upper bounds for the sum $\sum_{\substack{n\geq 1 \\ n\ square-free}} \alpha_n w(\frac{n}{x})$ to get a bound for the first sign change. As before, w is a compactly supported smooth function as in the section 3.

We begin by applying the (2.3) to f. We have

$$\sum_{n\geq 1} \alpha_n w(\frac{n}{x}) = \sum_j \frac{a_j}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \check{w}(s) x^s L(f_j, s) ds \ll \sum_j |a_j| q^{\frac{1}{2}+\epsilon} x^{1/2}. \tag{4.1}$$

On the other hand, let us consider the sum $\sum_{n\geq 1} \alpha_n^2 w(\frac{n}{x})$. Since $\alpha_n = \sum a_j \lambda_j(n)$ it is easily seen that

$$\sum_{n\geq 1} \alpha_n^2 w(\frac{n}{x}) = \sum_j a_j^2 \sum_{n\geq 1} \lambda_j^2(n) w(\frac{n}{x}) + \sum_{l\neq m} a_l a_m \sum_{n\geq 1} \lambda_l(n)\lambda_m(n) w(\frac{n}{x}).$$

For the first sum, we can obtain the bound $\sum_{n\geq 1} \lambda_j^2(n) \gg xq^{-\epsilon}$ using the Perron type formula (2.3), hence we know that

$$\sum_j a_j^2 \sum_{n\geq 1} \lambda_j^2(n) w(\frac{n}{x}) \gg \sum a_j^2 x q^{-\epsilon},$$

for $x \gg q$.

And for the second term, note that $L(f_l \otimes f_m, s)$ doesn't have a pole if $l \neq m$, since the fact that $L(f_l \otimes f_m)$ is entire if $< f_l, f_m >= 0$ was proved by Rankin [9]. Then we have

$$\sum_{l\neq m} a_l a_m \sum_{n\geq 1} \lambda_l(n)\lambda_m(n) w(\frac{n}{x}) = O(q^{1+\epsilon} x^{1/2}).$$

Suppose that $\alpha \geq 0$ for all $n < x$. Then $|\lambda_j(n)| \ll n^\epsilon$ for each j, we have

$$\alpha_n \leq (\sum |a_j|) \max_j |\lambda_j(n)| \ll x^\epsilon (\sum_j |a_j|).$$

Then by combining above estimates, for $x \gg q$, the following inequality holds:

$$(\sum_j |a_j| x^\epsilon)(\sum_{n\geq 1} \alpha_n w(\frac{n}{x})) \gg \sum_{n\geq 1} \alpha_n^2 w(\frac{n}{x}) \gg (\sum a_j^2) q^{-\epsilon} x + O(q^{1+\epsilon} x^{1/2} (\sum |a_j|)^2).$$

On the other hand, from Cauchy inequality, we have $\sum_j 1 \sum a_j^2 \geq (\sum_j |a_j|)^2$. Hence using $m_1 + m_2 \sim kp$ we see that the first term dominates the right hand side of the inequaility when $x \gg q^{2+2\epsilon} k^2 p^2$, and moreover we get the inequality

$$\sum_{n\geq 1} \alpha_n w(\frac{n}{x}) \gg \frac{\sum_j a_j^2}{\sum_j |a_j|} q^{-\epsilon} x^{1-\epsilon}. \tag{4.2}$$

Then combining (4.1), (4.2) we conclude that $x \ll (kp)^{2+\epsilon} q^{1+\epsilon}$ must be satisfied. But this is contradictory to the bound $x \gg q^{2+2\epsilon} k^2 p^2$ since $q \sim k^2 p$, hence there is an $n \ll q^{2+2\epsilon} k^2 p^2$ such that $\alpha_n < 0$.

We summarize the result below.

Theorem 4.1. *Let p be a prime number. Then for a general cusp form $f = \sum_{n=1}^{\infty} a(n) e(nz)$ of weight k and level $\Gamma_0(p)$ there is an $n \ll_\epsilon k^{6+\epsilon} p^{4+\epsilon}$ such that $a(n)$ is negative(or positive).*

Remark 1. *For a general level N, to obtain an orthogonal basis is non-trivial because it is not in general easy to get the orthogonality of the spaces of oldforms, and in the square-free level case, Choie-Kohnen [1] used a special orthogonal basis to get the first sign change bound for general cusp forms. But in this paper we have chosen a prime level p to avoid this issue.*

Acknowledgment

We appreciate Professor Jakyung Koo for encouraging us to study this problem and showing us the continuous interest. We also appriciate Junehyuk Jung for considerably helping us to attack this problem, especially by teaching the basic arguments of the analytic number theory. Seokho Jin was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIP)(No. 2014001824).

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