Unobstructedness of deformations of holomorphic maps onto Fano manifolds of Picard number 1

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Abstract

We show that deformations of a surjective morphism onto a Fano manifold of Picard number 1 are unobstructed and rigid modulo the automorphisms of the target, if the variety of minimal rational tangents of the Fano manifold is non-linear or finite. The condition on the variety of minimal rational tangents holds for practically all known examples of Fano manifolds of Picard number 1, except the projective space. When the variety of minimal rational tangents is non-linear, the proof is based on an earlier result of N. Mok and the author on the birationality of the tangent map. When the varieties of minimal rational tangents of the Fano manifold is finite, the key idea is to factorize the given surjective morphism, after some transformation, through a universal morphism associated to the minimal rational curves.

Key words: deformation of holomorphic maps, Fano manifolds, variety of minimal rational tangents 2000MSC: 14J45, 32H02

1 Introduction

We will work over the complex numbers. A variety or (a manifold) will be assumed to be irreducible except when we say 'the variety of minimal rational tangents', which may have finitely many components. See Section 2 for the definition. For a complex manifold Y, $T(Y)$ denotes its tangent bundle and $T_y(Y)$ denotes the tangent space at a point $y \in Y$. For two projective varieties X and Y, denote by $\mathrm{Hom}^s(Y, X)$ the space of surjective holomorphic maps $Y \to X$ and by $\mathrm{Aut}_o(X)$ the identity component of the group of biregular automorphisms of X. In [HM3] and [HM4], the following result was proved. **Theorem 1.1** *Let X be a Fano manifold of Picard number 1 whose variety of minimal rational tangents is non-linear or finite. Then for any projective variety Y, each component of the reduction* $\mathrm{Hom}^s(Y, X)_{red}$ *is a principal homogeneous space under the affine algebraic group* $\mathrm{Aut}_o(X)$. Theorem 1.1 was first proved for the rational homogeneous space $X = G/P$ in [HM1]. It was proved when the variety of minimal rational tangents has non-degenerate Gauss map in [HM2]. This was surpassed by [HM4] which proves it when the variety of minimal rational tangents is non-linear. The proofs in these three papers are of the same nature. The proof when the variety of minimal rational tangents is finite is quite different and appeared in [HM3]. The condition that the variety of minimal rational tangents is non-linear or finite holds for practically all known examples of Fano manifolds of Picard number 1, except the projective space. In fact, we have the following non-linearity conjecture: **Conjecture 1.2** *Let X be a Fano manifold of Picard number 1 whose variety of minimal rational tangents is linear and of positive dimension. Then X is biregular to the projective space.* There are some

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partial results toward Conjecture 1.2. For example, it was proved for Fano manifolds of index $\geq \frac{\dim X+3}{2}$ in [H1, Corollary 2.3]. For the projective space, the assertion in Theorem 1.1 certainly does not hold. In this sense, Theorem 1.1 is a reasonably satisfactory result, except that it does not say whether $\mathrm{Hom}^s(Y,X)$ is reduced. In other words, it does not address the unobstructedness of infinitesimal deformations. The goal of this paper is precisely to remedy this. Our main result is the following, which also gives an alternative proof of Theorem 1.1.

Theorem 1.3 *Let X be a Fano manifold of Picard number 1 whose variety of minimal rational tangents is non-linear or finite and let Y be a projective variety. If $f: Y \to X$ is a surjective morphism, then*

$$H^0(Y, f^*T(X)) = f^*H^0(X, T(X)).$$

In particular, all deformations of surjective morphisms $Y \to X$ are unobstructed and each component of $\mathrm{Hom}^s(Y,X)$ is a reduced principal homogeneous space of the affine algebraic group $\mathrm{Aut}_o(X)$. Note that $H^0(Y, f^*T(X))$ is the Zariski tangent space to $\mathrm{Hom}^s(Y,X)$ at $[f]$ and $H^0(X, T(X))$ is the Zariski tangent space to $\mathrm{Aut}_o(X)$ at the identity. Thus the identity

$$H^0(Y, f^*T(X)) = f^*H^0(X, T(X))$$

implies that the natural morphism $\mathrm{Aut}_o(X) \to \mathrm{Hom}^s(Y,X)$ sending each $g \in \mathrm{Aut}_o(X)$ to $g \circ f \in \mathrm{Hom}^s(Y,X)$ is bijective, implying the last sentence of Theorem 1.3. The proof of Theorem 1.3 when the variety of minimal rational tangents is non-linear is rather simple modulo the main result of [HM4] on the tangent map. In retrospect, this proof is the culmination of successive refinements of the arguments in [HM2] and [HM4]. The final formulation is much simpler than the old proofs and will be given in Section 2. The difficult case is when the variety of minimal rational tangents is finite. The key idea of the proof in that case is to show that, after a certain transformation, the morphism $f: Y \to X$ can be factorized through the universal morphism for the family of minimal rational curves. This factorization is established in Section 4. Combining this with an idea from [H1] on the behavior of minimal rational curves near the branch locus of f explained in Section 5, the proof is completed in Section 6 by using an argument in [H2].

2 Proof of Theorem 1.3 when the variety of minimal rational tangents of X is non-linear

Throughout this paper, we will denote by X a Fano manifold of Picard number 1. We refer the readers to [K] for basics on the space of rational curves on X. An irreducible component $\mathcal{K}$ of the space of rational curves on X is called a *minimal component* if for a general point $x \in X$, the subscheme $\mathcal{K}_x$ of $\mathcal{K}$ consisting of members passing through x is non-empty and complete. In this case, the subvariety $\mathcal{C}_x$ of the projectivized tangent space $\mathbf{P}T_x(X)$ consisting of the tangent directions at x of members of $\mathcal{K}_x$ is called the *variety of minimal rational tangents* at x (see [HM4] for more details). We say that the variety of minimal rational tangents of X is *non-linear* if $\dim \mathcal{C}_x > 0$ and some component of $\mathcal{C}_x$ is not a linear subspace in $\mathbf{P}T_x(X)$. Otherwise, we say that the variety of minimal rational tangents is linear. For a general member C of $\mathcal{K}$, the normalization $\nu: \mathbf{P}_1 \to C \subset X$ is an immersion and

$$\nu^*T(X) = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$$

where $p = \dim \mathcal{C}_x$ for a general $x \in X$ and $p+q = \dim X - 1$. Denote by $H^0(C, T^*(X))$ the vector space

$$H^0(\mathbf{P}_1, \nu^* T^*(X)) = H^0(\mathbf{P}_1, \mathcal{O}^q).$$

For a non-singular point $x \in C$, denote by $H^0(C, T^*(X))_x \subset T^*_x(X)$ the q-dimensional subspace of the cotangent space at x given by evaluating the elements of $H^0(C, T^*(X))$ at the point x. **Proposition 2.1** *Let X and $\mathcal{K}$ be as above and let $x \in X$ be a general point. Let $\mathcal{K}_1$ be an irreducible component of $\mathcal{K}_x$. Suppose that there exists a non-zero vector $v \in T_x(X)$ annihilating $H^0(C, T^*(X))_x \subset T^*_x(X)$ for any general member C of $\mathcal{K}_1$. Then the variety of minimal rational tangents of X is linear.* *Proof.* From the irreducibility of $\mathcal{K}$, it suffices to show that the component $\mathcal{C}_1$ of $\mathcal{C}_x$ corresponding to $\mathcal{K}_1$ is a linear subspace. For a general member C of $\mathcal{K}_1$, x is a non-singular point of C. Denote by

$$\mathbf{P}H^0(C, T^*(X))_x \subset \mathbf{P}T^*_x(X)$$

the projectivization of $H^0(C, T^*(X))_x$. The closure of the union of $\mathbf{P}H^0(C, T^*(X))_x$ as C varies over general points of $\mathcal{K}_1$, is the dual variety of $\mathcal{C}_1 \subset \mathbf{P}T_x(X)$ by [HR, Corollary 2.2]. Thus the existence of v implies that the dual variety of $\mathcal{C}_1$ is linearly degenerate in $\mathbf{P}T^*_x(X)$, i.e., $\mathcal{C}_1$ is a cone. Thus Proposition 2.1 follows from [HM4, Proposition 13], which says that $\mathcal{C}_1$ cannot be a cone unless it is a linear subspace. □ The next proposition is [HM2, Lemma 4.2]. **Proposition 2.2** *Let X and $\mathcal{K}$ be as above. Let Y be a projective variety and $f : Y \to X$ be a generically finite morphism of degree > 1. Given a general member $C \subset X$ of $\mathcal{K}$, there exists a component C' of $f^{-1}(C)$ such that the restriction $f|_{C'} : C' \to C$ is finite of degree > 1.* **Proposition 2.3** *In the situation of Proposition 2.2, let $x \in C$ be a non-singular point and let $y_1, y_2 \in C'$ be two distinct points with $f(y_1) = f(y_2) = x$. For a given $\sigma \in H^0(Y, f^*T(X))$, regard its value $\sigma_{y_i} \in (f^*T(X))_{y_i}$ as a vector in $T_x(X)$ for each $i = 1, 2$. Then $\sigma_{y_1} - \sigma_{y_2} \in T_x(X)$ annihilates $H^0(C, T^*(X))_x$.* *Proof.* Let $\nu : \hat{C} \to C$ be the normalization of C and let $\varphi \in H^0(\hat{C}, \nu^* T^*(X))$ be a section of the cotangent bundle of X on $\hat{C}$. Let $\nu' : \hat{C}' \to C'$ be the normalization of C' and $\hat{f} : \hat{C}' \to \hat{C}$ be the lifting of f. Let $\varphi' \in H^0(\hat{C}', (\nu \circ \hat{f})^* T^*(X))$ be the section induced by φ and $\hat{\sigma} \in H^0(\hat{C}', (f \circ \nu')^* T(X))$ be the section induced by σ. Since $\nu \circ \hat{f} = f \circ \nu'$, the pairing $\varphi'(\hat{\sigma})$ is a holomorphic function on $\hat{C}'$, hence is constant. It follows that $\varphi'(\sigma_{y_1}) = \varphi'(\sigma_{y_2})$. Thus $\sigma_{y_1} - \sigma_{y_2}$ is annihilates the evaluation of φ at x. □

Now we can prove Theorem 1.3 when the variety of minimal rational tangents of X is non-linear. **Proposition 2.4** *Let X and $\mathcal{K}$ be as above. Suppose that there exists a surjective morphism $f : Y \to X$ from a projective variety Y with*

$$H^0(Y, f^*T(X)) \neq f^* H^0(X, T(X)).$$

Then the variety of minimal rational tangents of X is linear. *Proof.* Fix an element $\sigma \in H^0(Y, f^*T(X)) \setminus f^* H^0(X, T(X))$. For each $y \in Y$, let $\sigma_y \in T_{f(y)}(X)$ be the corresponding tangent vector of X. Associated to σ, we have the projective subvariety $\Sigma \subset T(X)$ defined by

$$\Sigma := \{\sigma_y \in T_{f(y)}(X), y \in Y\}.$$

Since $\sigma \notin f^*H^0(X, T(X))$, the natural projection $\pi : \Sigma \to X$ is a finite morphism of degree > 1 and σ induces a natural section σ' of $\pi^*T(X)$ with $\sigma' \notin \pi^*H^0(X, T(X))$. Thus replacing Y by Σ, we may assume that $f : Y \to X$ is a finite morphism and for any $x \in X$,

$$\sigma_{y_1} \neq \sigma_{y_2} \text{ as vectors in } T_x(X) \text{ for each } y_1 \neq y_2 \in f^{-1}(x).$$

Let x be a general point of X and $\mathcal{K}_1$ be an irreducible component of $\mathcal{K}_x$. By Proposition 2.2, there exist two distinct points $y_1, y_2 \in f^{-1}(x)$ such that for each general member C of $\mathcal{K}_1$, there exists an irreducible component C' of $f^{-1}(C)$ with $\{y_1, y_2\} \subset C'$. Then by Proposition 2.3, $H^0(C, T^*(X))_x$ is annihilated by $\sigma_{y_1} - \sigma_{y_2}$ for all general members C of $\mathcal{K}_1$. Applying Proposition 2.1 with $v = \sigma_{y_1} - \sigma_{y_2}$, we conclude that the variety of minimal rational tangents of X is linear. $\square$

3 Free curves with trivial normal bundle

It is convenient to introduce the following notion. Let Y be a projective manifold of dimension n and $C \subset Y$ be an irreducible curve. We say that C is a *free curve with trivial normal bundle* if the following holds. (i) Under the normalization $\nu : \hat{C} \to C$, we have an exact sequence of vector bundles on $\hat{C}$

$$0 \longrightarrow T(\hat{C}) \longrightarrow \nu^*T(Y) \longrightarrow N_C \longrightarrow 0$$

where the second arrow is the differential of $\nu : \hat{C} \to Y$ and N_C is a trivial bundle of rank $= (n-1)$ on $\hat{C}$. (ii) Deformations of C with constant geometric genus cover an open subset of X. The germ of the space of deformations of C with constant geometric genus must have dimension $\geq n-1$. The Zariski tangent space to this space at the point corresponding to C is $H^0(\hat{C}, N_C)$, which has dimension $n-1$ from the triviality of the normal bundle. Thus the germ of this space of deformations of C, which we denote by $\mathcal{M}_C$, is non-singular. The following is obvious from the deformation theory of submanifolds. **Proposition 3.1** *Let $C \subset Y$ be a free curve with trivial normal bundle. Let ϑ be a germ of nowhere-vanishing holomorphic vector fields on $\mathcal{M}_C$ given by some element*

$$\vartheta_{[C_s]} \in H^0(\hat{C}_s, N_{C_s}) \textit{ for each } [C_s] \in \mathcal{M}_C.$$

Denote by Δ the complex unit disc. The integral curve of ϑ through $[C]$ defines a deformation $\{[C_t] \in \mathcal{M}_C, t \in \Delta, C = C_0\}$ of C. Let $x \in C$ be a non-singular point. Suppose there exists a germ θ of holomorphic vector fields of Y at x such that θ modulo $T(C_s)$ agrees with the germ of ϑ_{C_s} at x for each $[C_s] \in \mathcal{M}_C$. Then the integral curve of θ through x defines a deformation $\{x_t \in Y, t \in \Delta, x = x_0\}$ of x such that $x_t \in C_t$ for each t, up to reparametrization.

From now throughout the rest of this paper, we will fix a Fano manifold X of Picard number 1 and a minimal component $\mathcal{K}$ such that the variety of minimal rational tangents at a general point is finite. Then a general member C of $\mathcal{K}$ is a free curve with trivial normal bundle and the germ $\mathcal{M}_C$ can be realized by an open neighborhood of $[C] \in \mathcal{K}$. By desingularizing the universal family over $\mathcal{K}$ (see [K, II.2.12] for the definition of the universal family), we have the following. The proof, which is quite standard, will be omitted.

Proposition 3.2 *There exist a projective manifold X' with a generically finite morphism $\mu : X' \to X$ of degree > 1 and a proper surjective morphism $\rho : X' \to Z$ onto a projective manifold Z with the following properties. (a) ρ is a $\mathbf{P}_1$-bundle over a Zariski open dense subset*

$Z_o \subset Z$. (b) μ is unramified on $\rho^{-1}(Z_o)$. (c) Each member of $\mathcal{K}_x$ for a general $x \in X$ is the image of a fiber of ρ through $\mu^{-1}(x)$.

(d) For each $\zeta \in Z_o$, $\mu|_{\rho^{-1}(\zeta)}$ is the normalization of $P_\zeta := \mu(\rho^{-1}(\zeta))$. (e) For two distinct points $\zeta_1 \neq \zeta_2 \in Z_o$, the two curves P_{ζ_1} and P_{ζ_2} are distinct.

Let us denote by $T^\rho \subset T(\rho^{-1}(Z_o))$ the relative tangent bundle of ρ over $\rho^{-1}(Z_o)$. Let $\mathcal{C} \subset \mathbf{P}T(X)$ be the closure of the union of $\mathcal{C}_x$'s for general points $x \in X$. Let $\hat{\mathcal{C}} \subset T(X)$ be the cone over $\mathcal{C}$. Denote by $O \subset T(X)$ the zero section and by $\pi : T(X) \to X$ the natural projection. The following is immediate. **Proposition 3.3** *In the setting of Proposition 3.2, there exists a Zariski open dense subset $X_o \subset X$ such that $\mu^{-1}(X_o) \subset \rho^{-1}(Z_o)$ and the restriction of π to $(\hat{\mathcal{C}} \setminus O) \cap \pi^{-1}(X_o)$ is a smooth morphism. For each point $x \in X_o$ and $\mu^{-1}(x) = \{x_1, \ldots, x_m\}, m =$ degree of μ, we have a disjoint union*

$$\pi^{-1}(x) \cap (\hat{\mathcal{C}} \setminus O) = d\mu(T^\rho_{x_1} \setminus \{0\}) \cup \cdots \cup d\mu(T^\rho_{x_m} \setminus \{0\}).$$

In particular, we have a natural smooth morphism

$$\chi : \ (\hat{\mathcal{C}} \setminus O) \cap \pi^{-1}(X_o) \longrightarrow \mu^{-1}(X_o)$$

such that $\pi = \mu \circ \chi$ on $(\hat{\mathcal{C}} \setminus O) \cap \pi^{-1}(X_o)$.

Proposition 3.4 *Let Y be a projective manifold and $f : Y \to X$ be a generically finite surjective morphism. For a general member $C \subset X$ of $\mathcal{K}$, C intersects each component of the branch divisor of f transversally. Each irreducible component C' of $f^{-1}(C)$ is a free curve with trivial normal bundle and when $\hat{C}$ (resp. $\hat{C}'$) is the normalization of C (resp. C') and $\hat{f} : \hat{C}' \to \hat{C}$ is the morphism induced by f, there are canonical isomorphisms*

$$T_{[C']}(\mathcal{M}_{C'}) := H^0(\hat{C}', N_{C'}) \cong H^0(\hat{C}', \hat{f}^* N_C)) \cong H^0(\hat{C}, N_C) =: T_{[C]}(\mathcal{M}_C)$$

and a biholomorphic equivalence of germs $\mathcal{M}_C \cong \mathcal{M}_{C'}$. *Proof.* That C intersects the branch divisor transversally is obvious from Proposition 3.2 (b). The fact that C' is a free curve with trivial normal bundle is precisely [HM3, Proposition 6]. The canonical isomorphisms and the equivalence of germs are obvious from the isomorphism of two trivial vector bundles $N_{C'} \cong \hat{f}^* N_C$ induced by the differential $df : T(Y) \to \hat{f}^* T(X)$. □

Proposition 3.5 *Let Y be a projective variety and $f : Y \to X$ be a generically finite surjective morphism. Let C be a general member of $\mathcal{K}$ and let C' be a component of $f^{-1}(C)$. Pick a non-singular point $x \in C$ outside the branch loci. Let $\sigma \in H^0(Y, f^* T(X))$. For any two points $y_1, y_2 \in f^{-1}(x) \cap C'$, regard σ_{y_1} and σ_{y_2} as vectors in $T_x(X)$. Then $\sigma_{y_1} - \sigma_{y_2} \in T_x(C)$. In particular, σ induces a unique element in $H^0(\hat{C}, N_C)$, up to a choice of C'.* *Proof.* This is a consequence of Proposition 2.3. Since C has trivial normal bundle, $H^0(C, T^*(X))_x$ is the conormal space of C at x. Thus $\sigma_{y_1} - \sigma_{y_2} \in T_x(C)$. □

4 Factorization through μ

In the setting of Theorem 1.3, given a section $\sigma \in H^0(Y, f^* T(X))$, the values of σ define a projective variety in $T(X)$ dominant over X, as explained in the proof of Proposition 2.4. In fact, Theorem 1.3 is equivalent to the statement that a projective variety in $T(X)$ dominant over X

must be a section of $T(X)$. In other words, we have to prove that there do not exist projective varieties of $T(X)$ which have degree > 1 over X. The goal of this section is to show that given a projective variety $\Sigma \subset T(X)$ of degree > 1 over X, the difference transform of Σ contains an irreducible component that has very special properties with respect to the morphisms μ, ρ of Proposition 3.2. It should be mentioned that all the propositions proved from now on, except Proposition 5.1, are under the assumption of the existence of Σ of degree > 1, which will lead eventually to contradiction. In this sense all these propositions are of hypothetical nature.

Proposition 4.1 *Suppose there exists a projective variety $\Sigma \subset T(X)$ which is dominant over X of degree > 1. Let $T(X) \times_X T(X)$ be the fiber product of two copies of the projection $\pi : T(X) \to X$ and let*

$$\Sigma \times_X \Sigma \subset T(X) \times_X T(X)$$

be the fiber product of two copies of $\pi|_\Sigma : \Sigma \to X$. Then there exists an irreducible component $\Sigma^\sharp$ of $\Sigma \times_X \Sigma$ with the following property: for a general $\zeta \in Z_o$ and a general point $x \in P_\zeta$, there exists an irreducible component P'_ζ of $\pi^{-1}(P_\zeta) \cap \Sigma$ and two distinct points $a_1 \neq a_2 \in P'_\zeta \cap \pi^{-1}(x)$ such that $\Sigma^\sharp$, regarded as a subvariety of $\Sigma \times \Sigma$, contains the point (a_1, a_2). *Proof.* For a general $\zeta \in Z_o$, there exists an irreducible component P'_ζ of $\pi^{-1}(P_\zeta) \cap \Sigma$ such that the projection $P'_\zeta \to P_\zeta$ is finite of degree > 1 by Proposition 2.2. Thus we can choose two $a_1 \neq a_2$ on P'_ζ over $x \in P_\zeta$. The point $(a_1, a_2) \in \Sigma \times \Sigma$ lies in $\Sigma \times_X \Sigma$. From the generality of the choice of ζ and x, there is a unique component $\Sigma^\sharp$ containing (a_1, a_2). Certainly, $\Sigma^\sharp$ satisfies the required property from the irreducibility of Z_o. □

Proposition 4.2 *In the situation of Proposition 4.1, let $\delta : T(X) \times_X T(X) \to T(X)$ be the difference morphism defined by*

$$\delta(v_1, v_2) := v_1 - v_2 \ \ \textit{for} \ \ v_1, v_2 \in T_x(X) \ \ \textit{for } x \in X.$$

Then in the notation of Proposition 3.3,

$$\delta(\Sigma^\sharp) \subset \hat{\mathcal{C}}, \ \ \delta(\Sigma^\sharp) \not\subset O,$$

and the dominant rational map $\chi^\sharp : \delta(\Sigma^\sharp) \longrightarrow X'$, induced by the morphism χ, is generically finite.

Proof. We will apply Proposition 3.5 with $Y = \Sigma$, $f = \pi|_\Sigma$ and $C = P_\zeta$. There is a tautological section $\sigma \in H^0(Y, f^*T(X))$ defined by

$$\sigma_a = a \in T_x(X) \ \text{ for each } a \in \Sigma \cap T_x(X).$$

By Proposition 3.5,

$$a_1 - a_2 \ \in \ T_x(P_\zeta) \ \subset \ \hat{\mathcal{C}}.$$

As ζ varies over general points of Z_o, the element $a_1 - a_2$ varies over an open subset in the irreducible $\delta(\Sigma^\sharp)$. It follows that $\delta(\Sigma^\sharp) \subset \hat{\mathcal{C}}$. Since $a_1 \neq a_2$, $\delta(\Sigma^\sharp)$ is not contained in the zero section O. The dominant rational map $\chi^\sharp$ is certainly generically finite. □ **Proposition 4.3** *In the situation of Proposition 4.2, there exists a projective manifold $\tilde{\Sigma}$, a generically finite morphism $g : \tilde{\Sigma} \to X'$ and a section $\theta \in H^0(\tilde{\Sigma}, (\mu \circ g)^* T(X))$ with the following properties. (1) For a general point $x \in X$ and any two distinct $y_1, y_2 \in (\mu \circ g)^{-1}(x)$, $\theta_{y_1} \neq \theta_{y_2}$ as vectors in $T_x(X)$, (2) For a* *general point $x \in X$ and any $y \in (\mu \circ g)^{-1}(x)$, θ_y regarded as a vector in $T_{g(y)}(X') = T_x(X)$, belongs to $T^\rho_{g(y)}$ where T^ρ is as in Proposition 3.3.* *Proof.* Choose a desingularization $\alpha : \tilde{\Sigma} \to \delta(\Sigma^\sharp)$ which eliminates the indeterminacy of the generically finite rational map $\chi^\sharp$ such that $\chi^\sharp \circ \alpha$ defines a generically finite morphism $g : \tilde{\Sigma} \to X'$. Denote by τ the natural projection $\delta(\Sigma^\sharp) \to X$. Then $\tau \circ \alpha = \mu \circ g$. Since $\delta(\Sigma^\sharp) \subset T(X)$, there exists a tautological section $\kappa \in H^0(\delta(\Sigma^\sharp), \tau^* T(X))$ defined by $\kappa(a) = a \in T_{\tau(a)}(X)$ for each $a \in \delta(\Sigma^\sharp)$. Let

$$\theta \in H^0(\tilde{\Sigma}, (\mu \circ g)^* T(X)) = H^0(\tilde{\Sigma}, (\tau \circ \alpha)^* T(X))$$

be the pull-back of κ by α. Then θ satisfies property (1), because α is birational and the tautological section κ satisfies an analog of (1). It satisfies property (2) from $\delta(\Sigma^\sharp) \subset \hat{\mathcal{C}}$. □

5 Univalence of $\mathcal{K}$ on the branch divisor of μ

In the setting of Proposition 3.2, we say that $\mathcal{K}$ is *univalent on an irreducible hypersurface $B \subset X$* if (i) there exists only one irreducible component E of $\mu^{-1}(B)$ that is dominant over both Z and B, and (ii) the morphism $\mu|_E : E \to B$ is birational. This is equivalent to saying that at a general point $z \in B$, there exists exactly one member C of $\mathcal{K}$ passing through z with $C \not\subset B$ and C is non-singular at z. The following is essentially the same as [H1, Proposition 3.2]. **Proposition 5.1** *In the setting of Proposition 3.2, suppose that there exists an irreducible hypersurface $B \subset X$, such that $\mathcal{K}$ is not univalent on B. Then given a general point $x \in B$ and an open neighborhood $W \subset X$ of x, there exists a point $y \in W$ and two distinct points $\zeta_1, \zeta_2 \in Z_o$ with $y \in P_{\zeta_1} \cap P_{\zeta_2}$ and $T_y(P_{\zeta_1}) \neq T_y(P_{\zeta_2})$ such that irreducible components of $W \cap P_{\zeta_1}$ and $W \cap P_{\zeta_2}$ through y intersect B transversally at some point of $B \cap W$.* *Proof.* By assumption, there exist a union D of components of $\mu^{-1}(B)$ each of which is dominant over Z and B, and the morphism $\mu|_D : D \to B$ has degree > 1. Let y_1, y_2 be two distinct points of $\mu^{-1}(x) \cap D$. Since x is general, both $\rho(y_1)$ and $\rho(y_2)$ lie in Z_o. There exist open neighborhoods $W_1 \subset \rho^{-1}(Z_o)$ of y_1, $W_2 \subset \rho^{-1}(Z_o)$ of y_2 and $W_0 \subset W$ of x with the following properties (1) $\mu(W_1) = \mu(W_2) = W_0$, (2) $\mu|_{W_1}$ and $\mu|_{W_2}$ are biholomorphic, (3) $W_1 \cap D$ and $W_2 \cap D$ are non-singular and transversal to the fibers of ρ. There exist open neighborhood $W_1' \subset W_1$ of y_1 and $W_2' \subset W_2$ of y_2 such that for any $y \in W_1'$ (resp. $y \in W_2'$) $\rho^{-1}(\rho(y)) \cap W_1'$ (resp. $\rho^{-1}(\rho(y)) \cap W_2'$) is connected. Let y be a general point in $\mu(W_1') \cap \mu(W_2')$. Let $y_1' = W_1' \cap \mu^{-1}(y)$ and $y_2' = W_2' \cap \mu^{-1}(y)$. Then $\zeta_1 := \rho(y_1')$ and $\zeta_2 := \rho(y_2')$ give the desired two distinct points. □ The idea of the proof of the following proposition is the same as that of [H1, Proposition 3.3]. **Proposition 5.2** *In the setting of Proposition 4.3, $\mathcal{K}$ is univalent on each component B of the branch divisor of μ.* *Proof.* Suppose that $\mathcal{K}$ is not univalent on some component B. Set $Y := \tilde{\Sigma}$ and $f = \mu \circ g$. Then B is a component of the branch divisor of $f : Y \to X$. Let $R \subset Y$ be an irreducible component of the ramification divisor of f such that $B = f(R)$. Let $z \in R$ be a general point and let r be the local sheeting number of f at z. We can choose a holomorphic coordinate neighborhood V of z with coordinates $(w_1, \ldots, w_n)$ at z and a holomorphic coordinate neighborhood W of $f(z)$ with coordinates $(z_1, \ldots, z_n)$ such that $B \cap W$ is defined by $z_n = 0$ and f is given by

$$z_1 = w_1, \ldots, z_{n-1} = w_{n-1}, z_n = w_n^r.$$

Let $x \in W \setminus B$ and $\zeta_1, \zeta_2 \in Z_o$ be as in Proposition 5.1. Setting $C_1 = P_{\zeta_1}$ (resp. $C_2 = P_{\zeta_2}$), an easy coordinate computation in the above coordinate systems (see e.g. [HM3, p.636, Lemma1]) shows that there exists a unique irreducible component C'_1 (resp. C'_2) of $f^{-1}(C_1)$ (resp. $f^{-1}(C_2)$) intersecting V such that an irreducible component of $C'_1 \cap V$ (resp. $C'_2 \cap V$) contains $f^{-1}(x) \cap V$. In particular, $C'_1 \cap C'_2$ contains the r distinct points $f^{-1}(x) \cap V$. Let $y_1 \neq y_2$ be two distinct points in $f^{-1}(x) \cap V \cap C'_1 \cap C'_2$. Applying Proposition 3.5 to C'_1 and C_1,

$$\theta_{y_1} - \theta_{y_2} \in T_{x_o}(C_1).$$

Applying Proposition 3.5 to C'_2 and C_2,

$$\theta_{y_1} - \theta_{y_2} \in T_{x_o}(C_2).$$

Since $T_{x_o}(C_1) \cap T_{x_o}(C_2) = 0$, we get $\theta_{y_1} = \theta_{y_2}$, a contradiction to Proposition 4.3 (1). □ **Proposition 5.3** *In the setting of Proposition 5.2, let $B \subset X$ be a component of the branch divisor of μ and let D be the unique irreducible component of $\mu^{-1}(B)$ which is dominant over Z and B. Then for a general member C of $\mathcal{K}$, any component C' of $\mu^{-1}(C)$ which is finite over C of degree > 1 is disjoint from D.* *Proof.* Suppose not. Since C' is a free curve with trivial normal bundle by Proposition 3.4, we may assume that C' intersects D at a general point x' of D. Then through a general point x' of D, we have two distinct curves, C' and a fiber of ρ, neither of which are contained in D. Since μ is unramified at x' by Proposition 3.2 (b), the images of these curves under μ are of the form P_{ζ_1}, P_{ζ_2} with $\zeta_1 \neq \zeta_2$. Since these two curves pass through $x = \mu(x')$, which is a general point of B, $\mathcal{K}$ is not univalent on B, a contradiction to Proposition 5.2. □

6 Completion of the proof of Theorem 1.3

In this section, we will complete the proof of Theorem 1.3. The strategy is to establish some analogs of [H2, Section 5]. **Proposition 6.1** *In the setting of Proposition 4.3, let $C \subset X$ be a general member of $\mathcal{K}$. By Proposition 2.2, there exists a component C' of $\mu^{-1}(C)$ such that $\mu|_{C'} : C' \to C$ is finite of degree > 1, and C' is a free curve with trivial normal bundle by Proposition 3.4. Fix a choice of a component $C^\flat$ of $g^{-1}(C')$. Then θ modulo $T(\hat{C}')$ defines a non-zero element $\vartheta_{C'} \in H^0(\hat{C}', N_{C'})$.* *Proof.* By Proposition 3.5, $C^\flat$ determines a unique element in $H^0(\hat{C}, N_C)$. By the isomorphism in Proposition 3.4, this determines an element $\vartheta_{C'} \in H^0(\hat{C}', N_{C'})$. It cannot be zero because of Proposition 4.3 (2) and the generality of C. □ The next proposition is an analog of [H2, Lemma 5.5], although their proofs are of different nature. **Proposition 6.2** *In the setting of Proposition 6.1, denoting by Δ the complex unit disc, there exist a family of members of $\mathcal{K}$*

$$\{C_t, t \in \Delta, C = C_0\}$$

and the associated deformation

$$\{C'_t, t \in \Delta, C' = C'_0\}$$

such that for each $t \in \Delta$, (i) C'_t is a component of $\mu^{-1}(C_t)$; (ii) $\mu|_{C'_t} : C'_t \to C_t$ is finite of degree > 1; (iii) $\rho(C'_t) = \rho(C')$. *Proof.* For a deformation $[C'_s] \in \mathcal{M}_{C'}$ of C', we get a deformation $[C^\flat_s] \in \mathcal{M}_{C^\flat}$ with $C^\flat_s \subset g^{-1}(C'_s)$. By applying Proposition 6.1 to $C^\flat_s$, we get an element $\vartheta_{C'_s} \in H^0(C'_s, N_{C'_s})$. Thus the choice of $C^\flat$ determines a germ of holomorphic vector fields ϑ on $\mathcal{M}_{C'}$. By Proposition 4.3 (2), this is a germ of non-vanishing vector fields. Let

$$\{C'_t, t \in \Delta, C' = C'_0\} = \{[C'_t] \in \mathcal{M}_{C'}, t \in \Delta\}$$

be a local analytic arc integrating the vector field ϑ. (i) and (ii) are obvious from the local equivalence $\mathcal{M}_C \cong \mathcal{M}_{C'}$ in Proposition 3.4. It suffices to check (iii). Let $x \in C'$ be a general point and $y \in C^\flat$ be the point over x. Then the germ of holomorphic vector fields defined by θ at y induces a germ θ' of holomorphic vector fields in a neighborhood of x. Applying 3.1, we see that the integral curve of θ' through x, $\{x_t \in X', t \in \Delta\}$ with $x = x_0$, satisfies $x_t \in C'_t$ up to reparametrization. Since θ' is a section of T^ρ by Proposition 4.3 (2), $x_t \in \rho^{-1}(\rho(x))$. It follows that

$$\rho^{-1}(z) \cap C'_t \neq \emptyset \text{ for general } z \in \rho(C') \text{ and each } t \in \Delta.$$

This implies (iii). $\square$

The proof of the next proposition is, modulo Proposition 5.3 and Proposition 6.2, almost verbatim that of [H2, Proposition 5.3]. Since the terms and the notation are slightly different, we reproduce the proof for the reader's convenience. **Proposition 6.3** *Let us assume the situation of Proposition 6.1. Given C and C' as in Proposition 6.1, let*

$$h : \hat{C}' \longrightarrow \widehat{\rho(C')}$$

be the lift of

$$\rho|_{C'} : C' \longrightarrow \rho(C')$$

to the normalizations of C' and $\rho(C')$. Then h has a ramification point $z \in \hat{C}'$ such that the image of $h(z)$ in $\rho(C')$ lies in Z_o.

Proof. Suppose not. Then h is unramified over $\rho(C') \cap Z_o$. Let us use the deformation C_t constructed in Proposition 6.2. By the generality of C, we may assume that for each $t \in \Delta$ the holomorphic map

$$h_t : \hat{C}'_t \to \widehat{\rho(C'_t)} = \widehat{\rho(C')}, \quad h_0 = h,$$

which is the lift of $\rho|_{C'_t}$ to the normalization, is unramified over $\rho(C') \cap Z_o$. Since h_t is a continuous family of coverings of the Riemann surface $\widehat{\rho(C')}$ with fixed branch locus, we can find a biholomorphic map

$$(\clubsuit) \quad \psi_t : \hat{C}' \to \hat{C}'_t, \;\; \psi_0 = \mathrm{Id}_{\hat{C}'} \text{ with } h = h_t \circ \psi_t,$$

which depends holomorphically on t (e.g. [S, p. 32, Corollary 1].). By Proposition 6.2 (ii), there are at least two distinct points in $\hat{C}_t$, say $a_t \neq b_t \in \hat{C}_t$, such that the corresponding points in C_t lie in the branch divisor of μ in X. Let $\{0, \infty\} \subset \mathbf{P}_1$ be two distinct points on the projective line. We can choose a family of biholomorphic maps $\{\sigma_t : \hat{C}_t \to \mathbf{P}_1,\ t \in \Delta\}$ such that $\sigma_t(a_t) = 0$ and $\sigma_t(b_t) = \infty$ for each $t \in \Delta$. Denote by $\mu_t : \hat{C}'_t \to \hat{C}_t$ the lift of $\mu|_{C'_t}$ to the normalization. Then

$$\{\varphi_t : \hat{C}' \longrightarrow \mathbf{P}_1, \quad \varphi_t := \sigma_t \circ \mu_t \circ \psi_t,\ t \in \Delta\}$$

is a family of meromorphic functions on the compact Riemann surface $\hat{C}'$. By Proposition 5.3, for each component E of the branch divisor of μ, the intersection of C'_t with $\mu^{-1}(E)$ has a fixed image in $\rho(C') = \rho(C'_t)$, independent of $t \in \Delta$. This implies that there is a finite subset $Q \subset \widehat{\rho(C')}$, independent of t, such that

$$\mu_t^{-1}(a_t) \cup \mu_t^{-1}(b_t) \subset h_t^{-1}(Q)$$

for any $t \in \Delta$. Then

$$\varphi_t^{-1}(0) = \psi_t^{-1} \circ \mu_t^{-1} \circ \sigma_t^{-1}(0) = \psi_t^{-1}(\mu_t^{-1}(a_t)) \subset \psi_t^{-1}(h_t^{-1}(Q))$$

for all $t \in \Delta$. Since $\psi_t^{-1}(h_t^{-1}(Q)) = h^{-1}(Q)$ by the choice of ψ_t in ($\clubsuit$), $\varphi_t^{-1}(0) \subset h^{-1}(Q)$ for any $t \in \Delta$. Consequently, $\varphi_t^{-1}(0) = \varphi_0^{-1}(0)$ for all $t \in \Delta$. By the same argument we get $\varphi_t^{-1}(\infty) = \varphi_0^{-1}(\infty)$ for all $t \in \Delta$. In other words, the family of meromorphic functions φ_t have the same zeroes and the same poles on the Riemann surface $\hat{C}'$. This implies that for any $z \in \mathbf{P}_1$ and $t \in \Delta$, $\varphi_t^{-1}(z) = \varphi_0^{-1}(z)$. It follows that for any $w_1, w_2 \in \hat{C}'$ and any $t \in \Delta$,

$$(\diamondsuit) \quad \varphi_t(w_1) = \varphi_t(w_2) \text{ if and only if } \varphi_0(w_1) = \varphi_0(w_2).$$

Since $\mu|_{C'}$ is finite of degree > 1 by our assumption, we can choose two points $\alpha \neq \beta \in \hat{C}'$ such that $\varphi_0(\alpha) = \varphi_0(\beta)$. Furthermore, denoting by $\bar{\alpha} \in \rho(C')$ (resp. $\bar{\beta} \in \rho(C')$) the point corresponding to $h_0(\alpha) \in \widehat{\rho(C')}$ (resp. $h_0(\beta) \in \widehat{\rho(C')}$) under the normalization, we may assume that

$$(\heartsuit) \quad \bar{\alpha} \text{ and } \bar{\beta} \text{ are two distinct points in } Z_o.$$

From ($\diamondsuit$), we have $\varphi_t(\alpha) = \varphi_t(\beta)$ for all $t \in \Delta$. Since $\varphi_t = \sigma_t \circ \mu_t \circ \psi_t$ and σ_t is biholomorphic, we see that

$$(\spadesuit) \quad \mu_t \circ \psi_t(\alpha) = \mu_t \circ \psi_t(\beta) \text{ for all } t \in \Delta.$$

Denote by

$$\alpha_t \in C'_t \subset X' \quad (\text{resp. } \beta_t \in C'_t \subset X')$$

the point corresponding to $\psi_t(\alpha) \in \hat{C}'_t$ (resp. $\psi_t(\beta) \in \hat{C}'_t$) under the normalization. Then the locus

$$A := \{\alpha_t \in X', t \in \Delta\} \quad (\text{resp. } B := \{\beta_t \in X', t \in \Delta\})$$

covers a non-empty open subset in the fibre $\rho^{-1}(\bar{\alpha})$ (resp. $\rho^{-1}(\bar{\beta})$). Thus $\mu(A)$ (resp. $\mu(B)$) covers a non-empty open subset in

$$P_{\bar{\alpha}} := \mu(\rho^{-1}(\bar{\alpha})) \quad (\text{resp. } P_{\bar{\beta}} := \mu(\rho^{-1}(\bar{\beta}))).$$

Since $\mu(A)$ (resp. $\mu(B)$) is the locus of points corresponding to $\mu_t \circ \psi_t(\alpha)$ (resp. $\mu_t \circ \psi_t(\beta)$) by the normalization $\hat{C}_t \to C_t$, the equality ($\spadesuit$) above implies that $\mu(A) = \mu(B)$. Consequently,

$$P_{\bar{\alpha}} = P_{\bar{\beta}},$$

a contradiction to Proposition 3.2 via ($\heartsuit$). □ Now we are ready to finish the proof of Theorem 1.3 as follows. *End of the proof of Theorem 1.3.* As explained at the beginning of Section 4, we may assume the situation of Proposition 4.3 and get a contradiction. From Proposition 6.3, let $z \in C'$ be the image of a ramification point of h such that $\rho(z) \in Z_o$. Then a component of the germ of C' at z must be tangent to T_z^{ρ}. Choose $C^{\flat}$ as in Proposition 6.1. The value of θ at a point of $C^{\flat}$ over z determines $\theta_z \in T_z(X')$ which is in T_z^{ρ} by Proposition 4.3 (2). Thus θ_z is tangent to a component of the germ of C' at z. This means that the non-zero element $\vartheta_{C'} \in H^0(\hat{C}', N_{C'})$ in Proposition 6.1 vanishes at z, a contradiction to the triviality of $N_{C'}$. $\square$

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