ON SOME RAY CLASS INVARIANTS OVER IMAGINARY QUADRATIC FIELDS

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Abstract. We prove that the singular values of certain Siegel functon generate the ray class fields over almost all imaginary quadratic fields by extending the idea of our previous work([8]). These generators are not only the simplest ones conjectured by Schertz([13]), but also quite useful in the matter of computation of class polynomials. We indeed give an algorithm to find all the conjugates of such generators by virtue of [5].

1. Introduction

Let K be an imaginary quadratic field and $\mathcal{O}_K = \mathbb{Z}[\theta]$ be the ring of integers with θ in the complex upper half plane $\mathfrak{H}$. We denote the Hilbert class field and the ray class field modulo N of K for a positive integer $N \geq 2$ by H and $K_{(N)}$, respectively. Historically it was first found in 1927 by Hasse([7]) that for any integral ideal $\mathfrak{a}$ of K, $K_{(N)}$ is generated over H by adjoining the value of the Weber function for the elliptic curve $\mathbb{C}/\mathfrak{a}$ at a generator of the cyclic $\mathcal{O}_K$-module $\frac{1}{N}\mathfrak{a}/\mathfrak{a}$. It requires good understanding of arithmetic of elliptic curves, which is formulated by the main theorem of complex multiplication([14], [11]). Together with the Shimura reciprocity law which reveals remarkable relationship between class field theory and modular function fields, the theory of Shimura's canonical model allows us to generate $K_{(N)}$ over K by the specialization of certain modular function field. In particular, Cho-Koo([1]) showed that the singular value of a Hauptmodul with rational Fourier coefficients on some modular curve generates $K_{(N)}$ over K. For instance, Cho-Koo-Park([2]) considered the case $N = 6$ in terms of the Ramanujan's cubic continued fraction. And Koo-Shin further provided in [9] appropriate Hauptmoduli for this purpose.

After Hasse it seems to be a difficult problem to construct a ray class invariant (as a primitive generator of $K_{(N)}$) over K by means of evaluation of a transcendental function which can be applied to all K and $N \geq 2$. In 1964 Ramachandra([12]) at last found universal generators of ray class fields of arbitrary moduli by applying the Kronecker limit formula concerning L-series. However his invariants involve too complicated products of high powers of singular values of the Klein forms and singular values of the discriminant delta function. On the other hand, Schertz([13]) attempted to find simple and better answers for practical use with similar ideas. The simplest generators conjectured by Schertz are the singular values of a Siegel function, and Jung-Koo-Shin([8]) showed that his conjectural generators are the right ones at least over H for $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$.

Since the primitive element theorem guarantees the existence of a simple generator of $K_{(N)}$ over K, one might try to combine Hasse's two generators to get one ray class invariant. And Cho-Koo([1]) recently succeeded in obtaining such generator by showing that the singular value of a Weber function is an algebraic integer and then applying the result of Gross-Zagier([6] or [3] Theorem 13.28). Koo-Shin([9]) further investigated the problem over K in completely different

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point of view by utilizing both the singular values of the elliptic modular function j and Siegel functions.

For any pair $(r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we define a *Siegel function* $g_{(r_1,r_2)}(\tau)$ on $\tau \in \mathfrak{H}$ by the following Fourier expansion

$$g_{(r_1,r_2)}(\tau) = -q_\tau^{\frac{1}{2}\mathbf{B}_2(r_1)} e^{\pi i r_2(r_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}) \tag{1.1}$$

where $\mathbf{B}_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial, $q_\tau = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. Then it is a modular unit in the sense of [10]. Since its Fourier coefficients are quite small, we are able to estimate and compare the values of the function in order to derive our main theorem.

Let $\mathfrak{a} = [\omega_1, \omega_2]$ be a fractional ideal of K not containing 1, where ω_1 and ω_2 is an oriented basis so that $\frac{\omega_1}{\omega_2} \in \mathfrak{H}$. Writing $1 = r_1\omega_1 + r_2\omega_2$ for some $(r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we denote

$$g(1, [\omega_1, \omega_2]) = g_{(r_1,r_2)}\left(\tfrac{\omega_1}{\omega_2}\right).$$

When a product of these values becomes a unit, we call it an *elliptic unit*. By taking 12-th power the above value depends only on $\mathfrak{a}$ itself. So we may write $g^{12}(1, \mathfrak{a})$ instead of $g^{12}(1, [\omega_1, \omega_2])$.

For a given nontrivial integral ideal $\mathfrak{f}$ of K the ideal class group $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})$ is isomorphic to the Galois group of the ray class field $K(\mathfrak{f})$ modulo $\mathfrak{f}$ over K([11]). Now we consider the value

$$g^{12N(\mathfrak{f})}(1, \mathfrak{f})$$

where $N(\mathfrak{f})$ is the smallest positive integer in $\mathfrak{f}$. For a ray class C in $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})$ the action of the Artim map $\sigma(C)$ satisfies the rule

$$g^{12N(\mathfrak{f})}(1, \mathfrak{f})^{\sigma(C)} = g^{12N(\mathfrak{f})}(1, \mathfrak{f}\mathfrak{c}^{-1}) \tag{1.2}$$

where $\mathfrak{c}$ is a representative integral ideal of C by the main theorem of complex multiplication([10]). In our case we take $\mathfrak{f} = N\mathcal{O}_K$ for a positive integer $N \geq 2$. In this paper, as Schertz conjectured, we shall show that only the singular value

$$g^{12N}(1, N\mathcal{O}_K) = g^{12N}_{(0,\frac{1}{N})}(\theta)$$

or its any power generates $K_{(N)}$ over K for $K \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ except possibly finitely many cases(Theorem 4.5, Remark 4.6). While the formula (1.2) provides all the conjugates of $g^{12N}_{(0,\frac{1}{N})}(\theta)$, it is inconvenient for practical use because we can hardly describe the bases of representative ideals in general. Therefore, rather than working with the actions of $\mathrm{Gal}(K_{(N)}/K)$ directly by (1.2) we will manipulate the actions of $\mathrm{Gal}(H/K)$ and $\mathrm{Gal}(K_{(N)}/H)$ separtely by following Gee's idea([5]).

2. Fields of modular functions

This section will be devoted to reviewing briefly the modular function fields and the actions of Galois groups in terms of Siegel functions. For the full description of modularity of Siegel functions we refer to [10] or [9].

For a positive integer N, let $\zeta_N = e^{\frac{2\pi i}{N}}$ and

$$\Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

be the principal congruence subgroup of level N of $\mathrm{SL}_2(\mathbb{Z})$. The group Γ_N acts on $\mathfrak{H}$ by linear fractional transformation, and the orbit space $Y(N) = \Gamma_N \backslash \mathfrak{H}$ can be given a Riemann surface structure. Furthermore, $Y(N)$ can be compactified by adding the cusps so that $X(N) = \Gamma_N \backslash \mathfrak{H}^*$

with $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{P}^1(\mathbb{Q})$ becomes a compact Riemann surface (or an algebraic curve), which we call the *modular curve of level* N([14], [4]).

The meromorphic functions on $X(N)$ are said to be the *modular functions of level* N. In particular, we are interested in the field of modular functions of level N defined over the N-th cyclotomic field $\mathbb{Q}(\zeta_N)$, denoted by $\mathcal{F}_N$. Then it is well-known that the extension $\mathcal{F}_N/\mathcal{F}_1$ is Galois and

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$$

whose action is given as follows: We can decompose an element $\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ into $\alpha = \alpha_1 \cdot \alpha_2$ for some $\alpha_1 \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ and $\alpha_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$. The action of α_1 is defined by composition as linear fractional transformation. And, α_2 acts by the rule

$$\sum_{n=-\infty}^{\infty} c_n q_\tau^{\frac{n}{N}} \mapsto \sum_{n=-\infty}^{\infty} c_n^{\sigma_d} q_\tau^{\frac{n}{N}}$$

where $\sum_{n=-\infty}^{\infty} c_n q_\tau^{\frac{n}{N}}$ is the Fourier expansion of a function in $\mathcal{F}_N$ and σ_d is the automorphism of $\mathbb{Q}(\zeta_N)$ defined by $\zeta_N^{\sigma_d} = \zeta_N^d$([14], [11]).

Usually the fields $\mathcal{F}_N$ are described by $j(\tau)$ and the Fricke functions. However, we restate these fields in terms of Siegel functions for later use. First, we need some transformation formulas and modularity criterions for Siegel functions.

Theorem 2.1. *Let $N \geq 2$. For $(r_1, r_2) \in \frac{1}{N}\mathbb{Z}^2 \setminus \mathbb{Z}^2$ the function $g_{(r_1,r_2)}^{12N}(\tau)$ satisfies*

$$g_{(r_1,r_2)}^{12N}(\tau) = g_{(-r_1,-r_2)}^{12N}(\tau) = g_{(\langle r_1\rangle,\langle r_2\rangle)}^{12N}(\tau)$$

where $\langle X\rangle$ is the fractional part of $X \in \mathbb{R}$ such that $0 \leq \langle X\rangle < 1$. It belongs to $\mathcal{F}_N$ and $\alpha \in GL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1_2\}$ acts on it by

$$\left(g_{(r_1,r_2)}^{12N}(\tau)\right)^\alpha = g_{(r_1,r_2)\alpha}^{12N}(\tau).$$

And, we have

$$\mathcal{F}_N = \mathbb{Q}(\zeta_N)\big(j(\tau),\ g_{(\frac{1}{N},0)}^{12N}(\tau),\ g_{(0,\frac{1}{N})}^{12N}(\tau)\big).$$

Proof. See [9] Proposition 2.4, Theorem 2.5 and Theorem 4.3. □

We set

$$\mathcal{F} = \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

Passing to the projective limit of the exact sequence

$$1 \longrightarrow \{\pm 1_2\} \longrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \longrightarrow 1,$$

we obtain an exact sequence

$$1 \longrightarrow \{\pm 1_2\} \longrightarrow \prod_{p\ :\ \text{prime}} \mathrm{GL}_2(\mathbb{Z}_p) \longrightarrow \mathrm{Gal}(\mathcal{F}/\mathcal{F}_1) \longrightarrow 1. \tag{2.1}$$

For every $u = (u_p)_p \in \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ and $N \geq 1$, there exists an integral matrix α in $\mathrm{GL}_2^+(\mathbb{Q})$ with $\det\alpha > 0$ such that $\alpha \equiv u_p \pmod{N\mathbb{Z}_p}$ for all p dividing N by the Chinese remainder theorem. The action of u on $\mathcal{F}_N$ is understood as the action of α([14]).

3. Shimura reciprocity law

We shall develop an algorithm for finding all the conjuates of a singular value of a modular function, from which we can determine all the conjugates of $g^{12N}_{(0,\frac{1}{N})}(\theta)$. To this end we adopt Gee's idea([5]) which explains the Shimura reciprocity law explicitly for practical use.

Let $\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}} = \prod'_p \mathbb{Q}_p$ denote the ring of finite ideles. Here, the restricted product is taken with respect to the subgroups $\mathbb{Z}_p \subset \mathbb{Q}_p$. Every $x \in \mathrm{GL}_2(\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}})$ can be written as

$$x = u \cdot \alpha \quad \text{with } u \in \prod_p \mathrm{GL}_2(\mathbb{Z}_p) \text{ and } \alpha \in \mathrm{GL}_2^+(\mathbb{Q}).$$

Such a decomposition $x = u \cdot \alpha$ determines a group action of $\mathrm{GL}_2(\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}})$ on $\mathcal{F}$ by

$$h^x = h^u \circ \alpha$$

where h^u is given by the exact sequence (2.1). Then we have the following *Shimura exact sequence*

$$1 \longrightarrow \mathbb{Q}^* \longrightarrow \mathrm{GL}_2(\mathbb{A}^{\mathrm{f}}_{\mathbb{Q}}) \longrightarrow \mathrm{Aut}(\mathcal{F}) \longrightarrow 1$$

([14], [11]).

Let K be an imaginary quadratic field with the ring of integers $\mathcal{O}_K = \mathbb{Z}[\theta]$ such that $\theta \in \mathfrak{H}$. We use the notation $K_p = K \otimes_p \mathbb{Q}_p$ for each rational prime p and denote the group of finite ideles of K by $\mathbb{A}^{\mathrm{f}}_K = \prod'_p K_p^*$ where the restricted product is taken with respect to the subgroups $\mathcal{O}_p = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$ of K_p^*. Let $[\ ,K]$ denote the Artin map on $\mathbb{A}^{\mathrm{f}}_K$. Then the class field theory on K is summarized in the following exact sequence

$$1 \longrightarrow K^* \longrightarrow \mathbb{A}^{\mathrm{f}}_K \xrightarrow{[\ ,K]} \mathrm{Gal}(K^{\mathrm{ab}}/K) \longrightarrow 1$$

where K^{ab} is the maximal abelian extension of K([15]). The *main theorem of complex multiplication* states that the value $j(\theta)$ generates H over K, and the sequence

$$1 \longrightarrow \mathcal{O}^* \longrightarrow \prod_p \mathcal{O}_p^* \xrightarrow{[\ ,K]} \mathrm{Gal}\big(K^{\mathrm{ab}}/K(j(\theta))\big) \longrightarrow 1 \tag{3.1}$$

is exact. Furthemore, $K_{(N)}$ is none other than the field $K\big(\mathcal{F}_N(\theta)\big)$ which is the extension field of K obtained by adjoining all the singular values $h(\theta)$ for which $h \in \mathcal{F}_N$ is defined and finite at θ([14], [11]).

For a prime p we define

$$(g_\theta)_p : K_p^* \longrightarrow \mathrm{GL}_2(\mathbb{Q}_p)$$

as the injection that sends $x_p \in K_p^*$ to the matrix in $\mathrm{GL}_2(\mathbb{Q}_p)$ which represents multiplication by x_p with respect to the $\mathbb{Q}_p$-basis $[\theta, 1]$ for K_p. More precisely, if $\mathrm{irr}(\theta, \mathbb{Q}) = X^2 + BX + C$, then for $s_p, t_p \in \mathbb{Q}_p$ we can describe the map as

$$(g_\theta)_p : s_p\theta + t_p \mapsto \begin{pmatrix} t_p - Bs_p & -Cs_p \\ s_p & t_p \end{pmatrix}.$$

On $\mathbb{A}^{\mathrm{f}}_K$ we achieve an injection

$$g_\theta = \prod_p (g_\theta)_p : \mathbb{A}^{\mathrm{f}}_K \longrightarrow {\prod_p}' \mathrm{GL}_2(\mathbb{Q}_p).$$

Combining (2.1) and (3.1) we get the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{O}^* & \longrightarrow & \prod_p \mathcal{O}_p^* & \stackrel{[\ ,K]}{\longrightarrow} & \mathrm{Gal}\big(K^{\mathrm{ab}}/K(j(\theta))\big) & \longrightarrow & 1 \\ & & & & \downarrow g_\theta & & & & \\ 1 & \longrightarrow & \{\pm 1_2\} & \longrightarrow & \prod_p \mathrm{GL}_2(\mathbb{Z}_p) & \longrightarrow & \mathrm{Gal}(\mathcal{F}/\mathcal{F}_1) & \longrightarrow & 1. \end{array} \tag{3.2}$$

Then the *Shimura reciprocity law* says that for $h \in \mathcal{F}$ and $x \in \prod_p \mathcal{O}_p^*$

$$h(\theta)^{[x^{-1},K]} = h^{(g_\theta(x))}(\theta) \tag{3.3}$$

([14], [11]).

Let $Q = [a,b,c] = aX^2+bXY+cY^2$ be a primitive positive definite quadratic form of discriminant d_K. Under the properly equivalent relation these forms determine a group $C(d_K)$, called the *form class group of discriminant* d_K. In particular, the inverse of the class containing $[a,b,c]$ is the class containing $[a,-b,c]$. We identify $C(d_K)$ with the set of all *reduced quadratic forms*, which are characterized by the conditions

$$-a < b \le a < c \quad \text{or} \quad 0 \le b \le a = c. \tag{3.4}$$

Note that the above two condtions for reduced quadratic forms induce

$$a \le \sqrt{\tfrac{-d_K}{3}} \tag{3.5}$$

([3]).

It is well-known that $C(d_K)$ is isomorphic to $\mathrm{Gal}(H/K)$. In [5] Gee found an idele $x_Q \in \mathbb{A}_K^{\mathrm{f}}$ such that

$$[x_Q, K]|_H = [a,b,c].$$

Theorem 3.1. *Let $Q = [a,b,c]$ be a primitive positive definite quadratic form of discriminant d_K. We define*

$$\theta_Q = \frac{-b+\sqrt{d_K}}{2a}.$$

Furthermore, for every rational prime p we define x_p as

$$x_p = \begin{cases} a & \text{if} \quad p \nmid a \\ a\theta_Q & \text{if} \quad p \mid a \quad \text{and} \quad p \nmid c \\ a(\theta_Q - 1) & \text{if} \quad p \mid a \quad \text{and} \quad p \mid c. \end{cases}$$

Then, for $x_Q = (x_p)_p \in \mathbb{A}_K^f$ the Galois action of the Artin symbol $[x_Q, K]$ satisfies

$$j(\theta)^{[a,b,c]} = j(\theta)^{[x_Q,K]}.$$

Proof. See [5] Lemma 19. □

The next theorem gives the action of $[x_Q^{-1}, K]$ on K^{ab} by using the Shimura reciprocity law (3.3).

Theorem 3.2. *Let $Q = [a,b,c]$ be a primitive positive definite quadratic form of discriminant d_K. Define*

$$\theta = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{for} \quad d_K \equiv 0 \pmod 4 \\ \frac{-1+\sqrt{d_K}}{2} & \text{for} \quad d_K \equiv 1 \pmod 4 \end{cases}$$

and let θ_Q be as in Theorem 3.1. Furthermore, define $u_Q = (u_p)_p \in \prod_p GL_2(\mathbb{Z}_p)$ as

$$u_p = \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases} \tag{3.6}$$

for $d_K \equiv 0 \pmod 4$ and

$$u_p = \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \\ \begin{pmatrix} \frac{-b-1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \end{cases} \tag{3.7}$$

for $d_K \equiv 1 \pmod 4$. Then for $h \in \mathcal{F}$ we have

$$h(\theta)^{[x_Q^{-1}, K]} = h^{u_Q}(\theta_Q).$$

Proof. See [5] Lemma 20. □

Analyzing the diagram (3.2) and using the Shimura reciprocity law (3.3), she could express $\mathrm{Gal}(K_{(N)}/H)$ quite explicitly.

Theorem 3.3. *Assume $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$. Considering the group*

$$W_{N,\theta} = \left\{ \begin{pmatrix} t - Bs & -Cs \\ s & t \end{pmatrix} \in GL_2(\mathbb{Z}/N\mathbb{Z}) \ : \ t, s \in \mathbb{Z}/N\mathbb{Z} \right\},$$

we have a surjection

$$\begin{array}{ccl} W_{N,\theta} & \longrightarrow & Gal(K_{(N)}/H) \\ \alpha & \longmapsto & \big(h(\theta) \mapsto h^{\alpha}(\theta)\big) \text{ where } h \in \mathcal{F}_N \text{ is defined and finite at } \theta \end{array}$$

with kernel $\{\pm 1_2\}$ by (3.2) and (3.3).

Proof. See [5]. □

Finally we obtain an assertion which we shall use to resolve our main problem.

Theorem 3.4. *Assume $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$ and $h \in \mathcal{F}_N$ for some $N \geq 2$. Then*

$$\left\{ h^{\alpha \cdot u_Q}(\theta_Q) \ : \ \alpha \in W_{N,\theta}/\{\pm 1_2\} \text{ and } Q \text{ is any reduced quadratic form of discriminant } d_K \right\}$$

is the set of all the conjugates of $h(\theta)$ over K. Here, we follow the notations in Theorem 3.2 and Theorem 3.3.

Proof. Observe the following diagram:

$$\begin{array}{lll}
\underline{\text{Fields}} & \underline{\text{Galois groups}} & \\
K_{(N)} & & \\
\quad\Big| & \Big) \ \mathrm{Gal}(K_{(N)}/H) \cong W_{N,\theta}/\{\pm 1_2\} & \text{by Theorem 3.3} \\
H & & \\
\quad\Big| & \Big) \ \mathrm{Gal}(H/K) = \big\{[x_Q, K]|_H \ : \ Q \in C(d_K)\big\} & \text{by Theorem 3.1.} \\
K & &
\end{array}$$

Now the conclusion follows from Theorem 3.2. □

4. Ray class invariants

In this last section we will prove that the singular value $g^{12N}_{(0,\frac{1}{N})}(\theta)$ generates $K_{(N)}$ by showing that the only automorphism of $K_{(N)}$ over K which fixes the value is the identity. Then Galois theory guarantees our theorem. Although we leave out finitely many cases, one can readily verify that the remaining cases are indeed generators of $K_{(N)}$ over K, which can be checked out by computing minimal polynomials of the singular values if he wants.

Throughout this section we let $K \neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field with discriminant d_K so that $-d_K \geq 7$. We put

$$D = \sqrt{\tfrac{-d_K}{3}}$$

and define θ, θ_Q and u_Q for each primitive positive definite quadratic form $Q = [a, b, c]$ as in Theorem 3.2. If we set

$$B = |q_\theta| = |e^{2\pi i\theta}| = e^{-\pi\sqrt{-d_K}},$$

then we have

$$B \leq e^{-\sqrt{7}\pi} \quad \text{and} \quad B^{\frac{1}{D}} = e^{-\sqrt{3}\pi}. \tag{4.1}$$

In what follows we shall often use the following basic inequality

$$1 + X < e^X \quad \text{for } X > 0. \tag{4.2}$$

Lemma 4.1. *We have the following inequalities:*
(i) If $N \geq 21$, *then* $\left|\frac{1-\zeta_N}{1-B^{\frac{1}{DN}}}\right| < 1.306$.
(ii) If $N \geq 2$, *then* $\left|\frac{1-\zeta_N}{1-\zeta_N^s}\right| \leq 1$ *for all* $s \in \mathbb{Z} \setminus N\mathbb{Z}$.
(iii) If $N \geq 4$, *then* $\left|\frac{1-\zeta_N}{1-\zeta_N^s}\right| \leq \frac{1}{\sqrt{2}}$ *for* $2 \leq s \leq \frac{N}{2}$.
(iv) If $N \geq 2$, *then* $B^{\frac{1}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{1}{N}))}\left|\frac{1-\zeta_N}{1-B^{\frac{1}{N}}}\right| < 0.76$.
(v) $\frac{1}{1-B^{\frac{X}{D}}} < 1 + B^{\frac{X}{1.03D}}$ *for all* $X \geq \frac{1}{2}$.
(vi) $\frac{1}{1-B^X} < 1 + B^{\frac{X}{1.03}}$ *for all* $X \geq \frac{1}{2}$.

Proof. (i) It is routine to check that $\left|\frac{1-\zeta_N}{1-B^{\frac{1}{DN}}}\right| = \frac{2\sin\frac{\pi}{N}}{1-e^{-\frac{\sqrt{3}\pi}{N}}}$ is a decreasing function for $N \geq 21$. Hence its value is maximal when $N = 21$, which is less than 1.306.

(ii) $\left|\frac{1-\zeta_N}{1-\zeta_N^s}\right| = \left|\frac{\sin\frac{\pi}{N}}{\sin\frac{\pi s}{N}}\right| \leq 1$ for all $s \in \mathbb{Z} \setminus N\mathbb{Z}$.
(iii) Now that

$$\begin{aligned} 2|1-\zeta_N|^2 &= 4 - 4\cos\tfrac{2\pi}{N} \\ |1-\zeta_N^s|^2 &= 2 - 2\cos\tfrac{2s\pi}{N} \geq 2 - 2\cos\tfrac{4\pi}{N}, \end{aligned}$$

it is enough to prove

$$4 - 4\cos\tfrac{2\pi}{N} \leq 2 - 2\cos\tfrac{4\pi}{N}.$$

Observe that

$$\begin{aligned} & 2 - 2\cos\tfrac{4\pi}{N} - \big(4 - 4\cos\tfrac{2\pi}{N}\big) = 2\big(-1 - \cos\tfrac{4\pi}{N} + 2\cos\tfrac{2\pi}{N}\big) \\ = \ & 2\big(-2\cos^2\tfrac{2\pi}{N} + 2\cos\tfrac{2\pi}{N}\big) = 4\cos\tfrac{2\pi}{N}\big(-\cos\tfrac{2\pi}{N} + 1\big) \geq 0 \end{aligned}$$

due to the fact $N \geq 4$.
(iv) Observe that

$$B^{\frac{1}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{1}{N}))}\left|\frac{1-\zeta_N}{1-B^{\frac{1}{N}}}\right| \leq e^{-\frac{\sqrt{7}\pi}{2}(\frac{1}{N}-\frac{1}{N^2})}\frac{2\sin\frac{\pi}{N}}{1-e^{-\frac{\sqrt{7}\pi}{N}}} \quad \text{by (4.1)}.$$

It is also routine to check the last term on $N(\geq 2)$ is less than 0.76.
(v) By (4.1) the inequality is equivalent to $e^{-\sqrt{3}\pi X} + e^{-\frac{3\sqrt{3}\pi}{103}X} < 1$, which obviously holds for $X \geq \frac{1}{2}$.
(vi) The given inequality is equivalent to $B^X + B^{\frac{3}{103}X} < 1$. By (4.1) it suffices to show $e^{-\sqrt{7}\pi X} + e^{-\frac{3\sqrt{7}\pi}{103}X} < 1$, which is true for all $X \geq \frac{1}{2}$. □

Lemma 4.2. *Let $N \geq 21$ and $Q = [a, b, c]$ be a reduced quadratic form of discriminant d_K. If $a \geq 2$, then the inequality*

$$\left|g_{(0,\frac{1}{N})}(\theta)\right| < \left|g_{(\frac{r}{N},\frac{s}{N})}(\theta_Q)\right|$$

holds for $(r, s) \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2$.

Proof. We may assume $0 \leq r \leq \frac{N}{2}$ by Theorem 2.1. And, note that $2 \leq a \leq D$ by (3.5). From the definition (1.1) we obtain that

$$\begin{aligned} \left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(\frac{r}{N},\frac{s}{N})}(\theta_Q)}\right| &= \left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(\frac{r}{N},\frac{s}{N})}\big(\frac{-b+\sqrt{d_K}}{2a}\big)}\right| \\ &\leq B^{\frac{1}{2}(\mathbf{B}_2(0)-\frac{1}{a}\mathbf{B}_2(\frac{r}{N}))}\left|\frac{1-\zeta_N}{1-e^{2\pi i(\frac{r}{N}\cdot\frac{-b+\sqrt{d_K}}{2a}+\frac{s}{N})}}\right|\prod_{n=1}^{\infty}\frac{(1+B^n)^2}{(1-B^{\frac{1}{a}(n+\frac{r}{N})})(1-B^{\frac{1}{a}(n-\frac{r}{N})})}. \end{aligned}$$

If $r \neq 0$, then by the fact $2 \leq a \leq D$ and Lemma 4.1(i) we derive

$$\left|\frac{1-\zeta_N}{1-e^{2\pi i(\frac{r}{N}\cdot\frac{-b+\sqrt{d_K}}{2a}+\frac{s}{N})}}\right| \leq \left|\frac{1-\zeta_N}{1-B^{\frac{r}{Na}}}\right| \leq \left|\frac{1-\zeta_N}{1-B^{\frac{1}{ND}}}\right| < 1.306.$$

If $r = 0$, then we get by Lemma 4.1(ii)

$$\left|\frac{1-\zeta_N}{1-e^{2\pi i(\frac{r}{N}\cdot\frac{-b+\sqrt{d_K}}{2a}+\frac{s}{N})}}\right| = \left|\frac{1-\zeta_N}{1-\zeta_N^s}\right| \leq 1.$$

Therefore we acheive that

$$\begin{aligned}
&\left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(\frac{r}{N},\frac{s}{N})}(\theta_Q)}\right| \\
<\ & B^{\frac{1}{2}(\mathbf{B}_2(0)-\frac{1}{2}\mathbf{B}_2(0))}\cdot 1.306\cdot\prod_{n=1}^{\infty}\frac{(1+B^n)^2}{(1-B^{\frac{n}{D}})(1-B^{\frac{1}{D}(n-\frac{1}{2})})} \quad \text{by } 2\le a\le D,\ 0\le r\le \tfrac{N}{2}\\
<\ & 1.306B^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+B^n)^2(1+B^{\frac{n}{1.03D}})(1+B^{\frac{1}{1.03D}(n-\frac{1}{2})}) \quad \text{by Lemma 4.1(v)}\\
<\ & 1.306B^{\frac{1}{24}}\prod_{n=1}^{\infty}e^{2B^n+B^{\frac{n}{1.03D}}+B^{\frac{1}{1.03D}(n-\frac{1}{2})}} = 1.306B^{\frac{1}{24}}e^{\frac{2B}{1-B}+\frac{B^{\frac{1}{1.03D}}+B^{\frac{1}{2.06D}}}{1-B^{\frac{1}{1.03D}}}} \quad \text{by (4.2)}\\
\le\ & 1.306e^{-\frac{\sqrt{7}\pi}{24}}\,e^{\frac{2e^{-\sqrt{7}\pi}}{1-e^{-\sqrt{7}\pi}}+\frac{e^{-\frac{\sqrt{3}\pi}{1.03}}+e^{-\frac{\sqrt{3}\pi}{2.06}}}{1-e^{-\frac{\sqrt{3}\pi}{1.03}}}} < 1 \quad \text{by (4.1).}
\end{aligned}$$

This proves the lemma. □

Lemma 4.3. *Let $N\ge 2$ and $Q=[1,b,c]$ be a reduced quadratic form of discriminant d_K. Then we have the inequality*

$$\left|g_{(0,\frac{1}{N})}(\theta)\right| < \left|g_{(\frac{r}{N},\frac{s}{N})}(\theta_Q)\right|$$

for $r,s\in\mathbb{Z}$ with $r\not\equiv 0 \pmod N$.

Proof. We may assume $1\le r\le \frac{N}{2}$ by Theorem 2.1. Then we obtain

$$\begin{aligned}
&\left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(\frac{r}{N},\frac{s}{N})}(\theta_Q)}\right| \\
<\ & B^{\frac{1}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{r}{N}))}\left|\frac{1-\zeta_N}{1-B^{\frac{r}{N}}}\right|\prod_{n=1}^{\infty}\frac{(1+B^n)^2}{(1-B^{n+\frac{r}{N}})(1-B^{n-\frac{r}{N}})} \quad \text{by (1.1)}\\
<\ & B^{\frac{1}{2}(\mathbf{B}_2(0)-\mathbf{B}_2(\frac{1}{N}))}\left|\frac{1-\zeta_N}{1-B^{\frac{1}{N}}}\right|\prod_{n=1}^{\infty}\frac{(1+B^n)^2}{(1-B^{n})(1-B^{n-\frac{1}{2}})}\\
<\ & 0.76\prod_{n=1}^{\infty}(1+B^n)^2(1+B^{\frac{n}{1.03}})(1+B^{\frac{1}{1.03}(n-\frac{1}{2})}) \quad \text{by Lemma 4.1(iv) and (vi)}\\
<\ & 0.76\prod_{n=1}^{\infty}e^{2B^n+B^{\frac{n}{1.03}}+B^{\frac{1}{1.03}(n-\frac{1}{2})}} = 0.76e^{\frac{2B}{1-B}+\frac{B^{\frac{1}{1.03}}+B^{\frac{1}{2.06}}}{1-B^{\frac{1}{1.03}}}} \quad \text{by (4.2)}\\
\le\ & 0.76e^{\frac{2e^{-\sqrt{7}\pi}}{1-e^{-\sqrt{7}\pi}}+\frac{e^{-\frac{\sqrt{7}\pi}{1.03}}+e^{-\frac{\sqrt{7}\pi}{2.06}}}{1-e^{-\frac{\sqrt{7}\pi}{1.03}}}} < 1 \quad \text{by (4.1).}
\end{aligned}$$

□

Lemma 4.4. *Let $N\ge 2$ and $Q=[1,b,c]$ be a reduced quadratic form of discriminant d_K. Then we deduce*

$$\left|g_{(0,\frac{1}{N})}(\theta)\right| < \left|g_{(0,\frac{s}{N})}(\theta_Q)\right|$$

for $s\in\mathbb{Z}$ with $s\not\equiv 0\pm 1 \pmod N$.

Proof. If $N = 2$ or 3, there is nothing to prove. Thus, let $N \geq 4$. Here we may assume that $2 \leq s \leq \frac{N}{2}$ by Theorem 2.1. Observe that

$$\begin{aligned}
&\left|\frac{g_{(0,\frac{1}{N})}(\theta)}{g_{(0,\frac{s}{N})}(\theta_Q)}\right| \\
\leq\ & \left|\frac{1-\zeta_N}{1-\zeta_N^s}\right| \prod_{n=1}^{\infty} \frac{(1+B^n)^2}{(1-B^n)^2} \\
<\ & \tfrac{1}{\sqrt{2}} \prod_{n=1}^{\infty} (1+B^n)^2 (1+B^{\frac{n}{1.03}})^2 \quad \text{by Lemma 4.1 (iii) and (vi)} \\
<\ & \tfrac{1}{\sqrt{2}} \prod_{n=1}^{\infty} e^{2B^n + 2B^{\frac{n}{1.03}}} \quad \text{by (4.2)} \\
=\ & \tfrac{1}{\sqrt{2}} e^{\frac{2B}{1-B} + \frac{2B^{\frac{1}{1.03}}}{1-B^{\frac{1}{1.03}}}} \leq \tfrac{1}{\sqrt{2}} e^{\frac{2e^{-\sqrt{7}\pi}}{1-e^{-\sqrt{7}\pi}} + \frac{2e^{-\frac{\sqrt{7}\pi}{1.03}}}{1-e^{-\frac{\sqrt{7}\pi}{1.03}}}} < 1 \quad \text{by (4.1)},
\end{aligned}$$

which proves the lemma. □

Now we are ready to prove our main theorem.

Theorem 4.5. *Let $N \geq 21$. For an imaginary quadratic field $K \neq \mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ we let θ be as in Theorem 3.2. Then the value*

$$g^{12Nn}(1, N\mathcal{O}_K) = g_{(0,\frac{1}{N})}^{12Nn}(\theta) \qquad \text{for} \quad n = 1, 2, \cdots$$

generates $K_{(N)}$ over K. It is a real algebraic integer and its minimal polynomial has integer coefficients. In particular, if N has at least two prime factors, then it is an elliptic unit.

Proof. For simplicity we put $g(\tau) = g_{(0,\frac{1}{N})}^{12Nn}(\tau)$. Since g belongs to $\mathcal{F}_N$ by Theorem 2.1, the value $g(\theta)$ lies in $K_{(N)}$ by the main theorem of complex multiplication. Hence, if we show that the only element of $\mathrm{Gal}(K_{(N)}/K)$ fixing the value $g(\theta)$ is the identity, then we can conclude that it generates $K_{(N)}$ over K by Galois theory.

By Theorem 3.4 any conjugate of $g(\theta)$ is of the form

$$g^{\alpha \cdot u_Q}(\theta_Q)$$

for some $\alpha = \begin{pmatrix} t-Bs & -Cs \\ s & t \end{pmatrix} \in W_{N,\theta}$ and a reduced quadratic form $Q = [a, b, c]$ of discriminant d_K. Assume that $g(\theta) = g^{\alpha \cdot u_Q}(\theta_Q)$. Then Lemma 4.2 leads us to the fact $a = 1$, which yields

$$u_Q = \begin{cases} \begin{pmatrix} 1 & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \text{for} \quad d_K \equiv 0 \pmod 4 \\ \begin{pmatrix} 1 & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} & \text{for} \quad d_K \equiv 1 \pmod 4 \end{cases}$$

as an element of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ by definition (3.6) and (3.7). It follows from Theorem 2.1 that

$$g(\theta) = g^{\alpha \cdot u_Q}(\theta_Q) = g_{(0,\frac{1}{N})\alpha u_Q}^{12Nn}(\theta_Q) = \begin{cases} g_{(\frac{s}{N}, \frac{s}{N}\cdot\frac{b}{2} + \frac{t}{N})}^{12Nn}(\theta_Q) & \text{for} \quad d_K \equiv 0 \pmod 4 \\ g_{(\frac{s}{N}, \frac{s}{N}\cdot\frac{b-1}{2} + \frac{t}{N})}^{12Nn}(\theta_Q) & \text{for} \quad d_K \equiv 1 \pmod 4), \end{cases}$$

from which we get $s \equiv 0 \pmod N$ by Lemma 4.3. Now Lemma 4.4 enables us to have $t \equiv \pm 1 \pmod N$, which shows that α is the identity. Finally we derive from the condition (3.4) for the reduced quadratic form Q and the relation $b^2-4ac=d_K$ that

$$Q=[a,b,c]=\begin{cases}[1,0,-\frac{d_K}{4}] & \text{for} \quad d_K\equiv 0 \pmod 4\\ [1,1,\frac{1-d_K}{4}] & \text{for} \quad d_K\equiv 1 \pmod 4),\end{cases}$$

which yields $u_Q=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Therefore $\alpha\cdot u_Q$ represents the identity on $K_{(N)}$, and hence $g(\theta)$ actually generates $K_{(N)}$ over K.

From the definition (1.1) we have

$$\begin{aligned} g(\theta) &= \big(q_\theta^{\frac{1}{12}}(1-\zeta_N)\prod_{n=1}^{\infty}(1-q_\theta^n\zeta_N)(1-q_\theta^n\zeta_N^{-1})\big)^{12Nn}\\ &= q_\theta^{Nn}\big(2\sin\tfrac{\pi}{N}\big)^{12Nn}\prod_{n=1}^{\infty}\big(1-(\zeta_N+\zeta_N^{-1})q_\theta^n+q_\theta^{2n}\big)^{12Nn},\end{aligned}$$

and this claims that $g(\theta)$ is a real number. Furthermore, we see from [9] §3 that the function $g(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$. Since $j(\theta)$ is a real algebraic integer([14], [11]), so is the value $g(\theta)$, and its minimal polynomial over K has integer coefficients. In particular, if N has at least two prime factors, the function $1/g(\tau)$ is also integral over $\mathbb{Z}[j(\tau)]$([9]); hence $g(\theta)$ becomes a unit. □

Remark 4.6. If we assume that

$$\big(N\geq 4,\ -d_K\geq 31\big)\quad\text{or}\quad\big(N=3,\ -d_K\geq 39\big)\quad\text{or}\quad\big(N=2,\ -d_K\geq 43\big), \tag{4.3}$$

then the upper bounds of the inequalities appeared in Lemma 4.1 should be slightly changed. But it is routine to check that Lemma 4.2, 4.3 and 4.4 are also true. Therefore we can establish Theorem 4.5 again under the condition (4.3), however, we shall not repeat the redudant proofs.

Remark 4.7. For $N\geq 2$ and $(r_1,r_2)\in\frac{1}{N}\mathbb{Z}^2\setminus\mathbb{Z}^2$, the function $g_{(r_1,r_2)}^{\frac{12N}{\gcd(6,N)}}(\tau)$ belongs to $\mathcal{F}_N$ and satisfies the same transformation formulas as in Theorem 2.1 by [9] Theorem 2.5 and Proposition 2.4. Hence we are able to replace the value $g_{(0,\frac{1}{N})}^{12Nn}(\theta)$ in Theorem 4.5 by

$$g_{(0,\frac{1}{N})}^{\frac{12Nn}{\gcd(6,N)}}(\theta)\quad\text{for any}\quad n=1,2,\cdots$$

with smaller exponent, which enables us to have class polynomials with relatively small coefficients.

Now we close this section by presenting an example which illustrates Theorem 4.5, Remark 4.6 and Remark 4.7.

Example 4.8. Let $K=\mathbb{Q}(\sqrt{-10})$ and $N=6(=2\cdot 3)$. Then $d_K=-40$ and $\theta=\sqrt{-10}$. The reduced quadratic forms of discriminant d_K are

$$Q_1=[1,0,10]\quad\text{and}\quad Q_2=[2,0,5],$$

so we have

$$\theta_{Q_1}=\sqrt{-10},\ u_{Q_1}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\text{and}\quad\theta_{Q_2}=\frac{\sqrt{-10}}{2},\ u_{Q_2}=\begin{pmatrix}4&-3\\3&2\end{pmatrix}.$$

Furthermore, one can compute the group

$$\begin{aligned} W_{6,\theta}/\{\pm 1_2\} = \Big\{&\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&2\\1&1\end{pmatrix},\begin{pmatrix}1&4\\2&1\end{pmatrix},\begin{pmatrix}1&0\\3&1\end{pmatrix},\\ &\begin{pmatrix}1&2\\4&1\end{pmatrix},\begin{pmatrix}1&4\\5&1\end{pmatrix},\begin{pmatrix}3&2\\1&3\end{pmatrix},\begin{pmatrix}3&4\\2&3\end{pmatrix}\Big\}.\end{aligned}$$

Thus the class polynomial would be

$$\begin{aligned}
&\mathrm{irr}\big(g^{12}_{(0,\frac{1}{6})}(\theta), K\big)\\
= &\prod_{r=1}^{2}\prod_{\alpha\in W_{6,\theta}/\{\pm 1_2\}}\big(X - g^{12}_{(0,\frac{1}{6})\alpha u_{Q_r}}(\theta_{Q_r})\big)\\
= &\big(X - g^{12}_{(0,\frac{1}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{1}{6},\frac{1}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{2}{6},\frac{1}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{3}{6},\frac{1}{6})}(\sqrt{-10})\big)\\
&\big(X - g^{12}_{(\frac{4}{6},\frac{1}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{5}{6},\frac{1}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{1}{6},\frac{3}{6})}(\sqrt{-10})\big)\big(X - g^{12}_{(\frac{2}{6},\frac{3}{6})}(\sqrt{-10})\big)\\
&\big(X - g^{12}_{(\frac{3}{6},\frac{2}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{1}{6},\frac{5}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{5}{6},\frac{2}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{3}{6},\frac{5}{6})}(\tfrac{\sqrt{-10}}{2})\big)\\
&\big(X - g^{12}_{(\frac{1}{6},\frac{2}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{5}{6},\frac{5}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{1}{6},\frac{3}{6})}(\tfrac{\sqrt{-10}}{2})\big)\big(X - g^{12}_{(\frac{5}{6},0)}(\tfrac{\sqrt{-10}}{2})\big)\\
= &X^{16} + 20560X^{15} - 1252488X^{14} - 829016560X^{13} - 8751987701092X^{12}\\
&+217535583987600X^{11} + 181262520621110344X^{10} + 43806873084101200X^{9}\\
&-278616280004972730X^{8} + 139245187265282800X^{7} - 8883048242697656X^{6}\\
&+352945014869040X^{5} + 23618989732508X^{4} - 1848032773840X^{3}\\
&+49965941112X^{2} - 425670800X + 1,
\end{aligned}$$

which shows that $g^{12}_{(0,\frac{1}{6})}(\theta)$ is also a unit.

References

1. B. Cho and J. K. Koo, *Constructions of ray class fields over imaginary quadratic fields and applications*, submitted.
2. B. Cho, J. K. Koo, and Y. K. Park, *On Ramanujan's cubic continued fraction as modular function*, submitted.
3. D. A. Cox, *Primes of the form $x^2 + ny^2$: Fermat, Class Field, and Complex Multiplication*, John Wiley & Sons, Inc., 1989.
4. F. Diamond and J. Shurman, *A First Course in Modular Forms*, Springer, 2005.
5. A. Gee, *Class invariants by Shimura's reciprocity law*, J. Theor. Nombres Bordeaux 11 (1999), no. 1, 45-72.
6. B. Gross and D. Zagier, *On singular moduli*, J. reine angew. Math., 355 (1985), 191-220.
7. H. Hasse, *Neue Bergründung der Komplexen Multiplikation*, Teil I, J. für Math., 157 (1927), 115-139, Teil II, ibid. 165 (1931), 64-88.
8. H. Y. Jung, J. K. Koo and D. H. Shin, *Generation of ray class field by elliptic units*, submitted.
9. J. K. Koo and D. H. Shin, *On some arithmetic properties of Siegel functions*, submitted.
10. D. Kubert and S. Lang, *Modular Units*, Grundlehren der mathematischen Wissenschaften 244, Spinger-Verlag, 1981.
11. S. Lang, *Elliptic Functions, 2nd edition*, Spinger-Verlag, 1987.
12. K. Ramachandra, *Some applications of Kronecker's limit formula*, Ann. of Math. (2) 80(1964), 104-148.
13. R. Schertz, *Construction of ray class fields by elliptic units*, Journal de Théorie des Nombres de Bordeaux 9 (1997), 383-394.
14. G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton University Press, 1971.
15. J. H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.

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