THE MAXIMUM NUMBER OF SINGULAR POINTS ON RATIONAL HOMOLOGY PROJECTIVE PLANES

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Abstract. A normal projective complex surface is called a rational homology projective plane if it has the same Betti numbers with the complex projective plane $\mathbb{CP}^2$. It is known that a rational homology projective plane with quotient singularities has at most 5 singular points. So far all known examples have at most 4 singular points. In this paper, we prove that a rational homology projective plane S with quotient singularities such that K_S is nef has at most 4 singular points except one case. The exceptional case comes from Enriques surfaces with a configuration of 9 smooth rational curves whose Dynkin diagram is of type $3A_1 \oplus 2A_3$.

We also obtain a similar result in the differentiable case and in the symplectic case under certain assumptions which all hold in the algebraic case.

1. Introduction

A normal projective complex surface is called a rational homology projective plane if it has the same Betti numbers with the complex projective plane $\mathbb{CP}^2$. A normal projective complex surface with quotient singularities is a rational homology projective plane, if its second Betti number is equal to 1 ([11], p. 2). If a rational homology projective plane is smooth, then it is either $\mathbb{CP}^2$ or a fake projective plane, i.e. a smooth projective surface of general type with $p_g = q = 0$, $K^2 = 9$.

Now let S be a rational homology projective plane with quotient singularities. Assume that S is singular. L. Brenton constructed such surfaces [4], and all examples produced by his method have at most 4 singular points [3]. On the other hand, from the orbifold Bogomolov-Miyaoka-Yau inequality ([22], [18], [17]), one can derive that S has at most 5 singular points, see Corollary 3.4. However, there has been no known examples with 5 singular points. Our main result is :

Theorem 1.1. *Let S be a rational homology projective plane with quotient singularities. Assume that K_S is nef. Then S has at most 4 singular points except the following case:*

S has 5 singular points of type $3A_1 \oplus 2A_3$, and its minimal resolution S' is an Enriques surface.

An example of the exceptional case is given in Example 7.3.

One of the main ingredients in our proof is the orbifold Bogomolov-Miyaoka-Yau inequality. This is the reason why we need the nefness of K_S. The case where $-K_S$

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is ample has been recently dealt with by G. B. Belousov [1]. He has proved that log-Del Pezzo surfaces of Picard number 1 with quotient singularities have at most 4 singular points. Thus, Theorem 1.1 holds true without the nefness of K_S.

Corollary 1.2. *The following hold true.*

(1) *Rational cohomology projective planes with quotient singularities have at most* 4 *singular points except the case given in Theorem 1.1.*
(2) *Integral homology projective planes with quotient singularities have at most* 4 *singular points.*

Here, a rational cohomology projective plane is a normal projective complex surface having the same rational cohomology ring with $\mathbb{CP}^2$. A rational homology projective plane with quotient singularities is a rational cohomology projective plane. As regards integral cohomology projective planes with quotient singularities, D. Bindschadler and L. Brenton [2] have proved that they have at most one singular point of type E_8.

The problem of determining the maximum number of singular points on rational homology projective planes with quotient singularities is related to the algebraic Montgomery-Yang problem ([19], [11]).

We remark that if a rational homology projective plane S is allowed to have rational singularities, then there is no bound for the number of singular points. In fact, there are rational homology projective planes with an arbitrary number of rational singularities. Such examples can be constructed by modifying Example 5 from [11]: take a minimal ruled surface $X \to \mathbb{P}^1$ with negative section E, blow up m distinct fibres into m strings of 3 rational curves (-2) — (-1) — (-2), then contract the proper transform of E with the m adjacent (-2)-curves, and also the m remaining (-2)-curves, to get a rational homology projective plane with $m+1$ rational singularities.

We now present a brief outline of the proof of Theorem 1.1. Assume that our surface S has 5 singular points. Then from the weak version of orbifold Bogomolov-Miyaoka-Yau inequality (see Theorem 3.2) we get one of the following cases for the 5-tuple consisting of the orders of local fundamental groups of singular points:

$$\begin{array}{llll} (2,2,3,3,3), & (2,2,2,4,4), & & \\ (2,2,2,3,3), & (2,2,2,3,4), & (2,2,2,3,5), & (2,2,2,3,6), \\ (2,2,2,2,q) & \text{for } q \geq 2. & & \end{array}$$

Given its minimal resolution $f : S' \to S$, the exceptional curves and the canonical class $K_{S'}$ span a sublattice $R + \langle K_{S'} \rangle$ of the unimodular lattice $H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z})/\text{torsion}$, where R is the sublattice spanned by the exceptional curves. We note that $\text{rank}(R + \langle K_{S'} \rangle) = \text{rank}(R)$ if and only if K_S is numerically trivial (Lemma 3.3). The list above gives an infinite list of possible cases for R. We reduce this infinite list for R by using the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 3.1) together with detailed information about quotient singularities (e.g. Lemmas 2.6, 2.7, 3.6, 3.7, Table 2). Here, we also use the fact that $|\det(R + \langle K_{S'} \rangle)|$ is a square number if K_S is not numerically trivial (Lemma 3.3). The reduced list (Propositions 4.1, 5.1) is still an infinite list, but the infinite part comes from singularities of special type called singularities of type T_6. For each of these cases for R, we then use lattice theoretic arguments to show that, except the two cases $R = 3A_1 \oplus 2A_3$ or $4A_1 \oplus D_5$, either the lattice R or $R + \langle K_{S'} \rangle$ cannot be embedded into the unimodular lattice $H^2(S', \mathbb{Z})_{free}$ (see §6). Finally, in §7 we show that the case $R = 3A_1 \oplus 2A_3$ is supported by an example, and the case $R = 4A_1 \oplus D_5$ can be ruled out by an argument from the classification theory of algebraic surfaces and the theory of discriminant quadratic forms.

To prove that the lattice R or $R+\langle K_{S'}\rangle$ cannot be embedded into the unimodular lattice $H^2(S', \mathbb{Z})_{free}$, we consider the lattice $M = R + \langle K_{S'}\rangle$ when it is of the same rank as the unimodular lattice, and $M = R \oplus R^{\perp}$ otherwise, where $R^{\perp}$ is the orthogonal complement of R in the unimodular lattice. Then we use the Local-Global Principle together with computation of ϵ-invariants (in our case ϵ_3-invariants) to show that M is not isomorphic over $\mathbb{Q}$ to the unimodular lattice. The most complicated cases are the cases for R coming from singularities of type T_6. We note that in these cases $\mathrm{rank}(R + \langle K_{S'}\rangle) = \mathrm{rank}(R)$, hence we have to consider $M = R \oplus R^{\perp}$. We handle this infinite case by using induction on the rank of R (Lemma 6.7). There is an alternative equivalent method: one may compute the discriminant group of M and proceed to show that this group does not contain an isotropic subgroup of order the square root of its order. The latter can be done by showing that the 3-adic part of the discriminant group of M does not contain an isotropic subgroup of order the square root of the order of the 3-adic part. We do not give a detailed write-up of this computation. It takes about the same length of computation as that for ϵ_3-invariants.

Besides using the theory of algebraic surfaces to analyze the two cases in §7, we only use topological facts about algebraic surfaces and quotient singularities. So we can restate Theorem 1.1 in the differentiable case as well as in the symplectic case under certain assumptions which all hold in the algebraic case, see §8.

The first six sections of this paper are as follows. In §2, we review the classification theory of cyclic quotient surface singularities, and prove some properties of Hirzebruch-Jung continued fractions, which play a key role in reducing the list of possible cases for R. In §3, we review the orbifold Bogomolov-Miyaoka-Yau inequality and give some information regarding the sublattice $R + \langle K_{S'}\rangle$. In §4-§5, we obtain a reduced list for R. In §6, we prove that only two cases for R may occur.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

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2. Cyclic quotient singularity and T-singularity

In this section, we briefly review the classification theory of cyclic quotient surface singularities. We also prove some properties of Hirzebruch-Jung continued fractions, which will be used later.

Let $p \in Sing(S)$ be a cyclic quotient singularity. The irreducible components lying over the point p in its minimal resolution form a string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$, and their intersection matrix is given by

$$M(-n_1,\ldots,-n_l) = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -n_2 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & -n_3 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n_{l-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix}$$

It is known that the order of the local fundamental group G_p is equal to the absolute value of the determinant of the matrix $M(-n_1,\ldots,-n_l)$.

A string of smooth rational curves $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$ is also represented by a continued fraction

$$[n_1, n_2, ..., n_l] = n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \cfrac{1}{n_l}}}$$

called the Hirzebruch-Jung continued fraction.

Definition 2.1. .

(1) For rational numbers $n_1, n_2, ..., n_l$, we define

$$q := |\det(M(-n_1,\ldots,-n_l))|$$

$$q_{a_1,a_2,\ldots,a_m} := |\det(M')|$$

where M' is the $(l-m)\times(l-m)$ matrix obtained by deleting $-n_{a_1}, -n_{a_2}, \ldots, -n_{a_m}$ from $M(-n_1,\ldots,-n_l)$. For example,

$$q_1 = |\det(M(-n_2,\ldots,-n_l))| \text{ and } q_{1,l} = |\det(M(-n_2,\ldots,-n_{l-1}))|.$$

(2) For convenience, we also define $q_{1,\ldots,l} = |\det(M(\emptyset))| = 1$.
Note that

$$q_1 q_l = q_{1,l}q + 1, \quad [n_1, n_2, ..., n_l] = \frac{q}{q_1}.$$

The following fact from linear algebra will be used frequently.

Lemma 2.2 ([17]). *For rational numbers $n_1, n_2, ..., n_l$, the solution of the matrix equation*

$$\begin{pmatrix} -n_1 & 1 & 0 & \cdots & 0 \\ 1 & -n_2 & 1 & \cdots & 0 \\ 0 & 1 & -n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -n_l \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{l-1} \\ a_l \end{pmatrix} = - \begin{pmatrix} n_1 - 2 + u \\ n_2 - 2 \\ \vdots \\ n_{l-1} - 2 \\ n_l - 2 + v \end{pmatrix}$$

is given by

$$a_i = 1 - \frac{(1-u)|\det(M(-n_{i+1},\ldots,-n_l))|}{|\det(M(-n_1,\ldots,-n_l))|} - \frac{(1-v)|\det(M(-n_1,\ldots,-n_{i-1}))|}{|\det(M(-n_1,\ldots,-n_l))|}$$

$$= 1 - \frac{(1-u)q_{1,2,\ldots,i}}{q} - \frac{(1-v)q_{i,i+1,\ldots,l}}{q}$$

for $i = 1, 2, ..., l$ if $q \neq 0$.

There is a special type of quotient singularity, called T-singularity. A quotient singularity which admits a $\mathbb{Q}$-Gorenstein smoothing is called a singularity of class T.

Definition 2.3 ([14]). Let $\mathcal{H}$ be the set of all Hirzebruch-Jung continued fractions $[n_1, n_2, \dots, n_l]$,

$$\mathcal{H} = \bigcup_l \{[n_1, n_2, \dots, n_l] \mid \text{all } n_j \text{ are integers} \geq 2\}.$$

(1) A function $\tau : \mathcal{H} \to \mathcal{H}$ defined by

$$\tau([n_1, n_2, \dots, n_l]) = [2, n_1, n_2, \dots, n_{l-1}, n_l + 1]$$

is called a τ-operation.

(2) A reverse operation is a function $r : \mathcal{H} \to \mathcal{H}$ defined by

$$r([n_1, n_2, \dots, n_l]) = [n_l, \dots, n_2, n_1].$$

Theorem 2.4 ([14], [12], [15]). *For an integer $d > 0$, let $T_d \subset \mathcal{H}$ be the following set of continued fractions, or singularities*

$$T_d = \left\{ [n_1, n_2, \dots, n_l] = \frac{dn^2}{dna - 1} \in \mathcal{H} \mid n, a, \textit{integers}, n > a > 0, \gcd(n, a) = 1 \right\}.$$

Then

(1) $[4] \in T_1$, $[3, 3] \in T_2$, $[3, 2, 3] \in T_3$, *and* $[3, 2, 2, \dots, 2, 3]$ (d *vertices*) $\in T_d$.
(2) *If $x \in T_d$, then $r(x) \in T_d$ and $\tau(x) \in T_d$.*
(3) *Every element of T_d is obtained by starting with one of the singularities described in* (1) *and iterating τ-operations and reverse operations.*
(4) *If $[n_1, \dots, n_l] \in T_d$, then $\sum n_j = 3l + 2 - d$.*
(5) *Every singularity of class T is either a rational double point or a singular point of class T_d for some d.*

Furthermore, Looijenga and Wahl proved that a cyclic quotient singularity is of class T if and only if $\dfrac{q_1 + q_l + 2}{q}$ is an integer. More precisely,

Lemma 2.5 ([14], Proposition 5.9). *Let $[n_1, \dots, n_l] \in \mathcal{H}$.*

(1) *$q_1 + q_l + 2 = 2q$ if and only if $[n_1, \dots, n_l]$ corresponds to a rational double point of type A_l.*
(2) *$q_1 + q_l + 2 = q$ if and only if $[n_1, \dots, n_l] \in T_d$ for some d.*

We will also use the following properties of Hirzebruch-Jung continued fractions.

Lemma 2.6. *The value of the formula $q_1 + q_l - q$ is preserved under the τ-operation and the reverse operation, where l denotes the length of the corresponding continued fraction.*

Proof. Clearly, $q_1 + q_l - q$ is preserved under the reverse operation.

Let $v = [n_1, \dots, n_l]$. Then $\tau(v) = [2, n_1, \dots, n_{l-1}, n_l + 1]$. We use the small letter q for v and the capital letter Q for $\tau(v)$. We see that

$$\frac{Q_1}{Q_{1,l+1}} = [n_l + 1, n_{l-1}, \dots, n_1] = [n_l, n_{l-1}, \dots, n_1] + 1 = \frac{q}{q_l} + 1$$

and $Q_{1,l+1} = q_l$, thus $Q_1 = q + q_l$. Similarly,

$$\frac{Q_{l+1}}{Q_{1,l+1}} = [2, n_1, \dots, n_{l-1}] = 2 - \frac{q_{1,l}}{q_l},$$

$$\frac{Q_{1,2}}{Q_{1,2,l+1}} = [n_l + 1, n_{l-1}, \dots, n_2] = \frac{q_1}{q_{1,l}} + 1,$$

hence

$$Q_{l+1} = 2q_l - q_{1,l} \quad \text{and} \quad Q_{1,2} = q_1 + q_{1,l}.$$

Now we have

$$Q_1 + Q_{l+1} - Q = Q_1 + Q_{l+1} - (2Q_1 - Q_{1,2}) = q_1 + q_l - q.$$

□

Lemma 2.7. *Assume that $l \geq 3$. Let $V_l = \{[n_1, \dots, n_l] \in \mathcal{H} \mid -1 \leq q_1 + q_l - q \leq 1\}$. Then, the following hold true:*

1. *$[2, n_2, \dots, n_{l-1}, 2] \notin V_l$,*
2. *If $n_1 \geq 3$ and $n_l \geq 3$, then $[n_1, n_2, \dots, n_{l-1}, n_l] \notin V_l$,*
3. *No element of V_l satisfies $\sum_{j=1}^{l} n_j = 3l - 4$.*

Proof. (1) Suppose that $[2, n_2, \dots, n_{l-1}, 2] \in V_l$. Then $q_1 + q_l - q \leq 1$.
Since $q = n_1 q_1 - q_{1,2} = 2q_1 - q_{1,2}$,
$q_l - q_1 = (q_1 + q_l - q) - q_{1,2} \leq 1 - q_{1,2} < 0$.
On the other hand, since $q = n_l q_l - q_{l-1,l} = 2q_l - q_{l-1,l}$,
$q_1 - q_l = (q_1 + q_l - q) - q_{l-1,l} \leq 1 - q_{l-1,l} < 0$,
which is a contradiction.

(2) Suppose that $[n_1, n_2, \dots, n_{l-1}, n_l] \in V_l$. Then $q_1 + q_l - q \geq -1$.
Thus
$q_l - q_1 = (q_1 + q_l - q) + (n_1 - 2)q_1 - q_{1,2} \geq -1 + (n_1 - 2)q_1 - q_{1,2} \geq -1 + q_1 - q_{1,2} > 0$.
Here, if $-1 + q_1 - q_{1,2} = 0$, then $n_2 = n_3 = \cdots = n_l = 2$, which violates the condition $n_l \geq 3$.
On the other hand,
$q_1 - q_l = (q_1 + q_l - q) + (n_l - 2)q_l - q_{l-1,l} \geq -1 + q_l - q_{l-1,l} \geq 0$,
a contradiction.

(3) If $\sum n_j = 3l - 4$, then $l \geq 4$. Thus no element of V_3 satisfies $\sum n_j = 3l - 4$. We use induction on l. Assume that $l \geq 4$. Assume also that no element of V_{l-1} satisfies $\sum_{j=1}^{l-1} n_j = 3(l-1) - 4$. If $v = [n_1, n_2, \dots, n_l] \in V_l$, then by (1) and (2) either $n_1 = 3$ and $n_l = 2$, or $n_1 = 2$ and $n_l = 3$. Thus $v = \tau(v')$ for some $v' \in \mathcal{H}$. Then by Lemma 2.6, $v' \in V_{l-1}$. But if v satisfies $\sum n_j = 3l - 4$, so does v'. □

3. The orbifold Bogomolov-Miyaoka-Yau inequality

Let S be a surface with quotient singularities and $f : S' \to S$ be a minimal resolution of S.

It is well-known that quotient singularities are log-terminal singularities. Thus one can write

$$K_{S'} = f^* K_S - \sum D_p$$

where $D_p = \sum(a_j E_j)$ is an effective $\mathbb{Q}$-divisor supported on $f^{-1}(p) = \cup E_j$ and $0 \leq a_j < 1$. It implies that

$$K_S^2 = K_{S'}^2 - \sum_{p \in Sing(S)} D_p^2.$$

We also recall the orbifold Euler characteristic

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \Big(1 - \frac{1}{|G_p|}\Big)$$

where G_p is the local fundamental group of p.

The following theorem, called the orbifold Bogomolov-Miyaoka-Yau inequality, is one of the main ingredients in the proof of our main theorem.

Theorem 3.1 ([22], [18], [9], [17])**.** *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \leq 3e_{orb}(S).$$

We also need the following weaker inequality, which also holds when K_S is nef.

Theorem 3.2 ([6])**.** *Let S be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then*

$$0 \leq e_{orb}(S).$$

We know that the torsion free part of the second cohomology group,

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z})/\text{torsion},$$

has a lattice structure which is unimodular. For a singular point $p \in S$, let R_p be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the components of $f^{-1}(p)$. Let

$$R = \oplus_{p \in Sing(S)} R_p$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$. We also consider the sublattice $R + \langle K_{S'} \rangle$ of $H^2(S', \mathbb{Z})_{free}$ spanned by R and the canonical class $K_{S'}$. Note that

$$\text{rank}(R) \leq \text{rank}(R + \langle K_{S'} \rangle) \leq \text{rank}(R) + 1.$$

Lemma 3.3. *The following hold true.*

1. $\text{rank}(R + \langle K_{S'} \rangle) = \text{rank}(R)$ *if and only if K_S is numerically trivial.*
2. $\det(R + \langle K_{S'} \rangle) = \det(R) \cdot K_S^2$ *if K_S is not numerically trivial.*
3. *If S is a rational homology projective plane with quotient singularities, and if K_S is not numerically trivial, then $R + \langle K_{S'} \rangle$ is a sublattice of finite index in the unimodular lattice $H^2(S', \mathbb{Z})_{free}$, in particular $|\det(R + \langle K_{S'} \rangle)|$ is a square number.*

Proof. (1) follows from the equality $K_{S'} = f^*K_S - \sum D_p$.
(2) follows from the fact that $\sum D_p$ is a $\mathbb{Q}$-linear combination of generators of R, and f^*K_S is orthogonal to R.
(3) follows from (1). □

The following corollary is well-known.

Corollary 3.4. *A rational homology projective plane S with quotient singularities has at most 5 singular points.*

Proof. Since $b_2(S) = 1$, either K_S is nef or $-K_S$ is ample. Let $f : S' \to S$ be a minimal resolution of S. Quotient singularities are rational, so $p_g(S') = q(S') = 0$. Thus, by the Noether formula, $e(S') + K_{S'}^2 = 12$. Theorem 3.1 or 3.2 imply that

$$0 \leq e_{orb}(S) = 3 - \sum_{p \in Sing(S)} \Big(1 - \frac{1}{|G_p|}\Big).$$

Thus S has at most 6 singular points. Assume that S has exactly 6 singular points. Then, $|G_p| = 2$ for all $p \in Sing(S)$ and $b_2(S') = b_2(S) + 6 = 7$. Thus $K_{S'}^2 = 3$ by the Noether formula. The lattice $R + \langle K_{S'} \rangle$ is of finite index in $H^2(S', \mathbb{Z})_{free}$. Its discriminant $\det(R + \langle K_{S'} \rangle) = 2^6 3$ is not a square, so it cannot be embedded into a unimodular lattice of the same signature, a contradiction. □

Lemma 3.5. . *Let S be a rational homology projective plane with quotient singularities. Assume that S has 5 singular points. Then the 5-tuple consisting of the orders of local fundamental groups of singular points is one of the following:*

$$\begin{array}{llll} (2,2,3,3,3), & (2,2,2,4,4), & & \\ (2,2,2,3,3), & (2,2,2,3,4), & (2,2,2,3,5), & (2,2,2,3,6), \\ (2,2,2,2,q) & \text{for } q \geq 2. & & \end{array}$$

Proof. Theorem 3.1 implies that

$$0 \leq e_{orb}(S) = -2 + \sum_{p \in Sing(S)} \frac{1}{|G_p|}$$

from which we obtain the list. □

The list above gives an infinite list of possible cases for R. In the next two sections we will reduce this infinite list for R by using the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 3.1) together with detailed information about quotient singularities (e.g. Lemmas 2.6, 2.7, 3.3). The following two lemmas, useful to calculate K_S^2, are also part of such information.

Lemma 3.6 ([14], Proposition 5.9 (iii))**.** *Let p be a cyclic quotient singular point of S. Assume that $f^{-1}(p)$ has l components $E_1, \ldots, E_l$ with ${E_i}^2 = -n_i$ forming a string of smooth rational curves* $\overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$.

(1) *If $l = 1$, then* $D_p^2 = -\dfrac{(n_1-2)^2}{n_1}$.
(2) *If $l \geq 2$, then* $D_p^2 = 2l - \sum n_j + a_1 + a_l = 2l - \sum n_j + 2 - \dfrac{q_1 + q_l + 2}{q}$.

Lemma 3.7. *Let p be a non-cyclic quotient singular point of type D_{q,q_1} with the dual graph given by $< b; 2, 1; 2, 1; q, q_1 >$ (see Table 1 for the notion of dual graph). Let l be the length of the string* $< q, q_1 > = \overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$. *Assume that $l \geq 2$. Then we have the following:*

(1) $\det(R_p) = (-1)^{l+3} 4\{(b-1)q - q_1\}$.
(2) $a_l = 1 - \dfrac{(b-1)q_l - q_{1,l}}{(b-1)q - q_1}$
(3) $D_p^2 = 2l - \Sigma n_j + a_l - (b-2)$

Proof. (1) is just a linear algebra computation.
(2) Since $E_j.K_{S'} = n_j - 2$ by the adjunction formula, we have the following matrix equation:

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & -b & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -n_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -n_2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -n_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -n_l \end{pmatrix} \begin{pmatrix} a_{l+1} \\ a_{l+2} \\ a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_l \end{pmatrix} = - \begin{pmatrix} 0 \\ 0 \\ b-2 \\ n_1 - 2 \\ n_2 - 2 \\ n_3 - 2 \\ \vdots \\ n_l - 2 \end{pmatrix}$$

We see that $2a_{l+1} - a_0 = 0 = 2a_{l+2} - a_0$, hence $a_{l+1} = a_{l+2} = \frac{1}{2}a_0$. So the third row can be rewritten by $-(b-1)a_0 + a_1 = -(b-2)$. Thus the above matrix equation can be simplified to the following.

$$\begin{pmatrix} -(b-1) & 1 & 0 & \cdots & 0 \\ 1 & -n_1 & 1 & \cdots & 0 \\ 0 & 1 & -n_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -n_l \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_l \end{pmatrix} = - \begin{pmatrix} (b-1) - 2 + 1 \\ n_1 - 2 \\ n_2 - 2 \\ \vdots \\ n_l - 2 \end{pmatrix}$$

Since $l \geq 2$, by Lemma 2.2,

$$a_l = 1 - \frac{|\det(M(-(b-1), -n_1, -n_2, \ldots, -n_{l-1}))|}{|\det(M(-(b-1), -n_1, -n_2, \ldots, -n_l))|} = 1 - \frac{(b-1)q_l - q_{1,l}}{(b-1)q - q_1}.$$

From the matrix equation, we observe that

$$\begin{aligned} D_p^2 &= -\sum_{j=1}^{l} a_j(n_j - 2) - a_0(b-2) \\ &= -\sum_{j=1}^{l} (n_j - 2) - a_0 + a_1 + a_l - a_0(b-2) \\ &= -\sum_{j=1}^{l} n_j + 2l + a_l - (b-2). \end{aligned}$$

□

4. Case: S with only cyclic quotient singularities

Let S be a rational homology projective plane with quotient singularities. In this section we consider the case when S admits only cyclic quotient singularities.

By A_n, D_n, E_n we denote the negative definite root lattices.

Proposition 4.1. *Let S be a rational homology projective plane with only cyclic quotient singularities. Assume that K_S is nef. Assume that S has 5 singular points. Then we get one of the the following cases for $R = \oplus_{p \in Sing(S)} R_p$:*

$R = 3A_1 \oplus 2A_3,\ 3A_1 \oplus A_2 \oplus \langle -5\rangle,\ 3A_1 \oplus A_2 \oplus A_4,\ 3A_1 \oplus 2A_2,\ 4A_1 \oplus A_5,$ or $4A_1 \oplus [n_1, n_2, \ldots, n_l]$ for any $[n_1, n_2, \ldots, n_l] \in T_6$.

Proof. We will use the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 3.1) together with detailed information about quotient singularities (e.g. Theorem 2.4, Lemmas 2.5, 2.6, 2.7, 3.6, 3.7). Here, we also use the fact that $|\det(R + \langle K_{S'}\rangle)|$ is a square number if K_S is not numerically trivial (Lemma 3.3). We consider each of the cases given in Lemma 3.5.

(1) The case $(2, 2, 3, 3, 3)$
The lattice R is one of the following:

$$2A_1 \oplus 3\langle -3\rangle,\ 2A_1 \oplus A_2 \oplus 2\langle -3\rangle,\ 2A_1 \oplus 2A_2 \oplus \langle -3\rangle,\ 2A_1 \oplus 3A_2,$$

and $K^2_{S'} = 4, 3, 2, 1$, respectively. Using Lemma 3.6, we get $K^2_S = 5, \frac{11}{3}, \frac{7}{3}, 1$, respectively. Thus in each case, $K^2_S \neq 0$, hence K_S is not numerically trivial. Furthermore, $\det(R + \langle K_{S'}\rangle) = \det(R) \cdot K^2_S = (-2^2 3^3)5$, $(2^2 3^3)(\frac{11}{3})$, $(-2^2 3^3)(\frac{7}{3})$, $2^2 3^3$, respectively. None of these discriminants is a square number modulo $\pm$ sign, so the lattice $R + \langle K_{S'}\rangle$ cannot be embedded into a unimodular lattice of the same signature, a contradiction.

(2) The case $(2, 2, 2, 4, 4)$
Here, the lattice R is one of the following:

$$3A_1 \oplus 2\langle -4\rangle,\ 3A_1 \oplus A_3 \oplus \langle -4\rangle,\ 3A_1 \oplus 2A_3,$$

and $K^2_{S'} = 4, 2, 0$, respectively. In the first two cases, by Lemma 3.6 we see that $K^2_S \neq 0$, and $\det(R + \langle K_{S'}\rangle) = \det(R) \cdot K^2_S = (-2^3 4^2)6$, $(-2^3 4^2)3$, respectively. None of these is a square number modulo sign. Thus, the lattice $R + \langle K_{S'}\rangle$ cannot be embedded into a unimodular lattice of the same signature, a contradiction.

In the last case, $b_2(S') = 10$, $K_{S'} = f^*(K_S)$, and hence by Noether formula $K^2_S = K^2_{S'} = 0$. In particular, K_S is numerically trivial. This gives the first case for R.

(3) The case $(2, 2, 2, 3, 6)$
The lattice R is one of the following:

$$3A_1 \oplus \langle -3\rangle \oplus \langle -6\rangle,\ 3A_1 \oplus A_2 \oplus \langle -6\rangle,\ 3A_1 \oplus \langle -3\rangle \oplus A_5,\ 3A_1 \oplus A_2 \oplus A_5,$$

and $K^2_{S'} = 4, 3, 0, -1$, respectively. In the first three cases, we see that $K^2_S \neq 0$, and $\det(R + \langle K_{S'}\rangle) = \det(R) \cdot K^2_S$ is not a square number modulo sign. Thus, the lattice $R + \langle K_{S'}\rangle$ cannot be embedded into a unimodular lattice of the same signature, a contradiction.

In the last case $K^2_S = K^2_{S'} = -1$, a contradiction.

(4) The case $(2, 2, 2, 3, 5)$
The lattice R is one of the following:
$3A_1 \oplus \langle -3\rangle \oplus \langle -5\rangle,\ 3A_1 \oplus A_2 \oplus \langle -5\rangle,\ 3A_1 \oplus -3 \oplus [3, 2],\ 3A_1 \oplus A_2 \oplus [3, 2],$
$3A_1 \oplus \langle -3\rangle \oplus A_4,\ 3A_1 \oplus A_2 \oplus A_4,$
and $K^2_{S'} = 4, 3, 3, 2, 1, 0$, respectively. Except the second and the last case, we see that $K^2_S \neq 0$, and $\det(R + \langle K_{S'}\rangle) = \det(R) \cdot K^2_S$ is not a square number modulo sign. Thus, the lattice $R + \langle K_{S'}\rangle$ cannot be embedded into a unimodular lattice of the same signature.

In the second case, it can be checked that $K^2_S \neq 0$ and $\det(R + \langle K_{S'}\rangle) = \det(R) \cdot K^2_S$ is a square. This gives the second case for R.

In the last case, $b_2(S') = 10$ and $K_{S'}^2 = K_S^2 = 0$. This gives the third case.

(5) The case $(2,2,2,3,4)$
There are 4 possible cases for R. In each case, we see that $K_S^2 \neq 0$, and $\det(R + \langle K_{S'} \rangle) = \det(R) \cdot K_S^2$ is not a square number modulo sign. Thus, the lattice $R + \langle K_{S'} \rangle$ cannot be embedded into a unimodular lattice of the same signature.

(6) The case $(2,2,2,3,3)$
There are 3 possible cases for R. In each case, we see that $K_S^2 \neq 0$. The absolute value of the discriminant of $R + \langle K_{S'} \rangle$ is a square only if $R = 3A_1 \oplus 2A_2$ with $K_{S'}^2 = 2$. The latter gives the fourth case.

(7) The case $(2,2,2,2,q),\ q \geq 2$
In this case, we use the nefness of K_S. We see that $R = 4A_1 \oplus R_p$ where $|G_p| = q$ and Theorem 3.1 says that $0 \leq K_S^2 \leq \dfrac{3}{q}$. Let l be the number of irreducible components of $f^{-1}(p)$. Note that $b_2(S') = 1 + rank(R) = 5 + l$ and $K_{S'}^2 = 5 - l$.
If $l = 1$, then, by Lemma 3.6,

$$K_S^2 = K_{S'}^2 - D_p^2 = 4 + \frac{(q-2)^2}{q} \geq 4,$$

which is a contradiction.
Now assume that $l \geq 2$, and let $[n_1, n_2, \ldots, n_l]$ be the Hirzebruch-Jung continued fraction of R_p. In this case, also by Lemma 3.6,

$$K_S^2 = K_{S'}^2 - D_p^2 = \sum_{j=1}^{l} n_j - 3l + 5 - (a_1 + a_l).$$

So

$$3l - 5 + (a_1 + a_l) \leq \sum_{j=1}^{l} n_j \leq \frac{3}{q} + 3l - 5 + (a_1 + a_l).$$

Since $0 \leq a_1 + a_l < 2$, we see that $\sum\limits_{j=1}^{l} n_j = 3l-5, 3l-4$, or $3l-3$.

(7-1) Assume that $\sum\limits_{j=1}^{l} n_j = 3l - 5$. Then $a_1 = a_l = 0$, and $K_S^2 = 0$. Since $a_1 = 1 - \dfrac{q_1+1}{q}$, we see that $R_p = A_l$. Then $\sum n_j = 2l = 3l - 5$. Thus, $l = 5$ and $R = 4A_1 \oplus A_5$. This gives the fifth exceptional case.

(7-2) Assume that $\sum\limits_{j=1}^{l} n_j = 3l - 3$. Then by Lemma 2.2,

$$0 \leq K_S^2 = 2 - (a_1 + a_l) = \frac{q_1 + q_l + 2}{q} \leq \frac{3}{q}.$$

So $0 \leq q_1 + q_l + 2 \leq 3$, which is impossible.

(7-3) Now assume that $\sum\limits_{j=1}^{l} n_j = 3l - 4$. First note that $\sum n_j = 3l - 4 \geq 2l$, so $l \geq 4$. By Lemma 2.2,

$$0 \leq K_S^2 = 1 - (a_1 + a_l) = \frac{q_1 + q_l + 2}{q} - 1 \leq \frac{3}{q}.$$

Thus

$$q-2 \le q_1 + q_l \le q+1.$$

Hence, by Lemma 2.7,

$$q_1 + q_l = q-2.$$

Then by Lemma 2.5, $[n_1, n_2, \dots, n_l] \in T_d$ for some d, and by Theorem 2.4, $d=6$. Furthermore $K_S^2 = 0$. This gives the last infinite case for R. □

Remark 4.2. Except the case $(2,2,2,2,q)$, the argument above works without the nefness of K_S, i.e. works even in the case when $-K_S$ is ample.

Remark 4.3. Except the two cases $R = 3A_1 \oplus A_2 \oplus \langle -5 \rangle$ and $3A_1 \oplus 2A_2$, we have shown that $\mathrm{rank}(R + \langle K_{S'} \rangle) = \mathrm{rank}(R)$.

5. Case: S with a non-cyclic quotient singularity

Let S be a rational homology projective plane with quotient singularities. In this section we consider the case when S admits a non-cyclic quotient singular point.

First we recall Brieskorn's classification of finite subgroups of $GL(2,\mathbb{C})$ without quasi-reflections [5]. These are generalizations of the famous subgroups of $SL(2,\mathbb{C})$, i.e. cyclic or binary polyhedral groups. The result is summarized in Table 1.

Here we only explain the notation for dual graph.

$$< q, q_1 > \quad := \quad \text{the dual graph of the singularity of type } \frac{1}{q}(1, q_1),$$

$$< b; s_1, t_1; s_2, t_2; s_3, t_3 > \quad := \quad \text{the tree of the form}$$

$$\begin{array}{ccccc} & & < s_2, t_2 > & & \\ & & | & & \\ < s_1, t_1 > & - & \underset{-b}{\circ} & - & < s_3, t_3 > \end{array}$$

For more information about the table, we refer to the original paper of Brieskorn[5], Matsuki's exposition[16], or Riemenschneider's work[21].

Proposition 5.1. *Let S be a rational homology projective plane with quotient singularities. Assume that K_S is nef. Assume that S has 5 singular points including at least one non-cyclic quotient singular point. Then $R = 4A_1 \oplus D_5$.*

Proof. Since S has a non-cyclic quotient singular point, the possible 5-tuples are $(2,2,2,2,h)$. In particular, S has only one non-cyclic quotient singular point, and Theorem 3.1 gives the inequality

$$0 \le K_S^2 \le \frac{3}{h} \le \frac{3}{8}. \tag{5.1}$$

Let $p \in S$ be the non-cyclic quotient singular point.

(1) The case : p is of type D_{q,q_1}

Let l be the length of the long arm $< q, q_1 > = \overset{-n_1}{\circ} - \overset{-n_2}{\circ} - \cdots - \overset{-n_l}{\circ}$ of the dual graph of $f^{-1}(p)$. Then $f^{-1}(p)$ has $l+3$ irreducible components, $b_2(S') = l+8$ and $K_{S'}^2 = 2-l$.

If $l=1$, then $K_{S'}^2 = 1$, thus $1 \le K_S^2$, a contradiction to (5.1).

Assume that $l \ge 2$.

By Lemma 3.7,

$$K_S^2 = K_{S'}^2 - D_p^2 = \sum n_j - 3l + b - a_l.$$

Table 1

Type	G	$\lvert G \rvert$	Dual Graph Γ_G	
A_{q,q_1}	C_{q,q_1}	q	$< q, q_1 >$	$q_1 < q,\ \gcd(q, q_1) = 1$
D_{q,q_1}	$(Z_{2m}, Z_{2m}; D_q, D_q)$	$4mq$	$< b; 2, 1; 2, 1; q, q_1 >$	$m = (b-1)q - q_1$ odd
D_{q,q_1}	$(Z_{4m}, Z_{2m}; D_q, C_{2q})$	$4mq$	$< b; 2, 1; 2, 1; q, q_1 >$	$m = (b-1)q - q_1$ even
T_m	$(Z_{2m}, Z_{2m}; T, T)$	$24m$	$< b; 2, 1; 3, 2; 3, 2 >$	$m = 6(b-2) + 1$
			$< b; 2, 1; 3, 1; 3, 1 >$	$m = 6(b-2) + 5$
T_m	$(Z_{2m}, Z_{2m}; T, D_2)$	$24m$	$< b; 2, 1; 3, 1; 3, 2 >$	$m = 6(b-2) + 3$
O_m	$(Z_{2m}, Z_{2m}; O, O)$	$48m$	$< b; 2, 1; 3, 2; 4, 3 >$	$m = 12(b-2) + 1$
			$< b; 2, 1; 3, 1; 4, 3 >$	$m = 12(b-2) + 5$
			$< b; 2, 1; 3, 2; 4, 1 >$	$m = 12(b-2) + 7$
			$< b; 2, 1; 3, 1; 4, 1 >$	$m = 12(b-2) + 11$
I_m	$(Z_{2m}, Z_{2m}; I, I)$	$120m$	$< b; 2, 1; 3, 2; 5, 4 >$	$m = 30(b-2) + 1$
			$< b; 2, 1; 3, 2; 5, 3 >$	$m = 30(b-2) + 7$
			$< b; 2, 1; 3, 1; 5, 4 >$	$m = 30(b-2) + 11$
			$< b; 2, 1; 3, 2; 5, 2 >$	$m = 30(b-2) + 13$
			$< b; 2, 1; 3, 1; 5, 3 >$	$m = 30(b-2) + 17$
			$< b; 2, 1; 3, 2; 5, 1 >$	$m = 30(b-2) + 19$
			$< b; 2, 1; 3, 1; 5, 2 >$	$m = 30(b-2) + 23$
			$< b; 2, 1; 3, 1; 5, 1 >$	$m = 30(b-2) + 29$

By (5.1), $\sum n_j - 3l + b = 0$ or 1.

If $\sum n_j - 3l + b = 0$, then $a_l = 0$ and hence by Zariski lemma (see e.g. [17], Lemma 1.3) all components of $f^{-1}(p)$ are (-2)-curves, i.e. p is a rational double point. Thus $K_S^2 = K_{S'}^2 = 0$. It follows that $l = 2$ and p is of type D_5. This gives the case $R = 4A_1 \oplus D_5$.

If $\sum n_j - 3l + b = 1$, then

$$K_S^2 = 1 - a_l = \frac{(b-1)q_l - q_{1,l}}{(b-1)q - q_1} \geq \frac{1}{(b-1)q - q_1} = \frac{4q}{4mq} \geq \frac{8}{4mq} = \frac{8}{h},$$

which is a contradiction to the inequality (5.1).

(2) The case : p is of type T_m, O_m or I_m

By calculating K_S^2 explicitly, we can check that K_S^2 does not satisfy the inequality (5.1) for every possible case. The result of exact computation is summarized in Table 2. □

TABLE 2

Type	Dual Graph	K_S^2
T_m	$< b;2,1;3,2;3,2 >$	$\frac{6b^2-30b+35}{6b-11}\begin{cases}\le -\frac{1}{7} & \text{if } b\le 3\\ \ge \frac{11}{13} & \text{if } b\ge 4\end{cases}$
	$< b;2,1;3,1;3,1 >$	$\frac{6b^2-6b-1}{6b-7}\ge\frac{11}{5}$
T_m	$< b;2,1;3,1;3,2 >$	$\frac{18b^2-54b+41}{18b-27}\ge\frac{5}{9}$
O_m	$< b;2,1;3,2;4,3 >$	$\frac{12b^2-72b+94}{12b-23}\begin{cases}\le -\frac{2}{25} & \text{if } b\le 4\\ \ge \frac{34}{37} & \text{if } b\ge 5\end{cases}$
	$< b;2,1;3,1;4,3 >$	$\frac{12b^2-48b+46}{12b-19}\begin{cases}= -\frac{2}{5} & \text{if } b= 2\\ \ge \frac{10}{17} & \text{if } b\ge 3\end{cases}$
	$< b;2,1;3,2;4,1 >$	$\frac{12b^2-24b+10}{12b-17}\ge\frac{10}{7}$
	$< b;2,1;3,1;4,1 >$	$\frac{12b^2-14}{12b-13}\ge\frac{34}{11}$
I_m	$< b;2,1;3,2;5,4 >$	$\frac{30b^2-210b+297}{30b-59}\begin{cases}\le -\frac{3}{91} & \text{if } b\le 5\\ \ge \frac{117}{121} & \text{if } b\ge 6\end{cases}$
	$< b;2,1;3,2;5,3 >$	$\frac{30b^2-126b+129}{30b-53}\begin{cases}= -\frac{3}{7} & \text{if } b= 2\\ \ge \frac{21}{37} & \text{if } b\ge 3\end{cases}$
	$< b;2,1;3,1;5,4 >$	$\frac{30b^2-150b+165}{30b-49}\begin{cases}\le -\frac{15}{41} & \text{if } b\le 3\\ \ge \frac{45}{71} & \text{if } b\ge 4\end{cases}$
	$< b;2,1;3,2;5,2 >$	$\frac{30b^2-114b+105}{30b-47}\begin{cases}= -\frac{3}{13} & \text{if } b= 2\\ \ge \frac{33}{43} & \text{if } b\ge 3\end{cases}$
	$< b;2,1;3,1;5,3 >$	$\frac{30b^2-66b+33}{30b-43}\ge\frac{21}{17}$
	$< b;2,1;3,2;5,1 >$	$\frac{30b^2-30b-15}{30b-41}\ge\frac{45}{19}$
	$< b;2,1;3,1;5,2 >$	$\frac{30b^2-54b+21}{30b-37}\ge\frac{33}{23}$
	$< b;2,1;3,1;5,1 >$	$\frac{30b^2+30b-63}{30b-31}\ge\frac{117}{29}$

6. QUADRATIC FORMS

In this section we prove that the cases for R given in Proposition 4.1 cannot actually occur except the first case $R = 3A_1 + 2A_3$. We use the Local-Global Principle together with computation of ϵ-invariants to show that, except the first case, either the lattice R or $R + \langle K_{S'}\rangle$ cannot be embedded into the unimodular lattice $H^2(S', \mathbb{Z})_{free}$.

Theorem 6.1. [23] *(Local-Global Principle) Let f and f' be two quadratic forms over $\mathbb{Q}$. For f and f' to be equivalent over $\mathbb{Q}$ it is necessary and sufficient that they are equivalent over each p-adic field $\mathbb{Q}_p$ or the field $\mathbb{Q}_\infty$ of real numbers.*

Let f be a quadratic form in n variables over the p-adic field $\mathbb{Q}_p$ such that $f = a_1{X_1}^2 + a_2{X_2}^2 + \ldots + a_n{X_n}^2$. Define discriminant $d_p(f)$ and ϵ-invariant $\epsilon_p(f)$ of f as follows:

$$d_p(f) = a_1 \ldots a_n \in \mathbb{Q}_p/\mathbb{Q}_p^{*2}$$
$$\epsilon_p(f) = \prod_{i<j}(a_i, a_j)_p$$

where $(-,-)_p$ is the Hilbert symbol on $\mathbb{Q}_p$.

Let f, f' be two quadratic forms over the p-adic field $\mathbb{Q}_p$. Then these invariants have the following obvious properties.

$$d_p(f \oplus f') = d_p(f) \cdot d_p(f')$$
$$\epsilon_p(f \oplus f') = \epsilon_p(f)\epsilon_p(f')(d_p(f), d_p(f'))_p$$

We set $\epsilon_p(f) = 1$ if f is a quadratic form in 1 variable.

Theorem 6.2. [23] *Let k be a p-adic field. Then two quadratic forms over k are equivalent if and only if they have the same rank, the same discriminant, and the same ϵ-invariant.*

Every non-zero element of the p-adic field $\mathbb{Q}_p$ can be written uniquely in the form $p^\alpha u$ for some integer α and some p-adic unit u. For any prime number p and integers α and x with $1 \leq x < p$, we define

$$\bar{x} \cdot p^\alpha := \left\{ p^\alpha u \;\middle|\; u = x + \sum_{i \geq 1} a_i p^i \text{ is a } p\text{-adic unit } \right\}.$$

Theorem 6.3. [23] *(Computation of Hilbert symbol) Let $p > 2$ be a prime number and let $a, b \in \mathbb{Q}_p$.*
If $a \in \bar{u} \cdot p^\alpha,\ b \in \bar{v} \cdot p^\beta$, then the Hilbert symbol $(a,b)_p$ can be computed as

$$(a,b)_p = (-1)^{\alpha\beta\rho(p)} \left(\frac{u}{p}\right)^\beta \left(\frac{v}{p}\right)^\alpha$$

where $\left(\frac{u}{p}\right)$ denotes the Legendre symbol, $\rho(p)$ denotes the class modulo 2 of $\dfrac{p-1}{2}$.

Lemma 6.4. *Let L be the integral lattice corresponding to a Hirzebruch-Jung continued fraction $[n_1, n_2, \ldots, n_l]$ with standard basis $\{e_1, ..., e_l\}$. Let $(L \otimes \mathbb{Q}, f)$ be the quadratic form over $\mathbb{Q}$ defined by L. Then we can take an orthogonal basis $\{v_1, ..., v_l\}$ with ${v_i}^2 = -[n_i, ..., n_1]$ so that the quadratic form is given by $f = \sum {v_i}^2 {X_i}^2$.*

Proof. It is Gram-Schmidt process, essentially. □

Lemma 6.5. *Let L be the integral lattice corresponding to a Hirzebruch-Jung continued fraction $[n_1, n_2, \ldots, n_l]$. Let $(L \otimes \mathbb{Q}, f_L)$ be the quadratic from with $f_L = \sum_{i=1}^{l} c_i {X_i}^2$ where $c_i = -[n_i, \ldots, n_1]$ for $i = 1, \ldots, l$. Let $(\tau(L) \otimes \mathbb{Q}, f_{\tau(L)})$ be the quadratic form corresponding to $\tau([n_1, n_2, \ldots, n_l])$. Then we can choose an orthogonal basis $\{v_1, \ldots, v_{l+1}\}$ such that we can write $f_{\tau(L)} = \sum d_i {X_i}^2$ with $d_i = c_i$ for $i = 1, \ldots, l-1$, $d_l = c_l - 1$, and $d_{l+1} = -2 - \sum_{j=1}^{l} \dfrac{d_j}{(d_1 d_2 \cdots d_j)^2} = -2 + \dfrac{q_1 + q_{1,l}}{q + q_l}$, where $q = |\det(M(-n_1, -n_2, \ldots, -n_l))|$.*

In particular, if $[n_1, \dots, n_l] \in T_6$, then the 3-adic valuation of d_{l+1} is a positive odd integer, more precisely, $d_{l+1} \in \bar{2} \cdot 3^\alpha$ for a positive odd integer α.

Proof. Recall that $\tau([n_1, \dots, n_l]) = [2, n_1, \dots, n_{l-1}, n_l + 1]$. With respect to a suitable basis $\{e_1, e_2, \dots, e_{l+1}\}$, we can write the corresponding intersection matrix as follows:

$$M_{\tau(L)} = \begin{pmatrix} -n_1 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & -n_2 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -n_3 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -n_{l-1} & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 1 & -n_l - 1 & 0 \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & -2 \end{pmatrix}$$

Then, by Gram-Schmidt process, we can write $v_1 = e_1$ and for $i = 2, \dots, l+1$,

$$v_i = e_i - \sum_{j=1}^{i-1} \frac{\langle v_j, e_i \rangle}{\langle v_j, v_j \rangle} v_j.$$

Then $d_i = v_i^2$ for $i = 1, 2, \dots, l+1$. It is easy to see that

$$d_i = c_i \text{ for } i = 1, \dots, l-1,$$

$$d_l = -[n_l + 1, n_{l-1}, \dots, n_1] = -(1 + [n_l, \dots, n_1]) = -1 + c_l,$$

and

$$d_{l+1} = e_{l+1}^2 - \sum_{j=1}^{l} \frac{\langle v_j, e_{l+1} \rangle^2}{\langle v_j, v_j \rangle} = -2 - \sum_{j=1}^{l} \frac{d_j}{(d_1 \cdots d_j)^2}.$$

Write $c_j = -\dfrac{y_j}{y_{j-1}}$ where $y_0 = 1$ and $y_j = |\det(M(-n_1, -n_2, \dots, -n_j))|$. Clearly $y_l = q$ and $y_{l-1} = q_l$. Note that

$$\frac{d_j}{(d_1 \cdots d_j)^2} = \frac{c_j}{(c_1 \cdots c_j)^2} = -\frac{1}{y_{j-1} y_j}$$

for $j = 1, \dots, l-1$. Claim that

$$\sum_{j=1}^{k} \frac{1}{y_{j-1} y_j} = \frac{|\det(M(-n_2, -n_3, \dots, -n_k))|}{y_k}.$$

We prove the claim by using induction. If $k = 2$, then

$$\frac{1}{y_0 y_1} + \frac{1}{y_1 y_2} = \frac{y_2 + 1}{y_1 y_2} = \frac{n_2}{y_2}.$$

Now assume that the claim holds for $k < m$. Then

$$\begin{aligned} \sum_{j=1}^{m} \frac{1}{y_{j-1} y_j} &= \sum_{j=1}^{m-1} \frac{1}{y_{j-1} y_j} + \frac{1}{y_{m-1} y_m} \\ &= \frac{|\det(M(-n_2, -n_3, \dots, -n_{m-1}))| y_m + 1}{y_{m-1} y_m} \\ &= \frac{|\det(M(-n_2, -n_3, \dots, -n_m))|}{y_m}, \end{aligned}$$

which proves the claim. Thus

$$\begin{aligned} d_{l+1} &= -2 - \sum_{j=1}^{l-1} \frac{c_j}{(c_1 \cdots c_j)^2} - \frac{1}{(c_1 \cdots c_{l-1})^2 d_l} \\ &= -2 + \sum_{j=1}^{l-1} \frac{1}{y_{j-1} y_j} - \frac{1}{y_{l-1}^2 (c_l - 1)} \\ &= -2 + \frac{q_{1,l}}{q_l} + \frac{1}{q_l(q+q_l)} = -2 + \frac{q_1 + q_{1,l}}{q+q_l}. \end{aligned}$$

Now assume that $[n_1, \ldots, n_l] \in T_6$. Then

$$q = 6n^2,\ q_1 = 6na - 1,\ q_l = 6nb - 1$$

for some integers n, a, b with $n > a > 0$, $\gcd(n, a) = 1$, $a + b = n$.
Since $q_{1,l} q = q_1 q_l - 1$, we see that $q_{1,l} = 6ab - 1$. Using these we get

$$d_{l+1} = \frac{-6(n+b)^2}{6n^2 + 6nb - 1}.$$

Now it is easy to see that $d_{l+1} \in \bar{2} \cdot 3^{\alpha}$ for a positive odd integer α. □

Lemma 6.6. (1) *Let $I_{1,m} := \langle 1 \rangle \oplus m\langle -1 \rangle$ be the odd unimodular lattice of signature $(1, m)$. Then $\epsilon_p(I_{1,m}) = 1$ for all $p > 2$.*

(2) *Let $II_{1,8m+1} := H \oplus mE_8$ be the even unimodular lattice of signature $(1, 8m+1)$, where H is the even unimodular lattice of signature $(1, 1)$, and E_8 the even unimodular lattice of signature $(0, 8)$. Then $\epsilon_3(II_{1,8m+1}) = 1$.*

Proof. (1) follows from a direct calculation.

(2) It is easy to see that $\epsilon_3(H) = 1$. By a suitable change of basis, we can write the quadratic form of $E_8 \otimes \mathbb{Q}$ as follows:

$$f = -2X_1^2 - \frac{3}{2}X_2^2 - \frac{4}{3}X_3^2 - \frac{5}{4}X_4^2 - \frac{6}{5}X_5^2 - \frac{7}{6}X_6^2 - \frac{8}{7}X_7^2 - \frac{1}{8}X_8^2.$$

A direct calculation shows that $\epsilon_3(E_8) = 1$. Hence

$$\epsilon_3(H \oplus E_8) = \epsilon_3(H)\epsilon_3(E_8)(d(H), d(E_8))_3 = 1.$$

Now, use induction. □

Lemma 6.7. *Let $l = m - 4$ be an integer ≥ 6, and R_p be the lattice of rank l corresponding to a singularity p of class T_6. Then the negative definite lattice $N := 4A_1 \oplus R_p$ of rank m cannot be embedded into the lattice $I_{1,m}$.*

Proof. Assume that N is embedded to $I_{1,m}$. Let $N^{\perp}$ be the orthogonal complement of N in $I_{1,m}$. Then $(N \oplus N^{\perp}) \otimes \mathbb{Q}_3 \cong I_{1,m} \otimes \mathbb{Q}_3$. Thus by Lemma 6.6,

$$\epsilon_3(N \oplus N^{\perp}) = \epsilon_3(I_{1,m}) = 1.$$

To get a contradiction, we will show that $\epsilon_3(N \oplus N^{\perp}) = -1$. Note that $\det(N) = (-1)^l 2^4 6n^2$ and $\det(N^{\perp}) = 6n'^2$ for some n, n'. Hence by Theorem 6.3

$$(d_3(N), d_3(N^{\perp}))_3 = ((-1)^l 6, 6)_3 = (-1)^{l+1}.$$

It is easy to see that $\epsilon_3(N) = \epsilon_3(R_p)$. Thus

$$\epsilon_3(N \oplus N^{\perp}) = \epsilon_3(N)\epsilon_3(N^{\perp})(d_3(N), d_3(N^{\perp}))_3 = (-1)^{l+1}\epsilon_3(R_p)$$

It is enough to show that

$$\epsilon_3(R_p) = (-1)^l.$$

To do this we use induction on l.

If $l = 6$, then R_p corresponds to the Dynkin diagram $\overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ}$, and by Lemma 6.4 the quadratic form $(R_p \otimes \mathbb{Q}, f)$ over $\mathbb{Q}$ is given by

$$f = -3X_1^2 - \frac{5}{3}X_2^2 - \frac{7}{5}X_3^2 - \frac{9}{7}X_4^2 - \frac{11}{9}X_5^2 - \frac{24}{11}X_6^2.$$

A direct calculation shows that $\epsilon_3(R_p) = 1$.

It is clear that the epsilon invariant does not change under a reverse operation. Since the τ-operation increases $\mathrm{rank}(R_p)$ by 1, it is sufficient to show that

$$\epsilon_3(R_p)\epsilon_3(\tau(R_p)) = -1.$$

By Lemma 6.5 and notation there,

$$\epsilon_3(R_p)\epsilon_3(\tau(R_p)) = (c_l d_l d_{l+1}, c_1 \cdots c_{l-1})_3 (d_{l+1}, d_l)_3.$$

Recall that $[n_1, \ldots, n_l] = \dfrac{q}{q_1} = \dfrac{6n^2}{6na-1}$, $c_l = -[n_l, \ldots, n_1] = -\dfrac{q}{q_l} = -\dfrac{6n^2}{6nb-1}$, for some integers $n > a > 0$, $n > b > 0$ with $\gcd(n, a) = \gcd(n, b) = 1$. It implies that $q = 6n^2$, $q_1 = 6na - 1 \equiv 2 \mod 3$, $q_l = 6nb - 1 \equiv 2 \mod 3$, and $c_l \in \bar{2} \cdot 3^\alpha$ for some odd integer $\alpha > 0$. Note that

$$c_1 \cdots c_{l-1} = (-1)^{l-1} q_l.$$

Case 1: l is odd.
In this case $c_1 \cdots c_{l-1} \in \bar{2} \cdot 3^0$ and $d_l \in \bar{2} \cdot 3^0$. Thus $(c_l d_l d_{l+1}, c_1 \cdots c_{l-1})_3 = 1$ and $(d_{l+1}, d_l)_3 = -1$. Hence $\epsilon_3(R_p)\epsilon_3(\tau(R_p)) = -1$, as desired.

Case 2: l is even.
In this case $c_1 \cdots c_{l-1} \in \bar{1} \cdot 3^0$ and $d_l \in \bar{2} \cdot 3^0$. Thus $(c_l d_l d_{l+1}, c_1 \cdots c_{l-1})_3 = 1$ and $(d_{l+1}, d_l)_3 = -1$. Hence $\epsilon_3(R_p)\epsilon_3(\tau(R_p)) = -1$, as desired.
This completes the proof. ☐

Lemma 6.8. *Let $R = 3A_1 \oplus A_2 \oplus A_4$, or $4A_1 \oplus A_5$.*
Then the lattice R can be embedded into neither the lattice $I_{1,9}$ nor $II_{1,9}$.
In particular, neither the case $R = 3A_1 \oplus A_2 \oplus A_4$ nor $4A_1 \oplus A_5$ in Proposition 4.1 occurs.

Proof. Suppose that R can be embedded into $L := I_{1,9}$ or $II_{1,9}$. Let $R^\perp$ be the orthogonal complement of R in L. By Lemma 6.6, it suffices to show that $\epsilon_3(R \oplus R^\perp) = -1$.

Case 1. $R = 3A_1 \oplus A_2 \oplus A_4$.
Since $d(R) = -2^3 \cdot 3 \cdot 5$, we see that $d(R^\perp) = 30$. By a direct calculation, it is easy to see that $\epsilon_3(R) = -1$, so

$$\epsilon_3(R \oplus R^\perp) = \epsilon_3(R)\epsilon_3(R^\perp)(d_3(R), d_3(R^\perp))_3 = -1.$$

Case 2. $R = 4A_1 \oplus A_5$.
Similar to Case 1. Since $d(R) = -2^4 \cdot 6$, we see that $d_3(R^\perp) = 6$. A direct calculation shows that $\epsilon_3(R) = -1$, so $\epsilon_3(R \oplus R^\perp) = -1$. ☐

Corollary 6.9. *There is no Enriques surface with a configuration of 9 smooth rational curves whose Dynkin diagram is of type* $3A_1 \oplus A_2 \oplus A_4$ *or* $4A_1 \oplus A_5$.

Proof. The second cohomology group, modulo torsion, of any Enriques surface has a lattice structure isomorphic to $II_{1,9} = H \oplus E_8$. □

Lemma 6.10. *The two cases* $R = 3A_1 \oplus A_2 \oplus \langle -5 \rangle$ *and* $R = 3A_1 \oplus 2A_2$ *in Proposition 4.1 do not occur.*

Proof. It suffices to show that the lattice $R = 3A_1 \oplus A_2 \oplus \langle -5 \rangle$ (resp. $3A_1 \oplus 2A_2$ cannot be embedded into the unimodular lattice $H^2(S', \mathbb{Z})_{free}$ which is isomorphic to the lattice $I_{1,6}$ (resp. $I_{1,7}$). Note that

$$(R + \langle K_{S'} \rangle) \otimes \mathbb{Q} \cong (R + \langle f^*K_S \rangle) \otimes \mathbb{Q}.$$

Thus

$$\epsilon_3(R + \langle K_{S'} \rangle) = \epsilon_3(R + \langle f^*K_S \rangle).$$

In case $R = 3A_1 \oplus A_2 \oplus \langle -5 \rangle$, it can be checked that $\epsilon_3(R + \langle f^*K_S \rangle) = -1$, so by Lemma 6.6 the lattice $R + \langle K_{S'} \rangle$ cannot be embedded into $I_{1,6} = \langle 1 \rangle \oplus 6\langle -1 \rangle$.

Similarly, in case $R = 3A_1 \oplus 2A_2$, it can be checked that $\epsilon_3(R + \langle K_{S'} \rangle) = -1$, so by Lemma 6.6 the lattice $R + \langle K_{S'} \rangle$ cannot be embedded into $I_{1,7} = \langle 1 \rangle \oplus 7\langle -1 \rangle$. □

Now by Corollary 3.4, Lemmas 6.7, 6.8, 6.10, we can combine Propositions 4.1 and 5.1 into the following form:

Proposition 6.11. *Let S be a rational homology projective plane with quotient singularities. Assume that K_S is nef. Then S has at most* 4 *singular points except the following two cases:*

S has 5 *singular points of type* $3A_1 \oplus 2A_3$ *or* $4A_1 \oplus D_5$.

Proposition 6.12. *In either case* $R = 3A_1 \oplus 2A_3$ *or* $4A_1 \oplus D_5$, *S' is an Enriques surface.*

Proof. In either case, we have shown in the proof of Propositions 4.1 and 5.1 that K_S is numerically trivial. Since S has only rational double points, $K_{S'} = f^*K_S$, hence $K_{S'}$ is numerically trivial. We know that $p_g(S') = q(S') = 0$. Thus by the classification theory of algebraic surfaces S' is an Enriques surface. □

7. Enriques surfaces

In this section we show that the case $R = 3A_1 \oplus 2A_3$ is supported by an example, and the case $R = 4A_1 \oplus D_5$ can be ruled out by an argument from the classification theory of algebraic geometry and the theory of discriminant quadratic forms.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \subset L^* = Hom(L, \mathbb{Z})$. The factor group L^*/L, which is denoted by $disc(L)$, is an abelian group of order $|\det(L)|$. We denote by $l(L)$ the number of minimal generators of $disc(L)$. We extend the bilinear form on L to the one on L^*, taking value in $\mathbb{Q}$, and define

$$q_L : disc(L) \to \mathbb{Q}/2\mathbb{Z}, \quad q_L(x + L) = \langle x, x \rangle + 2\mathbb{Z} \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L. A subgroup A of $disc(L)$ is said to be isotropic if q_L takes value identically 0 on A.

For a non-degenerate odd lattice, its discriminant quadratic form can be defined similarly.

Let L be a sublattice of a lattice M. The lattice L is said to be primitive if M/L is torsion free. The minimal primitive sublattice of M containing L is called the primitive closure of L, and is denoted by $\bar{L}$. The orthogonal complement of L in M is denoted by $L_M^\perp$, or simply by $L^\perp$. The following is well known (see e.g. [20]).

Lemma 7.1. *Let L be a non-degenerate even lattice.*

(1) *If an even lattice M is an over-lattice of L, i.e. M has the same rank as L and contains L, then the group $A := M/L$ is an isotropic subgroup of $disc(L)$, and $disc(M) \cong A^\perp/A$.*
(2) *Conversely, every isotropic subgroup A of $disc(L)$ defines a unique over-lattice $M \subset L^*$ with $disc(M) \cong A^\perp/A$.*
(3) *If L is primitive in a unimodular even lattice, then*

$$(disc(L^\perp), q_{L^\perp}) \cong (disc(L), -q_L).$$

Proposition 7.2. *There is no Enriques surface with a configuration of 9 smooth rational curves whose Dynkin diagram is of type $4A_1 \oplus D_5$.*

Proof. Suppose that there is such an Enriques surface W. The Néron-Severi group modulo torsion, $H^2(W,\mathbb{Z})_{free} := H^2(W,\mathbb{Z})/\text{torsion}$, has a lattice structure isomorphic to $H \oplus E_8$. Here, the torsion is generated by the canonical class K_W. Let $R = 4A_1 \oplus D_5$ be the sublattice of $H^2(W,\mathbb{Z})_{free}$ generated by the 9 smooth rational curves on W. Let E_1, E_2, E_3, E_4 be the smooth rational curves corresponding to $4A_1$. Note that

$$disc(R) = \Big(\overset{4}{\underset{i=1}{\oplus}} (\mathbb{Z}/2)\langle e_i\rangle \Big) \oplus \Big((\mathbb{Z}/4)\langle v\rangle \Big),$$

where $\langle\cdot\rangle$ is the generator of the group, e.g. $e_i = \dfrac{[E_i]}{2}$. The quadratic form on $disc(D_5) \cong (\mathbb{Z}/4)\langle v\rangle$ is given by $v^2 = -\dfrac{5}{4}$. Since $\text{rank}R^\perp = 1$, $disc(R^\perp)$ is a cyclic group. Hence by Lemma 7.1, we see that $disc(\bar{R}) \cong -disc(R^\perp)$ is a cyclic group and $\bar{R}/R$ is an isotropic subgroup of $disc(R)$. Since $l(R) = 5$, this is possible only if $disc(\bar{R}) \cong \mathbb{Z}/4$ and $\bar{R}/R = \overset{2}{\underset{i=1}{\oplus}} (\mathbb{Z}/2)$. Finding generators of $\bar{R}/R$, we see that it is generated by two elements $e_1+e_2+e_3+e_4$ and e_i+e_j+2f for some $i \neq j$. In any case, $e_1+e_2+e_3+e_4 \in \bar{R}/R$. This means that $E_1+E_2+E_3+E_4$ is divisible by 2 in $H^2(W,\mathbb{Z})_{free}$, i.e. either $E_1+E_2+E_3+E_4$ or $E_1+E_2+E_3+E_4+K_W$ is divisible by 2 in $H^2(W,\mathbb{Z}) = \text{Pic}(W)$. Let X be the algebraic K3 cover of W. Then it follows that the pre-images in X of the 4 curves E_1, E_2, E_3, E_4 are 8 smooth rational curves whose sum is divisible by 2 in $\text{Pic}(X)$. Let $X \to X'$ be the contraction of these 8 curves. Note that away from these singular points, X' contains 10 smooth rational curves whose Dynkin diagram is of type $2D_5$. Then there is a double cover Y of X' branched exactly along the 8 singular points. The surface Y is an algebraic K3 surface (cf. [8], Theorem 1 and 2). Then Y contains 20 smooth rational curves whose Dynkin diagram is of type $4D_5$. This implies that Y has Picard number ≥ 21, which is impossible. □

The following example was mentioned in Theorem 1.1.

Example 7.3. There is an Enriques surface with a configuration of 9 smooth rational curves whose Dynkin diagram is of type $3A_1 \oplus 2A_3$. See Example III, [13]. This Enriques surface has an elliptic fibration with 2 double fibres of type I_4, 2 fibres of type I_2, and a special 2-section intersecting only one component in each fibre.

Let S be a rational homology projective plane with 5 singularities of type $3A_1 \oplus 2A_3$. Then S is not an integral homology projective plane, because $H_1(S,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \neq 0$. But S and $\mathbb{CP}^2$ have the isomorphic rational cohomology ring, although $H^2(S,\mathbb{Q})$ does not contain an element of self-intersection 1.

Now Theorem 1.1 follows from Propositions 6.11, 6.12, 7.2 and Example 7.3.

8. The differentiable case

Let M be a smooth, compact 4-manifold whose boundary components are spherical, that is, they are links. One can then attach cones to each boundary component to get a 4-dimensional orbifold S. As in the algebraic case, there is a minimal resolution $f : S' \to S$, where S' is a smooth, compact 4-manifold without boundary. To each singular point $p \in S$ (the vertex of each cone), we assign a uniquely defined class $D_p = \sum(a_j E_j) \in H^2(S',\mathbb{Q})$ such that $D_p \cdot E_i = 2 + E_i^2$ for each component E_i of $f^{-1}(p)$.
We always assume that S and S' satisfy the following two conditions:

(1) S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$, i.e. $H^1(S,\mathbb{Q}) = 0$ and $H^2(S,\mathbb{Q}) \cong \mathbb{Q}$.
(2) The intersection form on $H^2(S',\mathbb{Q})$ is indefinite, and is negative definite on the subspace generated by the classes of the exceptional curves of f.

If there is a class $K_{S'} \in H^2(S',\mathbb{Q})$ satisfying both the Nöther formula

$$K_{S'}^2 = 10 - b_2(S')$$

and the adjunction formula

$$K_{S'} \cdot E + E^2 = -2$$

for each exceptional curve E of $f : S' \to S$, we call it a *formal canonical class* of S'.

Theorem 8.1. *Let M, S, and S' be the same as above satisfying the conditions* (1) *and* (2). *Assume that S' admits a formal canonical class $K_{S'}$. Assume further that*

$$K_{S'}^2 - \sum_{p\in Sing(S)} D_p^2 \leq 3e_{orb}(S).$$

Then M has at most 4 *boundary components except the following two cases:*
M has 5 boundary components of type $3A_1 + 2A_3$ or $4A_1 + D_5$.

Note that the assumptions in Theorem 8.1 all hold for algebraic $\mathbb{Q}$-homology projective planes with quotient singularities such that the canonical divisor is nef.

Proof. In our proof up to Proposition 6.11 for the algebraic orbifold case, the canonical class K_S appears several times, but can be replaced by f^*K_S. Given a formal canonical class $K_{S'}$ in the differentiable case, the class $K_{S'} + \sum D_p \in H^2(S',\mathbb{Q})$ plays exactly the same role as f^*K_S. The words "K_S is numerically trivial" is now replaced by "$K_{S'} = -\sum D_p \in H^2(S',\mathbb{Q})$", or by "$K_{S'} \in R \otimes \mathbb{Q}$". □

Theorem 8.2. *Let M, S, and S' be the same as above satisfying the conditions* (1) *and* (2). *Assume that S' admits a formal canonical class $K_{S'}$. Assume further that*

$$0 \leq e_{orb}(S).$$

Then M has at most 5 boundary components. The bound is sharp.

The assumptions in Theorem 8.2 all hold for algebraic $\mathbb{Q}$-homology projective planes with quotient singularities.

If S is a symplectic orbifold, then S' is a symplectic manifold and the symplectic canonical class $K_{S'}$ gives a formal canonical class.

Corollary 8.3. *Let M, S, and S' be the same as above satisfying the conditions (1) and (2). Assume that S is a symplectic orbifold. Assume further that*

$$K_{S'}^2 - \sum_{p\in Sing(S)} D_p^2 \leq 3e_{orb}(S).$$

Then M has at most 4 boundary components except the following two cases: M has 5 boundary components of type $3A_1 + 2A_3$ or $4A_1 + D_5$.

Corollary 8.4. *Let M, S, and S' be the same as above satisfying the conditions (1) and (2). Assume that S is a symplectic orbifold. Assume further that*

$$0 \leq e_{orb}(S).$$

Then M has at most 5 boundary components. The bound is sharp.

Remark 8.5. In the differentiable case, if a formal canonical class $K_{S'}$ is given, then a *formal canonical class* of S can be defined as the class $K_{S'} + \sum D_p \in H^2(S', \mathbb{Q})$.

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