TORIC COHOMOLOGICAL RIGIDITY OF SIMPLE CONVEX POLYTOPES

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Abstract. A simple convex polytope P is *cohomologically rigid* if its combinatorial structure is determined by the cohomology ring of a quasitoric manifold over P. Not every P has this property, but some important polytopes such as simplices or cubes are known to be cohomologically rigid. In this article we investigate the cohomological rigidity of polytopes and establish it for several new classes of polytopes including products of simplices. Cohomological rigidity of P is related to the *bigraded Betti numbers* of its *Stanley–Reisner ring*, another important invariants coming from combinatorial commutative algebra.

1. Introduction

Quasitoric manifolds were defined by Davis and Januszkiewicz in [7] as a topological analogue of nonsingular toric varieties. Namely, a *quasitoric manifold* over a simple convex polytope P is a closed $2n$-dimensional manifold M with a locally standard action of an n-torus $G = (S^1)^n$ (that is, the action locally looks like a faithful real $2n$-dimensional representation of G) and a surjective map $\pi\colon M \to P$ whose fibers are the G-orbits. The combinatorial structure of P is completely determined by the equivariant cohomology ring $H^*_G(M)$ because $H^*_G(M)$ is isomorphic to the Stanley-Reisner ring (or the face ring) of P. On the other hand the $2i$-th Betti number of M is equal to the i-th component of the h-vector of P. Therefore the usual cohomolgy $H^*(M)$ contains some combinatorial information of P.

In general cohomology ring of a quasitoric manifold does not contain sufficient information to determine the combinatorial structure of the base polytope P, as in Example 4.3 of [10], which we will discuss briefly for reader's convenience. To do this let us fix some notation. For an n-dimensional simple convex polytope P and a vertex v of it, let $\mathrm{vc}(P, v)$ denote the connected sum of P with the n-simplex Δ^n at the vertex v. Hence $\mathrm{vc}(P, v)$ is the simple convex polytope obtained from P by cutting a small n-simplex neighborhood of the vertex v. We call $\mathrm{vc}(P, v)$ the *vertex cut* of P at v. When

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the combinatorial structure of $\mathrm{vc}(P,v)$ does not depend on the vertex v, we simply denote it by $\mathrm{vc}(P)$. For example when P is a product of simplices, the vertex cut $\mathrm{vc}(P,v)$ does not depend on the choice of a vertex v.

The following example explains a phenomenon leading to our main definition.

Example 1.1. We consider $M = \mathbb{C}P^2 \times \mathbb{C}P^1$ with the standard $(S^1)^3$-action. It is a quasitoric manifold over the triangular prism $P = \Delta^2 \times \Delta^1$. The equivariant blow up M' of M at a fixed point x is a quasitoric manifold over $P' = \mathrm{vc}(P)$, which does not depend on the choice of a fixed point x. Now if we blow up M' equivariantly at a fixed point y in M', then the resulting manifold M'' is a quasitoric manifold over $P'' = \mathrm{vc}(P', v)$. Manifold M'' is no longer independent of a fixed point y; in fact there are three equivariantly different manifolds corresponding to three combinatorially different vertex cuts $\mathrm{vc}(P', v)$ (these correspond to the first three simplicial complexes in the second line in p. 192 of [11]).

On the other hand, the cohomology ring of M'' does not depend on the choice of a fixed point y, because M'' is homeomorphic to the connected sum of $\mathbb{C}P^2 \times \mathbb{C}P^1$ with two copies of $\mathbb{C}P^3$. We therefore are in the situation when the cohomology ring of a quasitoric manifold does not determine the combinatorial structure of the base polytope.

Nevertheless, in many cases the combinatorial type of P is determined by $H^*(M)$. We therefore naturally come to the following definition, firstly introduced in [10].

Definition 1.2. A simple polytope P is *cohomologically rigid* if there exists a quasitoric manifold M over P, and whenever there exists a quasitoric manifold N over another polytope Q with a graded ring isomorphism $H^*(M) \cong H^*(N)$ there is a combinatorial equivalence $P \approx Q$. We shall refer to such P simply as *rigid* throughout the paper.

We shall extend this definition to arbitrary Cohen-Macaulay complexes in Definition 3.9. In [10] the rigidity property is expressed in terms of toric manifolds, but here we modify the original definition to make use of a wider class of quasitoric manifolds. The interval I is trivially rigid. More generally, it is shown in [9] that any cube I^n is rigid. In this paper we give more classes of rigid polytopes, as described in the following results.

Theorem 1.3. *Let P be a simple polytope supporting a quasitoric manifold. If there is no other simple polytope with the same numbers of i-faces as those of P for all i, then P is rigid.*

Corollary 1.4. *Every polygon, i.e. 2-dimensional convex polytope, is rigid.*

A simple convex polytope is called *triangle-free* if it has no triangular 2-face.

Theorem 1.5. *Every triangle-free n-dimensional simple convex polytope with less than $2n+3$ facets is rigid.*

Since a cube I^n has $2n$ facets, Theorem 1.5 gives a different proof of the rigidity for cubes [9]. Furthermore, this may be generalized as follows.

Theorem 1.6. *Any finite product of simplices is rigid.*

From the argument in Example 1.1 one can see immediately that if the vertex cut of a polytope P depends on a choice of vertex, then all the vertex cuts of P are not rigid. So it is natural to ask whether $\mathrm{vc}(P)$ is rigid if the vertex cut of P is independent of a choice of vertex. The following theorem confirms this when P is a product of simplices.

Theorem 1.7. *If P is a finite product of simplices, then* $\mathrm{vc}(P)$ *is rigid.*

We can apply the above results to determine rigidity of 3-dimensional simple convex polytopes with facet numbers up to nine. This result is given in Section 7. We also prove that dodecahedron is rigid in Theorem 7.1.

The rigidity property for simple polytopes is closely related to the following interesting question on quasitoric manifolds.

Question 1.8. *Suppose M and N are two quasitoric manifolds such that $H^*(M) \cong H^*(N)$ as graded rings. Are M and N homeomorphic?*

We can also consider the following slightly weaker question, which can be considered as an intermediate step to answering Question 1.8.

Question 1.9. *Suppose M and N are two quasitoric manifolds over the same simple convex polytope P such that $H^*(M) \cong H^*(N)$ as graded rings. Are M and N homeomorphic?*

Question 1.8 for quasitoric manifolds whose cohomology rings are isomorphic to those of the product of $\mathbb{C}P^1$ is considered in [9], and it is shown there that these manifolds are actually homeomorphic to the product of $\mathbb{C}P^1$. This is done in two steps; firstly the result is proved under additional assumption that the quotient polytope is a cube I^n, and then the rigidity of I^n is established, see [9].

In [5] it is proved that if M is a quasitoric manifold over a product of simplices $\prod_{i=1}^t \Delta^{n_i}$ such that $H^*(M) \cong H^*(\prod_{i=1}^t \mathbb{C}P^{n_i})$, then M is homeomorphic to $\prod_{i=1}^t \mathbb{C}P^{n_i}$. Since a product of simplices is rigid by Theorem 1.6, we have the following theorem.

Theorem 1.10. *Suppose M is a quasitoric manifold such that $H^*(M) \cong H^*(\prod_{i=1}^t \mathbb{C}P^{n_i})$, then M is homeomorphic to $\prod_{i=1}^t \mathbb{C}P^{n_i}$.*

The main technical ingredient for the proofs of the results in this paper is the following proposition. For a polytope P let $\beta^{-i,2j}(P)$ be the bigraded Betti numbers of the Stanley-Reisner ring $\mathbb{Q}(P)$ of P, see Section 3 or [4] for details.

Proposition 1.11. *Let M (reps. N) be a quasitoric manifold over P (resp. Q). If $H^*(M:\mathbb{Q}) \cong H^*(N:\mathbb{Q})$ as graded rings, then $\beta^{-i,2j}(P) = \beta^{-i,2j}(Q)$ for all i and j.*

2. Proof of Theorem 1.3 and Corollary 1.4

For a convex n-dimensional polytope P let f_i denote the number of codimension $i+1$ faces of P, and let $f(P) = (f_0, \dots, f_{n-1})$ denote the f-vector of

P. Note that if P and Q are two 2-dimensional polytopes, then $f(P) = f(Q)$ implies $P \approx Q$. Recall that h-vector $h(P) = (h_0, \dots, h_n)$ of P is defined by

$$\sum_{i=0}^{n} h_i t^{n-i} = \sum_{j=0}^{n} f_{j-1}(t-1)^{n-j}.$$

The following theorem proved in [7] shows that f-vector of the base polytope P is determined by the cohomology ring of the quasitoric manifold M over P.

Theorem 2.1. [7] *For a quasitoric manifold M over P the $2i$-th Betti number $b_{2i}(M)$ of M is equal to the i-th component h_i of the h-vector of P.*

□

Now let P be a polytope and M a given quasitoric manifold over P. Suppose N is another quasitoric manifold over Q such that $H^*(M) \cong H^*(N)$ as graded rings. Then the cohomology isomorphism implies $b_{2i}(M) = b_{2i}(N)$ for all i. Hence $h(P) = h(Q)$ by Theorem 2.1, which implies $f(P) = f(Q)$. Since there is no other simple convex polytope with the same face numbers of P, $P \approx Q$ and hence Theorem 1.3 is proved. Moreover Corollary 1.4 follows immediately from Theorem 1.3.

3. Bigraded Betti numbers of polytopes

Let $A = \mathbb{Q}[x_1, \dots, x_m]$ be the polynomial graded ring in $x_1, \dots, x_m$ over the rationals with $\deg x_i = 2$ for all i. A *free resolution* $[R, d]$ of a finitely generated A-module M is an exact sequence

$$0 \longrightarrow R^{-n} \xrightarrow{d} R^{-n+1} \xrightarrow{d} \cdots \xrightarrow{d} R^0 \xrightarrow{d} M \longrightarrow 0\,, \tag{1}$$

where R^{-i} are finitely generated free graded A-modules and d are degree preserving homomorphisms. If we take R^{-i} to be the module generated by the minimal basis of $\mathrm{Ker}(d : R^{-i+1} \to R^{-i+2})$, then we get a *minimal resolution* of M. This also shows the existence of a resolution.

Dropping the last term M in the sequence (1) and tensoring it over A with another finitely generated A-module N, we obtain the following sequence:

(2)

$$0 \longrightarrow R^{-n} \otimes_A N \xrightarrow{d\otimes 1} R^{-n+1} \otimes_A N \xrightarrow{d\otimes 1} \cdots \xrightarrow{d\otimes 1} R^0 \otimes_A N \longrightarrow 0$$

This sequence is not necessarily exact, and its cohomology is known as the *Tor-modules*:

$$\mathrm{Tor}_A^{-i}(M, N) := H^i(R^{-*} \otimes_A N).$$

Since everything is graded, we actually have the grading

$$\mathrm{Tor}_A^{-i}(M, N) = \oplus_j \mathrm{Tor}_A^{-i,j}(M, N).$$

The following proposition is well-known, and we refer the reader to [4] for details.

Proposition 3.1. *The above defined Tor-modules satisfy the following properties.*

(1) $\mathrm{Tor}_A(M, N)$ *is independent of the choice of a resolution $[R, d]$ of M.*

(2) $\mathrm{Tor}_A(M,N)$ *is functorial in all three arguments, i.e., in A, in M, and in N.*
(3) $\mathrm{Tor}^0_A(M,N) = M \otimes_A N$.
(4) $\mathrm{Tor}^{-i}_A(M,N) = \mathrm{Tor}^{-i}_A(N,M)$.

□

We regard $\mathbb{Q}$ as an A-module via the ring map $A \to \mathbb{Q}$ sending each x_i to 0. Set $N = \mathbb{Q}$ and consider $\mathrm{Tor}_A(M,\mathbb{Q})$.

Definition 3.2. The *bigraded Betti numbers* of M are defined by

$$\beta^{-i}(M) = \dim_{\mathbb{Q}} \mathrm{Tor}^{-i}_A(M,\mathbb{Q}),$$

$$\beta^{-i,j}(M) = \dim_{\mathbb{Q}} \mathrm{Tor}^{-i,j}_A(M,\mathbb{Q}).$$

When $[R,d]$ is a minimal resolution of M, then the map

$$d \otimes 1 : R^{-i} \otimes_A \mathbb{Q} \to R^{-i+1} \otimes_A \mathbb{Q}$$

are the zero maps for all i. Hence $\beta^{-i,j} = \mathrm{rank}_{\mathbb{Q}} R^{-i,j}$.

We now consider the case when M is the Stanley-Reisner ring $\mathbb{Q}(P)$ of a simple convex polytope P, which is

$$\mathbb{Q}(P) = \mathbb{Q}[x_1,\dots,x_m]/I_P$$

where x_i are indeterminates corresponding to the facets F_i of P, m is the number of facets, and I_P is the homogeneous ideal generated by the monomials $x_{i_1}\cdots x_{i_\ell}$ whenever $F_{i_1} \cap \cdots \cap F_{i_\ell} = \emptyset$. This I_P is called the *Stanley-Reisner ideal* of P. Then $\mathbb{Q}(P)$ is a graded A-module with $\deg x_i = 2$ for all $i = 1,\dots,m$. The *bigraded Betti numbers* of P are defined to be $\beta^{-i,2j}(P) = \beta^{-i,2j}(\mathbb{Q}(P))$. Since $\deg x_i = 2$ we only have even index $2j$.

From the previous observation that $\beta^{-i,j} = \mathrm{rank}_{\mathbb{Q}} R^{-i,j}$ for a minimal resolution $[R,d]$, we can see easily that $\beta^{-1,2j}$ is equal to the number of degree $2j$ monomial elements in a minimal basis of the ideal I_P. For example, if $P = I^n$ then $x_i x_{n+i}$ for $i = 1,\dots n$ form a minimal basis for the Stanley-Reisner ideal I_P of P (here we assume that x_i and x_{n+i} are the generators corresponding to the opposite facets F_i and F_{n+i} of I^n). Hence

$$\beta^{-1,2j}(I^n) = \begin{cases} n, & j = 2; \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem of Hochster gives a nice formula for bigraded Betti numbers.

Theorem 3.3. [8] *Let P be a simple convex polytope with facets $F_1,\dots,F_m$. For a subset $\sigma \subset \{1,\dots,m\}$ let $P_\sigma = \cup_{i\in\sigma} F_i \subset P$. Then we have*

$$\beta^{-i,2j}(P) = \sum_{|\sigma|=j} \dim \widetilde{H}^{j-i-1}(P_\sigma).$$

Here $\dim \widetilde{H}^{j-i-1}(\emptyset) = 1$ by convention.

Bigraded Betti numbers also satisfy the following relations, see [4] for details.

Proposition 3.4. *Let P be an n-dimensional simple convex polytope with m facets, i.e., $f_0(P) = m$. Then*

(1) $\beta^{0,0}(P) = \beta^{-(m-n),2m}(P) = 1$,
(2) *(Poincaré duality)* $\beta^{-i,2j}(P) = \beta^{-(m-n)+i,2(m-j)}(P)$, *and*
(3) $\beta^{-i,2j}(P_1 \times P_2) = \sum_{i'+i''=i,j'+j''=j} \beta^{-i',2j'}(P_1)\beta^{-i'',2j''}(P_2)$.

□

Definition 3.5. A sequence $\lambda_1, \ldots, \lambda_p$ of homogeneous elements in $\mathbb{Q}(P)$ is a *regular sequence* if it is algebraically independent and $\mathbb{Q}(P)$ is a free module over $\mathbb{Q}[\lambda_1, \ldots, \lambda_p]$.

Let J be an ideal of $\mathbb{Q}(P)$ generated by a regular sequence $\lambda_1, \ldots, \lambda_p$. Let $\pi : A \to \mathbb{Q}(P)$ be the projection. Choose $t_i \in A$ such that $\pi(t_i) = \lambda_i$. Let J also denote the ideal of A generated by $t_1, \ldots, t_p$.

Lemma 3.6 (Lemma 3.35 in [4])**.** *Let J be an ideal generated by a regular sequence of $\mathbb{Q}(P)$. Then we have the following algebra isomorphism.*

$$\operatorname{Tor}^{*,*}_A(\mathbb{Q}(P), \mathbb{Q}) \cong \operatorname{Tor}^{*,*}_{A/J}(\mathbb{Q}(P)/J, \mathbb{Q})$$

□

Lemma 3.7. *Let P and P' be two n-dimensional simple convex polytopes. Let $J = (\lambda_1, \ldots, \lambda_n)$ (resp. $J' = (\lambda'_1, \ldots, \lambda'_n)$) be an ideal of $\mathbb{Q}(P)$ (resp. $\mathbb{Q}(P')$) generated by a regular sequence of degree 2 elements λ_i (resp. λ'_i). If there is a graded ring isomorphism $h \colon \mathbb{Q}(P)/J \xrightarrow{\cong} \mathbb{Q}(P')/J'$, then $f_0(P) = f_0(P')$ and*

$$\operatorname{Tor}^{*,*}_A\big(\mathbb{Q}(P), \mathbb{Q}\big) = \operatorname{Tor}^{*,*}_A\big(\mathbb{Q}(P'), \mathbb{Q}\big).$$

Proof. The equality $f_0(P) = f_0(P')$ follows immediately from the isomorphism $\mathbb{Q}(P)/J \cong \mathbb{Q}(P')/J'$. Thus we may assume that J and J' are both ideals of $A = \mathbb{Q}[x_1, \ldots, x_m]$ and h is an A-algebra isomorphism. By Lemma 3.6 we have $\operatorname{Tor}_A(\mathbb{Q}(P), \mathbb{Q}) = \operatorname{Tor}_{A/J}(\mathbb{Q}(P)/J, \mathbb{Q})$, and a similar equality holds for P'.

Now we claim that there is an A-algebra isomorphism $\bar{h} \colon A/J \to A/J'$ closing the commutative diagram

$$\begin{array}{ccc} A/J & \xrightarrow{\bar{h}} & A/J' \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q}(P)/J & \xrightarrow{h} & \mathbb{Q}(P')/J'. \end{array} \tag{3}$$

Note that both A/J and A/J' are isomorphic to $\mathbb{Q}[x_1, \ldots, x_{m-n}]$ where $m = f_0(P) = f_0(P')$. Also note that the projection maps $A/J \to \mathbb{Q}(P)/J$ and $A/J' \to \mathbb{Q}(P')/J'$ induce isomorphisms $(A/J)_2 \to (\mathbb{Q}(P)/J)_2$ and $(A/J')_2 \to (\mathbb{Q}(P')/J')_2$ on degree 2 subgroups. Therefore we have an isomorphism $(A/J)_2 \to (\mathbb{Q}(P)/J)_2 \to (\mathbb{Q}(P')/J')_2 \to (A/J')_2$. Since A/J and A/J' are generated in degree 2, we obtain the isomorphism $\bar{h} \colon A/J \to A/J'$ as necessary.

Finally, the required isomorphism

$$\operatorname{Tor}^{*,*}_{A/J}\big(\mathbb{Q}(P)/J, \mathbb{Q}\big) \cong \operatorname{Tor}^{*,*}_{A/J'}\big(\mathbb{Q}(P')/J', \mathbb{Q}\big)$$

follows from (3) and the functoriality of Tor in Proposition 3.1(2). □

We are now ready to prove Proposition 1.11.

Proof of Proposition 1.11. Recall that if M is a quasitoric manifold over a simple convex polytope P, then $H^*(M\colon \mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_m]/K$ where $K = I_P + J$ and I_P is the rational Stanley-Reisner ideal of P, and J is an ideal generated by some linear combinations $\lambda_{i1}x_1 + \cdots + \lambda_{im}x_m \in \mathbb{Q}[x_1,\dots x_m]$ for $i = 1,\dots,n$ which project to a regular sequence $\theta_1,\dots,\theta_n$ in $\mathbb{Q}[x_1,\dots,x_m]/I_P$, see [7]. Here m is the number of facets in P. Therefore we have the isomorphism

$$\mathbb{Q}(P)/J \cong H^*(M:\mathbb{Q}) \cong H^*(N:\mathbb{Q}) \cong \mathbb{Q}(P')/J'.$$

Hence the proposition follows from Lemma 3.7. □

Since $\beta^{-i,j} = \operatorname{rank}_{\mathbb{Q}} R^{-i,j}$ for a minimal resolution $[R,d]$, which are independent of the choice of a resolution, it is convenient to calculate $\beta^{-i,j}$ using a particular minimal resolution. For this purpose we will consider the minimal resolution of $\mathbb{Q}(P)$ corresponding to the *canonical minimal basis* of the rational Stanley-Reisner ideal I_P, which we define below. Let P have m facets. Consider all monomials in $x_1,\dots,x_m$. The canonical minimal basis $\mathcal{B}$ is defined as follows. First choose all degree four monomials x_ix_j such that $F_{x_i} \cap F_{x_j} = \emptyset$ where F_{x_k} is the facet of P corresponding to x_k. Denote by $\mathcal{B}_2$ the collection of all these monomials. Let $\mathcal{P}_3$ be the set of all monomials of degree greater than or equal to six minus the monomials divisible by the elements of $\mathcal{B}_2$. We now choose all degree six monomials $x_ix_jx_k$ in $\mathcal{P}_3$ such that $F_{x_i} \cap F_{x_j} \cap F_{x_k} = \emptyset$, and call this $\mathcal{B}_3$. We continue this selection inductively to define $\mathcal{B}_\ell$ for $\ell \geq 2$. This process will terminate in a finite step, and we set

$$\mathcal{B} := \bigcup_{\ell\geq 2} \mathcal{B}_\ell.$$

Then it is not difficult to see that $\mathcal{B}$ is a minimal basis of $\mathbb{Q}(P)$ because any element of degree 2ℓ in $\mathcal{B}$ can not be generated by the lower degree elements in $\mathcal{B}$ from the construction.

Example 3.8. 1. If $P = I^n$, the n-dimensional cube, then

$$\mathcal{B}(\mathbb{Q}(P)) = \{x_ix_{n+i} \mid i = 1,\dots,m\}.$$

2. If $P = \prod_{i=1}^t \Delta^{n_i}$, a product of simplices, then

$$\mathcal{B}(\mathbb{Q}(P)) = \{x_{i,0}\cdots x_{i,n_i} \mid i = 1,\dots,t\}.$$

We close this section by giving an algebraic version of rigidity. Recall that the rational Stanley-Reisner ring $\mathbb{Q}(K)$ of a simplicial complex K with m vertices $v_1,\dots,v_m$ is the quotient ring $\mathbb{Q}[x_1,\dots,x_m]/I_K$ where I_K is the ideal generated by the monomials $x_{i_1}\cdots x_{i_\ell}$ where the corresponding vertices $v_{i_1},\dots,v_{i_\ell}$ do not form a simplex on K. Then the rational Stanley-Reisner ring $\mathbb{Q}(P)$ of a simple convex polytope P is actually the rational Stanley-Reisner ring of the dual simplicial complex of ∂P, i.e., $\mathbb{Q}(P) = \mathbb{Q}((\partial P)^*)$. Since P is simple $(\partial P)^*$ is a simplicial complex.

The above constructed minimal basis $\mathcal{B}$ of I_P coincides with the canonical minimal basis of the ideal I_K (see [4, §3.4]) consisting of monomials corresponding to all *missing faces* of the simplicial complex K dual to the boundary of P (a missing face of a simplicial complex is its subset of vertices

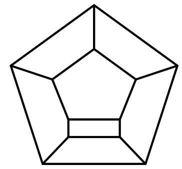


FIGURE 1. Schlegel diagram of Q

which does not span a simplex, but every whose proper subset does span a simplex).

A simplicial complex of dimension $n-1$ is called *Cohen-Macaulay* if there exists a length n regular sequence in $\mathbb{Q}(K)$. For any n-dimensional simple convex polytope P, its dual $(\partial P)^*$ is known to be Cohen-Macaulay. Therefore the definition of rigidity of a simple polytope can be generalized to that of a Cohen-Macaulay complex as follows:

Definition 3.9. An $(n-1)$-dimensional Cohen-Macaulay complex K is *rigid* if for any $(n-1)$-dimensional Cohen-Macaulay complex K' and for ideals $J \subset \mathbb{Q}(K)$ and $J' \subset \mathbb{Q}(K')$ generated by degree 2 regular sequences of length n, $\mathbb{Q}(K)/J \cong \mathbb{Q}(K')/J'$ implies $\mathbb{Q}(K) \cong \mathbb{Q}(K')$.

4. PROOF OF THEOREM 1.5

It is shown in [1] that if P is a triangle-free convex n-polytope then $f_i(P) \geq f_i(I^n)$ for all $i = 0, \dots, n-1$. Therefore the number of facets of P satisfies $f_0(P) \geq 2n$. Furthermore it is shown in [2] that if P is simple and

(1) if $f_0(P) = 2n$, then $P \approx I^n$,
(2) if $f_0(P) = 2n+1$, then $P \approx P_5 \times I^{n-2}$ where P_5 is a pentagon, and
(3) if $f_0 = 2n+2$, then $P \approx P_6 \times I^{n-2}$, $Q \times I^{n-3}$, or $P_5 \times P_5 \times I^{n-4}$ where P_6 is the hexagon and Q is 3-dimensional simple convex polytope obtained from pentagonal prism by cutting out one of the edges forming a pentagonal facet, see Figure 1.

Lemma 4.1. *Let P be an n-dimensional simple polytope. If $\beta^{-1,2j}(P) = 0$ for all $j \geq 3$, then P is triangle-free.*

Proof. Suppose otherwise. Namely, suppose there exists a triangular 2-face T of P. Then T is an intersection of $n-2$ facets of P. On the other hand each edge e_i of T for $i = 1, 2, 3$ is an intersection of $n-1$ facets of P. Thus there exists a unique facet, say F_i which contains the edge e_i but not the triangle T for $i = 1, 2, 3$. Then any two facets from F_1, F_2 and F_3 have nonempty intersection. However there exists a collection of facets including F_1, F_2 and F_3 whose intersection is empty but any proper subcollection of them has nonempty intersection. When we consider the canonical minimal basis $\mathcal{B}$ of I_P, the above collection of facets gives a monomial element $g \in \mathcal{B}$ divisible by $x_1x_2x_3$. But this contradicts to the hypothesis that $\beta^{-1,2j}(P) = 0$ for all $j \geq 3$. □

Note that the condition $\beta^{-1,2j}(P) = 0$ for all $j \geq 3$ means that the Stanley-Reisner ideal I_P of P is generated by quadratic monomials of the form x_ix_j, and this is equivalent to saying that the simplicial complex $K = (\partial P)^*$ is flag.

If the number of facets of P is less than or equal to $2n+2$, then the converse of Lemma 4.1 is true. Namely, we have

Lemma 4.2. *If P is a triangle-free n-dimensional simple convex polytope with $f_0(P) \le 2n+2$, then $\beta^{-1,2j}(P) = 0$ for all $j \ge 3$.*

Proof. Since $f_0(P) \le 2n+2$, we know that $P \approx I^n$, $P_5 \times I^{n-2}$, $P_6 \times I^{n-2}$, $Q \times I^{n-3}$, or $P_5 \times P_5 \times I^{n-4}$. Since $\beta^{-1,2j}$ is equal to the number of degree $2j$ monomial elements in a minimal basis of the Stanley-Reisner ideal of the polytope, we can see that

$$\beta^{-1,2j}(P_5) = \begin{cases} 5, & j=2 \\ 0, & j \ge 3 \end{cases}, \quad \beta^{-1,2j}(P_6) = \begin{cases} 9, & j=2 \\ 0, & j \ge 3 \end{cases},$$

$$\beta^{-1,2j}(Q) = \begin{cases} 10, & j=2 \\ 0, & j \ge 3 \end{cases}, \quad \beta^{-1,2j}(I^k) = \begin{cases} k, & j=2 \\ 0, & j \ge 3 \end{cases}.$$

By Proposition 3.4 (3), $\beta^{-1,2j}(P' \times I^k) = 0$ for $j \ge 3$ where $P' \approx I^2$, P_5, P_6, Q, or $P_5 \times P_5$. □

We now prepare for the proof of Theorem 1.5. By Theorem 3.3 we have $\beta^{-2,8}(P) = \sum_{|\sigma|=4} \dim \widetilde{H}^1(P_\sigma)$. Therefore

$$\begin{aligned} \beta^{-2,8}(P_5) &= \beta^{-2,8}(P_6) = 0, \\ \beta^{-2,8}(Q) &= 5, \\ \beta^{-2,8}(P_5 \times P_5) &= \beta^{-1,4}(P_5)\beta^{-1,4}(P_5) = 25 \end{aligned}$$

(note that since Q does not have triangular faces, $\beta^{-2,8}(Q)$ equals the number of 4-facet "belts" in Q). Hence we have

$$\begin{aligned} \beta^{-1,4}(P_6 \times I^{n-2}) &= \beta^{-1,4}(P_6) + \beta^{-1,4}(I^{n-2}) = n+7, \\ \beta^{-2,8}(P_6 \times I^{n-2}) &= \beta^{-1,4}(P_6) \cdot \beta^{-1,4}(I^{n-2}) + \beta^{0,0}(P_6) \cdot \beta^{-2,8}(I^{n-2}) \\ &\quad + \beta^{-2,8}(P_6) \cdot \beta^{0,0}(I^{n-2}). \end{aligned}$$

On the other hand, by an inductive application of Proposition 3.4 (3) we can see easily that $\beta^{-2,8}(I^{n-2}) = (n-2)(n-3)/2$. Therefore we have

$$\beta^{-1,4}(P_6 \times I^{n-2}) = n+7, \quad \beta^{-2,8}(P_6 \times I^{n-2}) = \frac{n^2+13n-30}{2}.$$

By a similar computation we have

$$\begin{aligned} &\beta^{-1,4}(Q \times I^{n-3}) = n+7, && \beta^{-2,8}(Q \times I^{n-3}) = \frac{n^2+13n-38}{2} \\ &\beta^{-1,4}(P_5 \times P_5 \times I^{n-4}) = n+6, && \beta^{-2,8}(P_5 \times P_5 \times I^{n-4}) = \frac{n^2+11n-10}{2}. \end{aligned}$$

Proof of Theorem 1.5. Let P be triangle-free with $f_0(P) \le 2n+2$, and let M be a quasitoric manifold over P. Let P' be another simple convex polytope and M' a quasitoric manifold over P'. If $H^*(M : \mathbb{Q}) \cong H^*(M' : \mathbb{Q})$ as graded rings, then by Proposition 1.11 we have the equality $\beta^{-i,2j}(P) = \beta^{-i,2j}(P')$ for all $1 \le i, j \le m$. Since P is triangle-free with $f_0(P) \le 2n+2$, $\beta^{-1,2j}(P) = 0$ for all $j \ge 3$ by Lemma 4.2. Hence $\beta^{-1,2j}(P') = 0$ for all

$j \geq 3$, and Lemma 4.1 implies that P' is triangle-free. Furthermore $H^*(M : \mathbb{Q}) \cong H^*(M' : \mathbb{Q})$ implies in particular $f_0(P) = f_0(P')$. If $f_0(P) = 2n$ or $2n+1$, then there is only one simple polytope with the given number of facets. So $P \approx P'$. When $f_0(P) = 2n+2$ then there are three possible polytopes, but the above computation shows that $\beta^{-i,2j}$ are distinct for these three polytopes. This shows that $P \approx P'$, which proves the theorem. The existence of quasitoric manifolds over P is clear because we know the existence of quasitoric manifolds over any two or three dimensional simpple convex polytopes and any n-simplex as well as any finite product of these polytopes. □

5. Proof of Theorem 1.6

We will make use of the following invariant in this and the next section.

Definition 5.1. Let $\sigma(P) = \sum_{j \geq 2} j\beta^{-1,2j}(P)$, and call it the *sigma invariant* of P.

Proposition 1.11 implies that $\sigma(P)$ is a cohomology invariant of quasitoric manifolds over P. As we observed in Section 3 the Betti number $\beta^{-1,2j}(P)$ is equal to the number of degree $2j$ elements in a minimal basis of the Stanley-Reisner ideal I_P of P. Therefore $2\sigma(P)$ is nothing but the sum of the degrees of all elements of a minimal basis of I_P.

Lemma 5.2. *Let P be a simple polytope with m facets. Then the following conditions are equivalent:*

(a) *$\sigma(P) = m$;*
(b) *the canonical minimal basis $\mathcal{B}$ of I_P forms a regular sequence;*
(c) *P is combinatorially equivalent to a product of simplices.*

Proof. (c) $\Rightarrow$ (a) Clear.

(a) $\Rightarrow$ (b) Let $\mathbb{Q}(P) = \mathbb{Q}[x_1, \dots, x_m]/I_P$, where x_i corresponds to a facet F_i of P. Let $\mathcal{B} = \{g_1, \dots, g_t\}$ be the canonical minimal basis of I_P. Since $\sigma(\mathbb{Q}(P)) = m$, each x_j must appear in exactly one element of $\mathcal{B}$ with exponent 1. It follows easily that $g_1, \dots, g_t$ is a regular sequence.

(b) $\Rightarrow$ (c). Let $\mathbb{Q}(P) = \mathbb{Q}[x_1, \dots, x_m]/(g_1, \dots, g_t)$, where $g_1, \dots, g_t$ is a monomial regular sequence. It is well known [4, §3.2] that $g_1, \dots, g_t$ is a regular sequence if and only if g_i is not a zero divisor in the quotient $\mathbb{Q}[x_1, \dots, x_m]/(g_1, \dots, g_{i-1})$ for $1 \leq i \leq t$ (this property is often taken as the definition of a regular sequence). Assume that some x_j appears in more than one of $g_1, \dots, g_t$, say in g_1 and g_2. Then g_2 is a zero divisor in $\mathbb{Q}[x_1, \dots, x_m]/(g_1)$, which leads to a contradiction. Therefore, each x_j appears in at most one of the monomials $g_1, \dots, g_t$. Since every x_j must appear in at least one element in I_P, we obtain that every x_j enters in exactly one of $g_1, \dots, g_t$. So we can rename $x_1, \dots, x_m$ by $y_{1,0}, \dots, y_{1,n_1}, \dots, y_{t,0} \dots, y_{t,n_t}$ such that $g_j = \prod_{k=0}^{n_j} y_{j,k}$ for $j = 1, \dots, t$. Therefore we can see immediately that

$$\begin{aligned}\mathbb{Q}(P) &\cong \mathbb{Q}[x_1,\dots,x_m]/I_P\\ &\cong \otimes_{i=1}^t \mathbb{Q}[y_{i,0},\dots,y_{i,n_i}]/(g_i)\\ &\cong \otimes_{i=1}^t \mathbb{Q}(\Delta^{n_i})\\ &\cong \mathbb{Q}\Big(\prod_{i=1}^t \Delta^{n_i}\Big).\end{aligned}$$

Since the Stanley-Reisner ring with $\mathbb{Q}$-coefficients determines the combinatorial type of a simple polytope, we have $P \approx \prod_{i=1}^t \Delta^{n_i}$. □

Note that (b) in Lemma 5.2 is equivalent to saying that $\mathbb{Q}(P)$ is a complete intersection ring.

Proof of Theorem 1.6. Let M be a $2n$-dimensional quasitoric manifold over $P = \prod_{i=1}^t \Delta^{n_i}$. Let N be an another quasitoric manifold over a simple convex polytope Q, such that $H^*(M:\mathbb{Z}) \cong H^*(N:\mathbb{Z})$. Then $H^*(M:\mathbb{Q}) \cong H^*(N:\mathbb{Q})$ and $f_i(P) = f_i(Q)$ for all i. In particular, $\sigma(\mathbb{Q}(P)) = f_0(P) = f_0(Q) = n+t$. Thus Q is a simple convex polytope with $\sigma(\mathbb{Q}(Q)) = f_0(Q)$. Therefore Q is also a product of simplices, i.e., $Q \approx \prod_{j=1}^s \Delta^{m_j}$. But $H^*(M:\mathbb{Q}) \cong H^*(N:\mathbb{Q})$ implies $\beta^{-1,2j}(P) = \beta^{-1,2j}(Q)$ for all J. This implies that $\{n_i\} = \{m_j\}$ and $t = s$. Thus $P \cong Q$. □

6. Proof of Theorem 1.7

Note that a vertex cut of P may depend on the choice of a vertex to be cut off, but the Betti numbers and the sigma invariants of vertex cuts of P are independent of the choice. The following proposition is easy to prove.

Proposition 6.1. *Let P be an n-dimensional simple convex polytope with m facets, which is different from the n-simplex Δ^n. Then we have*

(1) $$\beta^{-1,2j}(\mathrm{vc}(P)) = \begin{cases}\beta^{-1,2j}(P)+m-n, & j=2\\ \beta^{-1,2j}(P), & 3\le j\le n-1\\ \beta^{-1,2j}(P)+1, & j=n\end{cases}$$

(2) $\sigma(\mathrm{vc}(P)) = \sigma(P) + 2m - n.$

□

When $P = \prod_{i=1}^t \Delta^{n_i}$ with $t \neq 1$, we have $n = \sum_{i=1}^t n_i$, $m = n+t$ and $\sigma(P) = m$. Hence we have $\sigma(\mathrm{vc}(P)) = 3m - n$.

Let $\mathcal{F} = \{F_1,\dots,F_m\}$ be the set of facets in P. Let x_i be the corresponding generator to F_i in $\mathbb{Q}(P) = \mathbb{Q}[x_1,\dots,x_m]/I_P$. Let $\mathcal{B} = \{h_1,\dots,h_\ell\}$ be the canonical minimal basis for I_P. For each x_i the *frequency* $\mathfrak{f}(x_i)$ is the number of h_k in $\mathcal{B}$ divisible by x_i.

Lemma 6.2. *Let P be an n-dimensional simple convex polytope. Let $\mathcal{B}$ be the canonical minimal basis for I_P. If $\mathfrak{f}(x_i) = 1$ for some i, then $P \approx \Delta^k \times P'$ for some polytope P' of dimension $n-k$ and $k = \deg h/2 - 1$ where h is the unique element in $\mathcal{B}$ such that $x_i | h$.*

Proof. Let $\mathcal{B} = \{h_1, \dots, h_s\}$. Assume $\mathfrak{f}(x_1) = 1$ and $h_1 = x_1 \cdots x_t$ for simplicity. Hence h_1 is the unique element of $\mathcal{B}$ that is divisible by x_1. We claim that $\mathfrak{f}(x_2) = \cdots = \mathfrak{f}(x_t) = 1$. Assume otherwise, say $\mathfrak{f}(x_2) \geq 2$. Without loss of generality, we may assume $h_2 = x_2 x_{i_1} \cdots x_{i_k}$. Then $x_{i_j} \neq x_1$ and x_2 for all $j = 1, \dots, k$ because h_1 is the only element of $\mathcal{B}$ divisible by x_1. Since $h_2 \in \mathcal{B}$, if we let $T := F_{i_1} \cap \cdots \cap F_{i_k}$ then $T \neq \emptyset$ but $F_2 \cap T = \emptyset$. On the other hand, since $x_1 \nmid h_2$ we have $F_1 \cap T \neq \emptyset$.

If $k \geq n$, then $F_1 \cap T \neq \emptyset$ implies that more than n facets of P are intersecting, which is impossible because P is simple. Therefore $\dim T = n - k \geq 1$. Since $\dim(F_1 \cap T) = \dim T - 1$, there exists a vertex v of T which does not belong to F_1. Let v be the intersection of n facets $F_{\ell_1}, \dots, F_{\ell_n}$. Since $F_2 \cap T = \emptyset$, the vertex v does not belong to F_2, hence $F_{\ell_j} \neq F_2$ for all $j = 1, \dots, n$. Since v does not belong to F_1, we have $F_1 \cap F_{\ell_1} \cap \cdots \cap F_{\ell_n} = \emptyset$. Therefore there must exist an element $h \in \mathcal{B}$, which divides the monomial $x_1 x_{\ell_1} \cdots x_{\ell_n}$. But since $F_{\ell_1} \cap \cdots \cap F_{\ell_n} = v \neq \emptyset$, the element h must be divisible by x_1. Since $F_{\ell_j} \neq F_2$ for all $j = 1, \dots, n$, it follows that $x_2 \nmid h$. Thus $h_1 \nmid h$, which contradicts to the condition that $\mathfrak{f}(x_1) = 1$. This shows that $\mathfrak{f}(x_2) = 1$, and by a similar argument we can see that $\mathfrak{f}(x_i) = 1$ for all $i = 1, \dots, t$. Hence,

$$\mathbb{Q}(P) = \mathbb{Q}[x_1, \cdots, x_t]/h_1 \otimes \mathbb{Q}[x_{t+1}, \cdots, x_m]/I',$$

where I' is the ideal generated by $\widetilde{\mathcal{B}} \setminus \{h_1\}$.

Since $\mathbb{Q}[x_1, \dots x_t]/(h_1) \cong \mathbb{Q}(\Delta^t)$, it is enough to prove that there is an isomorphism $\mathbb{Q}[x_{t+1}, \dots, t_m]/I' \cong \mathbb{Q}(P')$ for some polytope P' of dimension $n-k$. (Indeed, then we instantly get $P \approx \Delta^k \times P'$ because rational Stanley-Reisner ring determines the combinatorial type of a simple polytope.) Let $P' := F_2 \cap \cdots \cap F_t$. Then every facets except F_1 intersects with P'. Let $G_j = F_j \cap P'$ for $j = t+1, \dots, m$. Then G_j's are facets of P'. This implies that the face poset structure of P' agrees the face poset structure of $\{F_{t+1}, \dots, F_m\}$. Thus $\widetilde{\mathcal{B}} \setminus \{h_1\} = \{h_2, \dots, h_s\}$ is the canonical minimal basis for $I_{P'}$. Hence $\mathbb{Q}[x_{t-1}, \dots, x_n]/I' \cong \mathbb{Q}(P')$. □

Theorem 6.3. *Let Q be an n-dimensional simple convex polytope with $m+1$ facets. If $\sigma(Q) = 3m - n$ and $\beta^{-1,2n}(Q) \neq 0$, then Q is a vertex cut of a product of simplices.*

Proof. We claim that one of the facets of Q is an $(n-1)$-simplex. Then Q is a vertex cut of some simple convex polytope P. By Proposition 6.1 we have

$$\sigma(P) = \sigma(Q) - 2m - n = (3m - n) - (2m - n) = m.$$

Thus by Lemma 5.2 P is a product of simplices, and we are done. We now prove the claim. Let $F_1, \dots, F_{m+1}$ be the facets of Q and let $x_1, \dots, x_{m+1}$ be the associated generators of $\mathbb{Q}(Q)$. Let $\mathcal{B}$ be the canonical minimal basis for the ideal I_Q. Since $\beta^{-1,2n}(Q) \geq 1$, there exists $\widetilde{h} \in \mathcal{B}$ with $\deg \widetilde{h} = 2n$. Without loss of generality we may assume $\widetilde{h} = x_1 \dots x_n$. Then we can see easily that $F_1 \cup \cdots \cup F_n$ is homeomorphic to $S^{n-2} \times I$, while $F_1 \cup \cdots \cup F_{m+1} \cong S^{n-1}$. Thus $F_1 \cup \cdots \cup F_{m+1} \backslash F_1 \cup \cdots \cup F_n = F_{n+1} \cup \cdots \cup F_{m+1}$ has two connected components. For simplicity let $F_{n+1} \cup \cdots \cup F_{n+k}$ and

$F_{n+k+1} \cup \cdots \cup F_{m+1}$ be the two components. Then $F_{n+i} \cap F_{n+j} = \emptyset$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, m+1-n$.

If $k = 1$ or $m-n$, then one of the components of $F_{n-1}, \ldots, F_{m+1}$ is a facet of Q, which is an $(n-1)$-simplex. This proves the claim. Assume otherwise, i.e., suppose $2 \leq k \leq [(m+1-n)/2]$. Let $\mathcal{B}_1 = \{x_{n+i}x_{n+j} | i = 1, \ldots, k \text{ and } j = k+1, \ldots, m+1-n\}$. Then we have

$$\sum_{h \in \mathcal{B}_1} \deg(h) = 4k(m+1-n-k) \geq 8(m-n-1) \tag{4}$$

because $k(m+1-n-k)$ is increasing for $2 \leq k \leq [(m+1-n)/2]$. Note that the frequency $\mathfrak{f}(x_i) \geq 2$ for all $i = 1, \ldots, n$ because otherwise Lemma 6.2 implies that $Q \approx \Delta^{n-1} \times \Delta^1$, but in this case $\sigma(Q) = n+2 \neq 3m-n = 2n+3$. Therefore for each x_i, there exits $h_i \in \widetilde{\mathcal{B}}$ such that $x_i | h_i$ and $h_i \neq \widetilde{h}$ for $i = 1, \ldots, n$. Note that some of $h_1, \ldots, h_n$ may coincide. So we let $\widetilde{h}_1, \ldots, \widetilde{h}_s$ denote all distinct elements among h_i's. If $s = 1$, then $\widetilde{h}_1$ is divisible by all x_i for $i = 1, \ldots, n$. Hence $\widetilde{h} | \widetilde{h}_1$ and therefore $\widetilde{h} = \widetilde{h}_1$, which is a contradiction. Therefore $s \geq 2$.

If $s \geq 3$, then

$$\begin{aligned} \sum_{h \in \mathcal{B} \setminus \mathcal{B}_1} \deg h &\geq \deg \widetilde{h} + \sum_{i=1}^{s} \deg \widetilde{h}_i \\ &\geq \deg \widetilde{h} + 2n + 6 = 4n + 6, \end{aligned} \tag{5}$$

where the last inequality follows from the conditions $s \geq 3$, $\deg \widetilde{h}_i \geq 4$ and $x_1 \cdots x_n \mid \widetilde{h}_1 \cdots \widetilde{h}_s$.

Suppose $s = 2$. Then without loss of generality we may assume that $\widetilde{h}_1 = g_1 x_1 \cdots x_\ell$ and $\widetilde{h}_2 = g_2 x_{\ell+1} \cdots x_n$ with $1 \leq \ell \leq n-1$ for some monomials g_1, g_2 in $x_{n+1}, \ldots, x_{m+1}$ of degree ≥ 2. If degree $g_2 \geq 4$, then

$$\deg \widetilde{h}_1 + \deg \widetilde{h}_2 = 2n + \deg g_1 + \deg g_2 \geq 2n + 6.$$

Therefore the inequality (5) holds in this case. Now suppose degree $g_2 = 2$. Then $g_2 = x_i$ for $n+1 \leq i \leq n+k$ or $g_2 = x_{n+j}$ for $k+1 \leq j \leq m+1-n$. We only prove the case when $g_2 = x_{n+k+1}$. The other cases are similar. In this case consider the monomial $q = \prod_{j=2}^{n+2} x_j$. By the assumption, $\widetilde{h}_i \nmid q$ for $i = 1, 2$. But q must vanish in $\mathbb{Q}(Q)$ because any set of $n+1$ facets has empty intersection in a simple polytope. Therefore there exists a monomial q' of degree ≥ 4 in $\mathcal{B} \setminus \mathcal{B}_1$, which divides q. Thus

$$\sum_{h \in \mathcal{B} \setminus \mathcal{B}_1} \deg h \geq \deg \widetilde{h} + \deg q' + \deg \widetilde{h}_1 + \deg \widetilde{h}_2 \geq 4n + 8 \geq 4n + 6.$$

We thus have proved that in all cases

$$2\sigma(Q) = \sum_{h \in \mathcal{B}} \deg h \geq 8(m-n-1) + 4n + 6 = 8m - 4n - 2. \tag{6}$$

On the other hand, by the assumption of the theorem, $\sigma(Q) = 3m - n$. Thus $3m - n \geq 4m - 2n - 1$, hence $n + 2 \geq m + 1$. Therefore Q is combinatorially equivalent to either $\Delta^{n_1} \times \Delta^{n_2}$ or Δ^n. But $\beta^{-1,2n}(Q) \neq 0$ implies that $Q \approx \Delta^{n-1} \times \Delta^1$, which implies $\sigma(Q) = m+1 \neq 3m-n$. This is a contradiction. Thus we have $k = 1$ or $m-n$, which proves the theorem. $\square$

Proof of Theorem 1.7. If P is an n-simplex, then $\mathrm{vc}(P) = \Delta^{n-1} \times \Delta^1$, which is rigid by Theorem 1.6. Assume otherwise. Let $Q = \mathrm{vc}(P)$, and let M be a quasitoric manifold over Q. Suppose N is quasitoric manifold over another simple convex polytope Q' such that $H^*(M : \mathbb{Q}) \cong H^*(N : \mathbb{Q})$ as graded rings. Then $\beta^{-1,2j}(Q) = \beta^{-1,2j}(Q')$, and hence $\sigma(Q') = \sigma(Q) = 3m - n$ and $\beta^{-1,2n}(Q') = \beta^{-1,2n}(Q) \neq 0$. By Theorem 6.3 $Q' = \mathrm{vc}(P')$ for $P' = \prod \Delta^{n_i}$. By Proposition 6.1(1), $\beta^{-1,2j}(Q) = \beta^{-1,2j}(Q')$ implies $\beta^{-1,2j}(P) = \beta^{-1,2j}(P')$ for all j. Both P and P' are products of simplices, thus $P \approx P'$. So $Q \approx Q'$, which proves the theorem. □

7. Rigidity of 3-dimensional simple convex polytopes

Since rigidity of 2-dimensional simple convex polytope is settled by Corollary 1.4, rigidity of 3-dimensional simple convex polytope is naturally the next target. Note that any 3-dimensional simple convex polytope supports a quasitoric manifolds. The four color problem gives an easy proof of this.

On page 192 and 193 of Appendix A.5 in [11] there is a list of 3-dimensional simple convex polytopes with ≤ 9 facets. In the list the polytopes are labeled in the form $\alpha^x\beta^y\gamma^z$ which means the polytope has x many α-gon facets, y many β-gon facets, and z many γ-gon facets. Polytope 3^4 is the tetrahedron, and $3^2 4^3$ is the triangular prism. So they are rigid by Theorem 1.6.

There are exactly two polytopes $3^2 4^2 5^2$ and 4^6 with $f_0 = 6$. Polytope 4^6 is rigid because it is the cube I^3, and polytope $3^2 4^2 5^2$ is the vertex cut of the triangular prism, so it is also rigid by Theorem 1.7.

There are five different polytopes with $f_0 = 7$, which are $3^2 4^3 6^2$, $3^3 5^3 6^1$, $3^2 4^2 5^2 6^1$, $3^1 4^3 5^3$, and $4^5 5^2$. The first three are the polytopes obtained from triangular prism $\Delta^2 \times I$ by taking vertex cuts twice. So by the argument of the example in Introduction they are all nonrigid. The polytope $3^1 4^3 5^3$ is the vertex cut of the cube I^3 and hence rigid by Theorem 1.7. Polytope $4^5 5^2$ is the pentagonal prism, which is rigid by Theorem 1.5.

There are 14 different polytopes with $f_0 = 8$. Seven of them are obtained from the triangular prism by taking vertex cuts three times, and so they are all nonrigid. They are $3^2 4^4 7^2$, $3^3 4^1 5^2 6^1 7^1$, $3^2 4^3 5^1 6^1 7^1$, $3^2 4^2 5^3 7^1$, $3^4 6^4$, $3^3 4^1 5^1 6^3$, and $3^2 4^2 5^2 6^2(i)$. There are four polytopes obtained from the cube by taking vertex cuts twice. They are $3^2 4^2 5^2 6^2(ii)$, $3^1 4^5 5^1 6^2$, $3^2 4^1 5^4 6^1$ and $3^2 5^6$, and all of them are nonrigid. The remaining polytopes are $3^1 4^3 5^3 6^1$ which is the vertex cut of the pentagonal prism, $4^6 6^2$ which is the hexagonal prism $P_6 \times I$, and $4^4 5^4$ which is obtained from the pentagonal prism by cutting out a triangular prism shaped neighborhood of an edge. Since the Betti numbers $\beta^{-1,2j}$ of $3^1 4^3 5^3 6^1$ are different from those of the other two and also different from the previous groups, it is rigid. The remaining polytopes are $4^6 6^2$ and $4^4 5^4$. These polytopes have $2n + 2 = 8$ facets. So by Theorem 1.5 they are rigid.

There are 50 different polytopes with $f_0 = 9$. There are only six rigid polytopes whose Betti numbers $\beta^{i,2j}$ are different from those of the other's. Five of them are triangle-free polytopes, $4^6 6^3$, $4^5 5^2 6^2$, $4^4 5^4 6^1$, $4^3 5^6$ and $4^7 7^2$, and the sixth is polytope $3^1 4^4 5^2 6^1 7^1(ii)$ which is the vertex cut of $P_6 \times I$.

Table 1 shows rigidity of simple convex 3-polytopes with facet numbers ≤ 8, and Table 2 shows the case for facet numbers $= 9$.

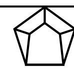
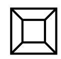
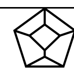
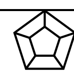
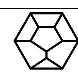
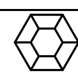
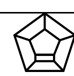
Type (The Betti number)	Simple polytopes	
3^4 ()	 3^4	rigid
$\mathrm{vc}(3^4)$ (1)	 $3^2 4^3$	rigid
$\mathrm{vc}^2(3^4)$ (3,2)	 $3^2 4^2 5^2$	rigid
4^6 (3,0)	 4^6	rigid
$\mathrm{vc}^3(3^4)$ (6,8,3)	$3^2 4^3 6^2$, $3^3 5^3 6^1$,$3^2 4^2 5^2 6^1$ (3 polytopes)	nonrigid
$\mathrm{vc}(4^6)$ (6,6,1)	 $3^1 4^3 5^3$	rigid
$4^5 5^2$ (6,5,0)	 $4^5 5^2$	rigid
$\mathrm{vc}^4(3^4)$ (10,20,15,4)	$3^2 4^4 7^2$, $3^3 4^1 5^2 6^1 7^1$, $3^2 4^3 5^1 6^1 7^1$, $3^2 4^2 5^3 7^1$, $3^4 6^4$ $3^3 4^1 5^1 6^3$, $3^2 4^2 5^2 6^2(i)$ (7 polytopes)	nonrigid
$\mathrm{vc}^2(4^6)$ (10,18,11,2)	$3^2 4^2 5^2 6^2(ii)$, $3^1 4^4 5^1 6^2$, $3^2 4^1 5^4 6^1$, $3^2 5^6$ (4 polytopes)	nonrigid
$\mathrm{vc}(4^5 5^2)$ (10,17,9,1)	 $3^1 4^3 5^3 6^1$	rigid
$4^6 6^2$ (10,16,9,0)	 $4^6 6^2$	rigid
$4^4 5^4$ (10,16,5,0)	 $4^4 5^4$	rigid

TABLE 1. Rigidity of simple 3-polytopes with $f_0 \leq 8$

In the tables let $\mathrm{vc}^k(P)$ denote a k-times vertex cut of the polytope P. The Betti numbers are listed in the form

$$(\beta^{-1,4}, \ldots, \beta^{-(j-1),2j}, \ldots, \beta^{-(m-4),2(m-3)}) \tag{7}$$

Note that the numbers in (7) completely determines all bigraded Betti numbers of a 3-dimensional polytope. Indeed, unless $(i,j) = (0,0)$ or $(m-3, m)$ the number $\beta^{-i,2j} = 0$ for $j - i \neq 1, 2$ by Theorem 3.3. By Proposition 3.4 $\beta^{0,0} = \beta^{-(m-3),2m} = 1$ and $\beta^{-(j-1),2j} = \beta^{-i',2j'}$ where $i' = (m-3) - (j-1)$ and $j' = m - j$. Note that $j' - i' = 2$. This implies the Betti numbers are perfectly determined only by $\beta^{-(j-1),2j}$'s for $j = 2, \ldots, m$. Moreover for a

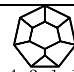
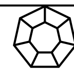
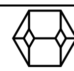
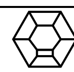
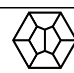
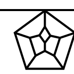
Type (The Betti number)	Simple polytopes	
$\mathrm{vc}^5(3^4)$ (15,40,45,24,5)	$3^24^58^2$, $3^34^25^27^18^1(i)$, $3^34^25^27^18^1(ii)$, $3^24^25^17^18^1$, $3^45^26^28^1$ $3^34^25^16^28^1(i)$, $3^34^25^16^28^1(ii)$, $3^24^46^28^1$, $3^34^15^36^18^1$, $3^24^35^26^18^1(i)$ $3^24^35^26^18^1(ii)$, $3^24^25^48^1$, $3^44^16^27^2$, $3^45^26^17^2$, $3^34^25^16^17^2(i)$ $3^34^25^16^17^2(ii)$, $3^34^15^37^2$, $3^24^35^27^2(ii)$, $3^34^26^37^1(i)$, $3^34^26^37^1(ii)$ $3^34^15^26^27^1(ii)$, $3^24^35^16^27^1(iii)$, $3^24^25^36^17^1(iii)$, $3^24^25^26^3(iii)$ (24 polytopes)	nonrigid
$\mathrm{vc}^3(4^6)$ (15,38,39,18,3)	$3^24^35^27^2(i)$, $3^14^55^17^2$, $3^34^15^26^27^1(i)$, $3^24^35^16^27^1(i)$, $3^24^35^16^27^1(ii)$ $3^24^25^36^17^1(ii)$, $3^14^45^26^17^1(iii)$, $3^35^36^3(i)$, $3^35^36^3(ii)$, $3^24^25^26^3(ii)$ $3^24^15^46^2(ii)$ (11 polytopes)	nonrigid
$\mathrm{vc}^2(4^55^2)$ (15,37,36,15,2)	$3^24^25^36^17^1(i)$, $3^14^45^26^17^1(i)$, $3^24^15^57^1$, $3^24^36^4$, $3^24^25^26^3(i)$ $3^14^45^16^3$, $3^24^15^46^2(i)$ (7 polytopes)	nonrigid
$\mathrm{vc}(4^66^2)$ (15,36,35,14,1)	 $3^14^45^26^17^1(ii)$	rigid
$\mathrm{vc}(4^45^4)$ (15,36,31,10,1)	$3^14^35^36^2$, $3^14^25^56^1$ (2 polytopes)	nonrigid
4^77^2 (15,35,35,14,0)	 4^77^2	rigid
$4^6\sharp 4^6$ (15,36,33,12,1)	 4^66^3	rigid
$4^55^26^2$ (15,35,29,8,0)	 $4^55^26^2$	rigid
$4^45^46^1$ (15,35,27,6,0)	 $4^45^46^1$	rigid
4^35^6 (15,35,24,3,0)	 4^35^6	rigid

TABLE 2. Rigidity of simple 3-polytopes with $f_0 = 9$

subset $\sigma \subset \{1, \dots, m\}$, if $|\sigma| > m - 3$, then P_σ is always connected. Thus we shall consider only $\beta^{-(j-1),2j}$'s for $j = 2, \dots, m-3$.

Finally we give a proof of rigidity of a dodecahedron.

Theorem 7.1. *A dodecahedron is rigid.*

Proof. Note that the $(-2, 8)$-th Betti number of a dodecahedron is 0. Let P be a simple 3-polytope with 12 facets whose Betti numbers are equal to those of a dodecahedron. Let x_k be the number of k-gonal facet of P. By euler equation $\sum_{k\geq 3} x_k(6-k) = 12$. Since the number of facets $\sum_{k\geq 3} x_k$ is 12, we have $\sum_{k\geq 3} x_k(5-k) = 0$. If P has triangular or quadrangular facets, then $\beta^{-2,8}(P) \neq 0$. Therefore, $x_3 = x_4 = 0$. Thus if $x_k \neq 0$ for $k \geq 6$, $\sum_{k\geq 3} x_k(5-k)$ must be negative. This implies $x_5 = 12$. Hence P is a dodecahedron. $\square$

8. Some variations of the definition of rigidity

There are several variations of the definition of cohomological rigidity. As is mentioned in Introduction, cohomological rigidity is first introduced in [10] in terms of toric varieties and simplical complexes. Namely, a simplicial complex Σ_X associated with a toric manifold X is *rigid* if $\Sigma_X \approx \Sigma_Y$ whenever $H^*(X) \cong H^*(Y)$ as graded rings. Therefore our definition is a variation of the original definition of rigidity.

Moreover we may consider cohomological rigidity of simple convex polytopes in terms of small covers. Namely, we may replace 'quasitoric manifolds' by 'small convers' and 'integral cohomology rings' by 'mod 2 cohomology rings' in Definition 1.2. Small cover is a mod 2 analogue of quasitoric manifold, which is a closed n-dimensional manifold with a locallly standard mod 2 torus $(\mathbb{Z}_2)^n$ action over a simple convex polytope. This gives another variation of the definition.

In the proof of our rigidity results we made essential use of bigraded Betti numbers which are purely combinatorial invariants of the polytopes. Considering this, Buchstaber asked the following question in his lecture note [3].

Question 8.1. *Let K and K' be simplicial complexes, and let $\mathcal{Z}_K$ and $\mathcal{Z}_{K'}$ be their respective moment angle complexes. When a cohomology ring isomorphism $H^*(\mathcal{Z}_K : k) \cong H^*(\mathcal{Z}_{K'} : k)$ implies a combinatorial equivalence $K \approx K'$ where k is field?*

Let us call the simplicial complexes giving the positive answer to the question *B-rigid*. Note that $H^*(\mathcal{Z}_K : k) \cong \mathrm{Tor}(k(K), k)$, see [4]. Let $K = (\partial P)^*$ (resp. $K' = (\partial P')^*$) be the dual of the boundary of a simple convex polytope P (resp. P'). Let M (resp. M') be a quasitoric manifold over P (resp. P') such that $H^*(M) \cong H^*(M)$. Then by Lemma 3.7 and the ring isomorphism $H^*(\mathcal{Z}_K : k) \cong \mathrm{Tor}(k(K), k)$, we have the isomorphism $H^*(\mathcal{Z}_K : k) \cong H^*(\mathcal{Z}_{K'} : k)$. Hence if P is cohomologically rigid, then K is B-rigid. Furthermore Example 1.1 still gives non B-rigid simplicial complexes. However at this moment we do not know whether cohomological rigidity and B-ridigity are equivalent for simple convex polytopes supporting quasitoric manifolds.

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