ON SOME RING CLASS INVARIANTS OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let K be an imaginary quadratic field different from $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ and let $\mathcal{O}_K = [\theta, 1]$ be its ring of integers with $\mathrm{Im}(\theta) > 0$. By $H_\mathcal{O}$ we mean the ring class field of the order $\mathcal{O} = [p^\ell\theta, 1]$ for a prime p and an integer $\ell \geq 1$. We show that if p is inert or ramified in $K/\mathbb{Q}$, then the real algebraic integer $p^{12}\frac{\Delta(p^\ell\theta)}{\Delta(p^{\ell-1}\theta)}$ generates $H_\mathcal{O}$ over K where Δ is the discriminant function.

1. INTRODUCTION

Given an imaginary quadratic field K with discriminant d_K and $\mathcal{O}_K = [\theta, 1]$, let $\mathcal{O} = [N\theta, 1]$ be the order of conductor $N(\geq 1)$ in K. Then it is a classical result by the main theorem of complex multiplication of elliptic curves that for any proper fractional $\mathcal{O}$-ideal $\mathfrak{a}$, the j-invariant $j(\mathfrak{a})$ generates the ring class field $H_\mathcal{O}$ over K as an algebraic integer([9]). Those ring class fields over imaginary quadratic fields play an important role in number theory, in particular in the study of certain quadratic Diophantine equations (refer to [3]).

Unlike the classical case, however, Chen-Yui([1]) constructed a generator of the ring class field of certain conductor in terms of the singular value of the Thompson series which is a Hauptmodul for $\Gamma_0(N)$ or $\Gamma_0^\dagger(N)$, where $\Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod N\}$ and $\Gamma_0^\dagger(N) = \langle \Gamma_0(N), \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{smallmatrix}\right)\rangle$ in $\mathrm{SL}_2(\mathbb{R})$. In like manner, Cox-Mckay-Stevenhagen([4]) showed that certain sigular value of a Hauptmodul for $\Gamma_0(N)$ or $\Gamma_0^\dagger(N)$ with rational Fourier coefficients generates $H_\mathcal{O}$ over K. And, Cho-Koo([2]) recently revisited and further extended their results by using the theory of Shimura's canonical models and his reciprocity law.

On the other hand, Jung-Koo-Shin([7]) tried to construct better invariants for practical use in terms of the singular values of Siegel functions under some conditions on discriminant and conductor and presented a systematic way of finding their minimal polynomials. More precisely, for any pair $(r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we define a *Siegel function* $g_{(r_1,r_2)}(\tau)$ by the following Fourier expansion

$$(1.1)\qquad g_{(r_1,r_2)}(\tau) = -q_\tau^{\frac{1}{2}\mathbf{B}_2(r_1)} e^{\pi i r_2(r_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n q_z^{-1}) \quad (\tau \in \mathfrak{H})$$

where $\mathbf{B}_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial, $q_\tau = e^{2\pi i\tau}$ and $q_z = e^{2\pi i z}$ with $z = r_1\tau + r_2$. Then it is a modular unit which has no zeros and poles on $\mathfrak{H}$([8]). Jung et al. showed in [7] that the singular value

$$(1.2)\qquad \prod_{\substack{1\leq w\leq \frac{N}{2}\\ \gcd(w,N)=1}} g_{(0,\frac{w}{N})}^{12N}(\theta)$$

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as a real algebraic integer generates $H_{\mathcal{O}}$ over K under the condition

$$d_K \leq -43 \quad \text{and} \quad 2 \leq N \leq \frac{-\sqrt{3}\pi}{\log\left(1 - 2.16e^{-\frac{\pi\sqrt{-d_K}}{24}}\right)}. \tag{1.3}$$

In this paper, however, we will prove that when $N = p^\ell(\ell \geq 1)$ for a prime p which is inert or ramified in $K/\mathbb{Q}$, the singular value in (1.2) generates $H_{\mathcal{O}}$ over $K(\neq \mathbb{Q}(\sqrt{-1}),\ \mathbb{Q}(\sqrt{-3}))$ without the condition (1.3). Thus by squaring and then taking N-th root (or, just taking square root if $N = 2$) of the value in (1.2) we will get the following ring class invariant

$$p^{12}\frac{\Delta(p^\ell\theta)}{\Delta(p^{\ell-1}\theta)} \tag{1.4}$$

where

$$\Delta(\tau) = (2\pi i)^{12} q_\tau \prod_{n=1}^{\infty}(1 - q_\tau^n)^{24} \quad (\tau \in \mathfrak{H}) \tag{1.5}$$

is the discriminant function(Theorem 2.6).

2. Generators of class fields with conductors of prime power

In what follows we let K be an imaginary quadratic field with discriminant d_K and $\mathcal{O}_K = [\theta, 1]$ be its ring of integers with $\theta \in \mathfrak{H}$. For a nonzero integral ideal $\mathfrak{f}$ of K we denote by $\mathrm{Cl}(\mathfrak{f})$ the ray class group of conductor $\mathfrak{f}$ and write C_0 for its unit class. If $\mathfrak{f} \neq \mathcal{O}_K$ and $C \in \mathrm{Cl}(\mathfrak{f})$, then we take an integral ideal $\mathfrak{c}$ in C so that $\mathfrak{f}\mathfrak{c}^{-1} = [z_1, z_2]$ with $z = \frac{z_1}{z_2} \in \mathfrak{H}$. Now we define the *Siegel-Ramachandra invariant* by

$$g_{\mathfrak{f}}(C) = g^{12N}_{(\frac{a}{N}, \frac{b}{N})}(z)$$

where N is the smallest positive integer in $\mathfrak{f}$ and $a,\ b \in \mathbb{Z}$ such that $1 = \frac{a}{N}z_1 + \frac{b}{N}z_2$. This value depends only on the class C and belongs to the ray class field $K_{\mathfrak{f}}$ modulo $\mathfrak{f}$ of K. Furthermore, we have a well-known transformation formula

$$g_{\mathfrak{f}}(C_1)^{\sigma(C_2)} = g_{\mathfrak{f}}(C_1C_2) \tag{2.1}$$

for $C_1,\ C_2 \in \mathrm{Cl}(\mathfrak{f})$ where σ is the Artin map([8] §11.1).

Let χ be a character of $\mathrm{Cl}(\mathfrak{f})$. We then denote by $\mathfrak{f}_\chi$ the conductor of χ and let χ_0 be the proper character of $\mathrm{Cl}(\mathfrak{f}_\chi)$ corresponding to χ. For a nontrivial character χ of $\mathrm{Cl}(\mathfrak{f})$ with $\mathfrak{f} \neq \mathcal{O}_K$ we define

$$S_{\mathfrak{f}}(\chi, g_{\mathfrak{f}}) = \sum_{C \in \mathrm{Cl}(\mathfrak{f})} \chi(C) \log|g_{\mathfrak{f}}(C)| \quad \text{and} \quad L_{\mathfrak{f}}(s, \chi) = \sum_{\substack{\mathfrak{a} \neq 0:\text{integral ideals} \\ \gcd(\mathfrak{a}, \mathfrak{f}) = \mathcal{O}_K}} \frac{\chi(\mathfrak{a})}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})^s} \quad (s \in \mathbb{C}).$$

If $\mathfrak{f}_\chi \neq \mathcal{O}_K$, then we see from the second Kronecker limit formula that

$$L_{\mathfrak{f}_\chi}(1, \chi_0) = T_0 S_{\mathfrak{f}_\chi}(\bar{\chi}_0, g_{\mathfrak{f}_\chi})$$

where T_0 is a nonzero constant depending on χ_0([9] §22.2 Theorem 2). Here we observe that the value $L_{\mathfrak{f}_\chi}(1, \chi_0)$ is nonzero([5] IV Proposition 5.7). Moreover, multiplying the above relation by the Euler factors we derive the identity

$$\prod_{\mathfrak{p}|\mathfrak{f},\ \mathfrak{p}\nmid\mathfrak{f}_\chi} (1 - \bar{\chi}_0(\mathfrak{p})) L_{\mathfrak{f}_\chi}(1, \chi_0) = T S_{\mathfrak{f}}(\bar{\chi}, g_{\mathfrak{f}}) \tag{2.2}$$

where T is a nonzero constant depending on $\mathfrak{f}$ and χ([8] p. 244).

Theorem 2.1. *Let L be an abelian extension of K with $[L:K] > 2h_K$ where h_K is the absolute class number of K. Assume that the conductor of the extension L/K is a power of prime ideal, namely $\mathfrak{f} = \mathfrak{p}^n (n \geq 1)$. Then the value*

$$\varepsilon = \mathbf{N}_{K_{\mathfrak{f}}/L}(g_{\mathfrak{f}}(C_0))$$

generates L over K.

Proof. We identify $\mathrm{Gal}(K_{\mathfrak{f}}/K)$ with $\mathrm{Cl}(\mathfrak{f})$ via the Artin map. Letting $F = K(\varepsilon)$ we deduce

$$\#\{\text{characters } \chi \text{ of } \mathrm{Cl}(\mathfrak{f}) : \ \chi|_{\mathrm{Gal}(K_{\mathfrak{f}}/L)} = 1 \quad \text{and} \quad \chi|_{\mathrm{Gal}(K_{\mathfrak{f}}/F)} \neq 1\} = [L:K] - [F:K]. \tag{2.3}$$

Furthermore, if we let H be the Hilbert class field of K, then we have

$$\#\{\text{characters } \chi \text{ of } \mathrm{Cl}(\mathfrak{f}) : \ \mathfrak{f}_\chi = \mathcal{O}_K\} = \#\{\chi \ : \ \chi|_{\mathrm{Gal}(K_{\mathfrak{f}}/H)} = 1\} = h_K. \tag{2.4}$$

Suppose that F is properly contained in L. Then we deduce

$$[L:K] - [F:K] = [L:K]\left(1 - \frac{1}{[L:F]}\right) > 2h_K\left(1 - \frac{1}{2}\right) = h_K$$

by the hypothesis $[L:K] > 2h_K$. Thus there exists a character ψ of $\mathrm{Cl}(\mathfrak{f})$ such that

$$\psi|_{\mathrm{Gal}(K_{\mathfrak{f}}/L)} = 1, \quad \psi|_{\mathrm{Gal}(K_{\mathfrak{f}}/F)} \neq 1 \quad \text{and} \quad \mathfrak{f}_\psi \neq \mathcal{O}_K$$

by (2.3) and (2.4). Moreover, since $\mathfrak{f} = \mathfrak{p}^n$, we get $\mathfrak{f}_\psi = \mathfrak{p}^m$ for some $1 \leq m \leq n$. Hence we obtain by (2.2) that

$$0 \neq L_{\mathfrak{f}_\psi}(1, \psi_0) = TS_{\mathfrak{f}}(\bar{\psi}, g_{\mathfrak{f}})$$

for a nonzero constant T and the proper character ψ_0 of $\mathrm{Cl}(\mathfrak{f}_\psi)$ corresponding to ψ. On the other hand, we get that

$$\begin{aligned}
S_{\mathfrak{f}}(\bar{\psi}, g_{\mathfrak{f}}) &= \sum_{C \in \mathrm{Cl}(\mathfrak{f})} \bar{\psi}(C) \log |g_{\mathfrak{f}}(C)| \\
&= \sum_{\substack{C_1 \in \mathrm{Cl}(\mathfrak{f}) \\ C_1 \ (\mathrm{mod}\ \mathrm{Gal}(K_{\mathfrak{f}}/F))}} \ \sum_{\substack{C_2 \in \mathrm{Gal}(K_{\mathfrak{f}}/F) \\ C_2 \ (\mathrm{mod}\ \mathrm{Gal}(K_{\mathfrak{f}}/L))}} \ \sum_{C_3 \in \mathrm{Gal}(K_{\mathfrak{f}}/L)} \bar{\psi}(C_1C_2C_3) \log |g_{\mathfrak{f}}(C_1C_2C_3)| \\
&= \sum_{C_1} \bar{\psi}(C_1) \sum_{C_2} \bar{\psi}(C_2) \log |\varepsilon^{\sigma(C_1C_2)}| \quad \text{by the fact } \psi|_{\mathrm{Gal}(K_{\mathfrak{f}}/L)} = 1 \text{ and (2.1)} \\
&= \sum_{C_1} \bar{\psi}(C_1) \left(\sum_{C_2} \bar{\psi}(C_2) \right) \log |\varepsilon^{\sigma(C_1)}| \quad \text{by the fact } \varepsilon \in F \\
&= 0 \quad \text{by the fact } \psi|_{\mathrm{Gal}(K_{\mathfrak{f}}/F)} \neq 1,
\end{aligned}$$

which is a contradiction. Therefore $L = F$ as desired. □

Remark 2.2. Schertz achieved in [10] a similar result for generators of the ray class fields. However, there seems to be some defect in his argument. For instance, in the proof of [10] Lemma1 he claimed that the conductor of a nontrivial character of $\mathrm{Cl}(\mathfrak{p}^n)$ is nontrivial. But one can see that his argument could be false if $h_K \geq 2$ because in this case the conductor of a character of $\mathrm{Cl}(\mathfrak{p}^n)$ induced from one of $\mathrm{Cl}(\mathcal{O}_K)$ is obviously trivial.

We apply this theorem to obtain the ring class invariants described in (1.4). To this end we are in need of certain efficient and explicit transformation formula besides the one in (2.1).

Lemma 2.3. *Let K be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. For $N \geq 2$, let $\mathfrak{f} = N\mathcal{O}_K$ and $\mathcal{O} = [N\theta, 1]$. Then the value $g_\mathfrak{f}(C_0) = g^{12N}_{(0,\frac{1}{N})}(\theta)$ which is a real algebraic integer (and a unit if N has at least two prime factors) generates $K_\mathfrak{f}$ over K. Furthermore, we have an isomorphism*

$$\mathrm{Gal}(K_\mathfrak{f}/H_\mathcal{O}) \cong \left\{ \left(\begin{smallmatrix} w & 0 \\ 0 & w \end{smallmatrix}\right) \ : \ w \in (\mathbb{Z}/N\mathbb{Z})^* \right\} / \left\{ \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right\}$$

whose action is given by

$$g^{12N}_{(0,\frac{1}{N})}(\theta)^{\left(\begin{smallmatrix} w & 0 \\ 0 & w \end{smallmatrix}\right)} = g^{12N}_{(0,\frac{w}{N})}(\theta).$$

Proof. See [6] Theorem 4.5, Remark 4.6 and [7] Remark 3.4. □

Remark 2.4. Note that

$$g^{12N}_{(0,\frac{w}{N})}(\theta) = g^{12N}_{(0,\frac{N-w}{N})}(\theta)$$

for $w \in \mathbb{Z} \setminus N\mathbb{Z}$.

Lemma 2.5. *Let $N \geq 1$. Then we have the relation*

$$\prod_{w=1}^{N-1} g^{12}_{(0,\frac{w}{N})}(\tau) = N^{12} \frac{\Delta(N\tau)}{\Delta(\tau)}$$

where the left hand side is understood to be 1 when $N = 1$.

Proof. Note the identity

$$\frac{1 - X^N}{1 - X} = 1 + X + \cdots + X^{N-1} = \prod_{w=1}^{N-1} (1 - e^{\frac{2\pi i w}{N}} X). \tag{2.5}$$

We then derive for $N \geq 2$ that

$$\begin{aligned} \prod_{w=1}^{N-1} g^{12}_{(0,\frac{w}{N})}(\tau) &= \prod_{w=1}^{N-1} \left(q_\tau^{\frac{1}{12}} e^{-\frac{\pi i w}{N}} (1 - e^{\frac{2\pi i w}{N}}) \prod_{n=1}^{\infty} (1 - q_\tau^n e^{\frac{2\pi i w}{N}})(1 - q_\tau^n e^{-\frac{2\pi i w}{N}}) \right)^{12} && \text{by (1.1)} \\ &= q_\tau^{N-1} N^{12} \prod_{n=1}^{\infty} \left(\frac{1 - q_\tau^{Nn}}{1 - q_\tau^n} \right)^{24} && \text{by the identity (2.5)} \\ &= N^{12} \frac{\Delta(N\tau)}{\Delta(\tau)} && \text{by (1.5).} \end{aligned}$$

□

Now we are ready to prove our main theorem about ring class invariants.

Theorem 2.6. *Let K be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. For a prime p which is inert or ramified in $K/\mathbb{Q}$, let $\mathcal{O} = [p^\ell\theta, 1] (\ell \geq 1)$. Then the real algebraic integer*

$$p^{12} \frac{\Delta(p^\ell\theta)}{\Delta(p^{\ell-1}\theta)}$$

generates $H_\mathcal{O}$ over K.

Proof. Let $\mathfrak{f} = p^\ell\mathcal{O}_K$. Then the conductor of the extension $H_\mathcal{O}/K$ is $\mathfrak{f}$ (for instance, see [3] Exercises 9.20-9.23) and

$$[H_\mathcal{O} : K] = \begin{cases} p^{\ell-1}(p+1)h_K & \text{if } p \text{ is inert in } K/\mathbb{Q} \\ p^\ell h_K & \text{if } p \text{ is ramified in } K/\mathbb{Q} \end{cases}$$

by the class number formula([3] Theorem 7.24).

If $p=2$ and $\ell=1$, then $K_{\mathfrak{f}}=H_{\mathcal{O}}$ by Lemma 2.3 and hence the real algebraic integer $g^{24}_{(0,\frac{1}{2})}(\theta)$ generates $H_{\mathcal{O}}$ over K. And, $g^{24}_{(0,\frac{1}{2})}(\theta)=(2^{12}\frac{\Delta(2\theta)}{\Delta(\theta)})^2$ by Lemma 2.5.

As for the other cases, since $\mathfrak{f}$ is a prime power and $[H_{\mathcal{O}}:K]>2h_K$, the value $\mathbf{N}_{K_{\mathfrak{f}}/H_{\mathcal{O}}}(g_{\mathfrak{f}}(C_0))$ generates $H_{\mathcal{O}}$ over K by Theorem 2.1. And we have

$$\begin{aligned}
\mathbf{N}_{K_{\mathfrak{f}}/H_{\mathcal{O}}}(g_{\mathfrak{f}}(C_0))^2 &= \prod_{\substack{1\leq w\leq p^\ell-1\\ \gcd(w,p)=1}} g^{12p^\ell}_{(0,\frac{w}{p^\ell})}(\theta) \quad \text{by Lemma 2.3}\\
&= \prod_{w=1}^{p^\ell-1} g^{12p^\ell}_{(0,\frac{w}{p^\ell})}(\theta) / \prod_{w=1}^{p^{\ell-1}-1} g^{12p^\ell}_{(0,\frac{pw}{p^\ell})}(\theta)\\
&= \left(p^{12\ell}\frac{\Delta(p^\ell\theta)}{\Delta(\theta)}/p^{12(\ell-1)}\frac{\Delta(p^{\ell-1}\theta)}{\Delta(\theta)}\right)^{p^\ell} \quad \text{by Lemma 2.5}\\
&= \left(p^{12}\frac{\Delta(p^\ell\theta)}{\Delta(p^{\ell-1}\theta)}\right)^{p^\ell}.
\end{aligned}$$

On the other hand, since both $\frac{\Delta(p^{\ell-1}\theta)}{\Delta(\theta)}$ and $\frac{\Delta(p^\ell\theta)}{\Delta(\theta)}$ are real algebraic integers which belong to $H_{\mathcal{O}}$([9] §12.1 Corollary to Theorem 1), we get the assertion. □

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