DIVISOR FUNCTIONS ARISING FROM q-SERIES

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Abstract. We investigate some of the properties of divisor functions arising from q-series and theta functions. Using these we obtain several new identities of divisor functions.

1. Introduction

Throughout this paper we use the standard notation

$$(a;q)_\infty := \prod_{n\geq 0}(1-aq^n).$$

If there is no confusion, we briefly write $(a)_\infty$ instead of $(a;q)_\infty$. In general, q will denote a fixed complex number of absolute value less than 1, so we may write $q=e^{\pi i t}$ with $\mathrm{Im}\ t>0$.

For $N, m, r, s, t, k, l \in \mathbb{Z}$, we define some necessary divisor functions for later use, which appear in many areas of number theroy:

$$\begin{aligned}
E_r(N;m) &= \sum_{\substack{d|N \\ d\equiv r \bmod m}} 1 - \sum_{\substack{d|N \\ d\equiv -r \bmod m}} 1,\\
E_{r,\cdots,s}(N;m) &= E_r(N;m)+\cdots+E_s(N;m),\\
\sigma_s^{\{l\}}(N;m) &= \sum_{\substack{d|N \\ d\equiv s \bmod m}} d^l,\\
\sigma_{s,\cdots,t}^{\{l\}}(N;m) &= \sigma_s^{\{l\}}(N;m)+\cdots+\sigma_t^{\{l\}}(N;m),\\
\sigma_s(N;m) &= \sigma_s^{\{1\}}(N;m) = \sum_{\substack{d|N \\ d\equiv s \bmod m}} d,\\
\sigma_{s,\cdots,t}(N;m) &= \sigma_s(N;m)+\cdots+\sigma_t(N;m),\\
\tau_{1;k,l}(N) &= \sum_{\substack{d|N \\ d\equiv (k-l)/2 \bmod k}} N/d + \sum_{\substack{d|N \\ d\equiv (k+l)/2 \bmod k}} N/d,\\
\tau_{2;k,l}(N) &= \sum_{\substack{d|N \\ d\equiv (k-l)/2 \bmod k}} (N/d)^2 - \sum_{\substack{d|N \\ d\equiv (k+l)/2 \bmod k}} (N/d)^2.
\end{aligned}$$

In the notations $\tau_{1;k,l}(N)$ and $\tau_{2;k,l}(N)$, we assume that k,l are positive integers such that $k\equiv l \bmod 2$, $k\geq 3$ and $l\leq k-2$. Let $r_Q(N)$ be the number of representations of a nonnegative integer N by a positive definite quadratic form Q. Finding explicit formulas for $r_Q(N)$ is an

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old and classical problem in number theory (see [4]), but even in some special cases we may use divisor functions rather than utilizing standard approach via local dencities. For example, Gauss (1801) showed that for $Q = x^2 + y^2$, $r_Q(N) = 4E_1(N;4)$ with $N \geq 1$. And, from the theory of theta functions we can further show that this is equivalent to

$$\frac{(q^2;q^2)_\infty^{10}}{(q)_\infty^4 (q^4;q^4)_\infty^4} = 1 + 4\sum_{N=1}^{\infty} E_1(N;4) q^N.$$

In §2 we consider various identities, that is, those whose coefficients in q-series are given by divisor functions. As a special example of Theorem 4, we can derive

$$\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_\infty^2(q;q^2)_\infty(q;q^2)_\infty} + 2\frac{(q^4;q^4)_\infty^2(-q;q^4)_\infty(-q^3;q^4)_\infty}{(-q^4;q^4)_\infty^2(q;q^4)_\infty(q^3;q^4)_\infty} = 3 + 8\sum_{N=1}^{\infty} E_1(N;8) q^N.$$

And, in §3 we will give several formulas of divisor functions for the Weierstrass $\wp$ function and the Eisenstein series g_2 and g_3.

Next, Farkas showed in [1] and [2] that for $N \geq 1$,

$$k \cdot \tau_{1;k,l}(N) = 2 \cdot \sigma_{0,\frac{k-l}{2},\frac{k+l}{2}}(N;k) + l \cdot E_{\frac{k-l}{2}}(N;k) + k \cdot \sum_{j=1}^{N-1} E_{\frac{k-l}{2}}(j;k) E_{\frac{k-l}{2}}(N-j;k). \tag{1}$$

In §4 we obtain a result analogous to Farkas' by adopting his idea, and in §5 by means of divisor functions we settle down the Sun's conjecture ([10]) on the number of representations of certain ternary quadratic forms and obtain some new properties of divisor functions.

2. SPECIAL EXAMPLES FOR DIVISOR FUNCTIONS

Among the Fine's list of identities for the basic hypergeometric series([3]) we introduce the following necessary and useful ones for later use, some of which seem to appear more than once on the list usually in a similar form.

$$\frac{(q)_\infty^9}{(q^3;q^3)_\infty^3} = 1 - 9\sum_{N=1}^{\infty}(\sigma_1^{\{2\}}(N;3) - \sigma_2^{\{2\}}(N;3))q^N. \quad ([3], \text{p.85}) \tag{2}$$

$$\begin{aligned} \frac{(q)_\infty^3}{(q^3;q^3)_\infty} &= 1 - 3\sum_{N=0}^{\infty}\{E_1(N;3) - 3E_1(N/3;3)\}q^N \\ &= 1 - 3\sum_{n=0}^{\infty} E_1(3n+1;3) q^{3n+1} + 6\sum_{n=1}^{\infty} E_1(3n;3) q^{3n} \\ &= 1 - 3\sum_{n=0}^{\infty} E_1(3n+1;3) q^{3n+1} + 6\sum_{n=1}^{\infty} E_1(n;3) q^{3n}. \quad ([3], \text{p.79}) \end{aligned} \tag{3}$$

$$\frac{(q^3;q^3)_\infty^3}{(q)_\infty} = \sum_{n=0}^{\infty} E_1(3n+1;3) q^n. \quad ([3], \text{p.79}) \tag{4}$$

$$\frac{(q^2;q^2)^{10}_\infty}{(q)^4_\infty(q^4;q^4)^4_\infty} = 1+4\sum_{N=1}^{\infty}E_1(N;4)q^N. \quad ([3],\text{p.78}) \tag{5}$$

$$\frac{(q^2;q^2)^{20}_\infty}{(q)^8_\infty(q^4;q^4)^8_\infty} = 1+8\sum_{N=1}^{\infty}q^N(2+(-1)^N)\sum_{\substack{\omega|N\\ \omega \text{ odd}}}\omega. \quad ([3],\text{ p.78}) \tag{6}$$

$$\frac{q(q^4;q^4)^8_\infty}{(q^2;q^2)^4_\infty} = \textstyle\sum_{N \text{ odd}}\sigma(N)q^N. \quad ([3],\text{ p.79}) \tag{7}$$

$$\frac{(q)^3_\infty(q^2;q^2)^3_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty} = 1-3\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6)-2\sigma_3(N;6)\}q^N. \quad ([3],\text{ p.84}) \tag{8}$$

$$\frac{q(q^3;q^3)^3_\infty(q^6;q^6)^3_\infty}{(q)_\infty(q^2;q^2)_\infty} = \sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6)+\frac{2}{3}\sigma_3(N;6)\}q^N. \quad ([3],\text{ p.86}) \tag{9}$$

$$\frac{(q)^4_\infty(q^3;q^3)^4_\infty}{(q^2;q^2)^2_\infty(q^6;q^6)^2_\infty} = 1-4\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6)-\sigma_{2,-2}(N;6)\}q^N. \quad ([3],\text{ p.85}) \tag{10}$$

$$\frac{q(q^2;q^2)^4_\infty(q^6;q^6)^4_\infty}{(q)^2_\infty(q^3;q^3)^2_\infty} = \sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;6)+\frac{1}{2}\sigma_{2,-2}(N;6)\}q^N. \quad ([3],\text{ p.87}) \tag{11}$$

$$\frac{(q^2;q^2)^3_\infty(q^3;q^3)^2_\infty(q^6;q^6)_\infty}{(q)^2_\infty(q^4;q^4)_\infty(q^{12};q^{12})_\infty} = 1+2\sum_{N=1}^{\infty}E_{1,5}(N;12)q^N. \quad ([3],\text{ p.80}) \tag{12}$$

$$\begin{aligned}\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)^2_\infty(q^6;q^6)^2_\infty}{(q^{12};q^{12})^2_\infty} \\ = 1-\sum_{N=1}^{\infty}\{\sigma_{1,-1}(N;12)-\sigma_5(N;12)\}q^N. \quad ([3],\text{ p.85})\end{aligned} \tag{13}$$

In [3] p.72 there is also an identity

$$\frac{(q^p;q^p)^2_\infty(-q^r;q^p)_\infty(-q^{p-r};q^p)_\infty}{(-q^p;q^p)^2_\infty(q^r;q^p)_\infty(q^{p-r};q^p)_\infty} = 1+2\sum_{N=1}^{\infty}E_{r,p-r}(N;2p)q^N. \tag{14}$$

Now, we deduce from (8) and (9) that

$$\begin{aligned}(15)\qquad & \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}+3\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}\\ =\ & 1+8\sum_{N=1}^{\infty}\sigma_3(N;6)q^N,\end{aligned}$$

and

$$\begin{aligned}(16)\qquad & \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}\\ =\ & 1-12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N.\end{aligned}$$

In particular, if $N=2^A3^Bp_1^{e_1}\cdots p_r^{e_r}q_1^{f_1}\cdots q_s^{f_s}$ for $p_i\equiv 1 \bmod 6$ and $q_j\equiv -1 \bmod 6$, then

$$\begin{aligned}(17)\qquad & \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}+3\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}\\ =\ & 1+8\sum_{N=1}^{\infty}\sigma_3(N;6)q^N\\ =\ & 1+8\sum_{N=1}^{\infty}q^N\frac{3^B-1}{3-1}\sigma(p_1^{e_1}\cdots p_r^{e_r}q_1^{f_1}\cdots q_s^{f_s}),\end{aligned}$$

and

$$\begin{aligned}(18)\qquad & \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}\\ =\ & 1-12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N\\ =\ & 1-12\sum_{N=1}^{\infty}q^N\sigma(p_1^{e_1}\cdots p_r^{e_r}q_1^{f_1}\cdots q_s^{f_s}).\end{aligned}$$

Similarly, by (10),(11),(13) and (15) we establish that

$$\begin{aligned}& \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}+12\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}{(q^{12};q^{12})_\infty^2}\\ =\ & 13-24\sum_{N=1}^{\infty}\sigma_{1,-1}(N;12)q^N,\end{aligned}$$

$$\begin{aligned}& \frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}-9\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}-12\frac{(q)_\infty(q^3;q^3)_\infty(q^4;q^4)_\infty^2(q^6;q^6)_\infty^2}{(q^{12};q^{12})_\infty^2}\\ =\ & -11-24\sum_{N=1}^{\infty}\sigma_{5,-5}(N;12)q^N,\end{aligned}$$

$$\frac{(q)_\infty^4(q^3;q^3)_\infty^4}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2} - 8\frac{q(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2}{(q)_\infty^2(q^3;q^3)_\infty^2} = 1 - 12\sum_{N=1}^{\infty}\sigma_{1,-1}(N;6)q^N,$$

and

$$(q^2;q^2)_\infty^4 = \frac{9}{8}(q)_\infty(q^3;q^3)^5(q^6;q^6)_\infty.$$

And, it follows from (3) and (8) that

$$\begin{aligned}
&\frac{(q)_\infty^3(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty}\\
= \;& 1 - 3\sum_{N\geq 1}\{\sigma_{1,-1}(N;6) - 2\sigma_3(N;6)\}q^N\\
= \;& (1 - 3\sum_{n\geq 0}E_1(3n+1;3)q^{3n+1} + 6\sum_{n\geq 1}E_1(3n;3)q^{3n})\\
& \times(1 - 3\sum_{n\geq 0}E_1(3n+1;3)q^{6n+2} + 6\sum_{n\geq 1}E_1(3n;3)q^{6n})\\
= \;& 1 - 3\sum_{n\geq 0}E_1(6n+1;3)q^{6n+1} - 3\sum_{n\geq 0}E_1(6n+4;3)q^{6n+4} + 6\sum_{n\geq 0}E_1(6n;3)q^{6n}\\
& +6\sum_{n\geq 0}E_1(6n+3;3)q^{6n+3} + 9\sum_{n\geq 0,\,N\geq 0}E_1(6n+1;3)E_1(3N+1;3)q^{6n+6N+3}\\
& +9\sum_{n\geq 0,\,N\geq 0}E_1(6n+4;3)E_1(3N+1;3)q^{6n+6N+6} - 18\sum_{n\geq 1,\,N\geq 0}E_1(6n;3)E_1(3N+1;3)q^{6n+6N+2}\\
& -18\sum_{n\geq 0,\,N\geq 0}E_1(6n+3;3)E_1(3N+1;3)q^{6n+6N+5} + 6\sum_{n\geq 1}E_1(3n;3)q^{6n} - 3\sum_{n\geq 0}E_1(3n+1;3)q^{6n+2}\\
& -18\sum_{n\geq 0,\,N\geq 1}E_1(6n+1;3)E_1(3N;3)q^{6n+6N+1} - 18\sum_{n\geq 0,\,N\geq 1}E_1(6n+4;3)E_1(3N;3)q^{6n+6N+4}\\
& +36\sum_{n\geq 1,\,N\geq 1}E_1(6n;3)E_1(3N;3)q^{6n+6N} + 36\sum_{n\geq 0,\,N\geq 1}E_1(6n;3)E_1(3N;3)q^{6n+6N+3}.
\end{aligned}$$

Therefore, we get some new identities of divisor functions from these results as follows.

Proposition 1. *For $M > 0$, we have*

(a) $\sigma_{1,5}(6M+1;6) = \sigma(6M+1) = E_1(6M+1;3) + 6\sum_{n=0}^{M-1}E_1(6n+1;3)E_1(M-n;3)$.

(b) $\sigma_{1,5}(6M+2;6) = E_1(3M+1;3) + 6\sum_{n=0}^{M-1}E_1(3n+1;3)E_1(2(M-n);3)$.

(c) $\sigma_{1,5}(6M+4;6) = E_1(6M+4;3) + 6\sum_{n=0}^{M-1}E_1(6n+4;3)E_1(M-n;3)$.

(d) $\sigma_{1,5}(6M+5;6) = \sigma(6M+5) = 6\sum_{n=0}^{M}E_1(2n+1;3)E_1(M-n;3)$.

(e) $2\sigma_3(6M+3;6) - \sigma_{1,5}(6M+3;6)$

$= 2E_1(2M+1;3) + 3\sum_{n=0}^{M}E_1(6n+1;3)E_1(3(M-n)+1;3)$
$+12\sum_{n=0}^{M}E_1(2n+1;3)E_1(M-n;3)$.

(f) $2\sigma_3(6M;6) - \sigma_{1,5}(6M;6)$

$= 2E_1(2M;3) + 2E_1(M;3) + 3\sum_{n=0}^{M-1}E_1(6n+4;3)E_1(3(M-n-1)+1;3)$
$+12\sum_{n=1}^{M-1}E_1(2n;3)E_1(M-n;3)$.

Next, we derive from (4) and (9) that

$$\begin{aligned}
&\frac{q(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty} \\
= &\sum_{N=1}^{\infty} q^N(\sigma_{1,5}(N;6)+\frac{2}{3}\sigma_3(N;6)) \\
= &q(\sum_{n=0}^{\infty} E_1(3n+1;3)q^n)(\sum_{m=0}^{\infty} E_1(3m+1;3)q^{2m}) \\
= &\sum_{N=0}^{\infty} q^{2N+1}\sum_{n=0}^{N} E_1(6n+1;3)E_1(3(N-n)+1;3) \\
&+\sum_{N=1}^{\infty} q^{2N}\sum_{n=0}^{N-1} E_1(6n+4;3)E_1(3(N-n-1)+1;3),
\end{aligned}$$

and this implies the following proposition.

Proposition 2. *(a)* $\sigma_{1,5}(6M+1;6)=\sigma(6M+1)=\sum_{n=0}^{3M} E_1(6n+1;3)E_1(9M-3n+1;3)$.

(b) $\sigma_{1,5}(6M+2;6)=\sum_{n=0}^{3M} E_1(6n+4;3)E_1(9M-3n+1;3)$.

(c) $\sigma_{1,5}(6M+4;6)=\sum_{n=0}^{3M+1} E_1(6n+4;3)E_1(9M-3n+4;3)$.

(d) $\sigma_{1,5}(6M+5;6)=\sigma(6M+5)=\sum_{n=0}^{3M+2} E_1(6n+1;3)E_1(9M-3n+7;3)$.

(e) $\frac{2}{3}\sigma_3(6M+3;6)+\sigma_{1,5}(6M+3;6)=\sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)$.

(f) $\frac{2}{3}\sigma_3(6M;6)+\sigma_{1,5}(6M;6)=\sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)$.

(g) $\sigma_3(6M+3;6)$

$$\begin{aligned}
= &\tfrac{3}{8}\{2E_1(2M+1;3)+3\sum_{n=0}^{M} E_1(6n+1;3)E_1(3(M-n)+1;3) \\
&+12\sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3)+\sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)\}.
\end{aligned}$$

(h) $\sigma_{1,5}(6M+3;6)$

$$\begin{aligned}
= &\tfrac{1}{4}\{-2E_1(2M+1;3)-3\sum_{n=0}^{M} E_1(6n+1;3)E_1(3(M-n)+1;3) \\
&-12\sum_{n=0}^{M} E_1(2n+1;3)E_1(M-n;3)+3\sum_{n=0}^{3M+1} E_1(6n+1;3)E_1(9M-3n+4;3)\}.
\end{aligned}$$

(i) $\sigma_3(6M;6)$

$$\begin{aligned}
= &\tfrac{3}{8}\{2E_1(2M;3)+2E_1(M;3)+3\sum_{n=0}^{M-1} E_1(6n+4;3)E_1(3(M-n-1)+1;3) \\
&+12\sum_{n=1}^{M-1} E_1(2n;3)E_1(M-n;3)+\sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)\}.
\end{aligned}$$

(j) For $M>1$, we have

$$\begin{aligned}
&\sigma_{1,5}(6M;6) \\
= &\tfrac{1}{4}\{-2E_1(2M;3)-2E_1(M;3)-3\sum_{n=0}^{M-1} E_1(6n+4;3)E_1(3(M-n-1)+1;3) \\
&-12\sum_{n=1}^{M-1} E_1(2n;3)E_1(M-n;3)+3\sum_{n=0}^{3M-1} E_1(6n+4;3)E_1(9M-3n+1;3)\}.
\end{aligned}$$

Here, we observe that

$$\sum_{\substack{d|N \\ d\equiv 1 \bmod 6}} 1-\sum_{\substack{d|N \\ d\equiv 5 \bmod 6}} 1 = \sum_{\substack{d|N \\ d\equiv 1 \bmod 12}} 1+\sum_{\substack{d|N \\ d\equiv 7 \bmod 12}} 1-\sum_{\substack{d|N \\ d\equiv 5 \bmod 12}} 1-\sum_{\substack{d|N \\ d\equiv 11 \bmod 12}} 1\ .$$

Then we readily see that

$$\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q)_\infty^3(q^6;q^6)_\infty^2} = 1+3\sum_{N=1}^\infty E_1(N;6)q^N = 1+3\sum_{N=1}^\infty E_{1,7}(N;12)q^N,$$

which gives the following proposition by (12).

Proposition 3. *(a)* $\frac{3}{5}\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}+\frac{2}{5}\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q)_\infty^3(q^6;q^6)_\infty^2} = 1+\frac{6}{5}\sum_{N=1}^\infty E_1(N;12)q^N$.

(b) $3\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}-2\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q)_\infty^3(q^6;q^6)_\infty^2} = 1+12\sum_{N=1}^\infty E_5(N;12)q^N.$

(c) $\frac{1}{3}\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}+\frac{2}{3}\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q)_\infty^3(q^6;q^6)_\infty^2} = 1-2\sum_{N=1}^\infty E_7(N;12)q^N.$

(d) $-\frac{1}{5}\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2(q^6;q^6)_\infty}{(q)_\infty^2(q^4;q^4)_\infty(q^{12};q^{12})_\infty}+\frac{6}{5}\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q)_\infty^3(q^6;q^6)_\infty^2} = 1.$

In general we can further derive the following identities.

Theorem 4. *For $n\geq 1$ we have*

(a)

$$\sum_{k=1}^n 2^{k-1}\frac{(q^{2^k};q^{2^k})_\infty^2(-q;q^{2^k})_\infty(-q^{2^k-1};q^{2^k})_\infty}{(-q^{2^k};q^{2^k})_\infty^2(q;q^{2^k})_\infty(q^{2^k-1};q^{2^k})_\infty} = (2^n-1)+2^{n+1}\sum_{N=1}^\infty E_1(N;2^{n+1})q^N.$$

(b)

$$2\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q;q)_\infty^3(q^6;q^6)_\infty^2}+3\sum_{k=1}^n 2^{k-1}\frac{(q^{3\cdot 2^k};q^{3\cdot 2^k})_\infty^2(-q;q^{3\cdot 2^k})_\infty(-q^{3\cdot 2^k-1};q^{3\cdot 2^k})_\infty}{(-q^{3\cdot 2^k};q^{3\cdot 2^k})_\infty^2(q;q^{3\cdot 2^k})_\infty(q^{3\cdot 2^k-1};q^{3\cdot 2^k})_\infty}$$

$$= (3\cdot 2^n-1)+3\cdot 2^{n+1}\sum_{N=1}^\infty E_1(N;3\cdot 2^{n+1})q^N.$$

Proof. Note that $E_r(N;p)=E_{r,p+r}(N;2p)$ and $E_{p+r}(N;2p)=-E_{p-r}(N;2p)$ imply $2E_r(N;2p)$ $=E_{r,p-r}(N;2p)+E_r(N;p)$. Hence, if we have a desired expression for $\sum_{N=1}^\infty E_r(N;p)q^N$, then we inductively get it for $\sum_{N=1}^\infty E_r(N;2p)q^N$ by the identity (14). Thus it suffices to obtain the initial expression. And, it follows from (14) with $p=2, r=1$ that

$$\frac{(q^2;q^2)_\infty^2(-q;q^2)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_\infty^2(q;q^2)_\infty(q;q^2)_\infty} = 1+4\sum_{N=1}^\infty E_1(N;4)q^N.$$

As for the second, we know by (32.42) in [3, p.80] that

$$\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q;q)_\infty^3(q^6;q^6)_\infty^2} = 1+3\sum_{N=1}^\infty E_1(N;6)q^N.$$

Therefore, we have the assertion inductively. □

By applying the Farkas' identity (1) to Theorem 4 we also have the following theorem.

Theorem 5. *For $n \geq 1$ we have*

$$\sum_{1\leq k,l\leq n} 2^{k+l-2}\frac{(q^{2^k};q^{2^k})_\infty^2(-q;q^{2^k})_\infty(-q^{2^k-1};q^{2^k})_\infty(q^{2^l};q^{2^l})_\infty^2(-q;q^{2^l})_\infty(-q^{2^l-1};q^{2^l})_\infty}{(-q^{2^k};q^{2^k})_\infty^2(q;q^{2^k})_\infty(q^{2^k-1};q^{2^k})_\infty(-q^{2^l};q^{2^l})_\infty^2(q;q^{2^l})_\infty(q^{2^l-1};q^{2^l})_\infty}$$

$$= (2^n-1)^2 + 2^{n+2}(2^n-1)q + 2^{n+2}\sum_{N=2}^{\infty}(2^n\tau_{1;2^{n+1},2^{n+1}-2}(N) - \sigma_{0,1,-1}(N;2^{n+1}))q^N.$$

Proof. The square of the right hand side of Theorem 4 (a) is equal to

$$(2^n-1)^2 + 2^{n+2}(2^n-1)\sum_{N=1}^{\infty}E_1(N;2^{n+1})q^N + 2^{2n+2}\sum_{N=2}^{\infty}\sum_{j=1}^{N-1}E_1(j;2^{n+1})E_1(N-j;2^{n+1})q^N.$$

And, the term $\sum_{j=1}^{N-1}E_1(j;2^{n+1})E_1(N-j;2^{n+1})$ can be replaced by using the identity (1). □

3. DIVISOR FUNCTIONS FOR WEIERSTRASS $\wp$ FUNCTION

Let $\Lambda_t = \mathbb{Z} + t\mathbb{Z}$ ($t \in \mathfrak{H}$ the complex upper half plane) be a lattice and $z \in \mathbb{C}$. The *Weierstrass $\wp$ function relative to Λ_t* is defined by the series

$$\wp(z;\Lambda_t) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda_t\\ \omega\neq 0}}\left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\},$$

and the *Eisenstein series of weight 2k for Λ_t* with $k > 1$ is the series

$$G_{2k}(\Lambda_t) = \sum_{\substack{\omega\in\Lambda_t\\ \omega\neq 0}}\omega^{-2k}.$$

We shall use the notations $\wp(z)$ and G_{2k} instead of $\wp(z;\Lambda_t)$ and $G_{2k}(\Lambda_t)$, respectively, when the lattice Λ_t has been fixed. Then the Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}.$$

As is customary, by setting

$$g_2(t) = g_2(\Lambda_t) = 60G_4 \quad\text{and}\quad g_3(t) = g_3(\Lambda_t) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(t)\wp(z) - g_3(t).$$

Proposition 6. *([8], [9]) (a)* $\wp\left(\frac{t}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2(q^2;q^2)_\infty^4(-q;q^2)_\infty^8.$

(b) $\wp\left(\frac{t+1}{2}\right) - \wp\left(\frac{1}{2}\right) = -\pi^2(q^2;q^2)_\infty^4(q;q^2)_\infty^8.$

(c) $\wp\left(\frac{t+1}{2}\right) - \wp\left(\frac{t}{2}\right) = 16\pi^2q(q^2;q^2)_\infty^4(-q^2;q^2)_\infty^8.$

Proposition 7. *([5],[6],[7]) (a)* $\wp\left(\frac{t}{2}\right) = -\frac{\pi^2}{3}((q^2;q^2)_\infty^4(-q;q^2)_\infty^8 + 16q(q^2;q^2)_\infty^4(q^2;q^2)_\infty^8)$.

(b) $\wp\left(\frac{t+1}{2}\right) = -\frac{\pi^2}{3}((q^2;q^2)_\infty^4(-q;q^2)_\infty^8 - 32q(q^2;q^2)_\infty^4(q^2;q^2)_\infty^8)$.

(c) $\wp\left(\frac{1}{2}\right) = \frac{2\pi^2}{3}((q^2;q^2)_\infty^4(-q;q^2)_\infty^8 - 8q(q^2;q^2)_\infty^4(q^2;q^2)_\infty^8)$.

(d) $g_2(t) = \frac{4\pi^4}{3}(q^2;q^2)_\infty^8((-q;q^2)_\infty^{16} - 16q(-q;q)_\infty^8 + 256q^2(-q^2;q^2)^{16})$.

(e)
$$\begin{aligned} g_3(t) &= \frac{8\pi^6}{27}(q^2;q^2)_\infty^{12}((-q;q^2)_\infty^{24} - 24q(-q;q^2)_\infty^{16}(-q^2;q^2)_\infty^8 \\ &\quad -384q^2(-q;q^2)_\infty^8(-q^2;q^2)^{16} + 4096q^3(-q^2;q^2)_\infty^{24}). \end{aligned}$$

Now, using (6), (7) and Proposition 7 we induce the following three identities for $\wp$:

$$\begin{aligned} K_1(q) &:= -\frac{3}{\pi^2}\wp\left(\frac{t}{2}\right) = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} + 16\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} \\ &= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N \\ &= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 16\sum_{N \text{ odd}}\sigma(N)q^N \\ &= 1 + 24\sum_{N=1}^{\infty} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega \\ &= 1 + 24\sum_{N=1}^{\infty}\sigma_1(N;2)q^N, \end{aligned} \tag{19}$$

$$\begin{aligned} K_2(q) &:= -\frac{3}{\pi^2}\wp\left(\frac{t+1}{2}\right) = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 32\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} \\ &= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega - 32\sum_{N \text{ odd}}\sigma(N)q^N \\ &= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N \\ \omega \text{ odd}}}\omega - 32\sum_{N \text{ odd}}\sigma(N)q^N \\ &= 1 + 24\sum_{N=1}^{\infty}(-1)^N q^N \sum_{\substack{\omega|N \\ \omega\text{odd}}}\omega \\ &= 1 + 24\sum_{N=1}^{\infty}(-1)^N\sigma_1(N;2)q^N, \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
K_3(q) &:= \frac{3}{2\pi^2}\wp\left(\frac{1}{2}\right) = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} \\
&= 1 + 8\sum_{N=1}^{\infty} q^N(2+(-1)^N)\sum_{\substack{\omega|N\\ \omega \text{ odd}}}\omega - 8\sum_{N \text{ odd}}\sigma(N)q^N \\
&= 1 + 8\sum_{N \text{ odd}} q^N \sum_{\substack{\omega|N\\ \omega \text{ odd}}}\omega + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N\\ \omega \text{ odd}}}\omega - 8\sum_{N \text{ odd}}\sigma(N)q^N \\
&= 1 + 24\sum_{N \text{ even}} q^N \sum_{\substack{\omega|N\\ \omega \text{ odd}}}\omega \\
&= 1 + 24\sum_{N \text{ even}} \sigma_1(N;2)q^N.
\end{aligned} \tag{21}
$$

Then, we achieve from (19)-(21) that

$$K_1(q^2) = K_3(q) \text{ and } K_1(-q) = K_2(q). \tag{22}$$

By (22) we see that

$$\frac{(q^4;q^4)_\infty^{20}}{(q^2;q^2)_\infty^8(q^8;q^8)_\infty^8} + 16\frac{q(q^8;q^8)_\infty^8}{(q^4;q^4)_\infty^4} = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}, \tag{23}$$

$$\frac{(q)_\infty^8}{(q^2;q^2)_\infty^4} + 48\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} = \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8}. \tag{24}$$

And, it follows from (23) and (24) that

$$\frac{(q^4;q^4)_\infty^{20}}{(q^2;q^2)_\infty^8(q^8;q^8)_\infty^8} + 16\frac{q(q^8;q^8)_\infty^8}{(q^4;q^4)_\infty^4} = \frac{(q)_\infty^8}{(q^2;q^2)_\infty^4} + 40\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}.$$

Therefore, we summarize the above as follows.

Theorem 8. *Let $S_1 := \sum_{N \text{ odd}} \sigma_1(N;2)q^N$ and $S_2 := \sum_{N \text{ even}} \sigma_1(N;2)q^N$. Then we get the followings:*

(a) $K_1(q) = 1 + 24S_1 + 24S_2$, $K_2(q) = 1 - 24S_1 + 24S_2$ *and* $K_3(q) = 1 + 24S_2$.

(b) $K_1(q^2) = K_3(q)$, $K_1(-q) = K_2(q)$.

(c)
$$
\begin{aligned}
\frac{(q^4;q^4)_\infty^{20}}{(q^2;q^2)_\infty^8(q^8;q^8)_\infty^8} + 16\frac{q(q^8;q^8)_\infty^8}{(q^4;q^4)_\infty^4} &= \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8} - 8\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4}, \\
\frac{(q)_\infty^8}{(q^2;q^2)_\infty^4} + 48\frac{q(q^4;q^4)_\infty^8}{(q^2;q^2)_\infty^4} &= \frac{(q^2;q^2)_\infty^{20}}{(q)_\infty^8(q^4;q^4)_\infty^8}.
\end{aligned}
$$

Using Proposition 7, (19), (20) and (21) we derive the identities for $g_2(t)$ and $g_3(t)$:

Corollary 9. *With the notations as above, we have*

(a)
$$
\begin{aligned}
\frac{3}{4\pi^4}g_2(t) &= \frac{(q^2;q^2)_\infty^{40}}{(q)_\infty^{16}(q^4;q^4)_\infty^{16}} - 16\frac{q(q^2;q^2)_\infty^{16}}{(q)_\infty^8} + 256\frac{q^2(q^4;q^4)_\infty^{16}}{(q^2;q^2)_\infty^8} \\
&= 1 + 192S_1^2 + 32S_2 + 384S_1S_2 + 448S_2^2.
\end{aligned}
$$

$$\text{(b)}\quad \begin{aligned}\frac{27}{8\pi^6}g_3(t) &= 1-576S_1^2-6144S_1^3-73728S_1^4-196608S_1^5+48S_2\\ &\quad -1152S_1S_2-39936S_1^2S_2-589824S_1^3S_2-2162688S_1^4S_2+192S_2^2\\ &\quad -61440S_1S_2^2-1622016S_1^2S_2^2-9043968S_1^3S_2^2-23552S_2^3\\ &\quad -1769472S_1S_2^3-17694720S_1^2S_2^3-663552S_2^4-15925248S_1S_2^4-5308416S_2^5.\end{aligned}$$

4. DIVISOR FUNCTIONS ARISING FROM THETA SERIES

As seen in the identity (1), Farkas provided a relation between the sums of divisors satisfying congruence conditions and the sums of numbers of divisors satisfying congruence conditions. In the proof he took logarithmic derivative to theta functions and used the heat equation. And we applied his result to prove Theorem 5. In this section, however, we obtain a similar result by differentiating further. More precisely,

Theorem 10. *For any $n \geq 1$, we have*

$$k\cdot\tau_{2;k,l}(n) = 2n\cdot E_{\frac{k-l}{2}}(n;k) + l\cdot\tau_{1;k,l}(n) + 2k\cdot\sum_{j=1}^{n-1}E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j).$$

Let $k=3, l=1, n=3m+2$. Then $\tau_{1;3,1}(3m+2)=\sigma(3m+2)$ and $\tau_{2;3,1}(3m+2)=\sum_{0<d|(3m+2)}d^2\chi(d)$, where σ is the usual divisor sum function and $\chi(d)$ is defined by $\chi(d)=1,-1,0$ according as $d\equiv 1,-1,0 \bmod 3$. Since $E_1(3j;3)=E_1(j;3)$ and $E_1(3j+2;3)=0$,

$$\begin{aligned}&3\tau_{2;3,1}(3m+2)\\ =\ &\tau_{1;3,1}(3m+2)+6\cdot\sum_{j=0}^{m}(E_1(3j;3)\tau_{1;3,1}(3(m-j)+2)\\ &+E_1(3j+1;3)\tau_{1;3,1}(3(m-j)+1)+E_1(3j+2;3)\tau_{1;3,1}(3(m-j)))\\ =\ &\sigma(3m+2)+6\sum_{j=0}^{m}E_1(j;3)\sigma(3(m-j)+2)+6\sum_{j=0}^{m}E_1(3j+1;3)\sigma(3(m-j)+1).\end{aligned}$$

Thus the above equation is equivalent to

$$\begin{aligned}&3\sum_{m=0}^{\infty}\tau_{2;3,1}(3m+2)q^m\\ =\ &\sum_{m=0}^{\infty}\sigma(3m+2)q^m+6(\sum_{n=0}^{\infty}E_1(n;3)q^n)\cdot(\sum_{n=0}^{\infty}\sigma(3n+2)q^n)\\ &+6(\sum_{n=0}^{\infty}E_1(3n+1;3)q^n)\cdot(\sum_{n=0}^{\infty}\sigma(3n+1)q^n).\end{aligned}$$

Proof. Let $u=e^{2\pi iz}$, $q=e^{2\pi it}$ with $z\in\mathbb{C}$ and $t\in\mathfrak{h}$, and $s=\frac{l}{k}\in\mathbb{Q}$ with positive integers l,k such that $k\equiv l \bmod 2$, $k\geq 3$, and $l\leq k-2$. Then we define theta functions as

$$\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)=e^{\pi is/2}q^{s^2/8}u^{s/2}(q)_\infty(q^{\frac{1+s}{2}}u)_\infty(q^{\frac{1-s}{2}}u^{-1})_\infty.$$

By taking logarithmic derivative with respect to z we have

$$\frac{1}{2\pi i}\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)=\frac{s}{2}+\sum_{n=0}^{\infty}\Big(\frac{-q^{n+\frac{1+s}{2}}u}{1-q^{n+\frac{1+s}{2}}u}+\frac{q^{n+\frac{1-s}{2}}u^{-1}}{1-q^{n+\frac{1-s}{2}}u^{-1}}\Big).$$

If we differentiate more with respect to z, we get

$$\begin{aligned}\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t) &= -\sum_{n=0}^{\infty}\Big(\frac{q^{n+\frac{1+s}{2}}u}{(1-q^{n+\frac{1+s}{2}}u)^2}+\frac{q^{n+\frac{1+s}{2}}u^{-1}}{(1-q^{n+\frac{1+s}{2}}u^{-1})^2}\Big),\\ \frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t) &= -\sum_{n=0}^{\infty}\Big(\frac{q^{n+\frac{1+s}{2}}u+(q^{n+\frac{1+s}{2}}u)^2}{(1-q^{n+\frac{1+s}{2}}u)^3}-\frac{q^{n+\frac{1-s}{2}}u^{-1}+(q^{n+\frac{1-s}{2}}u^{-1})^2}{(1-q^{n+\frac{1-s}{2}}u^{-1})^3}\Big).\end{aligned}$$

We now evaluate at $z=0$ to obtain

$$\begin{aligned}\frac{1}{2\pi i}\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,kt)|_{z=0} &= \frac{l}{2k}+\sum_{n=0}^{\infty}\Big(\sum_{m=1}^{\infty}q^{(kn+\frac{k-l}{2})m}-\sum_{m=1}^{\infty}q^{(kn+\frac{k+l}{2})m}\Big)\\ &= \frac{l}{2k}+\sum_{n=1}^{\infty}E_{\frac{k-l}{2}}(n;k)q^n,\end{aligned}\tag{25}$$

$$\begin{aligned}\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,kt)|_{z=0} &= -\sum_{n=0}^{\infty}\Big(\sum_{m=1}^{\infty}mq^{(kn+\frac{k-l}{2})m}+\sum_{m=1}^{\infty}mq^{(kn+\frac{k+l}{2})m}\Big)\\ &= -\sum_{n=0}^{\infty}\tau_{1;k,l}(n)q^n,\end{aligned}\tag{26}$$

and

$$\begin{aligned}\frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,kt)|_{z=0} &= -\sum_{n=0}^{\infty}\Big(\sum_{m=1}^{\infty}m^2q^{(kn+\frac{k+l}{2})m}-\sum_{m=1}^{\infty}m^2q^{(kn+\frac{k-l}{2})m}\Big)\\ &= \sum_{n=1}^{\infty}\tau_{2;k,l}(n)q^n.\end{aligned}\tag{27}$$

Here we recall that theta functions satisfy the heat equation, that is,

$$\frac{\partial^2}{\partial z^2}\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)=4\pi i\frac{\partial}{\partial t}\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t).$$

Hence we achieve

$$\begin{aligned}\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t) &= \frac{\theta''\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}-\Big(\frac{\theta'\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}\Big)^2\\ &= 4\pi i\frac{\partial}{\partial\tau}\frac{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}-\Big(\frac{\theta'\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}{\theta\left[\begin{smallmatrix}s\\1\end{smallmatrix}\right](z,t)}\Big)^2\\ &= 4\pi i\frac{\partial}{\partial t}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)-\Big(\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)\Big)^2.\end{aligned}$$

Here the symbol $'$ stands for the partial derivative with respect to z.

If we take logarithmic derivative to theta function with respect to τ, then we derive

$$\begin{aligned}&\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)\\ =\ &\frac{s^2}{8}-\sum_{n=0}^{\infty}\Big(\frac{(n+1)q^{n+1}}{1-q^{n+1}}+\frac{(n+\frac{1+s}{2})q^{n+\frac{1+s}{2}}u}{1-q^{n+\frac{1+s}{2}}u}+\frac{(n+\frac{1-s}{2})q^{n+\frac{1-s}{2}}u^{-1}}{1-q^{n+\frac{1-s}{2}}u^{-1}}\Big).\end{aligned}$$

So we establish

$$\begin{aligned}&\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)\\ =\ &-2(2\pi i)^3\sum_{n=0}^{\infty}\Big((n+\frac{1+s}{2})\frac{q^{n+\frac{1+s}{2}}u}{(1-q^{n+\frac{1+s}{2}}u)^2}-(n+\frac{1-s}{2})\frac{q^{n+\frac{1-s}{2}}u^{-1}}{(1-q^{n+\frac{1-s}{2}}u^{-1})^2}\Big)\\ &-2\frac{\partial}{\partial z}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t)\cdot\frac{\partial^2}{\partial z^2}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,t).\end{aligned}$$

Thus by evaluating at $z=0$ we have

$$\begin{aligned}&\frac{1}{(2\pi i)^3}\frac{\partial^3}{\partial z^3}\log\theta\begin{bmatrix}s\\1\end{bmatrix}(z,kt)|_{z=0}\\ =\ &2\Big(\frac{1}{k}\sum_{n=0}^{\infty}((kn+\frac{k-l}{2})\frac{q^{kn+\frac{k-l}{2}}}{(1-q^{kn+\frac{k-l}{2}})^2}-(kn+\frac{k+l}{2})\frac{q^{kn+\frac{k+l}{2}}}{(1-q^{kn+\frac{k+l}{2}})^2}\Big)\\ &+\Big(\frac{l}{2k}+\sum_{n=1}^{\infty}E_{\frac{k-l}{2}}(n;k)q^n\Big)\Big(\sum_{n=0}^{\infty}\tau_{1;k,l}(n)q^n\Big)\Big),\end{aligned}$$

which implies

$$\begin{aligned}&\sum_{n=1}^{\infty}\tau_{2;k,l}(n)q^n\\ =\ &\frac{2}{k}\Big(\sum_{n=0}^{\infty}n\big(\sum_{\substack{0<d\mid n\\ d\equiv\frac{k-l}{2}\ (k)}}1\big)q^n-\sum_{n=0}^{\infty}n\big(\sum_{\substack{0<d\mid n\\ d\equiv\frac{k+l}{2}\ (k)}}1\big)q^n\Big)+\Big(\frac{l}{2k}+\sum_{n=1}^{\infty}E_{\frac{k-l}{2}}(n;k)q^n\Big)\Big(\sum_{n=0}^{\infty}\tau_{1;k,l}(n)q^n\Big)\\ =\ &\frac{2}{k}\sum_{n=1}^{\infty}nE_{\frac{k-l}{2}}(n;k)q^n+\frac{l}{k}\sum_{n=1}^{\infty}\tau_{1;k,l}(n)q^n+2\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{n-1}E_{\frac{k-l}{2}}(j;k)\tau_{1;k,l}(n-j)\Big)q^n.\end{aligned}$$

This completes the proof of theorem. □

5. DIVISOR FUNCTIONS AND SUN'S CONJECTURE

We first recall some well-known facts (for instance, see [3]) about a quantitative problem of quadratic forms over $\mathbb{Z}$ in terms of divisor functions.

Proposition 11. *Let N be a positive integer.*

(1) When $Q_1(x,y)=x^2+3y^2$, $r_{Q_1}(N)=\begin{cases}2E_1(N;3) & \text{if } N \text{ is odd}\\ 6E_1(N;3) & \text{if } N \text{ is even.}\end{cases}$

(2) If $Q_2(x,y,z,w) = x^2+y^2+3z^2+3w^2$, *then* $r_{Q_2}(N) = (-1)^{N-1}\cdot 4\cdot \sum_{d|N} d\cdot\chi(d)$ *where* $\chi(d) = 1, -1,$ *and* 0 *according to* $d \equiv \pm 1 \bmod 6$, $d \equiv \pm 2 \bmod 6$, *and* $d \equiv 0 \bmod 3$, *respectively.*

(3) For $Q_3(x,y) = x^2+y^2$, $r_{Q_3}(N) = 4E_1(N;4)$.

(4) For $Q_1'(x,y) = x^2+2y^2$, $r_{Q_1'}(N) = 2E_{1,3}(N;8)$.

(5) If $Q_2'(x,y,z,w) = x^2+y^2+2z^2+2w^2$, *then* $r_{Q_2'}(N) = 4k(\alpha)\sum_{d|m} d$ *where* $N = 2^\alpha m$ *with* $(2,m)=1$ *and* $k(0)=1$, $k(1)=2$, $k(\alpha)=6$ *for* $\alpha \geq 2$.

Here we note that $E_1(N;3) = \sum_{d|N}(\frac{d}{3})$ is a multiplicative function, that is, for $(N,M)=1$, $E_1(NM;3) = E_1(N;3)E_1(M;3)$. Since $E_1(2^{2n};3) = 1$ and $E_1(2^{2n+1};3) = 0$ for any nonnegative integer n, we also get $E_1(4N;3) = E_1(N;3)$ by observing that E_1 is multiplicative. Now, we are ready to prove the Sun's conjecture on the number of representations of certain ternary quadratic forms without utilizing standard approach via local densities.

Proposition 12 (Sun's conjecture [10])**.** *Let* $Q(x,y,z) = x^2+y^2+3z^2$, *and* $Q'(x,y,z) = x^2+y^2+2z^2$. *Then we have*

(1) $r_Q(p^2) = 4(p+1-(\frac{p}{3}))$ *for any prime* $p \geq 5$.

(2) $r_{Q'}(p^2) = 4(p+1-(\frac{-2}{p}))$ *for any prime* $p \geq 3$.

Proof. (1) Since p is odd, $p^2 \equiv 1 \bmod 8$. Thus $p^2 = x^2+y^2+3z^2$ implies that $(x,y,z) =$ (odd, even, even) or (even, odd, even). Therefore we derive that

$$\begin{aligned}
r_Q(p^2) &= 4\sum_{\substack{1\le i\le p\\ i=\text{ odd}}} r_{Q_1}(p^2-i^2) \;=\; 4\Big(\sum_{\substack{1\le i<p\\ i=\text{ odd}}} r_{Q_1}(p^2-i^2)+1\Big)\\
&= 4\Big(6\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_1(p^2-i^2;3)+1\Big) \quad \text{by Lemma 11-(1)}\\
&= 4\Big(6\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_1(\tfrac{p-i}{2}\cdot\tfrac{p+i}{2};3)+1\Big), \quad \text{because } E_1(4N;3)=E_1(N;3)\\
&= 4\Big(6\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_1(\tfrac{p-i}{2};3)E_1(\tfrac{p+i}{2};3)+1\Big), \quad \text{since } (\tfrac{p-i}{2},\tfrac{p+i}{2})=1\\
&= 4\Big(6\sum_{j=1}^{\frac{p-1}{2}} E_1(j;3)E_1(p-j;3)+1\Big) \;=\; 4\Big(3\sum_{j=1}^{p-1} E_1(j;3)E_1(p-j;3)+1\Big)\\
&= \sum_{j=1}^{p-1} 12E_1(j;3)E_1(p-j;3)+4 \;=\; \sum_{j=1}^{p-1} r_{Q_1}(j)r_{Q_1}(p-j)+4\\
&= \sum_{j=0}^{p} r_{Q_1}(j)r_{Q_1}(p-j)+4-2r_{Q_1}(p) \;=\; r_{Q_2}(p)+4-2r_{Q_1}(p)\\
&= 4(p+1-(\tfrac{p}{3})) \quad \text{by Lemma 11-(1) and (2).}
\end{aligned}$$

(2) In a similar way we see that $E_{1,3}(N;8) = \sum_{0<d|N}(\frac{-2}{d})$ is a multiplicative function and $E_{1,3}(2^\alpha;8) = 1$. Then we claim that

$$\begin{aligned}
r_{Q'}(p^2) &= 4\sum_{\substack{1\le i\le p\\ i=\text{ odd}}} r_{Q_1'}(p^2-i^2) \;=\; 4\Big(\sum_{\substack{1\le i<p\\ i=\text{ odd}}} r_{Q_1'}(p^2-i^2)+1\Big)\\
&= 4\Big(2\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_{1,3}(p^2-i^2;8)+1\Big) \quad \text{by Lemma 11-(4)}\\
&= 4\Big(2\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_{1,3}(\tfrac{p-i}{2}\cdot\tfrac{p+i}{2};8)+1\Big), \quad \text{because } E_{1,3}(2^\alpha N;8)=E_{1,3}(N;8)\\
&= 4\Big(2\sum_{\substack{1\le i<p\\ i=\text{ odd}}} E_{1,3}(\tfrac{p-i}{2};8)E_{1,3}(\tfrac{p+i}{2};8)+1\Big) \;=\; 4\Big(2\sum_{j=1}^{\frac{p-1}{2}} E_{1,3}(j;8)E_{1,3}(p-j;8)+1\Big)\\
&= 4\Big(\sum_{j=1}^{p-1} E_{1,3}(j;8)E_{1,3}(p-j;8)+1\Big) \;=\; \sum_{j=1}^{p-1} r_{Q_1'}(j)r_{Q_1'}(p-j)+4\\
&= \sum_{j=0}^{p} r_{Q_1'}(j)r_{Q_1'}(p-j)+4-2r_{Q_1'}(p) \;=\; r_{Q_2'}(p)+4-2r_{Q_1'}(p)\\
&= 4(p+1-(\tfrac{-2}{p})).
\end{aligned}$$

□

As its application we have the following corollary.

Corollary 13. *Let $[r]$ be the greatest integer less than or equal to $r \in \mathbb{R}$.*
(1) For any prime $p \geq 5$,

$$\sum_{k=1}^{\frac{p-1}{2}} \big(3E_1(p^2-(2k-1)^2;3)+E_1(p^2-4k^2;3)\big) = \begin{cases} p-2, & p \equiv 1 \bmod 3 \\ p+1, & p \equiv 2 \bmod 3, \end{cases}$$

$$\sum_{k=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2-3k^2;4) = \begin{cases} \frac{p-3}{2}, & p \equiv 1 \bmod 12 \\ \frac{p-1}{2}, & p \equiv \pm 5 \bmod 12 \\ \frac{p+1}{2}, & p \equiv 11 \bmod 12. \end{cases}$$

(2) For any prime $p \geq 3$,

$$\sum_{k=1}^{p-1} E_{1,3}(p^2-k^2;8) = \begin{cases} p-2, & p \equiv 1,3 \bmod 8 \\ p+1, & p \equiv 5,7 \bmod 8, \end{cases}$$

$$\sum_{k=1}^{[\frac{p}{\sqrt{2}}]} E_1(p^2-2k^2;4) = \begin{cases} \frac{p-3}{2}, & p \equiv 1 \bmod 8 \\ \frac{p-1}{2}, & p \equiv \pm 3 \bmod 8 \\ \frac{p+1}{2}, & p \equiv -1 \bmod 8. \end{cases}$$

Proof. (1) It follows from by Lemma 11-(1) that

$$\begin{aligned} r_Q(p^2) &= \sum_{i=-p}^{p} r_{Q_1}(p^2-i^2) = r_{Q_1}(p^2) + 2\sum_{i=1}^{p-1} r_{Q_1}(p^2-i^2) + 2r_{Q_1}(0) \\ &= 2E_1(p^2;3) + 2 + 2\{6\sum_{\substack{i=1 \\ \text{odd}}}^{p-2} E_1(p^2-i^2;3) + 2\sum_{\substack{i=1 \\ \text{even}}}^{p-1} E_1(p^2-i^2;3)\}. \end{aligned}$$

On the other hand, we know by Proposition 12-(1) that $r_Q(p^2) = 4(p+1-(\frac{p}{3}))$. Hence we get that

$$\begin{aligned} \textstyle\sum_{k=1}^{\frac{p-1}{2}}\{3E_1(p^2-(2k-1)^2;3)+E_1(p^2-4k^2;3)\} &= (p+1-(\tfrac{p}{3})) - \tfrac{1}{2}E_1(p^2;3) - \tfrac{1}{2} \\ &= p - \tfrac{1}{2}(1+3(\tfrac{p}{3})), \end{aligned}$$

from which we derive the first statement.

Next, we obtain by Lemma 11-(3) and Proposition 12-(1) that

$$\begin{aligned} r_Q(p^2) &= \sum_{i=-[\frac{p}{\sqrt{3}}]+1}^{[\frac{p}{\sqrt{3}}]} r_{Q_3}(p^2-3i^2) = 4E_1(p^2;4) + 8\sum_{i=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2-3i^2;4) \\ &= 4(p+1-(\frac{p}{3})). \end{aligned}$$

Since $E_1(N;4) = \sum_{0<d|N} (\frac{-1}{d})$,

$$\begin{aligned}\sum_{k=1}^{[\frac{p}{\sqrt{3}}]} E_1(p^2 - 3k^2; 4) &= \tfrac{1}{2}(p + 1 - (\tfrac{p}{3}) - E_1(p^2; 4)) \\ &= \tfrac{1}{2}(p - 1 - (\tfrac{p}{3}) - (\tfrac{-1}{p})).\end{aligned}$$

This enables us to conclude the second statement.

(2) We first observe by Lemma 11-(4) and Proposition 12-(2) that

$$\begin{aligned}r_{Q'}(p^2) &= \sum_{i=-p}^{p} r_{Q'_1}(p^2 - i^2) = r_{Q'_1}(p^2) + 2\sum_{i=1}^{p-1} r_{Q'_1}(p^2 - i^2) + 2r_{Q'_1}(0) \\ &= 2E_{1,3}(p^2; 8) + 2 + 4\sum_{i=1}^{p-1} E_{1,3}(p^2 - i^2; 8) \\ &= 4(p + 1 - (\frac{-2}{p})),\end{aligned}$$

and by Lemma 11-(3) and Proposition 12-(2) that

$$\begin{aligned}r_{Q'}(p^2) &= \sum_{i=-[\frac{p}{\sqrt{2}}]+1}^{[\frac{p}{\sqrt{2}}]} r_{Q_3}(p^2 - 2i^2) \\ &= 4E_1(p^2; 4) + 8\sum_{i=1}^{[\frac{p}{\sqrt{2}}]} E_1(p^2 - 2i^2; 4) \\ &= 4(p + 1 - (\frac{-2}{p})).\end{aligned}$$

Therefore, we deduce the conclusions by similar arguments as in (1). □

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