MODULAR FORMS ARISING FROM DIVISOR FUNCTIONS

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ABSTRACT. For some infinite families of modular forms we provide the explicit formulas for their Fourier coefficients by using the theory of basic hypergeometric series (Proposition 1). By means of these modular forms we find the bases of the vector spaces of modular forms for some levels.

1. INTRODUCTION

Modular forms appear in many areas of number theory. In particular their Fourier coefficients provide us with many number theoretical properties such as the partition function, the number of representations of integers by quadratic forms, the number of points of an elliptic curve over a finite field, Fermat last theorem and so on. In this paper we investigate certain infinite families of modular forms whose Fourier coefficients are expressed in terms of divisor functions.

In the theory of basic hypergeometric series there are many useful identities between the q-products and the q-series. We first notice that most of the q-products under consideration are indeed modular forms up to trivial factors, and hence the identities can be restated as the formulas for the Fourier coefficients of such modular forms. In this way we get infinite families of modular forms whose Fourier coefficients are given by finite sums of divisor functions (Theorem 2). This result seems to be interesting in itself because these modular forms are holomorphic whose zeros are supported only at the cusps. In addition we can find by utilizing those modular forms the bases of the vector spaces of holomorphic modular forms for some small levels (Proposition 5) and its applications.

2. PRELIMINARIES

Let $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\ \tau > 0\}$ be the complex upper half plane, and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. Then $GL_2^+(\mathbb{Q})$ acts on $\mathfrak{H}^*$ by linear fractional transformation $\alpha(\tau) = \frac{a\tau+b}{c\tau+d}$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$.

For a positive integer N we define the principal congruence subgroup $\Gamma(N)$ of level N by

$$\Gamma(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N\},$$

and any subgroup of $SL_2(\mathbb{Z})$ containing some principal congruence subgroup is called a congruence subgroup. We usually use the congruence subgroups $\Gamma_0(N)$, $\Gamma^0(N)$, $\Gamma_1(N)$ and $\Gamma^1(N)$ defined as follows: $\Gamma_0(N)$ (respectively, $\Gamma^0(N)$) consists of all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that

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$c \equiv 0 \bmod N$ (respectively, $b \equiv 0 \bmod N$), and $\Gamma_1(N)$ (respectively, $\Gamma^1(N)$) consists of all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that $a \equiv d \equiv 1 \bmod N$ and $c \equiv 0 \bmod N$ (respectively, $a \equiv d \equiv 1 \bmod N$ and $b \equiv 0 \bmod N$).

Let k be an integer and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$. For a meromorphic function f on $\mathfrak{H}$, we define a meromorphic function $f|_k\alpha$ on $\mathfrak{H}$ by

$$(f|_k\alpha)(\tau) = (\det \alpha)^{\frac{k}{2}}(c\tau + d)^{-k} f(\alpha(\tau)).$$

Now for a congruence subgroup Γ and an integer k, a meromorphic function f on $\mathfrak{H}$ is called a modular form of weight k for Γ if the following two conditions hold:

(1) $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

(2) f is meromorphic at all cusps of Γ.

The $\mathbb{C}$-vector space of holomorphic modular forms (respectively, cusp forms) of weight k for Γ is denoted by $M_k(\Gamma)$ (respectively, $S_k(\Gamma)$). Throughout this paper we also use the standard notations $q = e^{2\pi i\tau}$, $q_N = e^{2\pi i\tau/N}$ and $\zeta_N = e^{2\pi i/N}$ for a positive integer N.

We first briefly recall the Klein forms which will be mainly used in this article. We refer to [2] for more details. For any lattice $L \subset \mathbb{C}$ and $z \in \mathbb{C}$, we define the Weierstrass σ-function by

$$\sigma(z; L) = z \prod_{\omega \in L - \{0\}} (1 - \frac{z}{\omega}) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2}$$

which is holomorphic with only simple zeros at all points $z \in L$. We further define the Weierstrass ζ-function by logarithmic derivative of the Weierstrass σ-function, i.e.,

$$\zeta(z; L) = \frac{\sigma'(z; L)}{\sigma(z; L)} = \frac{1}{z} + \sum_{\omega \in L - \{0\}} (\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2})$$

which is meromorphic with only simple poles at all points $z \in L$. It is easy to see that the Weierstrass σ-function (respectively, the Weierstrass ζ-function) is homogeneous of degree 1 (respectively, -1), that is, $\sigma(\lambda z; \lambda L) = \lambda \sigma(z; L)$ (respectively, $\zeta(\lambda z; \lambda L) = \lambda^{-1}\zeta(z; L)$) for any $\lambda \in \mathbb{C}^\times$. Note that $\zeta'(z; L) = -\wp(z; L)$ where

$$\wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} (\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2})$$

is the Weierstrass $\wp$-function. Since the Weierstrass $\wp$-function is an elliptic function, namely $\wp(z + \omega; L) = \wp(z; L)$ for $\omega \in L$, we derive that $\frac{d}{dz}(\zeta(z + \omega; L) - \zeta(z; L)) = 0$ for $\omega \in L$. This means that $\zeta(z + \omega; L) - \zeta(z; L)$ depends only on $\omega \in L$, not on $z \in \mathbb{C}$. Thus we may define $\eta(\omega; L) = \zeta(z + \omega; L) - \zeta(z; L)$ for all $\omega \in L$. Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. For $z = a_1\omega_1 + a_2\omega_2$ with $a_1, a_2 \in \mathbb{R}$ we define the Weierstrass η-function by

$$\eta(z; L) = a_1\eta(\omega_1; L) + a_2\eta(\omega_2; L).$$

Then it is easy to see that the Weierstrass η-function $\eta(z; L)$ is well-defined, in other words it does not depend on the choice of the basis $\{\omega_1, \omega_2\}$ of L, and $\eta(z; L)$ is $\mathbb{R}$-linear so that $\eta(rz; L) = r\eta(z; L)$ for $r \in \mathbb{R}$. Note that since the Weierstrass ζ-function is homogeneous of degree -1, so is the Weierstrass η-function. We now define the Klein form by

$$K(z; L) = e^{-\eta(z;L)z/2}\sigma(z; L).$$

Let $a = (a_1\ a_2) \in \mathbb{R}^2$ and $\tau \in \mathfrak{H}$. We further define $K_a(\tau) = K(a_1\tau + a_2; \mathbb{Z}\tau + \mathbb{Z})$ which is also called the Klein form by abuse of terminology. Obviously $K_a(\tau)$ for $a \in \mathbb{Z}^2$ is the constant function 0, thus we assume hereafter that $a \in \mathbb{R}^2 - \mathbb{Z}^2$ while considering the Klein forms. We also see that $K_a(\tau)$ for $a \in \mathbb{R}^2 - \mathbb{Z}^2$ is holomorphic and nonvanishing on $\mathfrak{H}$ and the Klein form is homogeneous of degree 1, i.e., $K(\lambda z; \lambda L) = \lambda K(z; L)$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $a = (a_1\ a_2) \in \mathbb{R}^2 - \mathbb{Z}^2$. Then the Klein form satisfies the following well-known properties (see [2]).

(**K0**) $K_{-a}(\tau) = -K_a(\tau)$.

(**K1**) $(K_a|_{-1}\gamma)(\tau) = K_{a\gamma}(\tau)$.

(**K2**) For $b = (b_1\ b_2) \in \mathbb{Z}^2$ we have

$$K_{a+b}(\tau) = \varepsilon(a, b)K_a(\tau),$$

where $\varepsilon(a, b) = (-1)^{b_1b_2+b_1+b_2}e^{\pi i(b_2a_1-b_1a_2)}$.

(**K3**) For $a = (\frac{r}{N}\ \frac{s}{N}) \in \frac{1}{N}\mathbb{Z}^2 - \mathbb{Z}^2$ and $\gamma \in \Gamma(N)$ with $N > 1$, we have

$$(K_a|_{-1}\gamma)(\tau) = \varepsilon_a(\gamma)K_a(\tau),$$

where $\varepsilon_a(\gamma) = -(-1)^{(\frac{a-1}{N}r+\frac{c}{N}s+1)(\frac{b}{N}r+\frac{d-1}{N}s+1)}e^{\pi i(br^2+(d-a)rs-cs^2)/N^2}$.

(**K4**) For $z = a_1\tau + a_2$ with $a = (a_1\ a_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, we let $q = e^{2\pi i\tau}$ and $q_z = e^{2\pi iz} = e^{2\pi ia_2}e^{2\pi ia_1\tau}$. Then we have

$$K_a(\tau) = -\frac{1}{2\pi i}e^{\pi ia_2(a_1-1)}q^{\frac{1}{2}a_1(a_1-1)}(1 - q_z)\prod_{n=1}^{\infty}\frac{(1 - q^nq_z)(1 - q^nq_z^{-1})}{(1 - q^n)^2}$$

and $ord_qK_a(\tau) = \frac{1}{2}\langle a_1\rangle(\langle a_1\rangle - 1)$, where $\langle a_1\rangle$ denotes the rational number such that $0 \le \langle a_1\rangle < 1$ and $a_1 - \langle a_1\rangle \in \mathbb{Z}$.

(**K5**) Let $f(\tau) = \prod_a K_a^{m(a)}(\tau)$ be a finite quotient of Klein forms with $a = (\frac{r}{N}\ \frac{s}{N}) \in \frac{1}{N}\mathbb{Z}^2 - \mathbb{Z}^2$ for $N > 1$, and let $k = -\sum_a m(a)$. Then $f(\tau)$ is a modular form of weight k for $\Gamma(N)$ if and only if

$$\begin{cases} \sum_a m(a)r^2 \equiv \sum_a m(a)s^2 \equiv \sum_a m(a)rs \equiv 0 \bmod N & \text{if } N \text{ is odd} \\ \sum_a m(a)r^2 \equiv \sum_a m(a)s^2 \equiv 0 \bmod 2N,\ \sum_a m(a)rs \equiv 0 \bmod N & \text{if } N \text{ is even.} \end{cases}$$

3. Explicit formulas for the Fourier coefficients

For nonzero integers n, m, r, r_1, $\cdots$, r_l, we define

$$E_r(n; m) = \sum_{\substack{d|n,\, d>0 \\ d\equiv r \bmod m}} 1 - \sum_{\substack{d|n,\, d>0 \\ d\equiv -r \bmod m}} 1,$$

$$E_{r_1,\cdots,r_l}(n; m) = \sum_{i=1}^{l} E_{r_i}(n; m).$$

First we recall the following necessary identities which can be derived by using the theory of basic hypergeometric series ([1]).

Proposition 1. *(1) For $p>1$, $0<r<p$ with $(r,2p)=1$, we have*

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2}{(1-q^{pn-r})(1-q^{pn-p+r})}=\sum_{n=0}^{\infty}E_r(2pn+r(p-r);2p)q^n.$$

(2) For $p>2$, $r>0$, $s>0$ with $r+s<p$, $(r,p)=1$, and for $\alpha=e^{2\pi ia/k}$ with $k>0$, $(a,k)=1$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2(1-\alpha^{-1}q^{pn-(r+s)})(1-\alpha q^{pn-(p-r-s)})}{(1-q^{pn-r})(1-q^{pn-p+r})(1-\alpha^{-1}q^{pn-s})(1-\alpha q^{pn-p+s})}$$

$$=\sum_{N=0}^{\infty}\left(\sum_{m=0}^{k-1}\alpha^m E_{mp+r}(pN+rs;kp)\right)q^N.$$

(3) For $p>1$, $r>0$ with $r<p$, $(r,p)=1$, and for $\alpha=e^{2\pi ia/k}$ with $k>0$, $(a,k)=1$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2(1-\alpha^{-1}q^{pn-r})(1-\alpha q^{pn-p+r})}{(1-q^{pn-r})(1-q^{pn-p+r})(1-\alpha^{-1}q^{pn})(1-\alpha q^{pn})}$$

$$=1+(1-\alpha)\sum_{N=1}^{\infty}\left(\sum_{m=0}^{k-1}\alpha^m E_{mp+r}(N;kp)\right)q^N.$$

(4) For $p>1$, $r>0$ with $r<p$, $(r,p)=1$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2(1-q^{pn-r})(1-q^{pn-p+r})}{(1+q^{pn})^2(1+q^{pn-r})(1+q^{pn-p+r})}$$

$$=1-2\sum_{N=1}^{\infty}\left(E_{r,p-r}(N;2p)-2E_{r,p-r}(\frac{N}{2};2p)\right)q^N.$$

(5) For $p>1$, $r>0$ with $r<p$, $(r,p)=1$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^{pn})(1-q^{2pn})(1-q^{4pn-p-2r})(1-q^{4pn-3p+2r})}{(1-q^{pn-r})(1-q^{pn-p+r})}$$

$$=1+\sum_{N=1}^{\infty}E_{r,p-r,p+r,2p-r}(N;4p)q^N.$$

(6) For $p>2$, $r>0$ with $r<p$, $(r,p)=1$, we have

$$q^r\prod_{n=1}^{\infty}\frac{(1-q^{pn})(1-q^{2pn})(1-q^{4pn-p+2r})(1-q^{4pn-3p-2r})}{(1-q^{pn-r})(1-q^{pn-p+r})}$$

$$=\sum_{N=1}^{\infty}E_{r,2p-r,3p-r,3p+r}(N;4p)q^N.$$

(7) For $p>3$, $\omega=e^{2\pi i/p}$ and $a\not\equiv \pm s \bmod p$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^4(1-\omega^{a+s}q^n)(1-\omega^{-a-s}q^n)(1-\omega^{a-s}q^n)(1-\omega^{s-a}q^n)}{(1-\omega^a q^n)^2(1-\omega^{-a}q^n)^2(1-\omega^s q^n)^2(1-\omega^{-s}q^n)^2}$$

$$=1+2\omega^{-s}\frac{(1-\omega^s)^2(1-\omega^a)^2}{(1-\omega^{a+s})(1-\omega^{a-s})}\sum_{n=1}^{\infty}\left(\cos\frac{2\pi sn}{p}-\cos\frac{2\pi an}{p}\right)\frac{nq^n}{1-q^n}.$$

(8) For $p>1$, $r>0$ with $r<p$, $(r,p)=1$, we have

$$q^r\left(\prod_{n=1}^{\infty}\frac{(1-q^{pn})^2}{(1-q^{pn-r})(1-q^{pn-p+r})}\right)^2=\frac{1}{p}\sum_{N=1}^{\infty}\left(\sum_{\substack{d|pN-r^2,\, d>0\\ d\equiv r \bmod p}}\left(d+\frac{pN-r^2}{d}\right)\right)q^N.$$

(9) For $p>2$, $\omega=e^{2\pi i/p}$, $2a\not\equiv 0 \bmod p$, we have

$$\prod_{n=1}^{\infty}\frac{(1-q^n)^6(1-\omega^{2a}q^n)(1-\omega^{-2a}q^n)}{(1-\omega^a q^n)^4(1-\omega^{-a}q^n)^4}$$

$$=1+2i\omega^{-a}\frac{(1-\omega^a)^4}{(1-\omega^{2a})}\sum_{n=1}^{\infty}\left(\sin\frac{2\pi an}{p}\right)\frac{n^2q^n}{1-q^n}.$$

Proof. We refer to [1, (10.6), (19.4), (19.5), (31.15), (32.48), (32.49), (18.85), (31.5) and (18.87)] respectively. □

Now we restate the above identities in terms of Klein forms. For example, the q-product in Proposition 1 (1) is, up to a trivial factor, equal to $1/K_{(\frac{r}{p}\ 0)}(p\tau)$ by (**K4**). Hence the corresponding identity between a q-product and a q-series can be readily interpreted as an explicit formula for the Fourier coefficients of the modular form $1/K_{(\frac{r}{p}\ 0)}(\tau)$ of weight 1 for some sufficiently large level.

Theorem 2. *(1) For $N>1$, let $0<r<N$ satisfy $(r,2N)=1$. Then we have*

$$K^{-1}_{(\frac{r}{N}\ 0)}(\tau)=-2\pi i\sum_{\substack{n\geq r(N-r)\\ n\equiv r(N-r) \bmod 2N}}E_r(n;2N)q^n_{2N^2}.$$

Furthermore, $K^{-1}_{(\frac{r}{N}\ 0)}(\tau)$ is a holomorphic modular form of weight 1 for $\Gamma^1(2N)\cap\Gamma^0(2N^2)$ if N is even, and for $\Gamma^1(N)\cap\Gamma^0(N^2)$ if N is odd.

(2) For $N>2$, let $r_1>0$, $r_2>0$ satisfy $\frac{N}{(N,r_1)}>2$, $r_1+r_2<N$, $(N,r_1)|r_2$. Then we have

$$\frac{K_{(\frac{r_1+r_2}{N}\ \frac{s}{N})}(\tau)}{K_{(\frac{r_1}{N}\ 0)}(\tau)K_{(\frac{r_2}{N}\ \frac{s}{N})}(\tau)}=-2\pi i e^{\pi i r_1 s/N^2}.$$

$$\sum_{\substack{n\geq\frac{r_1r_2}{(N,r_1)^2}\\ n\equiv\frac{r_1r_2}{(N,r_1)^2} \bmod \frac{N}{(N,r_1)}}}\left(\sum_{m=0}^{\frac{N}{(N,s)}-1}e^{2\pi i ms/N}E_{\frac{mN+r_1}{(N,r_1)}}(n;\frac{N^2}{(N,r_1)(N,s)})\right)q^n_{\frac{N^2}{(N,r_1)^2}}.$$

And, if we let $\alpha = \begin{pmatrix} \frac{N^2}{(N,r_1)(N,r_2)} & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\Gamma = \Gamma_1(\frac{N^2}{(N,r_1)(N,s)}) \cap \Gamma_0(\frac{N^3}{(N,r_1)(N,r_2)(N,s)})$, *then* $\frac{K_{(\frac{r_1+r_2}{N}\ \frac{s}{N})}(\tau)}{K_{(\frac{r_1}{N}\ 0)}(\tau)K_{(\frac{r_2}{N}\ \frac{s}{N})}(\tau)}$ *is a holomorphic modular form of weight* 1 *for* $\alpha\Gamma\alpha^{-1}$.

(3) Let $N > 1$, $0 < r < N$ *and* $s \not\equiv 0 \bmod N$. *Then we have*

$$\frac{K_{(\frac{r}{N}\ \frac{s}{N})}(\tau)}{K_{(\frac{r}{N}\ 0)}(\tau)K_{(0\ \frac{s}{N})}(\tau)} = -2\pi i e^{\pi i r s/N^2}\cdot$$

$$\left(\frac{1}{1-e^{2\pi i s/N}} + \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\frac{N}{(N,s)}-1} e^{2\pi i m s/N} E_{\frac{mN+r}{(N,r)}}(n;\frac{N^2}{(N,r)(N,s)})\right) q^n_{\frac{N}{(N,r)}}\right).$$

If we let $\alpha = \begin{pmatrix} \frac{N}{(N,r)} & 0 \\ 0 & 1 \end{pmatrix}$, *then* $\frac{K_{(\frac{r}{N}\ \frac{s}{N})}(\tau)}{K_{(\frac{r}{N}\ 0)}(\tau)K_{(0\ \frac{s}{N})}(\tau)}$ *is a holomorphic modular form of weight* 1 *for* $\alpha\Gamma_1(\frac{N^2}{(N,r)(N,s)})\alpha^{-1}$.

(4) For $N > 1$, *let* $0 < r < N$ *satisfy* $(N,r) = 1$. *Then we have*

$$\frac{K_{(\frac{r}{N}\ 0)}(\tau)}{K_{(0\ \frac{1}{2})}(\tau)K_{(\frac{r}{N}\ \frac{1}{2})}(\tau)} = -2\pi i e^{-\pi i r/(2N)}\left(-\frac{1}{2} + \sum_{n=1}^{\infty}\left(E_{r,N-r}(n;2N) - 2E_{r,N-r}(\frac{n}{2};2N)\right) q^n_N\right).$$

Moreover, if we let $\alpha = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\Gamma = \Gamma_1(2N) \cap \Gamma_0(4N)$, *then* $\frac{K_{(\frac{r}{N}\ 0)}(\tau)}{K_{(0\ \frac{1}{2})}(\tau)K_{(\frac{r}{N}\ \frac{1}{2})}(\tau)}$ *is a holomorphic modular form of weight* 1 *for* $\alpha\Gamma\alpha^{-1}$.

(5) For $N > 1$, *let* $0 < r < N$ *satisfy* $(N,r) = 1$. *Then we have*

$$\frac{K_{(\frac{1}{4}+\frac{r}{2N}\ 0)}(4\tau)}{K_{(\frac{1}{4}\ 0)}(4\tau)K_{(\frac{r}{N}\ 0)}(\tau)} = -2\pi i\left(1 + \sum_{n=1}^{\infty} E_{r,N-r,N+r,2N-r}(n;4N) q^n_N\right).$$

If we let $\alpha = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, *then* $\frac{K_{(\frac{1}{4}+\frac{r}{2N}\ 0)}(4\tau)}{K_{(\frac{1}{4}\ 0)}(4\tau)K_{(\frac{r}{N}\ 0)}(\tau)}$ *is a holomorphic modular form of weight* 1 *for* $\alpha\Gamma_1(4N)\alpha^{-1}$.

(6) For $N > 2$, *let* $0 < r < N$ *satisfy* $(N,r) = 1$. *Then we have*

$$\frac{K_{(\frac{1}{4}-\frac{r}{2N}\ 0)}(4\tau)}{K_{(\frac{1}{4}\ 0)}(4\tau)K_{(\frac{r}{N}\ 0)}(\tau)} = -2\pi i \sum_{n=1}^{\infty} E_{r,2N-r,3N-r,3N+r}(n;4N) q^n_N.$$

And, if we let $\alpha = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, *then* $\frac{K_{(\frac{1}{4}-\frac{r}{2N}\ 0)}(4\tau)}{K_{(\frac{1}{4}\ 0)}(4\tau)K_{(\frac{r}{N}\ 0)}(\tau)}$ *is a holomorphic modular form of weight* 1 *for* $\alpha\Gamma_1(4N)\alpha^{-1}$.

(7) Let $N > 3$, $s_1 \not\equiv 0 \bmod N$, $s_2 \not\equiv 0 \bmod N$ *and* $s_1 \not\equiv \pm s_2 \bmod N$. *Then we have*

$$\frac{K_{(0\ \frac{s_1+s_2}{N})}(\tau)K_{(0\ \frac{s_1-s_2}{N})}(\tau)}{K^2_{(0\ \frac{s_1}{N})}(\tau)K^2_{(0\ \frac{s_2}{N})}(\tau)} = (-2\pi i)^2\cdot$$

$$\left(e^{2\pi i s_2/N}\frac{(1-e^{2\pi i(s_1+s_2)/N})(1-e^{2\pi i(s_1-s_2)/N})}{(1-e^{2\pi i s_1/N})^2(1-e^{2\pi i s_2/N})^2}+2\sum_{n=1}^{\infty}\left(\sum_{d|n,\, d>0}\left(\cos\frac{2\pi s_2 d}{N}-\cos\frac{2\pi s_1 d}{N}\right)d\right)q^n\right).$$

Furthermore, $\frac{K_{(0\ \frac{s_1+s_2}{N})}(\tau)K_{(0\ \frac{s_1-s_2}{N})}(\tau)}{K^2_{(0\ \frac{s_1}{N})}(\tau)K^2_{(0\ \frac{s_2}{N})}(\tau)}$ *is a holomorphic modular form of weight* 2 *for* $\Gamma_1(N)$.

(8) For $N>1$*, let* $0<r<N$ *satisfy* $(N,r)=1$*. Then we have*

$$K^{-2}_{(\frac{r}{N}\ 0)}(\tau)=(-2\pi i)^2\frac{1}{N}\sum_{\substack{n\geq N-r^2\\ n\equiv -r^2 \bmod N}}\left(\sum_{\substack{d|n,\, d>0\\ d\equiv r \bmod N}}\left(d+\frac{n}{d}\right)\right)q^n_{N^2}.$$

And, $K^{-2}_{(\frac{r}{N}\ 0)}(\tau)$ *is a holomorphic modular form of weight* 2 *for* $\Gamma^1(N)\cap\Gamma^0(N^2)$.

(9) Let $N>2$ *and* $2s\not\equiv 0 \bmod N$*. Then we have*

$$\frac{K_{(0\ \frac{2s}{N})}(\tau)}{K^4_{(0\ \frac{s}{N})}(\tau)}=(-2\pi i)^3\left(e^{2\pi i s/N}\frac{(1-e^{4\pi i s/N})}{(1-e^{2\pi i s/N})^4}+2i\sum_{n=1}^{\infty}\left(\sum_{d|n,\, d>0}d^2\sin\frac{2\pi s d}{N}\right)q^n\right).$$

Furthermore, $\frac{K_{(0\ \frac{2s}{N})}(\tau)}{K^4_{(0\ \frac{s}{N})}(\tau)}$ *is a holomorphic modular form of weight* 3 *for* $\Gamma_1(N)$.

Proof. All the identities in the theorem can be obtained by Proposition 1 and (**K4**). So it is enough to prove whether the left hand sides are indeed modular forms.

(1) We restrict ourselves to the case when N is odd, because the proofs are similar. If we let $\gamma=\begin{pmatrix}1+aN & bN^2\\ c & 1+dN\end{pmatrix}\in\Gamma^1(N)\cap\Gamma^0(N^2)$, then by (**K1**) and (**K2**) we have

$$(K^{-1}_{(\frac{r}{N}\ 0)}|_1\gamma)(\tau)=K^{-1}_{(\frac{r}{N}+ar\ \ brN)}(\tau)=(-1)^{abr^2N+ar+brN+br^2}K^{-1}_{(\frac{r}{N}\ 0)}(\tau)=K^{-1}_{(\frac{r}{N}\ 0)}(\tau).$$

Since $K^{-1}_{(\frac{r}{N}\ 0)}(\tau)$ is holomorphic at all cusps by (**K4**), we achieve the assertion.

(2) For convenience, set $f(\tau)=\frac{K_{(\frac{r_1+r_2}{N}\ \frac{s}{N})}(\tau)}{K_{(\frac{r_1}{N}\ 0)}(\tau)K_{(\frac{r_2}{N}\ \frac{s}{N})}(\tau)}$. Let $\gamma=\begin{pmatrix}1+a' & b'\\ c' & 1+d'\end{pmatrix}\in\alpha\Gamma\alpha^{-1}$. Then $a'\equiv d'\equiv 0 \bmod \frac{N^2}{(N,r_1)(N,s)}$, $b'\equiv 0 \bmod \frac{N^2}{(N,r_1)(N,r_2)}$, $c'\equiv 0 \bmod \frac{N}{(N,s)}$ and $a'+d'+a'd'-b'c'=0$. By (**K1**) and (**K2**) we derive

$$(f|_1\gamma)(\tau)=e^{2\pi i(\frac{r_1r_2}{N^2}b'(a'+1)+\frac{r_1 s}{N^2}(b'c'-a'))}f(\tau)=f(\tau).$$

Now let $\rho=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$. Then we establish by (**K1**) and (**K4**)

$$ord_q(f|_1\rho)(\tau)=\begin{cases}\langle\frac{r_1}{N}a\rangle\langle\frac{r_2}{N}a+\frac{s}{N}c\rangle & \text{, if } \langle\frac{r_1}{N}a\rangle+\langle\frac{r_2}{N}a+\frac{s}{N}c\rangle<1\\ (1-\langle\frac{r_1}{N}a\rangle)(1-\langle\frac{r_2}{N}a+\frac{s}{N}c\rangle) & \text{, otherwise.}\end{cases}$$

Therefore $f(\tau)$ is holomorphic at all cusps.

(3) The statement concerning holomorphic modular form can be deduced from the argument as in (2) by taking $r_2=0$, because the condition $r_2>0$ is not necessary for the proof.

(4) For convenience, put $f(\tau)=\frac{K_{(\frac{r}{N}\ 0)}(\tau)}{K_{(0\ \frac{1}{2})}(\tau)K_{(\frac{r}{N}\ \frac{1}{2})}(\tau)}$. And, let $\gamma=\begin{pmatrix}1+a' & b'\\ c' & 1+d'\end{pmatrix}\in\alpha\Gamma\alpha^{-1}$. Then $a'\equiv d'\equiv 0 \bmod 2N$, $b'\equiv 0 \bmod N$, $c'\equiv 0 \bmod 4$ and $a'+d'+a'd'-b'c'=0$. By (**K2**) we get

$$(f|_1\gamma)(\tau)=e^{\pi i(\frac{r}{N}(b'c'-d')+\frac{c'}{2})}f(\tau)=f(\tau).$$

Now let $\rho=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$. Then it follows from (**K1**) and (**K4**) that

$$ord_q(f|_1\rho)(\tau)=\begin{cases}\langle\frac{c}{2}\rangle(1-\langle\frac{r}{N}a\rangle-\langle\frac{c}{2}\rangle) & \text{, if } \langle\frac{r}{N}a\rangle+\langle\frac{c}{2}\rangle<1\\ (1-\langle\frac{c}{2}\rangle)(\langle\frac{r}{N}a\rangle+\langle\frac{c}{2}\rangle-1) & \text{, otherwise.}\end{cases}$$

Thus $f(\tau)$ is holomorphic at all cusps.

(5) Let $f(\tau)=\frac{K_{(\frac{1}{4}+\frac{r}{2N}\ 0)}(4\tau)}{K_{(\frac{1}{4}\ 0)}(4\tau)K_{(\frac{r}{N}\ 0)}(\tau)}$ and $\gamma=\begin{pmatrix}1+a' & b'\\ 4c' & 1+d'\end{pmatrix}\in\alpha\Gamma_1(4N)\alpha^{-1}$. Then $a'\equiv d'\equiv 0 \bmod 4N$, $b'\equiv 0 \bmod N$. We see from (**K2**) that

$$(f|_1\gamma)(\tau)=(-1)^{-\frac{r}{N}b'}e^{\pi i\frac{r}{N}b'}f(\tau)=f(\tau).$$

Now let $\rho=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$, and let y', $w\in\mathbb{Z}$ satisfy $\frac{c}{(c,4)}y'+\frac{4}{(c,4)}w=d$. Furthermore we let $y=(c,4)b-ay'$. Then we deduce

$$\begin{pmatrix}4&0\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}(c,4)&y'\\0&\frac{4}{(c,4)}\end{pmatrix}^{-1}=\begin{pmatrix}\frac{4}{(c,4)}a & y\\ \frac{c}{(c,4)} & w\end{pmatrix}\in SL_2(\mathbb{Z}).$$

Since $(f|_1\rho)(\tau)=\frac{K_{(\frac{a}{(c,4)}+\frac{2}{(c,4)}\frac{r}{N}a\ \ *)}(\frac{(c,4)^2}{4}\tau+\frac{(c,4)}{4}y')}{K_{(\frac{a}{(c,4)}\ \ *)}(\frac{(c,4)^2}{4}\tau+\frac{(c,4)}{4}y')K_{(\frac{r}{N}a\ \ *)}(\tau)}$, we have

$$\begin{aligned}ord_q(f|_1\rho)(\tau) &= \tfrac{1}{2}\Big(\tfrac{(c,4)^2}{4}\Big(\langle\tfrac{a}{(c,4)}+\tfrac{2}{(c,4)}\tfrac{r}{N}a\rangle^2-\langle\tfrac{a}{(c,4)}+\tfrac{2}{(c,4)}\tfrac{r}{N}a\rangle-\langle\tfrac{a}{(c,4)}\rangle^2+\langle\tfrac{a}{(c,4)}\rangle\Big)\\ &\quad -\langle\tfrac{r}{N}a\rangle^2+\langle\tfrac{r}{N}a\rangle\Big).\end{aligned}$$

Thus we obtain

$$ord_q(f|_1\rho)(\tau)=\begin{cases}\frac{1}{4}\langle\frac{r}{N}a\rangle & \text{, if } (c,4)=1 \text{ and } \langle\frac{r}{N}a\rangle<\frac{1}{2}\\ \frac{1}{4}(1-\langle\frac{r}{N}a\rangle) & \text{, if } (c,4)=1 \text{ and } \langle\frac{r}{N}a\rangle\geq\frac{1}{2}\\ \frac{1}{2}\langle\frac{r}{N}a\rangle & \text{, if } (c,4)=2 \text{ and } \langle\frac{r}{N}a\rangle<\frac{1}{2}\\ \frac{1}{2}(1-\langle\frac{r}{N}a\rangle) & \text{, if } (c,4)=2 \text{ and } \langle\frac{r}{N}a\rangle\geq\frac{1}{2}\\ \frac{1}{2}\langle\frac{r}{N}a\rangle(4\langle\frac{a}{(c,4)}\rangle-1) & \text{, if } (c,4)=4 \text{ and } \langle\frac{a}{(c,4)}\rangle+\frac{1}{2}\langle\frac{r}{N}a\rangle<1\\ 1-\langle\frac{r}{N}a\rangle & \text{, if } (c,4)=4 \text{ and } \langle\frac{a}{(c,4)}\rangle+\frac{1}{2}\langle\frac{r}{N}a\rangle\geq 1.\end{cases}$$

Hence $f(\tau)$ is holomorphic at all cusps.

(6) The assertion concerning holomorphic modularity can be obtained from the argument of (5) by using (**K0**), because the condition $r>0$ is not necessary for the proof.

(7) For convenience, we let $f(\tau)=\frac{K_{(0\ \frac{s_1+s_2}{N})}(\tau)K_{(0\ \frac{s_1-s_2}{N})}(\tau)}{K^2_{(0\ \frac{s_1}{N})}(\tau)K^2_{(0\ \frac{s_2}{N})}(\tau)}$. By (**K5**) we see that $f(\tau)$ is a modular form of weight 2 for $\Gamma(N)$. And by (**K1**) we get $f|_2\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)=f$. Hence $f(\tau)$ is a

modular form of weight 2 for $\Gamma_1(N)$, because $\Gamma_1(N) = \langle \Gamma(N), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. Now let $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Since $(f|_2\rho)(\tau) = \frac{K_{(\frac{s_1+s_2}{N}c\ *)}(\tau)K_{(\frac{s_1-s_2}{N}c\ *)}(\tau)}{K^2_{(\frac{s_1}{N}c\ *)}(\tau)K^2_{(\frac{s_2}{N}c\ *)}(\tau)}$, we have by (**K4**)

$$ord_q(f|_2\rho)(\tau) = \begin{cases} min\{\langle \frac{s_1}{N}c\rangle, \langle \frac{s_2}{N}c\rangle\} & \text{, if } \langle \frac{s_1}{N}c\rangle + \langle \frac{s_2}{N}c\rangle < 1 \\ 1 - max\{\langle \frac{s_1}{N}c\rangle, \langle \frac{s_2}{N}c\rangle\} & \text{, otherwise.} \end{cases}$$

Therefore, $f(\tau)$ is holomorphic at all cusps.

(8) Let $\gamma = \begin{pmatrix} 1+aN & bN^2 \\ c & 1+dN \end{pmatrix} \in \Gamma^1(N) \cap \Gamma^0(N^2)$. Then by (**K1**) and (**K2**) we derive

$$(K^{-2}_{(\frac{r}{N}\ 0)}|_2\gamma)(\tau) = K^{-2}_{(\frac{r}{N}+ar\ brN)}(\tau) = e^{-2\pi i brN\cdot\frac{r}{N}}K^{-2}_{(\frac{r}{N}\ 0)}(\tau) = K^{-2}_{(\frac{r}{N}\ 0)}(\tau).$$

Thus we obtain the conclusion.

(9) For convenience, we let $f(\tau) = \frac{K_{(0\ \frac{2s}{N})}(\tau)}{K^4_{(0\ \frac{s}{N})}(\tau)}$. We see by (**K5**) that $f(\tau)$ is a modular form of weight 3 for $\Gamma(N)$. And by (**K1**) we have $f|_3\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = f$. Hence $f(\tau)$ is a modular form of weight 3 for $\Gamma_1(N)$, because $\Gamma_1(N) = \langle \Gamma(N), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$. Now let $\rho = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Since $(f|_3\rho)(\tau) = \frac{K_{(\frac{2s}{N}c\ *)}(\tau)}{K^4_{(\frac{s}{N}c\ *)}(\tau)}$, we establish by (**K4**)

$$ord_q(f|_3\rho)(\tau) = \begin{cases} \langle \frac{s}{N}c\rangle & \text{, if } \langle \frac{s}{N}c\rangle < \frac{1}{2} \\ 1 - \langle \frac{s}{N}c\rangle & \text{, otherwise.} \end{cases}$$

Hence we deduce the assertion. □

Here we remark that in the theory of elliptic functions Silverman ([5]) introduced certain generalizations of Theorem 2 (7), (9) as follows.

Theorem 3. *(1) Let $(r_1\ s_1)$, $(r_2\ s_2) \in \mathbb{Z}^2 - N\mathbb{Z}^2$ satisfy $(r_1\ s_1) \pm (r_2\ s_2) \in \mathbb{Z}^2 - N\mathbb{Z}^2$. Then we have*

$$\frac{K_{(\frac{r_1+r_2}{N}\ \frac{s_2+s_2}{N})}(\tau)K_{(\frac{r_1-r_2}{N}\ \frac{s_2-s_2}{N})}(\tau)}{K^2_{(\frac{r_1}{N}\ \frac{s_1}{N})}(\tau)K^2_{(\frac{r_2}{N}\ \frac{s_2}{N})}(\tau)} = (-2\pi i)^2\left(\sum_{n\in\mathbb{Z}}\frac{\zeta_N^{s_2}q_N^{r_2+Nn}}{(1-\zeta_N^{s_2}q_N^{r_2+Nn})^2} - \sum_{n\in\mathbb{Z}}\frac{\zeta_N^{s_1}q_N^{r_1+Nn}}{(1-\zeta_N^{s_1}q_N^{r_1+Nn})^2}\right).$$

(2) Let $(r\ s) \in \mathbb{Z}^2$ satisfy $(2r\ 2s) \in \mathbb{Z}^2 - N\mathbb{Z}^2$. Then we have

$$\frac{K_{(\frac{2r}{N}\ \frac{2s}{N})}(\tau)}{K^4_{(\frac{r}{N}\ \frac{s}{N})}(\tau)} = (-2\pi i)^3\sum_{n\in\mathbb{Z}}\frac{\zeta_N^s q_N^{r+Nn}(1+\zeta_N^s q_N^{r+Nn})}{(1-\zeta_N^s q_N^{r+Nn})^3}.$$

Proof. These are restatements of [5, Corollary 5.6 and Theorem 6.2 in Chapter I]. □

The modular forms described in Theorem 2 can be used when we construct a basis of $M_k(\Gamma_1(N))$ for some level N. To this end we first recall the dimension formulas of $M_k(\Gamma_1(N))$ for $k = 1, 2$.

Lemma 4. *Let $N > 2$ be an integer. Then we have the following dimension formulas.*

(1) If $12\sum_{d|N, d>0}\phi(d)\phi(\frac{N}{d}) > \phi(N)N\prod_{p|N}(1+\frac{1}{p})$, then we obtain

$$\dim M_1(\Gamma_1(N)) = \begin{cases} 1 & \text{, if } N = 4 \\ \frac{1}{4}\sum_{d|N, d>0}\phi(d)\phi(\frac{N}{d}) & \text{, otherwise.} \end{cases}$$

(2)

$$\dim M_2(\Gamma_1(N)) = \begin{cases} 1 & , \textit{ if } N=3 \\ 2 & , \textit{ if } N=4 \\ \frac{1}{4}\sum_{d|N,\, d>0}\phi(d)\phi(\frac{N}{d}) + \frac{\phi(N)N}{24}\prod_{p|N}(1+\frac{1}{p}) & , \textit{ if } N\geq 5. \end{cases}$$

Proof. See [4, §2.6]. □

Proposition 5. *(1) Let $(p,q) \in \{(3,2),(5,2),(7,2),(11,2),(13,2)\}$. Then $M_1(\Gamma_1(pq))$ has a basis $\{f_{pq}^{(r,s)}(\tau), g_{pq}^{(r,s)}(\tau) \mid 1 \leq r \leq \frac{p-1}{2}, 1 \leq s \leq q-1\}$, where*

$$\begin{aligned} f_{pq}^{(r,s)}(\tau) &= 1 + (1-e^{2\pi i s/q})\sum_{n=1}^{\infty}\left(\sum_{m=0}^{q-1} e^{2\pi i m s/q} E_{mp+r}(n;pq)\right) q^n \\ g_{pq}^{(r,s)}(\tau) &= 1 + (1-e^{2\pi i r/p})\sum_{n=1}^{\infty}\left(\sum_{m=0}^{p-1} e^{2\pi i m r/p} E_{mq+s}(n;pq)\right) q^n. \end{aligned}$$

(2) $M_1(\Gamma_1(3))$ has a basis $\{f_{3,1}(\tau)\}$, where

$$f_{3,1}(\tau) = 1 + 6\sum_{n=1}^{\infty} E_1(n;3)q^n.$$

(3) $M_1(\Gamma_1(4))$ has a basis $\{f_{4,1}(\tau)\}$, where

$$f_{4,1}(\tau) = 1 + 4\sum_{n=1}^{\infty} E_1(n;4)q^n.$$

(4) $M_1(\Gamma_1(8))$ has a basis $\{f_{8,1}(\tau), f_{8,2}(\tau), f_{8,3}(\tau)\}$, where

$$\begin{aligned} f_{8,1}(\tau) &= 1 + 4\sum_{n=1}^{\infty} E_1(n;4)q^n \\ f_{8,2}(\tau) &= \sum_{\substack{n\geq 1 \\ n\equiv 1 \bmod 4}} E_1(n;4)q^n \\ f_{8,3}(\tau) &= 1 + 2\sum_{n=1}^{\infty} E_{1,3}(n;8)q^n. \end{aligned}$$

(5) $M_1(\Gamma_1(9))$ has a basis $\{f_{9,1}(\tau), f_{9,2}(\tau), f_{9,3}(\tau), f_{9,4}(\tau)\}$, where

$$\begin{aligned}
f_{9,1}(\tau) &= 1+6\sum_{n=1}^{\infty} E_1(n;3)q^n \\
f_{9,2}(\tau) &= \sum_{\substack{n\geq 1\\ n\equiv 1 \bmod 3}} E_1(2n;6)q^n \\
f_{9,3}(\tau) &= 1+(1-e^{2\pi i/3})\sum_{n=1}^{\infty}\left(E_1(n;9)+e^{2\pi i/3}E_4(n;9)+e^{4\pi i/3}E_7(n;9)\right)q^n \\
f_{9,4}(\tau) &= 1+(1-e^{4\pi i/3})\sum_{n=1}^{\infty}\left(E_1(n;9)+e^{4\pi i/3}E_4(n;9)+e^{2\pi i/3}E_7(n;9)\right)q^n.
\end{aligned}$$

(6) $M_1(\Gamma_1(12))$ has a basis $\{f_{12,1}(\tau), f_{12,2}(\tau), f_{12,3}(\tau), f_{12,4}(\tau), f_{12,5}(\tau)\}$, where

$$\begin{aligned}
f_{12,1}(\tau) &= 1+4\sum_{n=1}^{\infty} E_1(n;4)q^n \\
f_{12,2}(\tau) &= 1+2\sum_{n=1}^{\infty} E_{1,2}(n;6)q^n \\
f_{12,3}(\tau) &= 1+3\sum_{n=1}^{\infty} E_1(n;6)q^n \\
f_{12,4}(\tau) &= 1+\sum_{n=1}^{\infty} E_{1,2,4,5}(n;12)q^n \\
f_{12,5}(\tau) &= \sum_{n=1}^{\infty} E_{1,5,8,10}(n;12)q^n.
\end{aligned}$$

(7) $M_2(\Gamma_1(4))$ has a basis $\{f(\tau), g(\tau)\}$, where

$$\begin{aligned}
f(\tau) &= \sum_{\substack{n\geq 1\\ n\equiv 1 \bmod 2}} \left(\sum_{d|n,\, d>0} d\right) q^n \\
g(\tau) &= 1+8\sum_{n=1}^{\infty}\left(\sum_{d|n,\, d>0}\left(\cos \pi d-\cos\frac{\pi d}{2}\right) d\right) q^n.
\end{aligned}$$

Proof. By Theorem 2 (3) we have $\frac{K_{(\frac{1}{2}\ \frac{1}{2})}(2\tau)}{K_{(\frac{1}{2}\ 0)}(2\tau)K_{(0\ \frac{1}{2})}(2\tau)} = -2\pi i e^{\pi i/4}(\frac{1}{2}+2\sum_{n=1}^{\infty} E_1(n;4)q^n) \in M_1(\Gamma_1(4))$. And it follows from Lemma 4 (1) that $\dim M_1(\Gamma_1(4)) = 1$, which immediately implies the assertion (3).

We see that $f_{4,1}(\tau) \in M_1(\Gamma_1(4)) \subset M_1(\Gamma_1(8))$, and we have two more modular forms in $M_1(\Gamma_1(8))$ by Theorem 2 (1) and (3) such as

$$\begin{aligned}
K^{-1}_{\left(\frac{1}{2}\ 0\right)}(8\tau) &= -2\pi i \sum_{\substack{n\geq 1 \\ n\equiv 1 \bmod 4}} E_1(n;4)q^n \\
&= -2\pi i(q+2q^5+\cdots) \\
\frac{K_{\left(\frac{1}{2}\ \frac{1}{4}\right)}(2\tau)}{K_{\left(\frac{1}{2}\ 0\right)}(2\tau)K_{\left(0\ \frac{1}{4}\right)}(2\tau)} &= -2\pi i(1+i)e^{\pi i/8}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\big(E_1(n;8)+E_3(n;8)\big)q^n\right) \\
&= -2\pi i(1+i)e^{\pi i/8}\left(\frac{1}{2}+q+q^2+2q^3+\cdots\right).
\end{aligned}$$

By inspecting their Fourier coefficients we are sure that these three modular forms are linearly independent over $\mathbb{C}$. Since $\dim M_1(\Gamma_1(8)) = 3$ by Lemma 4 (1), we obtain the assertion (4).

For two primes p, q with $p > q$, and $1 \leq r \leq \frac{p-1}{2}$, $1 \leq s \leq q-1$, we get by Theorem 2 (3) that

$$\begin{aligned}
\frac{K_{\left(\frac{r}{p}\ \frac{s}{q}\right)}(p\tau)}{K_{\left(\frac{r}{p}\ 0\right)}(p\tau)K_{\left(0\ \frac{s}{q}\right)}(p\tau)} &= -2\pi i e^{\pi irs/(pq)}\left(\frac{1}{1-e^{2\pi is/q}}+\sum_{n=1}^{\infty}\left(\sum_{m=0}^{q-1}e^{2\pi ims/q}E_{mp+r}(n;pq)\right)q^n\right) \\
\frac{K_{\left(\frac{s}{q}\ \frac{r}{p}\right)}(q\tau)}{K_{\left(\frac{s}{q}\ 0\right)}(q\tau)K_{\left(0\ \frac{r}{p}\right)}(q\tau)} &= -2\pi i e^{\pi irs/(pq)}\left(\frac{1}{1-e^{2\pi ir/p}}+\sum_{n=1}^{\infty}\left(\sum_{m=0}^{p-1}e^{2\pi imr/p}E_{mq+s}(n;pq)\right)q^n\right)
\end{aligned}$$

are holomorphic modular forms of weight 1 for $\Gamma_1(pq)$. For $(p,q) = (3,2)$, $(5,2)$, $(7,2)$, $(11,2)$ and $(13,2)$, we can verify with the aid of computer that the above $(p-1)(q-1)$ modular forms are linearly independent over $\mathbb{C}$. Thus for those (p,q) we have $\dim M_1(\Gamma_1(pq)) = (p-1)(q-1)$ by Lemma 4 (1), and hence we achieve the assertion (1).

For those $(p,2)$ in the hypothesis of (1), we might explicitly find a basis of $M_1(\Gamma_1(p))$ by means of a basis of $M_1(\Gamma_1(2p))$ together with the observation that $M_1(\Gamma_1(p)) \subset M_1(\Gamma_1(2p))$ and $\Gamma_1(p) = \langle\ \Gamma_1(2p), \begin{pmatrix} 1+p & 1 \\ p & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ p & 1+p \end{pmatrix}, \begin{pmatrix} 1+p & -p \\ p & 1-p \end{pmatrix}\ \rangle$. Indeed, for $p=3$ we have a basis $\{f_6^{(1,1)}(\tau), g_6^{(1,1)}(\tau)\}$ of $M_1(\Gamma_1(6))$, which are given by

$$\begin{aligned}
f_6^{(1,1)}(\tau) &= 1+2\sum_{n=1}^{\infty}(E_1(n;6)-E_4(n;6))q^n \\
&= 1+2q+4q^2+2q^3+\cdots \\
g_6^{(1,1)}(\tau) &= 1+3\sum_{n=1}^{\infty}E_1(n;6)q^n \\
&= 1+3q+3q^2+3q^3+\cdots.
\end{aligned}$$

Since $M_1(\Gamma_1(3)) \subset M_1(\Gamma_1(6))$ and $\dim M_1(\Gamma_1(3)) = 1$, it can be checked out that $M_1(\Gamma_1(3)) = \mathbb{C}f(\tau)$ where $f(\tau) = -3f_6^{(1,1)}(\tau) + 4g_6^{(1,1)}(\tau)$. So we get the assertion (2).

We know that $f_{3,1}(\tau) \in M_1(\Gamma_1(3)) \subset M_1(\Gamma_1(9))$, and we have three more modular forms in $M_1(\Gamma_1(9))$ by Theorem 2 (1) and (3) which are

$$\begin{aligned}
K^{-1}_{(\frac{1}{3}\ 0)}(9\tau) &= -2\pi i \sum_{\substack{n\geq 1\\ n\equiv 1 \bmod 3}} E_1(2n;6)q^n \\
&= -2\pi i(q+q^4+\cdots) \\
\frac{K_{(\frac{1}{3}\ \frac{1}{3})}(3\tau)}{K_{(\frac{1}{3}\ 0)}(3\tau)K_{(0\ \frac{1}{3})}(3\tau)} &= -2\pi i e^{\pi i/9}\left(\frac{1}{1-e^{2\pi i/3}}+\sum_{n=1}^{\infty}\left(\sum_{m=0}^{2} e^{2\pi i m/3}E_{3m+1}(n;9)\right)q^n\right) \\
&= -2\pi i e^{\pi i/9}\left(\frac{1}{1-e^{2\pi i/3}}+q+(1-e^{4\pi i/3})q^2+q^3+\cdots\right) \\
\frac{K_{(\frac{1}{3}\ \frac{2}{3})}(3\tau)}{K_{(\frac{1}{3}\ 0)}(3\tau)K_{(0\ \frac{2}{3})}(3\tau)} &= -2\pi i e^{2\pi i/9}\left(\frac{1}{1-e^{4\pi i/3}}+\sum_{n=1}^{\infty}\left(\sum_{m=0}^{2} e^{4\pi i m/3}E_{3m+1}(n;9)\right)q^n\right) \\
&= -2\pi i e^{2\pi i/9}\left(\frac{1}{1-e^{4\pi i/3}}+q+(1-e^{2\pi i/3})q^2+q^3+\cdots\right).
\end{aligned}$$

Since these four modular forms are linearly independent over $\mathbb{C}$ by checking their Fourier coefficients and $\dim M_1(\Gamma_1(9)) = 4$ by Lemma 4 (1), we conclude the assertion (5).

We have $f_{4,1}(\tau) \in M_1(\Gamma_1(4)) \subset M_1(\Gamma_1(12))$, and $f_6^{(1,1)}(\tau)$, $g_6^{(1,1)}(\tau) \in M_1(\Gamma_1(6)) \subset M_1(\Gamma_1(12))$, and further we have two modular forms in $M_1(\Gamma_1(12))$ by Theorem 2 (5) and (6), which are

$$\begin{aligned}
\frac{K_{(\frac{5}{12}\ 0)}(12\tau)}{K_{(\frac{1}{4}\ 0)}(12\tau)K_{(\frac{1}{3}\ 0)}(3\tau)} &= -2\pi i\left(1+\sum_{n=1}^{\infty}E_{1,2,4,5}(n;12)q^n\right) \\
\frac{K_{(\frac{1}{12}\ 0)}(12\tau)}{K_{(\frac{1}{4}\ 0)}(12\tau)K_{(\frac{1}{3}\ 0)}(3\tau)} &= -2\pi i\sum_{n=1}^{\infty}E_{1,5,8,10}(n;12)q^n.
\end{aligned}$$

These five modular forms are also linearly independent over $\mathbb{C}$ by investigating their Fourier coefficients, and so they form a basis of $M_1(\Gamma_1(12))$ because $\dim M_1(\Gamma_1(12)) = 5$. Therefore we have the assertion (6).

Note that we have two modular forms in $M_2(\Gamma_1(4))$ by Theorem 2 (8) and (7), which are

$$\begin{aligned}
K^{-2}_{(\frac{1}{2}\ 0)}(4\tau) &= (-2\pi i)^2\frac{1}{2}\sum_{\substack{n\geq 1\\ n\equiv 1 \bmod 2}}\left(\sum_{d|n,\, d>0}(d+\frac{n}{d})\right)q^n \\
\frac{K_{(0\ \frac{3}{4})}(\tau)K_{(0\ \frac{1}{4})}(\tau)}{K^2_{(0\ \frac{2}{4})}(\tau)K^2_{(0\ \frac{1}{4})}(\tau)} &= (-2\pi i)^2\left(-\frac{1}{4}+2\sum_{n=1}^{\infty}\left(\sum_{d|n,\, d>0}\left(\cos\frac{\pi d}{2}-\cos\pi d\right)d\right)q^n\right).
\end{aligned}$$

Since these are linearly independent over $\mathbb{C}$ again by inspecting their Fourier coefficients and $\dim M_2(\Gamma_1(4)) = 2$ by Lemma 4 (2), we derive the conclusion (7). $\square$

4. Some applications

Even though all the results in this section are well-known by means of either class field theory or standard approach via local density, we revisit the numbers of representations by quadratic forms as applications of Proposition 5.

We first briefly review theta functions. Let $A \in M_r(\mathbb{Z})$ be a positive definite symmetric matrix of size r over $\mathbb{Z}$ with even diagonal and Q be a quadratic form associated with A, namely

$$Q(x) = \frac{1}{2}\,{}^t\!xAx$$

for a column vector $x = {}^t(x_1 \ \cdots \ x_r) \in \mathbb{Z}^r$ of size r.

We then define the theta function $\theta_Q(\tau)$ associated with Q by

$$\theta_Q(\tau) = \sum_{m \in \mathbb{Z}^r} e^{2\pi i Q(m)\tau} = \sum_{n=0}^{\infty} r_Q(n) q^n,$$

where $r_Q(n)$ is the number of representations of a nonnegative integer n by a quadratic form Q, namely

$$r_Q(n) = |\{m \in \mathbb{Z}^r \mid Q(m) = n\}|.$$

Here we take a positive integer N satisfying $NA^{-1} \in M_r(\mathbb{Z})$.

Lemma 6. *With the notation as above, we further assume that r is even. If all the diagonal entries of NA^{-1} are even, then $\theta_Q(\tau)$ is a holomorphic modular form of weight $\frac{r}{2}$ for $\Gamma_1(N)$. In particular, $\theta_Q(\tau)$ is a holomorphic modular form of weight $\frac{r}{2}$ for $\Gamma_1(2N)$.*

Proof. We refer the reader to [3, Corollary 4.9.5]. □

If $Q(x) = x_1^2 + x_1x_2 + x_2^2$, then $\theta_Q(\tau) \in M_1(\Gamma_1(3))$ by Lemma 6. Since we know a basis of $M_1(\Gamma_1(3))$ by Proposition 5 (2), we conclude that

$$r_Q(n) = 6E_1(n;3) \qquad (n > 0).$$

In particular for a prime p, we deduce

$$p = x_1^2 + x_1x_2 + x_2^2 \Longleftrightarrow p = 3 \text{ or } p \equiv 1 \bmod 3.$$

Since $\frac{1}{6}r_Q(n) = E_1(n;3) = \sum_{d|n,\, d>0} \left(\frac{d}{3}\right)$ is multiplicative, we also have a different simple expression of $r_Q(n)$ as follows. Let $n = 3^t \prod_{i=1}^{a} p_i^{r_i} \prod_{j=1}^{b} q_j^{s_j}$ be the prime factorization of n with $p_i \equiv 1 \bmod 3$, $q_j \equiv 2 \bmod 3$ for all i, j. Then we have

$$r_Q(n) = 6\prod_{i=1}^{a}(r_i + 1)\prod_{j=1}^{b}\frac{1 + (-1)^{s_j}}{2} \qquad (n > 0).$$

Similarly as for $Q(x) = x_1^2 + x_2^2$ we have $\theta_Q(\tau) \in M_1(\Gamma_1(4))$; hence we obtain

$$r_Q(n) = 4E_1(n;4) \qquad (n > 0).$$

And, for $Q(x) = x_1^2 + 2x_2^2$ we see that $\theta_Q(\tau) = af_{8,1}(\tau) + bf_{8,2}(\tau) + cf_{8,3}(\tau) \in M_1(\Gamma_1(8))$ with some constants a, b, c. Since $r_Q(0) = 1$ and $r_Q(1) = r_Q(2) = 2$, we get that $a = b = 0$ and $c = 1$, and hence we derive

$$r_Q(n) = 2E_{1,3}(n; 8) \qquad (n > 0).$$

In like manner, if $Q(x) = x_1^2 + 3x_2^2$, then we have $\theta_Q(\tau) = -2f_{12,2}(\tau) + 2f_{12,3}(\tau) + f_{12,4}(\tau) - f_{12,5}(\tau) \in M_1(\Gamma_1(12))$; hence we get

$$r_Q(n) = 2E_1(n; 3) + 4E_1(\frac{n}{4}; 3) \qquad (n > 0).$$

Next, if $Q(x) = x_1^2 + x_1x_2 + 2x_2^2$, we have $\theta_Q(\tau) \in M_1(\Gamma_1(7))$. Since we have a basis of $M_1(\Gamma_1(14))$ by Proposition 5 (1), we can express $\theta_Q(\tau)$ as a linear combination of $f_{14}^{(1,1)}(\tau), \cdots, g_{14}^{(3,1)}(\tau)$. Indeed, we achieve by routine computation that $\theta_Q(\tau) = -f_{14}^{(1,1)}(\tau) + f_{14}^{(2,1)}(\tau) + f_{14}^{(3,1)}(\tau) - \frac{8}{7}(\cos\frac{\pi}{7} + \cos\frac{2\pi}{7})g_{14}^{(1,1)}(\tau) + \frac{8}{7}(\cos\frac{2\pi}{7} + \cos\frac{3\pi}{7})g_{14}^{(2,1)}(\tau) + \frac{8}{7}(\cos\frac{\pi}{7} - \cos\frac{3\pi}{7})g_{14}^{(3,1)}(\tau)$. Thus we deduce

$$r_Q(n) = 2E_{1,2,4}(n; 7) \qquad (n > 0).$$

In particular for a prime p, we have

$$p = x_1^2 + x_1x_2 + 2x_2^2 \Longleftrightarrow p = 7 \text{ or } p \equiv 1, 2, 4 \bmod 7.$$

Since $\frac{1}{2}r_Q(n) = E_{1,2,4}(n; 7) = \sum_{d|n,\, d>0} \left(\frac{d}{7}\right)$ is multiplicative, we can deduce other simple expression of $r_Q(n)$ as follows. Let $n = 7^t \prod_{i=1}^a p_i^{r_i} \prod_{j=1}^b q_j^{s_j}$ be the prime factorization of n with $p_i \equiv 1, 2, 4 \bmod 7$, $q_j \equiv 3, 5, 6 \bmod 7$ for all i, j. Then we have

$$r_Q(n) = 2\prod_{i=1}^{a}(r_i + 1)\prod_{j=1}^{b}\frac{1 + (-1)^{s_j}}{2} \qquad (n > 0).$$

As for $Q(x) = x_1^2 + x_1x_2 + 3x_2^2$ we have $\theta_Q(\tau) \in M_1(\Gamma_1(11))$ by Lemma 6. Since we have a basis of $M_1(\Gamma_1(22))$, we can express $\theta_Q(\tau)$ as a linear combination of $f_{22}^{(1,1)}, \cdots, g_{22}^{(5,1)}$ in Proposition 5. With the help of computer we can see that there exist somewhat complicated constants $a_1, \cdots, a_5$ such that

$$\begin{aligned}\theta_Q(\tau) &= -f_{22}^{(1,1)}(\tau) - f_{22}^{(2,1)}(\tau) - f_{22}^{(3,1)}(\tau) + f_{22}^{(4,1)}(\tau) - f_{22}^{(5,1)}(\tau) + \sum_{r=1}^{5} a_r g_{22}^{(r,1)}(\tau) \\ &= 1 + 2\sum_{n=1}^{\infty} E_{1,3,4,5,9}(n; 11)q^n.\end{aligned}$$

In particular for a prime p, we have

$$p = x_1^2 + x_1x_2 + 3x_2^2 \Longleftrightarrow p = 11 \text{ or } p \equiv 1, 3, 4, 5, 9 \bmod 11.$$

Since $\frac{1}{2}r_Q(n) = E_{1,3,4,5,9}(n; 11) = \sum_{d|n,\, d>0} \left(\frac{d}{11}\right)$ is multiplicative, we also have simple expression of $r_Q(n)$ as follows. Let $n = 11^t \prod_{i=1}^a p_i^{r_i} \prod_{j=1}^b q_j^{s_j}$ be the prime factorization of n with

$p_i \equiv 1, 3, 4, 5, 9 \bmod 11$, $q_j \equiv 2, 6, 7, 8, 10 \bmod 11$ for all i, j. Then we have

$$r_Q(n) = 2\prod_{i=1}^{a}(r_i+1)\prod_{j=1}^{b}\frac{1+(-1)^{s_j}}{2} \qquad (n>0).$$

Lastly, since $\theta_Q(\tau) \in M_2(\Gamma_1(4))$ for $Q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$, we see from Proposition 5 (7) that $\theta_Q(\tau) = 16f(\tau) + g(\tau)$. Therefore we rediscover that for $n \in \mathbb{Z}_{>0}$

$$r_Q(n) = \begin{cases} 8\sum_{d|n,\, d>0} d & \text{if } n \text{ is odd} \\ 8\sum_{d|n, d>0}(\cos \pi d - \cos \frac{\pi d}{2})d & \text{if } n \text{ is even.} \end{cases}$$

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