REMARKS ON SYZYGIES OF THE SECTION MODULES AND GEOMETRY OF PROJECTIVE VARIETIES

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Abstract. Let $X \subset \mathbb{P}(H^0(\mathcal{L}))$ be a smooth projective variety embedded by the complete linear system associated to a very ample line bundle $\mathcal{L}$ on X. We call $R_{\mathcal{L}} = \oplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^{\ell})$ the section module of $\mathcal{L}$. It has been known that the syzygies of $R_{\mathcal{L}}$ as $R = \mathrm{Sym}(H^0(\mathcal{L}))$-module play important roles in understanding geometric properties of X([2], [3], [5], [9], [10]) even if X is not projectively normal.

Generalizing the case of $N_{2,p}$([2], [10]), we prove some uniform theorems on higher normality and syzygies of a given linearly normal variety X and general inner projections when $R_{\mathcal{L}}$ satisfies property $N_{3,p}$ (Theorems 1.1, 1.2 and Proposition 3.1). In particular, our uniform bounds are sharp as hyperelliptic curves and elementary transforms of elliptic ruled surfaces show.

Keywords: linear syzygy, Castelnuovo-Mumford regularity, inner projection, property $N_{d,p}$, Eagon-Northcott complex.

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1. Introduction

Let $R = k[x_0, \dots, x_n]$ be a polynomial ring over an algebraically closed field k. Consider a minimal free resolution of a finitely generated graded R-module $M = \bigoplus_{j \geq 0} M_j$ as follows;

$$\cdots \to L_{i+1} \to L_i \to L_{i-1} \to \cdots \to L_1 \to L_0 \to M \to 0 \tag{1.1}$$

where $L_i = \bigoplus_j R(-i-j)^{\oplus \beta_{i,j}}$. Then, one can define that M satisfies property $N_{d,p}$ if $\beta_{i,j} = 0$ for $0 \leq i \leq p$ and all $j \geq d$ in the minimal free resolution (1.1). In particular, a reduced projective scheme X in $\mathbb{P}^n$ satisfies property $N_{d,p}$ ([5]) if the homogeneous coordinate ring R/I_X of X satisfies property $N_{d,p}$. This definition coincides with the classical notion N_p when $d = 2$ and X is projectively normal. Recall that M is d-regular if $\beta_{i,j} = 0$ for all $i \geq 0$ and $j \geq d$. Therefore, the regularity $\mathrm{reg}(M)$ of M is defined as the minimum of such d.

On the other hand, for an irreducible projective variety $X \subset \mathbb{P}^n = \mathbb{P}(H^0(\mathcal{L}))$ associated to a very ample line bundle $\mathcal{L}$ on X and a smooth point $q \in X$, consider an inner projection $\pi_q : X \dashrightarrow \mathbb{P}^{n-1}$. This rational map π_q can be extended to the blow-up morphism $\sigma : \mathrm{Bl}_q(X) \to X$ with the following diagram;

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$$\begin{array}{ccc} \tilde{X} = \mathrm{Bl}_q(X) & \longrightarrow & \mathbb{P}^n \times \mathbb{P}^{n-1} \\ \sigma \downarrow & \searrow^{\widetilde{\pi_q}} & \downarrow p_2 \\ X \subset \mathbb{P}^n & \overset{}{\underset{\pi_q}{\dashrightarrow}} & X_q = \overline{\pi_q(X)} \subset \mathbb{P}^{n-1} \end{array}$$

Let Trisec(X) be the union of all tri-secant lines ℓ or $\ell \subset X$. It is well known that if $q \in X \setminus \mathrm{Trisec}(X)$, then $\widetilde{\pi_q}$ given by the linear system $|\sigma^*\mathfrak{L} - E|$ is an embedding (see [6], pp.268 – 269).

However, it is very delicate to say $X \not\subseteq \mathrm{Trisec}(X)$ if codimension of X is small and there are strong obstructions to $X \not\subseteq \mathrm{Trisec}(X)$, see [1]. In the authors' previous paper ([2], Theorem 1.1), it was shown that if a smooth variety X satisfies property N_p then an embedding $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n-1}$ for $q \in X \setminus \mathrm{Trisec}(X)$ satisfies at least property N_{p-1}.

In this paper, first of all, we generalize Theorem 1.1 in [2] to the case of morphism $\widetilde{\pi_q} : \mathrm{Bl}_q(X) \to X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}^{n-1}$ for $q \in L \subset X$, L is a linear subspace. Even though $\widetilde{\pi_q}$ is not an embedding, we have the following main theorem.

Theorem 1.1. *Let $X \subset \mathbb{P}(H^0(\mathfrak{L})) = \mathbb{P}^n$ be a smooth variety with property N_p for $p \geq 1$. For any $q \in X$ (possibly q is contained in a linear space $L \subset X$), $\overline{\pi_q(X \setminus \{q\})}$ in $\mathbb{P}^{n-1}$ satisfies at least property N_{p-1}.*

Main idea in proving Theorem 1.1 is to use Corollary 2.2 and induction arguement from the related commutative diagram in the Main Lemma 3.3. As examples, we can consider property N_p for elliptic surface scrolls and their inner projections which are elementary transforms as the center q moves inside X.

Secondly, let $X \subset \mathbb{P}(H^0(\mathfrak{L}))$ is a projectively normal variety satisfying property $N_{3,p}$. Recently, property $N_{3,p}$ has been focussed on for higher secant varieties for varieties with the condition $N_{2,p}$ ([14], [15]). In this case, it is possible to control the higher normality, degree of defining equations and syzygies of inner projections.

Theorem 1.2. *Let $X \subset \mathbb{P}(H^0(\mathfrak{L}))$ be projectively normal and satisfy property $N_{3,p}, p \geq 2$. Let $\beta_{1,2}$ be the number of cubic generators of I_X. Then, for $q \in X$ such that there is no proper trisecant line through q, one has the following for an inner projection X_q;*

(a) $h^1(\mathfrak{I}_{X_q}(2)) \leq \beta_{1,2}$
(b) *X_q is m-normal for all $m \geq h^1(\mathfrak{I}_{X_q}(2)) + 2$;*
(c) *X_q is cut out by equations of degree at most $h^1(\mathfrak{I}_{X_q}(2)) + 3$ and further X satisfies property $N_{h^1(\mathfrak{I}_{X_q}(2))+3,p-1}$;*
(d) $\mathrm{reg}(X_q) \leq \max\{\mathrm{reg}(X), h^1(\mathfrak{I}_{X_q}(2)) + 3\}$.

Main idea in proving Theorem 1.2 is to use Eagon-Northcott complex arising from the property $N_{3,p}, p \geq 2$ (see Proposition 3.1) and vector bundle techniques used in [8], [10]. Proposition 3.1 is also very important in itself because it generalizes Theorem 1.2 in [10]. Note that our uniform bounds are sharp as many examples show.

In Section 2, notations and well-known preliminary results are introduced and in Section 3, we give proofs of main Theorems 1.1 ,1.2 and Proposition 3.1. Further interesting optimal examples, i.e. hyperelliptic curves with degree $2g+1$ and elliptic surface scrolls are also provided.

2. Notations and Preliminaries

For our convenience, we adopt the following notations:

- $R = k[x_0, \dots, x_n] = \mathrm{Sym}(V)$ where $V \subset H^0(X, \mathcal{L})$.
- $R_{\mathcal{L}} = \oplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^\ell)$: the graded R-module of twisted sections of $\mathcal{L}$.
- $\beta_{i,j} := \dim_k \mathrm{Tor}_i^R(R_{\mathcal{L}}, k)_{i+j}$.
- $\tilde{X} = \mathrm{Bl}_q(X)$: a blowing up of X at a point q with a morphism $\sigma : \tilde{X} \to X$.
- E : the exceptional divisor of $\tilde{X}$.
- $W = H^0(\tilde{X}, \sigma^*\mathcal{L}(-E)) = H^0(X, \mathcal{L}(-q))$.
- $S_W = \mathrm{Sym}(W)$: the homogeneous coordinate ring of $\mathbb{P}(W) = \mathbb{P}^{n-1}$.
- $R' = \oplus_{\ell \in \mathbb{Z}} H^0(\tilde{X}, (\sigma^*\mathcal{L} - E)^\ell)$: the graded S_W-module of twisted sections of $\sigma^*\mathcal{L} - E$.
- $\beta'_{i,j} := \dim_k \mathrm{Tor}_i^{S_W}(R', k)_{i+j}$.

2.1. **Criterion for property $N_{d,p}$.** Let $\mathcal{M}$ be the tautological rank-n subbundle on $\mathbb{P}^n = \mathbb{P}(V)$ which fits into the exact sequence $0 \to \mathcal{M} \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$. We have also an induced exact sequence for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$;

$$0 \to \wedge^{i+1}\mathcal{M} \otimes \mathcal{F}(j-1) \xrightarrow{\tau_{i,j}} \wedge^{i+1} V \otimes \mathcal{F}(j-1) \xrightarrow{\varphi_{i,j}} \wedge^i \mathcal{M} \otimes \mathcal{F}(j) \to 0.$$

Then, for the saturated R-module $F = \bigoplus_{n \geq 0} H^0(\mathcal{F}(n))$, one has the following useful theorem.

Theorem 2.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ with the section module $F = \bigoplus_{n \geq 0} H^0(\mathcal{F}(n))$. If $j \geq 1$, then there is an exact sequence*

$$0 \to \mathrm{Tor}_i^R(F, k)_{i+j} \to H^1(\wedge^{i+1}\mathcal{M} \otimes \mathcal{F}(j-1)) \xrightarrow{\tau_{i,j}} \wedge^{i+1} V \otimes H^1(\mathcal{F}(j-1))$$

where the map $\tau_{i,j}$ is induced by the inclusion $\mathcal{M} \subset V \otimes \mathcal{O}_{\mathbb{P}^n}$.

Proof. see [4], Theorem 5.7. □

Therefore, $F = \bigoplus_{n \geq 0} H^0(\mathcal{F}(n))$ satisfies property $N_{d,p}$ iff for $0 \leq i \leq p$ and $j \geq d$, the homomorphism

$$H^1(\mathbb{P}^n, \wedge^{i+1}\mathcal{M} \otimes \mathcal{F}(j-1)) \xrightarrow{\tau_{i,j}} \wedge^{i+1} V \otimes H^1(\mathbb{P}^n, \mathcal{F}(j-1))$$

is injective, equivalently the homomorphism

$$\wedge^{i+1} V \otimes H^0(\mathbb{P}^n, \mathcal{F}(j-1)) \xrightarrow{\varphi_{i,j}} H^0(\mathbb{P}^n, \wedge^i \mathcal{M} \otimes \mathcal{F}(j))$$

is surjective.

On the other hand, for a projective variety $X \subset \mathbb{P}(W)$, $W \subset H^0(\mathcal{L})$, we have an exact sequence $0 \to \mathcal{M}_W \to W \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \simeq \mathcal{L} \to 0$. Then, $\mathrm{Tor}_i^{S_W}(R_{\mathcal{L}}, k)_{i+j}$ fits similarly into the exact sequence

$$\begin{aligned} 0 \quad & \to \mathrm{Tor}_i^{S_W}(R_{\mathcal{L}}, k)_{i+j} \to H^1(X, \wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}^{j-1}) \to \wedge^{i+1} W \otimes H^1(X, \mathcal{L}^{j-1}) \\ & \to H^1(X, \wedge^i \mathcal{M}_W \otimes \mathcal{L}^j) \to \cdots \end{aligned}$$

and we have the following corollary:

Corollary 2.2. *For a projective variety $X \subset \mathbb{P}(W), W \subset H^0(X, \mathcal{L})$, let S_W be a projective coordinate ring of $\mathbb{P}(W)$. Then the section module $R_{\mathcal{L}} := \oplus_{n \geq 0} H^0(\mathcal{L}^{\otimes n})$ satisfies property $N_{d,p}$ as a graded S_W-module if and only if the homomorphism $\wedge^{i+1} W \otimes H^0(X, \mathcal{L}^{j-1}) \to H^0(X, \wedge^i \mathcal{M}_W \otimes \mathcal{L}^j)$ is surjective for $0 \leq i \leq p$ and $j \geq d$, equivalently the homomorphism*

$$H^1(X, \wedge^{i+1}\mathcal{M}_W \otimes \mathcal{L}^{j-1}) \to \wedge^{i+1} W \otimes H^1(X, \mathcal{L}^{j-1})$$

is injective for $0 \leq i \leq p$ and $j \geq d$.

3. Proofs of main results and examples

To begin with, let us recall the following known results.

Let $X \subset \mathbb{P}(V)$ be a projective variety with $R_{\mathcal{L}}$ satisfying property $N_{2,p}$ for $p \geq 1$ as a graded R-module where $V \subset H^0(\mathcal{L})$.

- If $t = h^1(\mathcal{I}_X(1)) = \mathrm{codim}(V, H^0(\mathcal{L}))$, then X is m-normal for all $m \geq t+1$ and cut out by equations of degree at most $t+2$. In addition, I_X satisfies property $N_{t+2,p-1}$ and $\mathrm{reg}(X) \leq \max\{\mathrm{reg}(\mathcal{O}_X)+1, t+2\}$ ([10], Theorem 1.2).
- If X is projectively normal, then an inner projection X_q from a smooth point $q \in X \setminus \mathrm{Trisec}(X)$ is also projectively normal and further satisfies N_{p-1}. Furthermore, $\mathrm{reg}(X_q) = \mathrm{reg}(X)$ ([2], Theorem 1.1).

We proceed with the following proposition which generalizes the first fact.

Proposition 3.1. *Let $X \subset \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^n$ be a reduced linearly normal variety. Suppose that the section module $R_{\mathcal{L}} = \oplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^\ell)$ satisfies property $N_{3,p}$ for $p \geq 1$. Then,*

(a) *X is m-normal for all $m \geq h^1(\mathcal{I}_X(2)) + 2$;*
(b) *X is cut out by equations of degree at most $h^1(\mathcal{I}_X(2)) + 3$ and further, I_X satisfies property $N_{h^1(\mathcal{I}_X(2))+3,p}$;*
(c) *$\mathrm{reg}(X) \leq \max\{\mathrm{reg}(\mathcal{O}_X)+1, h^1(\mathcal{I}_X(2))+3\}$.*

Proof. If $X \subset \mathbb{P}(H^0(\mathcal{L}))$ is quadratically normal, i.e., $h^1(\mathcal{I}_X(2)) = 0$, it is projectively normal since $R_{\mathcal{L}}$ satisfies property $N_{3,p}$. In this case, the conclusion is trivial. Now, we assume that $X \subset \mathbb{P}(H^0(\mathcal{L}))$ is not quadratically normal, i.e. $h^1(\mathcal{I}_X(2)) \neq 0$. Let $R = k[x_0, x_1 \ldots, x_n]$ be the coordinate ring of $\mathbb{P}^n = \mathbb{P}(H^0(\mathcal{L}))$. Since X is not projectively normal, we have the following basic sequence;

$$0 \longrightarrow R/I_X \longrightarrow R_{\mathcal{L}} = \oplus_{\ell \in \mathbb{Z}} H^0(X, \mathcal{L}^\ell) \longrightarrow H^1_*(\mathcal{I}_X) \longrightarrow 0$$

where $H^1_*(\mathcal{I}_X) = \bigoplus_{\ell \in \mathbb{Z}} H^1(\mathbb{P}^n, \mathcal{I}_X(\ell))$ is the Hartshorne-Rao module.

Since X is linearly normal but not quadratically normal, we have $\beta_{0,1}(R_{\mathcal{L}}) = 0$ and $\beta_{0,2}(R_{\mathcal{L}}) = h^1(\mathcal{I}_X(2))$. The property $N_{3,p}$ of $R_{\mathcal{L}}$ for $p \geq 1$ gives the following minimal free resolution of $R_{\mathcal{L}}$ as a graded R-module:

$$0 \to K_1 = \ker(\varphi_1) \to R(-3)^{\beta_{1,2}} \oplus R(-2)^{\beta_{1,1}} \xrightarrow{\varphi_1} R \oplus R(-2)^{\beta_{0,2}} \xrightarrow{\varphi_0} R_{\mathcal{L}} \to 0.$$

Letting $K_0 = \ker(\varphi_0)$ and by sheafification, we have the following two commutative diagrams (cf. [8],[10]);

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{I}_X & \to & \mathcal{O}_{\mathbb{P}^n} & \to & \mathcal{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{K}_0 & \to & \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & \to & \mathcal{O}_X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & = & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.1}$$

and in the first syzygies of $R_{\mathcal{L}}$, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathcal{I}_X & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & \mathcal{K}_1 & \to & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^n}(-3)^{\beta_{1,2}} & \to & \mathcal{K}_0 & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \mathcal{N} & \to & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^n}(-3)^{\beta_{1,2}} & \to & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathcal{I}_X & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} \tag{3.2}$$

Claim 3.2. From the commutative diagrams (3.1) and (3.2),

(a) $H^0_*(\mathcal{K}_0) = K_0 = \ker(\varphi_0)$ and $H^1(\mathcal{K}_0(m)) \simeq H^2(\mathcal{K}_1(m)) = 0$ for all $m \in \mathbb{Z}$,
(b) $H^0_*(\mathcal{K}_1) = K_1$, $H^1(\mathcal{K}_1(m)) = 0$ for all $m \in \mathbb{Z}$,
(c) $\mathrm{reg}(\mathcal{N}) \leq h^1(\mathcal{I}_X(2)) + 3$.

Proof. By taking global sections, we have the following sequence:

$$0 \to H^0_*(\mathcal{K}_0) \longrightarrow R \oplus R(-2)^{\beta_{0,2}} \xrightarrow{\varphi_0} R_{\mathcal{L}} \longrightarrow H^1_*(\mathcal{K}_0) \longrightarrow 0.$$

Therefore, we get $H^0_*(\mathcal{K}_0) = K_0$ and $H^1_*(\mathcal{K}_0) = 0$. On the other hand, from the following diagram

$$\begin{array}{ccccccccc} 0 & \to & H^0_*(\mathcal{K}_1) & \to & R(-2)^{\beta_{1,1}} \oplus R(-3)^{\beta_{1,2}} & \to & H^0_*(\mathcal{K}_0) & \to H^1_*(\mathcal{K}_1) & \to 0 \\ & & \uparrow & & \| & & \| & & \\ 0 & \to & K_1 & \to & R(-2)^{\beta_{1,1}} \oplus R(-3)^{\beta_{1,2}} & \xrightarrow{\varphi_1} & K_0 & \to 0, & \end{array}$$

we have $H^0_*(\mathcal{K}_1) = K_1$ and $H^1_*(\mathcal{K}_1) = \bigoplus_{m\in\mathbb{Z}} H^1(\mathcal{K}_1(m)) = 0$. In addition, from the sequence $0 \to \mathcal{K}_1 \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^n}(-3)^{\beta_{1,2}} \to \mathcal{K}_0 \to 0$, we obtain $H^1_*(\mathcal{K}_0) = H^2_*(\mathcal{K}_1) = 0$. The Castelnuovo-Mumford regularity of $\mathcal{N}$ in the second row of (3.2).

$$0 \to \mathcal{N} \to O_{\mathbb{P}^n}(-2)^{\beta_{1,1}} \oplus O_{\mathbb{P}^n}(-3)^{\beta_{1,2}} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} \to 0$$

can be controlled from the following diagram :

$$\begin{array}{cccccccc} & & 0 & & 0 & & & \\ & & \downarrow & & \downarrow & & & \\ & & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} & = & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} & & & \\ & & \downarrow & & \downarrow & & & \\ 0 \to & & \mathcal{N} & \to & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^n}(-3)^{\beta_{1,2}} & \xrightarrow{\widetilde{\varphi_1}} & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \| & \\ 0 \to & & \mathcal{N}_1 & \to & \mathcal{O}_{\mathbb{P}^n}(-3)^{\beta_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{0,2}} & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & & \\ & & 0 & & 0 & & & \end{array} \quad (3.3)$$

It is very important to note that in a second row, the restriction of $\widetilde{\varphi_1}$ to $\mathcal{O}_{\mathbb{P}^n}(-2)^{\beta_{1,1}}$ is a *zero* map because it is induced by the minimal free resolution of $R_{\mathcal{L}}$.

On the other hand, by using Eagon-Northcott complex associated to the exact sequence in the third row of (3.3)(cf. [8], [10],[11]), we get $\operatorname{reg}(\mathcal{N}_1) \leq \beta_{0,2} + 3$ and finally we have

$$\operatorname{reg}(\mathcal{N}) \leq \beta_{0,2} + 3 = h^1(\mathcal{I}_X(2)) + 3.$$

□

We now return to the proof of Proposition 3.1. From the exact sequence $0 \to \mathcal{K}_1 \to \mathcal{N} \to \mathcal{I}_X \to 0$, and by Claim 3.2 (a) and (b), we conclude that X is m-normal for all $m \geq h^1(\mathcal{I}_X(2)) + 2$.

For the syzygies of I_X, consider the exact sequence by taking global sections

$$0 \to K_1 = H^0_*(\mathcal{K}_1) \to H^0_*(\mathcal{N}) \to I_X \to 0 = H^1_*(\mathcal{K}_1).$$

Since $K_1 = H^0_*(\mathcal{K}_1)$ is the first syzygy module of $R_{\mathcal{L}}$, we have

$$\mathrm{Tor}_i^R(K_1,k)_{i+j} = 0 \quad for\ \ all \quad 0 \le i \le p-2,\ j \ge 3. \tag{3.4}$$

Now, consider the long exact sequence:

$$\mathrm{Tor}_i^R(K_1,k)_{i+j} \to \mathrm{Tor}_i^R(H^0_*(\mathcal{N}),k)_{i+j} \to \mathrm{Tor}_i^R(I_X,k)_{i+j} \to$$
$$\xrightarrow{\delta} \mathrm{Tor}_{i-1}^R(K_1,k)_{i+j} \to \mathrm{Tor}_{i-1}^R(H^0_*(\mathcal{N}),k)_{i+j} \to \mathrm{Tor}_{i-1}^R(I_X,k)_{i+j}.$$

Since we have (3.4) and $\operatorname{reg} H^0_*(\mathcal{N}) = \operatorname{reg}(\mathcal{N}) \le h^1(\mathcal{I}_X(2)) + 3$, we get $\mathrm{Tor}_i^R(I_X,k)_{i+j} = \mathrm{Tor}_{i+1}^R(R/I_X,k)_{i+j} = 0$ for $0 \le i \le p-1$ and $j \ge h^1(\mathcal{I}_X(2)) + 3$. Thus, we conclude that X is generated by equations of degree at most $h^1(\mathcal{I}_X(2)) + 3$ and further satisfies property $N_{h^1(\mathcal{I}_X(2))+3,p}$. □

The following Lemma is a refined version of theorem 4.6 in [2]. It gives a new inequality (Main Lemma 3.3 (b)). It is expected, but somewhat surprising that the syzygies of $R_{\mathcal{L}}$ control those of $R_{\mathcal{L}'}$ where $\mathcal{L}' = \sigma^*\mathcal{L} - E$.

Main Lemma 3.3. Suppose that X is a smooth linearly normal variety in $\mathbb{P}(H^0(\mathcal{L}))$ and $R_{\mathcal{L}} = \oplus_{\ell\in\mathbb{Z}} H^0(X,\mathcal{L}^\ell)$ satisfies property $N_{d,p},\ p \ge 1$. Then, we have the following;

(a) $R' = \oplus_{\ell\in\mathbb{Z}} H^0(\tilde{X},(\sigma^*\mathcal{L} - E)^\ell)$ is a finitely generated graded $\mathrm{Sym}(H^0(\sigma^*\mathcal{L} - E))$ module and satisfies property $N_{d,p-1}$, i.e. $\beta'_{i,j} = 0$ for $0 \le i \le p-1$ and $j \ge d$;

(b) $\beta'_{i,d-1} \le \beta_{i+1,d-1}$ for $0 \le i \le p-1$.

Proof. Note that in the case of $d = 2$, (a) was already proved in [2]. Without a loss of generality, we prove the case of $d = 3$. As in the proof of theorem 4.6 in [2], we have the following complicated but very useful inductive diagrams; let $\sigma : \tilde{X} = \mathrm{Bl}_q(X) \to X$ be the blow-up morphism with $W = H^0(\sigma^*\mathcal{L}(-E))$. Then, we have the following diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{M}_W & \to & W\otimes\mathcal{O}_{\tilde{X}} & \to & \sigma^*\mathcal{L}(-E) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \sigma^*\mathcal{M}_V & \to & V\otimes\mathcal{O}_{\tilde{X}} & \to & \sigma^*\mathcal{L} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{O}_{\tilde{X}}(-E) & \to & \mathcal{O}_{\tilde{X}} & \to & \mathcal{O}_E & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.5}$$

Taking wedge products and tensoring by $\sigma^*\mathcal{L}^{j-1}$ in the diagram (3.5), we have the following diagram on cohomology groups in order to prove the case of $p = 1$ (even when $p \ge 2$, we have the same proof):

(3.6)

$$\begin{array}{ccccccc}
\wedge^2 W\otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \to & H^0(\mathcal{M}_W\otimes\sigma^*\mathcal{L}^j(-E)) & \to & H^1(\wedge^2\mathcal{M}_W\otimes\sigma^*\mathcal{L}^{j-1}) & \to & \wedge^2 W\otimes H^1(\sigma^*\mathcal{L}^{j-1}) \\
\downarrow & & \downarrow & & \downarrow & & \\
\wedge^2 V\otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \stackrel{\varphi_{1,j}}{\to} & H^0(\sigma^*\mathcal{M}_V\otimes\sigma^*\mathcal{L}^j) & \to & H^1(\wedge^2\sigma^*\mathcal{M}_V\otimes\sigma^*\mathcal{L}^{j-1}) & \stackrel{\tau_{1,j}}{\to} & \wedge^2 V\otimes H^1(\sigma^*\mathcal{L}^{j-1}) \\
\downarrow\mu_{1,j} & & \downarrow\nu_{1,j} & & \downarrow\xi_{1,j} & & \\
W\otimes H^0(\sigma^*\mathcal{L}^{j-1}) & \stackrel{\gamma_{1,j}}{\to} & H^0(\mathrm{coker}\ \alpha_{1,j}) & \to & H^1(\mathcal{M}_W\otimes\sigma^*\mathcal{L}^{j-1}(-E)) & \stackrel{\delta_{1,j}}{\to} & W\otimes H^1(\sigma^*\mathcal{L}^{j-1}) \\
\downarrow & & \downarrow & & & & \\
0 & & H^1(\mathcal{M}_W\otimes\sigma^*\mathcal{L}^j(-E)) & \stackrel{\delta_{1,j+1}}{\longrightarrow} & W\otimes H^1(\sigma^*\mathcal{L}^j) & & \\
 & & \downarrow\omega_{1,j} & & \downarrow\rho_{1,j} & & \\
 & & H^1(\sigma^*\mathcal{M}_V\otimes\sigma^*\mathcal{L}^j) & \stackrel{\tau_{0,1+j}}{\longrightarrow} & V\otimes H^1(\sigma^*\mathcal{L}^j) & &
\end{array}$$

where coker $\alpha_{1,j}$ in the second column is defined as follows:

$$0\longrightarrow\mathcal{M}_W\otimes\sigma^*\mathcal{L}^j(-E)\stackrel{\alpha_{1,j}}{\longrightarrow}\sigma^*\mathcal{M}_V\otimes\sigma^*\mathcal{L}^j\longrightarrow\mathrm{coker}\ \alpha_{1,j}\longrightarrow 0\ .$$

The property $N_{3,1}$ of $R_{\mathcal{L}}$ implies that $\tau_{1,j}$ is always injective for all $j\geq 3$ because $\beta_{1,j}=0$ for $j\geq 3$. Note also that $\mu_{1,j}$ is surjective and $\rho_{1,j}$ is injective for all $j\geq 1$. By the inductive argument from the above diagram (cf. theorem 4.6 [2]), we can show that, for $j\geq 3$,

$$\delta_{1,j+1}\text{ is injective}\quad\Longrightarrow\quad\delta_{1,j}\text{ is injective.}$$

Indeed, $H^1(\mathcal{M}_W\otimes\sigma^*\mathcal{L}^j(-E))=H^1(\sigma_*\mathcal{M}_W(-E)\otimes\mathcal{L}^j)=0$ for $j\gg 0$ because $\mathcal{L}$ is very ample. So, $\delta_{1,j+1}$ is a zero map for $j\gg 0$. Since our inductive method works for all $j\geq 3$, we obtain

$$\delta_{1,j}\ \text{ is injective for all }\ j\geq 3.$$

Now look at the following commutative diagram

$$\begin{array}{ccc}
H^1(\mathcal{M}_W\otimes(\sigma^*\mathcal{L}-E)^{j-1}) & \stackrel{\widetilde{\tau_{1,j}}}{\longrightarrow} & W\otimes H^1((\sigma^*\mathcal{L}-E)^{j-1}) \\
\downarrow & & \downarrow\wr \\
H^1(\mathcal{M}_W\otimes\sigma^*\mathcal{L}^{j-1}(-E)) & \stackrel{\delta_{1,j}}{\longrightarrow} & W\otimes H^1(\sigma^*\mathcal{L}^{j-1}).
\end{array}\tag{3.7}$$

For $j\geq 2$, the left column map is always injective by lemma 4.4 in [2] and the right column map is an isomorphism by corollary 2.4 in [2]. Therefore, $\widetilde{\tau_{1,j}}$ is injective for $j\geq 3$ and equivalently, ${\beta'}_{0,j}=0$ for all $j\geq 3$. Therefore $R'=\oplus_{\ell\in\mathbb{Z}}H^0(\tilde{X},(\sigma^*\mathcal{L}-E)^\ell)$ satisfies property $N_{3,0}$ as a graded S_W-module.

Note that $\nu_{1,2}$ in the diagram (3.6) is surjective because $\delta_{1,3}$ is injective (so, $\omega_{1,2}$ is also injective). From the following commutative diagram for $j = 2$

(3.8)
$$\begin{array}{ccccccc}
0 \longrightarrow & \text{Coker } \varphi_{1,2} & \longrightarrow & H^1(\wedge^2\sigma^*\mathcal{M}_V \otimes \sigma^*\mathcal{L}) & \xrightarrow{\tau_{1,2}} & \wedge^2 V \otimes H^1(\sigma^*\mathcal{L}) \\
 & \downarrow \widetilde{\nu_{1,2}} & & \downarrow \xi_{1,2} & & \downarrow \\
0 \longrightarrow & \text{Coker } \gamma_{1,2} & \longrightarrow & H^1(\mathcal{M}_W \otimes \sigma^*\mathcal{L}(-E)) & \xrightarrow{\delta_{1,2}} & W \otimes H^1(\sigma^*\mathcal{L}) \\
 & \downarrow & & & & \\
 & 0 & & & &
\end{array}$$

we get $\text{Coker } \varphi_{1,2} \simeq \text{Tor}_1^R(R_{\mathcal{L}}, k)_3$ and by the isomorphism diagram (3.7) for $j = 2$, we also have

$$\text{Coker } \gamma_{1,2} \simeq \ker \ \delta_{1,2} \simeq \ker \ \widetilde{\tau_{1,2}} \simeq \text{Tor}_0^{S_W}(R', k)_2.$$

Therefore, $\beta'_{0,2} = \dim \text{Tor}_0^{S_W}(R', k)_2 \leq \dim \text{Tor}_1^R(R_{\mathcal{L}}, k)_3 = \beta_{1,2}$. This completes the Main Lemma for $i = 0$. For $i \geq 1$, the same inductive argument can be applied as in ([2]). So we are done. □

Note that if X is a projectively normal embedding in $\mathbb{P}(H^0(\mathcal{L}))$ with property $N_{3,p}$, then $\beta_{1,2}$ is the number of cubic generators of I_X and $\beta'_{0,2} = h^1(\mathcal{I}_{X_q}(2)) \leq \beta_{1,2}$.

Let us go back to the basic situation again. Let $X \subset \mathbb{P}(H^0(\mathcal{L})) = \mathbb{P}^n$ is a smooth projective variety, and L be a linear subspace such that $q \in L \subset X$. Then, $\sigma^*\mathcal{L}(-E)$ is not very ample but base-point free so that $\widetilde{\pi}_q : \widetilde{X} = \text{Bl}_q(X) \to X_q = \overline{\pi_q(X \setminus \{q\})} \subset \mathbb{P}(W) = \mathbb{P}^{n-1}$ is a morphism which is not an embedding. However, one can still get some syzygetic information about the section module $R_q = \oplus_{\ell \in \mathbb{Z}} H^0(X_q, \mathcal{O}_{X_q}(\ell))$ if X_q is a normal variety. In this situation, we proceed to prove Theorem 1.1.

• Proof of Theorem 1.1

Since $X \subset \mathbb{P}^n$ satisfies property N_p, there is no line $l \subset \mathbb{P}^n$ such that $\dim(l \cap X) = 0$ and length $(l \cap X) \geq 3$. Then the inverse image ${\widetilde{\pi}_q}^{-1}(y)$ is geometrically connected for all $y \in X_q$. By Stein factorization, we get $\pi_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{X_q}$. Note also that property N_{p-1} of X_q is equivalent to the vanishing $\text{Tor}_i^{S_W}(R_q, k)_{i+j} = 0$ for $0 \leq i \leq p-1$ and $j \geq 2$.

On the other hand, from the restricted Euler sequence

$$0 \to M_W \to W \otimes \mathcal{O}_{X_q} \to \mathcal{O}_{X_q}(1) \to 0,$$

we have the following commutative diagram by projection formula and $\pi_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{X_q}$:

$$\begin{array}{ccccccc}
\wedge^{i+1} W \otimes H^0(\mathcal{O}_{X_q}(j-1)) & \xrightarrow{\psi_{i,j}} & H^0(\wedge^i \mathcal{M}_W \otimes \mathcal{O}_{X_q}(j)) & \longrightarrow & \text{Tor}_i^{S_W}(R_q, k)_{i+j} & \to & 0. \\
\| & & \| & & & & \\
\wedge^{i+1} W \otimes H^0(\sigma^*\mathcal{L}(-E)^{j-1}) & \xrightarrow{\widetilde{\psi_{i,j}}} & H^0(\wedge^i \mathcal{M}_W \otimes \sigma^*\mathcal{L}(-E)^j) & & & &
\end{array}$$

By the Main Lemma 3.3 (a), the morphism $\widetilde{\psi_{i,j}}$ is surjective for $i \leq p-1$ and $j \geq 2$ because $R' = \oplus_{\ell \in \mathbb{Z}} H^0(\tilde{X}, (\sigma^*\mathcal{L} - E)^{\ell})$ satisfies property $N_{2,p-1}$. Thus, the morphism $\psi_{i,j}$ is also surjective, and equivalently (see Corollary 2.2) X_q satisfies property N_{p-1}. □

• **Proof of Theorem 1.2**

Let $R = k[x_0, x_1 \ldots, x_n]$ be the coordinate ring of $\mathbb{P}^n = \mathbb{P}(H^0(\mathcal{L}))$ and $S_W = k[x_1, x_2, \ldots, x_n]$ be a coordinate ring of $\mathbb{P}^{n-1} = \mathbb{P}(W)$ as in the Notations. By the same reason as in Theorem 1.1, we know that $\widetilde{\pi}_{q*}(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{X_q}$ and thus

$$R' := \oplus_{\ell\in\mathbb{Z}} H^0(\widetilde{X}, \sigma^*\mathcal{L}(-E)^\ell) = \oplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_{X_q}(\ell)) := R_q.$$

Since $R_\mathcal{L}$ satisfies property $N_{3,p}$, the section module R' also satisfies property $N_{3,p-1}$ for $p \geq 2$ by Main Lemma 3.3 (a), and we have the minimal free resolution of $R' = R_q$ as a graded S_W-module:

$$0 \to K_1 = \ker(\varphi_1) \to S_W(-3)^{\oplus\beta'_{1,2}} \xrightarrow{\varphi_1} S_W \oplus S_W(-2)^{\oplus\beta'_{0,2}} \xrightarrow{\varphi_0} R' = R_q \to 0.$$

First note that if X_q is projectively normal, then $\beta'_{0,2} = 0$ and our theorem is clearly true by Main Lemma 3.3 (a). Suppose that X_q is not projectively normal. Then, X_q is not quadratically normal with inequality $0 \neq h^1(\mathcal{I}_{X_q}(2)) = \beta'_{0,2} \leq \beta_{1,2}$ by Main Lemma 3.3 (b). Therefore, by applying Proposition 3.1 immediately, we are done. □

The following Corollary is also a generalization of Theorem 1.2 in [10] and Theorem 2 in [3] to the case of $N_{3,p}$.

Corollary 3.4. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^n, V \subset H^0(\mathcal{L})$ be a projective variety which is not necessary linearly normal. If the section module $R_\mathcal{L} = \oplus_{\ell\in\mathbb{Z}} H^0(X, \mathcal{L}^\ell)$ satisfies property $N_{3,p}$ for $p \geq 2$, then for $q \in X$ such that there is no proper trisecant line through q, $R_q := \oplus_{\ell\in\mathbb{Z}} H^0(\mathcal{O}_{X_q}(\ell))$ satisfies property $N_{3,p-1}$.*

Proof. As in the proof of Theorem 1.2, we have $\widetilde{\pi}_{q*}(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_{X_q}$ by Stein factorization and thus $H^0(\widetilde{X}, \sigma^*\mathcal{L}(-E)^\ell) = H^0(\mathcal{O}_{X_q}(\ell))$. So by Main Lemma 3.3 (a), we are done. □

Example 3.5 (hyperelliptic curves). Let $X \subset \mathbb{P}^{g+1}$ be a hyperelliptic curve of genus $g \geq 3$ and degree $2g+1$ which is embedded by a complete linear system $|(g-2)g^1_2 + p_1 + p_2 + p_3 + p_4 + q|$ where g^1_2 is an unique hyperelliptic involution. Then X is projectively normal but fails to satisfy property N_1. However, the homogeneous ideal I_C is 3-regular(i.e. $N_{3,p}$) and in particular generated by quadrics and g-number of cubic hypersurfaces. If $H^0(p_1+p_2+p_3+p_4-g^1_2) = 0$, then the projection X_q from q is a linearly normal embedding with 4-secant line because Span $\langle p_1, p_2, p_3, p_4, q\rangle = \mathbb{P}^2$. In addition, It can be computed that $h^1(\mathcal{I}_{X_q}(2)) = 1$ and $h^1(\mathcal{I}_{X_q}(\ell)) = 0$ for all $\ell \geq 3$. Thus, this is an optimal example which makes our uniform bound sharp in the main Theorem 1.2 (see [13] for details). □

Example 3.6 (surface scrolls over an elliptic curve). Let C be a smooth elliptic curve and let $\mathcal{E}$ be a normalized rank 2 vector bundle on C with $\mathbf{e} = \bigwedge^2 \mathcal{E}$ and $e = -\deg(\mathbf{e})$. Let $X = \mathbb{P}_C(\mathcal{E})$ be an associated ruled surface with projection morphism $\pi : X \to C$. We fix a section C_0 such that $\mathcal{O}_X(C_0) = \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Then, $C_0^2 = -e$. Denote $\mathbf{b}f$ by the pullback of $\mathbf{b} \in$ Pic C. Consider an elliptic scroll $X \subset \mathbb{P}^n$ embedded by a complete linear system $|C_0 + \mathbf{b}f|$. First note that by Theorem 1.4 in [12]

$$X \subset \mathbb{P}^n \text{ satisfies property } N_p \text{ if and only if } \deg \mathbf{b} \geq e + 3 + p. \tag{3.9}$$

Now, suppose $(X, C_0+\mathbf{b}f)$ satisfies property N_p. An inner projection X_q is an elementary transform $\mathbb{P}_C(\mathcal{E}')$ of $X = \mathbb{P}_C(\mathcal{E})$ over C because X has no proper trisecant line through q. By theorem 1.1, X_q satisfies at least property N_{p-1}. However, the syzygies of X_q depend on the point $q \in X$ as follows ([7], §4):

- Assume that q is contained in a minimal section D which is not necessary equal to C_0. One can easily check that the strict transformation of a minimal section on $\mathbb{P}(\mathcal{E})$ passing through q is again a minimal section D' on $X_q = \mathbb{P}_C(\mathcal{E}')$ such that $\mathcal{O}_{X_q}(D') = \mathcal{O}_{\mathbb{P}_C(\mathcal{E}')}(1)$ and $(D')^2 = -e-1$. Therefore, we have $-\deg(\bigwedge^2 \mathcal{E}') = e+1$ and $X_q \subset \mathbb{P}^{n-1}$ is embedded by a complete linear system $|D'+\mathbf{b}'f|$ where $\deg \mathbf{b}' = b$ because $\deg X_q = \deg X - 1$. Therefore X_q satisfies property N_{p-1} but fails to satisfy N_p by (3.9).
- Assume that q is not contained in any minimal section in X. In this case, the strict transformation $C_0{}'$ of a minimal section C_0 on $\mathbb{P}_C(\mathcal{E})$ is again a minimal section on $\mathbb{P}(\mathcal{E}')$ and $(C_0{}')^2 = -e+1$. Therefore $-\deg(\bigwedge^2 \mathcal{E}') = e-1$ and $X_q \subset \mathbb{P}^{n-1}$ is embedded by $|C_0' + \mathbf{b}'f|$ where $\deg \mathbf{b}' = b-1$. Therefore X_q satisfies property N_p.

Assume that $-\deg(\bigwedge^2 \mathcal{E}) = -1$. Then $\mathbb{P}_C(\mathcal{E})$ is covered by minimal sections. If not, there exists a point $q \in X$ which is not contained in any minimal section. Then, the projection X_q is an elliptic scroll $\mathbb{P}_C(\mathcal{E}')$ over C such that $-\deg(\bigwedge^2 \mathcal{E}') < -1$. But there is no such a vector bundle on an elliptic curve by Nagata's theorem.

References

[1] I.Bauer, *Inner projections of algebraic surfaces: a finiteness result*, J. reine angew. Math. 460, 1-13 (1995).

[2] Y.Choi, P-L. Kang, S. Kwak, *Higher linear syzygies of Inner projections*, J. Algebra 305 (2006), 859–876.

[3] Y.Choi, S. Kwak, E. Park *On syzygies of non-complete embedding of projective varieties*, Math. Z. 258 (2008), 463-475.

[4] D. Eisenbud, *The Geometry of Syzygies*, Springer-Verlag, New York, (2005).

[5] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restriction linear syzygies: algebra and geometry*, Compositio Math. 141 (2005), 1460-1478.

[6] H. Flenner, L. O'Carroll, W. Vogel, *Joins and intersections*, Springer-Verlag, Berlin, (1999).

[7] L. Fuentes-Garcia, M. Pedreira, *The projective theory of ruled surfaces*, Note Mat. 24 (2005), no. 1, 25–63.

[8] L.Gruson, R. Lazarsfeld and C.Peskine, *On a theorem of Castelnuovo and the equations defining projective varieties*, Inv. Math. 72 (1983), 491-506.

[9] M. Green, *Koszul cohomology and the geometry of projective variety*, J. Differential Geometry 19 (1984) 125–171.

[10] S. Kwak, E. Park, *Some effects of property N_p on the higher normality and defining equations of nonlinearly normal varieties*, J. Reine Angew. Math. 582 (2005), 87-105.

[11] A. Noma, *A bound on the Castelnuovo-Mumford regularity for curves*, Math.Ann.322, 69-74 (2002).

[12] E. Park, *On higher syzygies of ruled surfaces*, Trans. Amer. Math. Soc. 358 (2006), 3733-3749.

[13] E. Park, *Complete linear series on a hyperelliptic curve*, Preprint, arXiv:0808.0113.

[14] J. Sidman and P. Vermeire, *Syzygies of the Secant Variety*, arXiv:0806.3056

[15] P. Vermeire, *Generation and syzygies of the first secant variety*, arXiv:0809.4463.

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