THE COMPLETE DETERMINATION OF NARROW RICHAUD-DEGERT TYPE WHICH IS NOT 5 MODULO 8 WITH CLASS NUMBER TWO

JUNGYUN LEE

Abstract. In this paper, we will show that there are exactly 3 real quadratic fields of the form $K = \mathbb{Q}(\sqrt{n^2-1})$ with class number 2, where n^2-1 is a square free integer. This completely determines narrow Richaud-Degert type $d \not\equiv 5$ modulo 8 with class number 2.

1. Introduction and Statement of Result

Let $h(d)$ be the class number of the field $K = \mathbb{Q}(\sqrt{d})$, for a square free integer d. Let $d = n^2 + r$, $d \neq 5$, be a positive square free integer satisfying $r|4n$ and $-n < r \leq n$. Then we call $K = \mathbb{Q}(\sqrt{d})$ a real quadratic field of Richaud-Degert type. Specially if $|r| \in \{1, 4\}$, then d is called a narrow-Richaud-Degert type. Otherwise, it is called a wide-Richaud-Degert type.

Gauss conjectured that there are exactly nine imaginary quadratic fields with class number one. This was proved by Heegner[19], Stark[20] and Baker[18]. Analogously, we have a problem to determine all real quadratic fields of Richaud-Degert type with class number one. In [1] [2], Biró found all real quadratic fields of the form $\mathbb{Q}(\sqrt{n^2+1})$ and $\mathbb{Q}(\sqrt{n^2+4})$ with class number 1. Also, in [12] Byeon-Kim-Lee determine all real quadratic fields of the form $\mathbb{Q}(\sqrt{n^2-4})$ with class number 1. These completely determaine all narrow Richaud-Degert type with class number 1. Moreover, Lee[17] classified all wide-Richaud-Degert type $d \not\equiv 5$ modulo 8 with class number 1.

Many authors[16],[15],[14] studied to classify all Richaud-Degert type with class number 2. Meanwhile, Mollin and Williams[16] determine all real quadratic fields of Richaud-Degert type with class number 2 under the assumption of Generalized Riemann Hypothesis. A recent progress was made by Byeon-Lee[11]. They proved that there are exactly 4 real quadratic fields $\mathbb{Q}(\sqrt{n^2+1})$ with class number 2, where n^2+1 is a even square free integer, without G.R.H.

In this paper, we find all real quadratic fields $\mathbb{Q}(\sqrt{n^2-1})$ with class number 2, where n^2-1 is a squre free integer, without G.R.H. This classifies all narrow Richaus-Degert type $d \not\equiv 5$ modulo 8 with class number 2. The method of the proof is similar to that of [11]. The new ingredient is the skill of computing the special values of zeta functions associated with $\mathbb{Q}(\sqrt{n^2+1})$ whose fundamental units have the positive norm. Our main result is as follows:

Theorem 1.1. *Let $d = n^2 - 1$ be a square free integer. Then $h(d) = 2$ if and only if*

$$d = 15, 35, 143.$$

Theorem 1.2. *Let d be a narrow Richaud-Degert type with $d \not\equiv 5$ modulo 8. Then $h(d) = 2$ if and only if*

$$d = 10, 15, 26, 35, 65, 122, 143, 362$$

2. A calculation of special values of zeta functions associated with $\mathbb{Q}(\sqrt{n^2-1})$

Let $n^2 - 1$ be a square free integer and K be $\mathbb{Q}(\sqrt{n^2-1})$ and $\mathcal{O}(K)$ be the ring of integers in K. Then $\epsilon := n + \sqrt{n^2-1}$ is a fundamental unit for K, and $\{\epsilon,1\}$ is an integral basis for $\mathcal{O}(K)$. And $i(K^+)$ is a set of the principal ideals generated by an element in K^+ and $\mathbf{b} \cdot i(K^+) := \{\mathbf{b} \cdot \mathbf{a} | \mathbf{a} \in i(K^+)\}$ for an integral ideal $\mathbf{b}$ in K where K^+ be a set of totally positive elements in K. We define $R(\mathbf{b}) := \{a + b\epsilon \mid 0 < a \leq 1, 0 \leq b < 1 \text{ and } \mathbf{b} \cdot (a + b\epsilon) \in \mathcal{O}(K)\}$. And $N(\mathbf{b})$ is the number of the elements of $\mathcal{O}(K)/\mathbf{b}$ for an integral ideal $\mathbf{b}$ and we define $N_K(\alpha) := \alpha \cdot \bar{\alpha}$ for $\alpha \in K$.

Lemma 2.1. *An integral ideal $\mathbf{a}$ is in $\mathbf{b} \cdot i(K^+)$ if and only if*

$$\mathbf{a} = \mathbf{b} \cdot (a + b\epsilon + n_1 + n_2\epsilon)$$

for $a+b\epsilon \in R(\mathbf{b})$ and nonnegative integers n_1,n_2, where $\mathbf{b}$ is an integral ideal of K.

Proof: See Lemma 2.2 in [12] and Lemma 3.2 in [11] □

From the facts of $N_K(\epsilon) = 1$ and $N_K(\omega) = 2 - 2n < 0$, we have

Lemma 2.2. *If K has class number 2, then*

$$I(K) = (q) \cdot i(K^+) \cup (q)\mathbf{b} \cdot i(K^+) \cup (q\omega) \cdot i(K^+) \cup (q\omega)\mathbf{b} \cdot i(K^+).$$

where $\mathbf{b} = (2, \omega)$ and $\omega = -n + 1 + \sqrt{n^2-1}$.

Proof. See Propositon 2.2 in [11] and Lemma 2.1 in [12] □

Lemma 2.3. *If* $(q)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon)$ *and* $(q\omega)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon)$ *are the integral ideals, then*

$$N((q)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon)) = N((q)\mathbf{b}\cdot(x+y\epsilon^2)) \ (mod\ q),$$

$$N((q\omega)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon)) = N((q\omega)\mathbf{b}\cdot(x+y\epsilon^2)) \ (mod\ q),$$

for $0 < x \leq 1$, $0 \leq y < 1$ *and nonnegative integers* n_1, n_2 *and* $\mathbf{b} = (2,\omega)$ *and* $\omega = -n+1+\sqrt{n^2-1}$.

Proof: See Lemma 3.3 in [11]. □

Let χ be an odd primitive character with a conductor q. And we define

$$\zeta_K(s,\chi) := \sum_{\substack{\mathbf{a}\in I(K)\\ integral}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}.$$

If we assume $h(K)=2$, then from Lemma 2.1, 2.2, 2.3, we deduce that

(1)

$$\begin{aligned}
&\zeta_K(0,\chi)\\
&= \sum_{\substack{\mathbf{a}\in (q)\cdot i(K^+)\\ integral}} + \sum_{\substack{\mathbf{a}\in (q)\mathbf{b}\cdot i(K^+)\\ integral}} + \sum_{\substack{\mathbf{a}\in (q\omega)\cdot i(K^+)\\ integral}} + \sum_{\substack{\mathbf{a}\in (q\omega)\mathbf{b}\cdot i(K^+)\\ integral}} \frac{\chi(N(\mathbf{a}))}{N(\mathbf{a})^s}\Big|_{s=0}\\
&= \sum_{x+y\epsilon\in R((q\omega))} \chi(N((q\omega)\cdot(x+y\epsilon)) \sum_{n_1,n_2=0}^{\infty} N((q\omega)\cdot(x+y\epsilon+n_1+n_2\epsilon))^{-s}|_{s=0}\\
&\quad + \sum_{x+y\epsilon\in R((q))} \chi(N((q)\cdot(x+y\epsilon)) \sum_{n_1,n_2=0}^{\infty} N((q)\cdot(x+y\epsilon+n_1+n_2\epsilon))^{-s}|_{s=0}\\
&\quad + \sum_{x+y\epsilon\in R((q)\cdot\mathbf{b})} \chi(N((q)\mathbf{b}\cdot(x+y\epsilon))) \sum_{n_1,n_2=0}^{\infty} N((q)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon))^{-s}|_{s=0}\\
&\quad + \sum_{x+y\epsilon\in R((q\omega)\cdot\mathbf{b})} \chi(N((q\omega)\mathbf{b}\cdot(x+y\epsilon))) \sum_{n_1,n_2=0}^{\infty} N((q\omega)\mathbf{b}\cdot(x+y\epsilon+n_1+n_2\epsilon))^{-s}|_{s=0}
\end{aligned}$$

And Shintani in [7] [8], prove that

(2)

$$\sum_{n_1,n_2=0}^{\infty} N_K(x+y\epsilon+n_1+n_2\epsilon)^{-s}|_{s=0} = B_1(x)B_1(y)+\frac{1}{4}(\epsilon+\bar{\epsilon})(B_2(x)+B_2(y))$$

where $B_1(x)$ and $B_2(x)$ are the 1st and 2nd Bernoulli polynomials. So by defining

$$S(x,y) := B_1(x)B_1(y) + \frac{1}{4}(\epsilon+\bar{\epsilon})(B_2(x)+B_2(y)).$$

we can rewrite the equation (1) as follows:

(3)

$$\begin{aligned}
&\zeta_K(0,\chi) \\
&= \sum_{x+y\epsilon\in R((q))} \chi(N((q)\cdot(x+y\epsilon)))S(x,y) + \sum_{x+y\epsilon\in R((q)\cdot\mathbf{b})} \chi(N((q)\mathbf{b}\cdot(x+y\epsilon)))S(x,y) \\
&+ \sum_{x+y\epsilon\in R((q\omega))} \chi(N((q\omega)\cdot(x+y\epsilon)))S(x,y) + \sum_{x+y\epsilon\in R((q\omega)\cdot\mathbf{b})} \chi(N((q\omega)\mathbf{b}\cdot(x+y\epsilon)))S(x,y)
\end{aligned}$$

In the following Lemma, we find the complete set of (x,y) such that $x+y\epsilon \in R((q))$, $R((q)\cdot\mathbf{b})$, $R((q\omega)\cdot\mathbf{b})$ and $R((q\omega)\cdot\mathbf{b})$, to evaluate $\zeta_K(0,\chi)$.

Lemma 2.4.

(a) $x+y\epsilon \in R((q))$ *if and only if*

$$x = \begin{cases} 1, \text{ if } A=0 \\ \frac{A}{q}, \text{ if } A\neq 0 \end{cases} \quad \text{and} \quad y=\frac{B}{q} \text{ for } 0\le A,B\le q-1$$

(b) $x+y\epsilon \in R((q)\cdot\mathbf{b})$ *if and only if*

$$x = \begin{cases} 1, \text{ if } C=0 \\ \frac{C}{2q}, \text{ if } C\neq 0 \end{cases} \quad \text{and} \quad y=\frac{D}{2q}$$

for $0\le C,D\le 2q-1$ *and* $C\equiv D \pmod 2$

(c) $x+y\epsilon \in R((q\omega))$ *if and only if*

$$x=\frac{B}{q}-\frac{A+qi}{2q(n-1)}+\sigma_1(i) \text{ and } y=\frac{A+qi}{2q(n-1)}$$

for $0\le A,B\le q-1$ *and* $i=0,1,2,\cdots(2n-3)$

$$\text{where } \sigma_1(i) = \begin{cases} 0, \text{ if } 0\le i\le \lceil\frac{2(n-1)B-A}{q}\rceil-1 \\ 1, \text{ if } \lceil\frac{2(n-1)B-A}{q}\rceil \le i\le 2n-3 \end{cases}$$

(d) $x + y\epsilon \in R((q\omega)\mathbf{b})$ *if and only if*

$$x = \frac{D}{2q} - \frac{C+2qi}{4q(n-1)} + \sigma_2(i) \ \textit{and} \ \ y = \frac{C+2qi}{4q(n-1)}$$

for $0 \le C, D \le 2q-1, C \equiv 0 \ (\textit{mod } 2)$ *and* $i = 0, 1, 2, \cdots (2n-3)$

$$\textit{where } \sigma_2(i) = \begin{cases} 0, \ \textit{if } 0 \le i \le \lceil \frac{2(n-1)D-C}{2q} \rceil - 1 \\ 1, \ \textit{if } \lceil \frac{2(n-1)D-C}{2q} \rceil \le i \le 2n-3 \end{cases}$$

Proof:

(a) Since the set $\{A + B\epsilon | 0 \le A, B \le q-1\}$ is the complete representatives of $\mathcal{O}(K)/q\mathcal{O}(K)$, for $x + y\epsilon \in R((q))$, we have

$$q(x + y\epsilon) = A + B\epsilon + q(i + j\epsilon) \tag{4}$$

with $0 \le A, B \le q-1$ and $0 < x \le 1$, $0 \le y < 1$. By comparing the right and left hand sides, we obtain (a).

(b) The set $\{C + D\epsilon | 0 \le C, D \le 2q-1\}$ is the complete representatives of $\mathcal{O}(K)/2q\mathcal{O}(K)$. Thus if $x + y\epsilon \in R((q) \cdot \mathbf{b})$, then we have

$$2q(x + y\epsilon) = C + D\epsilon + 2q(i + j\epsilon), \tag{5}$$

where $0 \le C, D \le 2q-1$ and $0 < x \le 1$, $0 \le y < 1$. It directly follows that

$$x = \begin{cases} 1, \text{ if } C = 0 \\ \frac{C}{2q}, \text{ if } C \ne 0 \end{cases} \quad \text{and} \ \ y = \frac{D}{2q} \ \ \text{for} \ \ 0 \le C, D \le 2q-1. \tag{6}$$

Moreover, $\omega q(\frac{C}{2q} + \frac{D}{2q}\epsilon) \in \mathcal{O}(K)$ for $C \ne 0$ and $\omega q(1 + \frac{D}{2q}\epsilon) \in \mathcal{O}(k)$ for $C = 0$. Since

$$\omega q(\frac{C}{2q} + \frac{D}{2q}\epsilon) = \frac{(C+D)}{2}\omega + D(n-1)$$

and

$$\omega q(1 + \frac{D}{2q}\epsilon) = \omega q + (n-1)D + \frac{D}{2}\omega,$$

we have

$$C \equiv D(\text{ mod } 2). \tag{7}$$

From (6),(7), we can prove (b).

(c) The set $\{A + B\omega | 0 \le A, B \le q-1\}$ is the complete representatives of $\mathcal{O}(K)/q\mathcal{O}(K)$. Hence if $x + y\epsilon \in R((q\omega))$, then we have

$$q\omega(x + y\epsilon) = A + B\omega + q(i + j\omega) \tag{8}$$

for $0 \leq A, B \leq q-1$ and $0 < x \leq 1$, $0 \leq y < 1$. And (8) implies that

$$x + y\epsilon = \frac{B + qj}{q} + \frac{A + qi}{2q(1-n)} + \frac{A + qi}{2q(n-1)}\epsilon.$$

By above equation, we can induce (c), immediately.

(d) We also note that $\{C + D\omega | 0 \leq C, D \leq 2q-1\}$ is the complete representatives of $\mathcal{O}(K)/2q\mathcal{O}(K)$. Thus if $x + y\epsilon \in R((q\omega) \cdot \mathbf{b})$, we have

$$2q\omega(x + y\epsilon) = C + D\omega + 2q(i + j\omega) \tag{9}$$

for $0 \leq C, D \leq 2q-1$ and $0 < x \leq 1$, $0 \leq y < 1$. Then (9) implies that

$$x = \frac{D}{2q} - \frac{C + 2qi}{4q(n-1)} + \sigma_2(i) \text{ and } y = \frac{C + 2qi}{4q(n-1)}$$
for $i = 0, 1, 2, ..., (2n-3)$ and $0 \leq C, D \leq 2q-1$.

Moreover,
(10)
$$q\omega^2(x + y\epsilon) = q\omega^2\Big(\frac{D}{2q} - \frac{C + 2qi}{4q(n-1)} + \sigma_2(i) + (\frac{C + 2qi}{4q(n-1)})\epsilon\Big) \in \mathcal{O}(K).$$

Since

$$\frac{\omega^2}{2} = (n-1)(n - \sqrt{n^2 - 1}),$$

(10) implies that

$$C \equiv 0(\text{ mod } 2).$$

This complete the proof of (d) $\square$

From the equation (3) and Lemma 2.4, we deduce that

Proposition 2.5. *If h(K)=2, then*

$$\begin{aligned}
&\zeta_K(0,\chi)\\
&= \sum_{\substack{0\le A,B\le q-1\\ A\neq 0}} \chi(A^2+B^2+2nAB)S(\frac{A}{q},\frac{B}{q}) + \sum_{0\le B\le q-1}\chi(B^2)S(1,\frac{B}{q})\\
&+ \sum_{\substack{0\le C,D\le 2q-1, C\neq 0\\ C\equiv D \ (mod\ 2)}} \chi(\frac{C^2}{2}+\frac{D^2}{2}+nCD)S(\frac{C}{2q},\frac{D}{2q}) + \sum_{\substack{0\le D\le 2q-1\\ D\equiv 0\ (mod 2)}} \chi(\frac{D^2}{2})S(1,\frac{D}{2q})\\
&- \sum_{0\le A,B\le q-1}\chi(A^2+B^2(2-2n)+AB(2-2n))\sum_{i=0}^{2n-3} S(\frac{B}{q}-\frac{A+qi}{2q(n-1)}+\sigma_1(i),\frac{A+qi}{2q(n-1)})\\
&- \sum_{\substack{0\le C,D\le 2q-1\\ C\equiv 0 (mod\ 2\ )}} \chi(\frac{C^2}{2}+D^2(1-n)+CD(1-n))\sum_{i=0}^{2n-3} S(\frac{D}{2q}-\frac{C+2qi}{4q(n-1)}+\sigma_2(i),\frac{C+2qi}{4q(n-1)})
\end{aligned}$$

where σ_1 and σ_2 are defined in the previous Lemma.

Proof: From (4), we deduce that for $(x+y\epsilon)\in R((q))$,

$$N((q)\cdot(x+y\epsilon)) = N_K(q(x+y\epsilon)) \equiv A^2+B^2+2nAB \pmod q. \tag{11}$$

And (8) implies that for $(x+y\epsilon)\in R((q\omega))$,

$$N((q\omega)\cdot(x+y\epsilon)) = -N_K(q\omega(x+y\epsilon)) \equiv -A^2+B^2(2n-2)+AB(2n-2) \pmod q. \tag{12}$$

Moreover, (5) implies that for $(x+y\epsilon)\in R((q)\cdot\mathbf{b})$,

$$N((q)\mathbf{b}\cdot(x+y\epsilon)) = 2N_K(q(x+y\epsilon)) \equiv \frac{C^2}{2}+\frac{D^2}{2}+nCD \pmod q. \tag{13}$$

Also by (9), we have for $(x+y\epsilon)\in R((q\omega)\cdot\mathbf{b})$

$$\begin{aligned} N((q\omega)\cdot\mathbf{b}\cdot(x+y\epsilon)) &= -2N_K(q\omega(x+y\epsilon))\\ &\equiv -\frac{C^2}{2}+(n-1)D^2+(n-1)CD \pmod q. \end{aligned} \tag{14}$$

By (11)-(14), we can immediately prove above Proposition. $\square$

Finally, we have the following theorem.

Theorem 2.6. *If $h(K)=2$ then*

$$
\begin{aligned}
&\zeta_K(0,\chi)= \\
&\frac{1}{48q^2}\Big[-\sum_{\substack{0\le C,D\le 2q-1\\ C\equiv 0\ (mod\ 2)}}\chi(\frac{C^2}{2}-(n-1)D^2-(n-1)CD)\cdot\Big(3u_{C,D}(n)^2+(6q-12D)u_{C,D}(n) \\
&\quad +3C^2+12CD+12D^2-12D^2n-18Cq-24Dq+24Dnq+12q^2-8nq^2\Big) \\
&+\sum_{\substack{0\le C,D\le 2q-1, C\neq 0\\ C\equiv D\ (mod\ 2)}}\chi(\frac{C^2}{2}+\frac{D^2}{2}+nCD)\Big(12(1-n)CD-12q(1+n)(C+D)+6n(C+D)^2 \\
&\quad +4q^2(2n+3)\Big)+\sum_{\substack{0\le D\le 2q-1\\ D\equiv 0\ (mod 2)}}\chi(\frac{D^2}{2})\Big(24q(1-n)D-12q(1+n)(2q+D) \\
&\quad +6n(2q+D)^2+4q^2(2n+3)\Big) \\
&-\sum_{0\le A,B\le q-1}\chi(A^2+(2-2n)B^2+(2-2n)AB)\cdot 4\Big(3v_{A,B}(n)^2+(3q-12B)v_{A,B}(n)+3A^2 \\
&\quad +12AB+12B^2-12B^2n-9Aq-12Bq+12Bqn+3q^2-2nq^2\Big) \\
&+\sum_{0\le A,B\le q-1}\chi(A^2+B^2+2nAB)\Big(48(1-n)AB-24q(1+n)(A+B)+24n(A+B)^2 \\
&\quad +4q^2(2n+3)\Big)+\sum_{0\le B\le q-1}\chi(B^2)\Big(48q(1-n)B-24q(1+n)(q+B) \\
&\quad +24n(q+B)^2+4q^2(2n+3)\Big)\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
s_{C,D}(n) &= \left\lceil\frac{2(n-1)D-C}{2q}\right\rceil \\
t_{A,B}(n) &= \left\lceil\frac{2(n-1)B-A}{q}\right\rceil \\
u_{C,D}(n) &= C-2D(n-1)+2qs_{C,D}(n) \\
v_{A,B}(n) &= A-2B(n-1)+qt_{A,B}(n).
\end{aligned}
$$

Proof: By the computer work, we have the followings:

(15)

$$\sum_{i=0}^{2n-3} S(\frac{D}{2q} - \frac{C+2qi}{4q(n-1)} + \sigma_2(i), \frac{C+2qi}{4q(n-1)})$$

$$= \frac{1}{24q^2}(3C^2 + 6CD - 6CDn - 6D^2n + 6D^2n^2 - 6Cq - 6Dq + 6Dnq + 6q^2 - 4nq^2 + 6Cqs_{C,D}(n) - 12Dnqs_{C,D}(n) + 6q^2s_{C,D}(n) + 6q^2s_{C,D}(n)^2)$$

$$= \frac{1}{48q^2}(3u_{C,D}(n)^2 + (6q - 12D)u_{C,D}(n) + 3C^2 + 12CD + 12D^2 - 12D^2n - 18Cq - 24Dq + 24Dnq + 12q^2 - 8nq^2)$$

(16)

$$\sum_{i=0}^{2n-3} S(\frac{B}{q} - \frac{A+qi}{2q(n-1)} + \sigma_1(i), \frac{A+qi}{2q(n-1)})$$

$$= \frac{1}{12q^2}(6A^2 + 12AB - 12ABn - 12B^2n + 12B^2n^2 - 6Aq - 6Bq + 6Bnq + 3q^2 - 2nq^2 + 6Aqt_{A,B}(n) - 12Bnqt_{A,B}(n) + 3q^2t_{A,B}(n) + 3q^2t_{A,B}(n)^2)$$

$$= \frac{1}{12q^2}(3v_{A,B}(n)^2 + (3q - 12B)v_{A,B}(n) + 3A^2 + 12AB + 12B^2 - 12B^2n - 9Aq - 12Bq + 12Bqn + 3q^2 - 2nq^2)$$

And a simple computation induce the followings:

(17)

$$S(\frac{C}{2q}, \frac{D}{2q}) = \frac{1}{24q^2}(6(1-n)CD - 6q(1+n)(C+D) + 3n(C+D)^2 + 2q^2(2n+3))$$

(18)

$$S(1, \frac{D}{2q}) = \frac{1}{24q^2}(12q(1-n)D - 6q(1+n)(2q+D) + 3n(2q+D)^2 + 2q^2(2n+3))$$

(19)

$$S(\frac{A}{q}, \frac{B}{q}) = \frac{1}{12q^2}(12(1-n)AB - 6q(1+n)(A+B) + 6n(A+B)^2 + q^2(2n+3))$$

(20)

$$S(1, \frac{B}{q}) = \frac{1}{12q^2}(12q(1-n)B - 6q(1+n)(q+B) + 6n(q+B)^2 + q^2(2n+3))$$

From (15)-(20) and Propositon 2.5, we obtain above theorem. □

Also by defining $A_\chi(n) := 48q^2\zeta_K(0,\chi)$ for $K = \mathbb{Q}(\sqrt{n^2-1})$, we have

Corollary 2.7. *If $h(n^2-1)=2$ and $n=qk+r$ for $0\le r<q$ then*

$$\zeta_K(0,\chi)=\frac{1}{48q^2}(B_\chi(r)k+A_\chi(r))$$

where

$$\begin{aligned}
B_\chi(r) &= \\
&- \sum_{\substack{0\le C,D\le 2q-1\\ C\equiv 0 \ (mod\ 2)}} \chi\Big(\frac{C^2}{2}-(r-1)D^2-(r-1)CD\Big)\cdot\Big(-12D^2q-8q^3+24Dq^2\Big)\\
&+ \sum_{\substack{0\le C,D\le 2q-1, C\neq 0\\ C\equiv D \ (mod\ 2)}} \chi\Big(\frac{C^2}{2}+\frac{D^2}{2}+rCD\Big)\Big(-12qCD-12q^2(C+D)+6q(C+D)^2+8q^3\Big)\\
&+ \sum_{\substack{0\le D\le 2q-1\\ D\equiv 0 \ (mod\ 2\ )}} \chi\Big(\frac{D^2}{2}\Big)\Big(-24q^2D-12q^2(2q+D)+6q(2q+D)^2+8q^3\Big)\\
&- \sum_{0\le A,B\le q-1} \chi(A^2+(2-2r)B^2+(2-2r)AB)\cdot 4\Big(-12B^2q+12Bq^2-2q^3\Big)\\
&+ \sum_{0\le A,B\le q-1} \chi(A^2+B^2+2rAB)\Big(-48qAB-24q^2(A+B)+24q(A+B)^2+8q^3\Big)\\
&+ \sum_{0\le B\le q-1} \chi(B^2)\Big(-48q^2B-24q^2(q+B)+24q(q+B)^2+8q^3\Big)
\end{aligned}$$

Proof: We note that $s_{C,D}(qk+r)=Dk+s_{C,D}(r)$, $t_{C,D}(qk+r)=2Bk+t_{C,D}(qk+r)$,$u_{C,D}(qk+r)=u_{C,D}(r)$ and $v_{C,D}(qk+r)=v_{C,D}(r)$. And from the fact that χ has a conductor q, we directly prove the corollary. $\square$

3. Proof of theorem

Let n^2-1 be a positive square free integer for even n and $K=\mathbb{Q}(\sqrt{n^2-1})$. Let $q>2$ be an integer with $(q,4(n^2-1))=1$, χ an odd primitive character with conductor q, $\chi_D(\cdot)=(\frac{D}{\cdot})$ and L_χ the field generated over $\mathbb{Q}$ by the values $\chi(a)$ $(1\le a\le q)$. From [1], we know that

$$\zeta_K(0,\chi)=\frac{1}{4q^2(n^2-1)}\sum_{a=1}^{q}a\chi(a)\sum_{b=1}^{4q(n^2-1)}b\chi(b)\chi_{4(n^2-1)}(b). \tag{21}$$

And we can define the functions $T_1(r)$, $T_2(r)$ and $T_3(r)$ as follows

$$-175\frac{A_{\chi_1}(r)}{B_{\chi_1}(r)}+r+I_1=T_1(r)+61\mathbb{Z}$$
$$-61\frac{A_{\chi_2}(r)}{B_{\chi_2}(r)}+r+I_2=T_2(r)+1861\mathbb{Z}$$
$$-175\frac{A_{\chi_3}(r)}{B_{\chi_3}(r)}+r+I_3=T_3(r)+1861\mathbb{Z}$$

where the characters and ideals χ_i and I_i are defined in Example 1, Example 3 and Example 4 of Section 4 in [1], respectively for $i=1,2,3$.

Let $U_m=\{a\in\mathbb{Z}|(\frac{a^2-1}{p})=-1$,for any prime p dividing $m\}$. Now, we can find the residue of n modulo p such that $n=qk+r$ and $h(n^2-1)=2$ using (23). Let a_{175} be an residue modulo 175 for a_{175} with $B_{\chi_1}(a_{175})\notin I_1$ and b_{61} be the residue modulo 61 such that

$$b_{61}=T_1(a_{175}).$$

And for b_{61} with $B_{\chi_2}(b_{61})\notin I_2$, let c_{1861} be the residue modulo 1861 for which

$$c_{1861}=T_2(b_{61}).$$

And let d_{1861} be the residue modulo 1861 for which

$$d_{1861}=T_3(a_{175})$$

for a_{175} with $B_{\chi_3}(a_{175})\notin I_3$. Then by computer work, we can check that if $a_{175}\in U_{175}$ then $B_{\chi_1}(a_{175})\notin I_1$ and $B_{\chi_3}(a_{175})\notin I_3$. So we can calculate $T_1(a_{175})$ and $T_3(a_{175})$ for $a_{175}\in U_{175}$. Finally we obtain the following table:

By (21) and Corollary 2.7, if we assume $h(n^2-1)=2$ and $n=qk+r$, then
(22)

$$\frac{1}{48q^2}(B_\chi(r)k+A_\chi(r))=\frac{1}{4q^2(n^2-1)}\sum_{a=1}^{q}a\chi(a)\sum_{b=1}^{4q(n^2-1)}b\chi(b)\chi_{4(n^2-1)}(b).$$

By defining $m_\chi:=\sum_{a=1}^{q}a\chi(a)$, we can rewrite the equation (22) as follows

$$B_\chi(r)k+A_\chi(r)=48q\cdot m_\chi\cdot\Big(\frac{1}{4q(n^2-1)}\sum_{b=1}^{4q(n^2-1)}b\chi(b)\chi_{4(n^2-1)}(b)\Big).$$

Also from [1], we know that $\frac{1}{4q(n^2-1)}\sum_{b=1}^{4q(n^2-1)}b\chi(b)\chi_{4(n^2-1)}(b)$ is an algebraic integer in L_χ. So if we assume I is a prime ideal of L_χ for which $m_\chi\in I$, then

$$B_\chi(r)k+A_\chi(r)\equiv 0 \pmod{I}.$$

Assume that the positive integers q and p satisfy the following condition:

Condition(*): *The integer q is odd, p is an odd prime, and there is an odd prime character χ with conductor q and a prime ideal I of L_χ lying over p such that $m_\chi\in I$ and the residue field of I is the prime field.*

Then for r such that $B_\chi(r)\notin I$, we have

$$n\equiv -q\frac{A_\chi(r)}{B_\chi(r)}+r \pmod{I}.$$

And if the residue field of I is a prime field, then there exists an unique $T(r)\in\{0,1,2,.......p-1\}$ for each r with $B_\chi(r)\notin I$ such that

$$-q\frac{A_\chi(r)}{B_\chi(r)}+r+I=T(r)+p\mathbb{Z}.$$

Moreover

$$n\equiv T(r) \pmod{p} \tag{23}$$

We will denote by $q\to p$, if the integers q and p satisfy Condition(*). In the Section 4 in [1], we can find that

$$175\to 61,\quad 61\to 1861,\quad 175\to 1861.$$

$a_{175} \in U_{175}$	b_{61}	c_{1861}	d_{1861}
± 2	± 2	± 2	± 2
± 7	± 21	± 1214	± 1805
± 12	± 12	± 12	± 12
± 23	± 38	± 286	± 637
± 28	± 54	± 1277	± 412
± 33	± 27	± 34	± 1005
± 37	± 48	± 110	± 208
± 42	± 12	± 12	± 347
± 47	± 51	± 573	± 100
± 58	± 18	± 1388	± 376
± 63	± 35	± 696	± 1177
± 68	± 0	± 0	± 1517
± 72	± 12	± 12	± 262
± 77	± 13	± 1751	± 1034
± 82	± 17	± 709	± 943
± 93	± 44	± 1152	± 918

The following is a class number 2 criteria of $\mathbb{Q}(\sqrt{n^2-1})$ for even n which is needed to prove our main theorem

Lemma 3.1. *Let $4k^2-1$ be a square free with $k > 1$. Then $h(4k^2-1) = 2$ if and only if $2k^2 - 2t^2 - 2t - 1 (0 \le t \le k)$ are primes.*

Proof: See [10]. □

From above table and Lemma 3.1, we can deduce that

Proposition 3.2. *For even integer $n \in U_{175}$ with $n > 2016$, we have $h(n^2-1) > 2$.*

Proof: By above table, we can know that if $n \equiv a_{175} \pmod{175}$ with $a_{175} \in U_{175}$ and $a_{175} \ne \pm 2, \pm 12$, then the residues c_{1861} and d_{1861} modulus 1861 corresponding to a_{175} are not equal. So

$$h(n^2-1) > 2 \quad \text{for} \quad n \not\equiv \pm 2, \pm 12 \pmod{175} \text{ with } n \in U_{175}. \tag{24}$$

If $n = 2k \equiv \pm 2 \pmod{175}$ then from the table, we have $n = 2k \equiv \pm 2 \pmod{61}$. Hence $k = 61l + 1$ or $61l + 60$, for an integer l. So if we take $t_0 = 34$, then $2k^2 - 2t_0^2 - 2t_0 - 1$ is $61(122l^2 + 4l - 39)$ or $61(122l^2 + 240l + 79)$. So Lemma 3.1 implies that

$$h(n^2-1) > 2 \text{ for } n > 68 \text{ and } n \equiv \pm 2 \pmod{175}. \tag{25}$$

Also if $n = 2k \equiv \pm 12 \pmod{175}$ then we have $n = 2k \equiv \pm 12 \pmod{1861}$, by the table. So the integer k is $1861l + 6$ or $1861l + 1855$, for some integer l. Hence if we take $t_0 = 1008$ then $2k^2 - 2t_0^2 - 2t_0 - 1$ is

$1861(3722l^2+24l-1093)$ or $1861(3722l^2+7420l+2605)$. By applying Lemma 3.1, we obtain

$$h(n^2-1) > 2 \text{ for } n > 2016 \text{ and } n \equiv \pm 12 \ (\text{mod}175). \tag{26}$$

From (24)-(26), we complete the proof. $\square$

We also can find the upper bound of even n with $n \notin U_{175}$ and $h(n^2-2) = 2$.

Proposition 3.3. *$h(n^2-1) > 2$, for even n such that $n \notin U_{175}$ and $n > 20$.*

Proof: We note that $2k^2 - 2t_0^2 - 2t_0 - 1 = 0 \ (\text{mod } p)$ if and only if $(2k)^2 - 1 - (2t_0+1)^2 = 0 \ (\text{mod } p)$, for odd prime p. So if $n = 2k \notin U_{175}$ then there exists an integer $t_0 \geq 0$ such that $2k^2 - 2t_0^2 - 2t_0 - 1 = 0 \ (\text{mod } 5)$ or $2k^2 - 2t_0^2 - 2t_0 - 1 = 0 \ (\text{mod } 7)$. For example, if $k = 5l+2$ for integer l and $t_0 = 7$ then $2k^2 - 2t_0^2 - 2t_0 - 1 = 5(10l^2+8l-21)$ is not prime, since for any integer l,

$$10l^2 + 8l - 21 \neq \pm 1.$$

Thus Lemma 3.1 implies that if $k = 2 \ (\text{mod } 5)$ and $k > 7$ then $h((2k)^2-1) \neq 2$. Applying this method, we can prove other cases. $\square$

We conclude this section with the proofs of Theorem 1.1 and 1.2.

Proof of Theorem 1.1: By the Proposition 3.2 3.3, we have

$$h(n^2-1) > 2 \text{ for } n > 2016$$

where n^2-1 is square free. And in [16], Mollin and Williams shows that if $d = n^2-1$ is a square free integer with $n < 5000$, then

$$h(d) = 2, \quad \text{only for } d = 15, 35, 143.$$

By combining above two results, we prove the theorem.

Proof of Theorem 1.2: From Theorem 1.2 of [11], we inform that for a even square free integer $d = n^2+1$,

$$h(d) = 2 \text{ if and only if } d = 10, 26, 122, 362.$$

Also, by Corollary 3.11 of [10], we know that for a square free integer $d = n^2+1$ with $d \equiv 1$ modulo 8,

$$h(d) = 2 \text{ if and only if } d = 65.$$

Hence Theorem 1.2 is a direct consequence of Theorem 1.1 and Theorem 1.2 of [11] and Corollary 3.11 of [10].

4. Appendix

In this section, we will provide the MATHEMATICA program to evaluate the values $T_i(r)$ in section 3.
(The function `f[x_,y_]` computes the logarithm of x with base 2 modulo y. And `g[x_,y_]` computes the logarithm of x with base 3 modulo y.)

```
f[x_, y_] := ( j = 0; m = Mod[x, y];
          If [Mod[x, y] == 0, Return[0]];
          While[ Mod[m, y] >1, m = Mod[m*2, y] ; j = j + 1];
          Return[y - 1 - j]);
g[x_, y_] := (j = 0; m = Mod[x, y];
      If[Mod[x, y] == 0, Return[0]];
      While[ Mod[m, y] >1, m = Mod[m*3, y]; j = j + 1];
      Return[y - 1 - j]);
g7[x_] := g[x, 7];
f25[x_] := (j = 0; m = Mod[x, 25];
        If[ Mod[m, 5] == 0, Return[0]];
        While[Mod[m, 25] >1, m = Mod[m*2, 25]; j = j + 1];
        Return[20 - j]);
f61[x_] := f[x, 61];
```

(The function `iv[x_,y]` computes the multiplicative inverse of x modulo y.)

```
iv[x_, y_] := (
      i = 1;
      While[Mod[ i*x, y] >1, i++];
      Return[i] );
```

(The functions `chi[a_]` computes $\chi_i(a)$ modulo I_i, for $i = 1, 2, 3, 4$)

```
ch1[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[8, f25[Mod[a, 25]], 61]*
      PowerMod[47, g7[Mod[a, 7] ], 61], 61]]);
ch2[a_] := (If[Mod[a, 61] == 0, Return[0]];
      Return[PowerMod[1833, f61[Mod[a, 61]], 1861]]);
ch3[a_] := (If [Mod[a, 5] == 0 || Mod[a, 7] == 0, Return[0]];
      Return[Mod[PowerMod[380, f25[Mod[a, 25]], 1861]*
      PowerMod[1406, g7[Mod[a, 7] ], 1861], 1861]]);
```

(The followings are needed to compute $A_{\chi_i}(r)$ and $B_{\chi_i}(r)$ modulo I_i.)

```
s[c_, d_, n_, q_] := -Floor[(c - 2(n - 1)d)/(2q)]
t[a_, b_, n_, q_] := -Floor[(a - 2(n - 1)b)/(q)]
u[c_, d_, n_, q_] := c - 2d(n - 1) + 2 q s[c, d, n, q]
```

```
v[a_, b_, n_, q_] := a - 2b(n - 1) + q t[c, d, n, q]
SA1[c_, d_, n_, q_] :=
    3u[c, d, n, q]^2 + (6q - 12d)u[c, d, n, q] + 3c^2 + 12c d +
    12 d^2 - 12d^2n - 18c q - 24d q + 24 d n q + 12q^2 - 8n q^2
SA2[c_, d_, n_, q_] :=
 12(1 - n)c d - 12 q(1 + n)(c + d) + 6n(c + d)^2 + 4q^2(2n + 3)
SA3[c_, d_, n_, q_] :=
 24 q(1 - n)d - 12 q (1 + n)(2q + d) + 6n(2q + d)^2 + 4q^2(2n + 3)
SA4[a_, b_, n_, q_] := 4(3 v[a, b, n, q]^2 + (3q - 12b)v[a, b, n, q] +
  3a^2 + 12a b + 12 b^2 - 12b^2n - 9 a q - 12 b q + 12 b q n
  + 3 q^2 - 2n q^2)
SA5[a_, b_, n_, q_] := 48(1 - n)a b - 24q(1 + n)(a + b) +
     24 n(a + b)^2 + 4q^2(2n + 3)
SA6[a_, b_, n_, q_] := 48 q (1 - n)b - 24q(1 + n)(q + b) +
     24 n(q + b)^2 + 4q^2(2n + 3)
SB1[c_, d_, n_, q_] := -12d^2q + 24d q^2 - 8q^3
SB2[c_, d_, n_, q_] := -12 q c d - 12 q^2(c + d) + 6q(c + d)^2 + 8q^3
SB3[c_, d_, n_, q_] :=
  -24 q^2 d - 12 q^2(2q + d) + 6q(2q + d)^2 + 8q^3
SB4[a_, d_, n_, q_] :=
  (-12 b^2 q + 12 b q^2 - 2 q^3) 4
SB5[a_, b_, n_, q_] :=
  -48q a b - 24 q^2(a + b) + 24q(a + b)^2 + 8q^3
SB6[a_, b_, n_, q_] :=
  -48 q^2b - 24 q^2(q + b) + 24 q (q + b)^2 + 8q^3
```

(The functions `RAi[a_]` and `RBi[a_]` computes $A_{\chi_i}(a)$ and $B_{\chi_i}(a)$ modulo I_i respectively, where `q` is the conductor for the character χ_i for $i = 1, 2, 3, 4$.)

```
RA1[q_, n_] :=
-Mod[Sum[Mod[Sum[Mod[ch1[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
 SA1[2c, d, n, q], 61], {c, 0, q - 1}], 61], {d, 0, 2q - 1}], 61]
+Mod[Sum[Mod[Sum[Mod[ch1[2c^2 + 2d^2 + 4 n c d]*SA2[2c,
            2d, n, q], 61], {c, 1, q - 1}], 61], {d, 0, q - 1}], 61]
+Mod[Sum[Mod[Sum[Mod[ch1[2c^2 + 2d^2 + 2c + 2d + 1+
     n(2c + 1)(2d + 1)]*SA2[2c + 1, 2d + 1, n, q], 61],
     {c, 0, q - 1}], 61], {d, 0, q - 1}], 61]
+Mod[Sum[Mod[ch1[2 d^2] SA3[0, 2 d, n, q], 61], {d, 0, q - 1}], 61]
-Mod[Sum[Mod[Sum[Mod[ch1[a^2 + (2 - 2n)b^2 + (2 - 2n)a
    b]SA4[a, b, n, q], 61], {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
+Mod[Sum[Mod[Sum[Mod[ch1[a^2 + b^2 + 2n a b]*SA5[a, b, n, q],
               61], {a, 1, q - 1}], 61], {b, 0, q - 1}], 61]
```

```
+Mod[Sum[Mod[ch1[b^2]SA6[0, b, n, q], 61], {b, 0, q - 1}], 61]

RB1[q_, n_] :=
-Mod[Sum[Mod[Sum[Mod[ch1[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
   SB1[2c, d, n, q], 61], {c, 0, q - 1}], 61], {d, 0, 2q - 1}], 61]
+Mod[Sum[Mod[Sum[Mod[ch1[2c^2 + 2d^2 + 4 n c d]*
   SB2[2c, 2d, n, q], 61], {c, 1, q - 1}], 61], {d, 0, q - 1}], 61]
+Mod[Sum[Mod[Sum[
    Mod[ch1[2c^2 + 2d^2 + 2c + 2d +1+ n(2c + 1)(2d + 1)]*
    SB2[2c + 1, 2d + 1, n, q], 61], {c, 0, q - 1}], 61], {d, 0, q - 1}], 61]
+Mod[Sum[Mod[ch1[2 d^2] SB3[0, 2 d, n, q], 61], {d, 0, q - 1}], 61]
-Mod[Sum[Mod[Sum[Mod[ch1[a^2 + (2 - 2n)b^2 + (2 - 2n)a b]SB4[a, b,
    n, q], 61], {a, 0, q - 1}], 61], {b, 0, q - 1}], 61]
+Mod[Sum[Mod[Sum[Mod[ch1[a^2 + b^2 + 2n a b]*SB5[a, b, n, q], 61]
   , {a, 1, q - 1}], 61], {b, 0, q - 1}], 61]
+Mod[Sum[Mod[ch1[b^2]SB6[0, b, n, q], 61], {b, 0, q - 1}], 61]

RA2[q_, n_] :=
-Mod[Sum[Mod[Sum[Mod[ch2[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
   SA1[2c, d, n, q], 1861], {c, 0, q - 1}], 1861], {d, 0, 2q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch2[2c^2 + 2d^2 + 4 n c d]*SA2[2c,
            2d, n, q], 1861], {c, 1, q - 1}], 1861], {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[
      Sum[Mod[ch2[2c^2 + 2d^2 + 2c + 2d +1+ n(2c + 1)(2d + 1)]*
      SA2[2c + 1, 2d + 1, n, q], 1861], {c, 0, q - 1}], 1861], {d, 0,
              q - 1}], 1861]
+Mod[Sum[Mod[ch2[2 d^2] SA3[0, 2 d, n, q], 1861], {d, 0, q - 1}], 1861]
-Mod[Sum[Mod[Sum[Mod[ch2[a^2 + (2 - 2n)b^2 + (2 - 2n)a b]
    SA4[a, b, n, q], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch2[a^2 + b^2 + 2n a b]*SA5[a, b, n,
              q], 1861], {a, 1, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch2[b^2]SA6[0, b, n, q], 1861], {b, 0, q - 1}], 1861]

RB2[q_, n_] :=
-Mod[ Sum[Mod[Sum[Mod[ch2[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
   SB1[2c, d, n, q], 1861], {c, 0, q - 1}], 1861], {d, 0, 2q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch2[2c^2 + 2d^2 + 4 n c d]*SB2[2c,
    2d, n, q], 1861], {c, 1, q - 1}], 1861], {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch2[2c^2 + 2d^2 + 2c +
    2d +1+ n(2c + 1)(2d + 1)]*SB2[2c + 1, 2d + 1, n, q], 1861],
    {c, 0, q - 1}], 1861], {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch2[2 d^2] SB3[0, 2 d, n, q], 1861], {d, 0, q - 1}], 1861]
```

```
-Mod[Sum[Mod[Sum[Mod[ch2[a^2 + (2 - 2n)b^2 + (2 - 2n)a b]SB4[a,
      b, n, q], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch2[a^2 + b^2 + 2n a b]*SB5[a, b, n, q],
             1861], {a, 1, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch2[b^2]SB6[0, b, n, q], 1861], {b, 0, q - 1}], 1861]

RA3[q_, n_] :=
-Mod[Sum[Mod[Sum[Mod[ch3[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
    SA1[2c, d, n, q],1861], {c, 0, q - 1}], 1861], {d, 0, 2q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch3[2c^2 + 2d^2 + 4 n c d]*SA2[2c,
             2d, n, q], 1861], {c, 1, q - 1}], 1861], {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[
  Sum[Mod[ch3[2c^2 + 2d^2 + 2c + 2d +1+ n(2c + 1)(2d + 1)]*
  SA2[2c + 1, 2d + 1, n, q], 1861], {c, 0, q - 1}], 1861],
     {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch3[2 d^2] SA3[0, 2 d, n, q], 1861], {d, 0, q - 1}], 1861]
-Mod[Sum[Mod[Sum[Mod[ch3[a^2 + (2 - 2n)b^2 + (2 - 2n)a b]
    SA4[a, b, n, q], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch3[a^2 + b^2 + 2n a b]*SA5[a, b, n,
               q], 1861], {a, 1, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch3[b^2]SA6[0, b, n, q], 1861], {b, 0, q - 1}], 1861]

RB3[q_, n_] :=
-Mod[ Sum[Mod[Sum[Mod[ch3[2 c^2 - (n - 1) d^2 - 2(n - 1) c d]*
  SB1[2c, d, n, q],1861], {c, 0, q - 1}], 1861], {d, 0, 2q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch3[2c^2 + 2d^2 + 4 n c d]*SB2[2c,
    2d, n, q], 1861], {c, 1, q - 1}], 1861], {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[
   Sum[Mod[ch3[2c^2 + 2d^2 + 2c + 2d +1+ n(2c + 1)(2d + 1)]*
     SB2[2c + 1, 2d + 1, n, q], 1861], {c, 0, q - 1}], 1861],
     {d, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch3[2 d^2] SB3[0, 2 d, n, q], 1861], {d, 0, q - 1}], 1861]
-Mod[Sum[Mod[Sum[Mod[ch3[a^2 + (2 - 2n)b^2 + (2 - 2n)a b]SB4[a,
      b, n, q], 1861], {a, 0, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[Sum[Mod[ch3[a^2 + b^2 + 2n a b]*SB5[a, b, n, q],
             1861], {a, 1, q - 1}], 1861], {b, 0, q - 1}], 1861]
+Mod[Sum[Mod[ch3[b^2]SB6[0, b, n, q], 1861], {b, 0, q - 1}], 1861]
```

(The functions `Ti[r_]` compute $T_i(r)$, for i=1,2,3.)

```
T1[r_] := Mod[-RA1[175, r]*175*iv[RB1[175, r], 61] + r, 61];
T2[r_] := Mod[-RA2[61, r]*61*iv[RB2[61, r], 1861] + r, 1861];
T3[r_] := Mod[-RA3[175, r]*175*iv[RB3[175, r], 1861] + r, 1861];
```

(The followings are the result for our table)

```
Print[{T1[2], T1[7], T1[12], T1[23], T1[28], T1[33], T1[37], T1[42],
T1[47], T1[58], T1[63], T1[68], T1[72], T1[77], T1[82], T1[93]}]
{2, 21, 12, 38, 54, 27, 48, 12, 51, 18, 35, 0, 12, 13, 17, 44}
```

```
Print[{T2[2], T2[21], T2[12], T2[38], T2[54], T2[27], T2[48], T2[12],
T2[51], T2[18], T2[35], T2[0], T2[12], T2[13], T2[17], T2[44]}]
{2, 1214, 12, 286, 1277, 34, 110, 12, 573, 1388, 696, 0,
12, 1751, 709, 1152}
```

```
Print[{T3[2], T3[7], T3[12], T3[23], T3[28], T3[33], T3[37], T3[42],
T3[47], T3[58], T3[63], T3[68], T3[72], T3[77], T3[82], T3[93]}]
{2, 1805, 12, 637, 412, 1005, 208, 347, 100, 376, 1177, 1517, 262,
1034, 943, 918}
```

References

[1] A. Biró, *Yokoi's conjecture,* Acta Arith. **106** (2003), 85–104.

[2] A. Biró, *Chowla's conjecture,* Acta Arith. **107** (2003), 179–194.

[3] S. Chowla and J. Friedlandler, *Class numbers and quadratic residues,* Glasgow Math. J. **17** (1976), 47–52.

[4] G. Janusz, *Algebraic number fields,* Amer. Math. Soc., Graduate Studies in Math. **7** (1996).

[5] R. A. Mollin, *Class number one criteria for real quadratic fields. I,* Proc. Japan Acad. **63** (1987), 121–125.

[6] R. A. Mollin, *Class number one criteria for real quadratic fields. II,* Proc. Japan Acad. **63** (1987), 162–164.

[7] T. Shintani, *On evaulation of zeta funtions of totally real algebraic number fields at non-positive integers,* J. Fac. Sci. Univ. Tokyo. **63** (1976), 393–417.

[8] T. Shintani, *On special values of zeta functions of totally real algebraic number fields,* in: Proc. Internat. Congress of Math., Helsinki, 1978, 591–597.

[9] H. Yokoi, *Class number one problem for certain kind of real quadratic fields,* in: Proc. Internat. Conf. (Katata, 1986), Nagoya Univ., Nagoya, 1986, 125–137.

[10] Dongho Byeon and Hyun Kwang Kim, *Class Number 2 Criteria for Real Quadratic Fields of Richaud-Degert Type,* Journal of Number Theory. **62** (1997), 257–272.

[11] D. Byeon and J. Lee, *Class number 2 problem for certain real quadratic fields of Richaud-Degert type*, to appear in Journal of Number Theory

[12] D. Byeon and M. Kim, and J. Lee *Mollin's conjecture*, Acta Arithmetica (2007)

[13] Mollin, R.A., and Williams,H.C., *Prime producing quadratic polynomials and real quadratic fields of calss number one, in Number Theory (J.-M. DeKoninck and C.Levesque (eds.))*, Walter de Gruyter, Berlin (1989),654-663

[14] M. G. Leu, "On a determination of certain real quadratic fields of class number two," Journal of Number Theory **33** (1989), 101–106.

[15] R. A. Mollin, *Applications of a new class number two criterion for real quadratic fields,* Computational Number Theory, Walter de Gruyter & Co., Berlin. New York, (1991), 83–94.

[16] R. A. Mollin and H. C. Williams, *On a solution of a class number two problem for a family of real quadratic fields,* in "Computational Number Theory", Walter de Gruyter, Berlin. New York, (1991), 95–101.
[17] J. Lee, *A complete determination of wide Richaud-Degert type which is not 5 modulo 8 with class number 1*, Preprint.
[18] A.Baker, *Transcendental Number Theory*, Cambridge Univ. Press, 1990.
[19] K.Heegner, *Diophantische Analysis und Modulfunktionen*, Math. Zeit. **56** (1952),227-253
[20] H.Stark, *A complete determination of the complex quadratic fields with class number one*, Michigan Math.J.**14**(1967)1-27

E-mail address: `lee9311@snu.ac.kr`

Department of Mathematics, Seoul National University Seoul 151-747, Korea