GENERALIZED CASTELNUOVO-MUMFORD REGULARITY FOR AFFINE KAC-MOODY ALGEBRAS

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ABSTRACT. The graded modules over noncommutative algebras often have minimal free resolutions of infinite length, resulting in infinite *Castelnuovo-Mumford regularity*. While a generalized notion of regularity was developed in in [6] to address this problem, it is not easily computable. In this paper, we compute the generalized Castelnuovo-Mumford regularity for integrable highest weight representations of affine Kac-Moody algebras. It is shown that the generalized regularity depends only on the type and rank of algebras and the level of representations.

INTRODUCTION

For graded modules over commutative algebras, the *Castelnuovo-Mumford regularity* provides a computational invariant measuring the degree-complexity [1, 2]. However, in the case of graded modules over noncommutative algebras, the projective dimension is usually infinite, resulting in infinite Castelnuovo-Mumford regularity. In such a case, one cannot deduce any meaningful information out of it.

To overcome this difficulty, the notion of the *exponent of growth $e(M)$* and the *rate of growth $r(M)$* were introduced in [6], for a graded module M over a noncommutative algebra A. When the projective dimension of M is finite, we have $e(M) = 0$, and $r(M)$ coincides with the usual Castelnuovo-Mumford regularity. For this reason, the pair $(e(M), r(M))$ is called the *generalized Castelnuovo-Mumford regularity* of M. While the generalized Castelnuovo-Mumford regularity is often difficult to compute, it was successfully computed for several interesting classes. These include finite dimensional irreducible modules over finite dimensional simple Lie algebras, and integrable highest weight modules over affine Kac-Moody algebras of type $A_n^{(1)}$.

In this paper, we aim to compute the generalized Castelnuovo-Mumford regularity for integrable highest weight representations of *all* affine Kac-Moody algebras. By contrast to an *ad hoc* method used in the previous work [6], a unified approach is taken here that works for all affine cases. One of the key ingredients of the computation is a thorough understanding of the structure of affine Weyl groups. The Coxeter number and the dual Coxeter number appear naturally in the process of the

computation. As a result, it is shown that the exponents of growth are always 2 and that the rates of growth depend only on the type and rank of algebras and the level of representations.

1. GENERALIZED CASTELNUOVO-MUMFORD REGULARITIES

Let $A = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}^n} A_\alpha$ be a $\mathbb{Z}_{\geq 0}^n$-graded noncommutative $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$. An A-module M is said to be $\mathbb{Z}^n$-*graded* if M has a decomposition

$$M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta \quad \text{such that} \quad A_\alpha M_\beta \subset M_{\alpha+\beta} \quad \text{for all} \ \ \alpha, \beta \in \mathbb{Z}^n.$$

We assume that $M_\beta = 0$ for $\beta \ll 0$. An element m of M is *homogeneous of weight* β if $m \in M_\beta$ for some $\beta \in \mathbb{Z}^n$. We write $\mathrm{wt}(m) = \beta$ if $m \in M_\beta$. A homomorphism $\phi : M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha \to N = \bigoplus_{\alpha \in \mathbb{Z}^n} N_\alpha$ is called a *graded homomorphism of degree* β if $\phi(M_\alpha) \subset N_{\alpha+\beta}$ for all $\alpha \in \mathbb{Z}^n$. A free resolution $(F_i, \phi_i)_{i \geq 0}$ of M is said to be *graded* if ϕ_i are graded homomorphisms of degree 0 for all $i \geq 0$. It is called *minimal* if it cannot be pruned. It can be shown that the minimal graded free resolution is unique.

In [6], the following generalization of Caselnuovo-Mumford regularity was introduced for graded A-modules.

Definition 1.1. Let M be a $\mathbb{Z}^n$-graded A-module and let $(F_i, \phi_i)_{i \geq 0}$ be the minimal graded free resolution of M. For each $i \geq 0$, write $F_i = \bigoplus_{j \geq 0} A\varepsilon_{ij}$, and set

$$T_i = \max\{|\mathrm{wt}(\varepsilon_{ij})| \mid j \geq 0\},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$.

(a) The length of $(F_i, \phi_i)_{i \geq 0}$ is called the *projective dimension* of M and is denoted by $\mathrm{pdim}_A M$.

(b) We define the *exponent of growth* of M to be

$$e(M) := \begin{cases} 0 & \text{if } \mathrm{pdim}_A M < \infty; \\ \limsup_{i \to \infty} \dfrac{\log(T_i - i + 1)}{\log i} & \text{if } \mathrm{pdim}_A M = \infty. \end{cases}$$

(c) If $e(M) < \infty$, we define the *rate of growth* of M to be

$$r(M) := \begin{cases} \sup\{T_i - i \mid i \geq 0\} & \text{if } e(M) = 0; \\ \limsup_{i \to \infty} \dfrac{T_i - i}{i^{e(M)}} & \text{if } e(M) \neq 0. \end{cases}$$

(d) The *generalized Castelnuovo-Mumford regularity* of M is defined to be the pair

$$\mathrm{reg}_A M = (e(M), r(M)).$$

As we have seen in [6], if $\mathrm{pdim}_A M < \infty$, then $e(M) = 0$, and the rate of growth of M coincides with the usual Castelnuovo-Mumford regularity. If $\mathrm{pdim}_A M = \infty$, then we obtain the following asymptotic behavior of the degree twistings:

$$T_i - i \ \ \approx \ \ r(M) i^{e(M)} \ \ \text{for sufficiently large } i.$$

2. Affine Kac-Moody algebras

In this section, we recall the basic facts about affine Kac-Moody algebras. We follow the definitions and notations in [4, 5]. Let $I = \{0, 1, \ldots, n\}$ be the index set. The *affine Cartan datum of type* $X_N^{(r)}$ ($N \geq 1, r = 1, 2, 3$) consists of (i) the affine generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ of type $X_N^{(r)}$, (ii) dual weight lattice $P^\vee = \bigoplus_{i=0}^n \mathbb{Z}h_i \oplus \mathbb{Z}d$, (iii) the affine weight lattice $P = \bigoplus_{i=0}^n \mathbb{Z}\Lambda_i \oplus \mathbb{Z}(\delta/a_0) \subset \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$, $\Lambda_i(h_j) = \delta_{ij}$, $\Lambda_i(d) = 0$, $\delta(h_j) = 0$, $\delta(d) = 1$ $(i, j \in I)$, $a_0 = 2$ for type $A_{2n}^{(2)}$, $a_0 = 1$ otherwise, (iv) the set of simple coroots $\Pi^\vee = \{h_i \mid i \in I\}$, (v) the set of simple roots $\Pi = \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ such that $\alpha_j(h_i) = a_{ij}$, $\alpha_j(d) = \delta_{j,0}$ $(i, j \in I)$.

The free abelian group $Q = \bigoplus_{i=0}^n \mathbb{Z}\alpha_i$ is called the *root lattice* and the semigroup $Q_+ = \sum_{i=0}^n \mathbb{Z}_{\geq 0}\alpha_i$ is called the *positive root lattice*. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q$, we define the *height* of α to be $\mathrm{ht}(\alpha) := \sum_{i \in I} k_i$. We denote by

$$P_+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, \ i \in I\}$$

the set of *dominant integral weights*.

The *affine Weyl group* W is the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by the simple reflections $\{r_i\}_{i \in I}$, where $r_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$ and $i \in I$. The *length* $l(w)$ of $w \in W$ is defined to be the smallest $k \geq 0$ such that $w = r_{i_1} \cdots r_{i_k}$ $(i_j \in I)$. For each $k \in \mathbb{Z}_{\geq 0}$, we set

$$W(k) := \{w \in W \mid l(w) = k\}.$$

Definition 2.1. The *affine Kac-Moody algebra* $\mathfrak{g}$ of type $X_N^{(r)}$ is the Lie algebra over $\mathbb{C}$ generated by e_i, f_i, h_i $(i \in I)$ and d with the defining relations:

$$[h_i, h_j] = 0, \quad [h_i, d] = 0 \ \ (i, j \in I),$$
$$[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \ \ (h \in \mathfrak{h}, i \in I),$$
$$[e_i, f_j] = \delta_{ij}h_i \ \ (i, j \in I),$$
$$(\mathrm{ad}e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad}f_i)^{1-a_{ij}}(f_j) = 0 \ \ (i \neq j).$$

The affine Kac-Moody algebra $\mathfrak{g}$ has the *triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) is the subalgebra of $\mathfrak{g}$ generated by the elements e_i $(i \in I)$ (respectively, f_i $(i \in I)$). For each $\alpha \in Q$, $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is called the *root space* attached to α. If $\alpha \in Q \setminus \{0\}$ and $\mathfrak{g}_\alpha \neq 0$, then α is called a *root* of $\mathfrak{g}$. The set of all roots is denoted by Δ. We have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The elements in $\Delta_+ := \Delta \cap Q_+$ (respectively, $\Delta_- := -\Delta_+$) are called the *positive* (respectively, *negative*) roots. The sets of all long and short roots are denoted by Δ_L and Δ_S, respectively. A root $\alpha \in \Delta$ is called *real* if there exists $w \in W$ such that $w(\alpha) \in \Pi$. We denote by Δ^{re} the set of all real roots. The sets of all positive and negative real roots are denoted by $\Delta_+^{re} := \Delta^{re} \cap Q_+$ and $\Delta_-^{re} := -\Delta_+^{re}$, respectively.

Fix a dominant integral weight $\lambda \in P_+$ and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}_+$ the Borel subalgebra of $\mathfrak{g}$. We denote by $\mathbb{C}_\lambda := \mathbb{C}v_\lambda$ the 1-dimensional $\mathfrak{b}$-module defined by $hv_\lambda = \lambda(h)v_\lambda$ and $e_i v_\lambda = 0$ for $h \in \mathfrak{h}, i \in I$. Then the $U(\mathfrak{g})$-module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is called the *Verma module* with highest weight λ. Let $N(\lambda)$ be the submodule of $M(\lambda)$ generated by $f_i^{\lambda(h_i)+1}v_\lambda$ ($i \in I$). The irreducible quotient $V(\lambda) := M(\lambda)/N(\lambda)$ is called the *integrable highest weight module* with highest weight λ.

Choose an element $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i \in I$. In [3], Garland and Lepowsky constructed a natural free resolution of $V(\lambda)$ viewed as a $U(\mathfrak{g}_-)$-module:

$$(2.1) \qquad \cdots \to F_i(\lambda) \to \cdots \to F_1(\lambda) \to F_0(\lambda) \to V(\lambda) \to 0,$$

where $F_i(\lambda) = \bigoplus_{w \in W(i)} M(w(\lambda + \rho) - \rho)$. The free resolution 2.1 is called the *Berstein-Gelfand-Gelfand resolution* of $V(\lambda)$. As we have seen in [6], the Bernstein-Gelfand-Gelfand resolution is a minimal graded free resolution of $V(\lambda)$. We denote by ε_w^i the highest weight vector of the Verma module $M(w(\lambda + \rho) - \rho)$ with $w \in W(k)$. Then the Berstein-Gelfand-Gelfand resolution 2.1 can be written as

$$(2.2) \qquad \cdots \to \bigoplus_{w \in W(i)} U(\mathfrak{g}_-)\varepsilon_w^i \to \cdots \to \bigoplus_{w \in W(1)} U(\mathfrak{g}_-)\varepsilon_w^1 \to U(\mathfrak{g}_-)\varepsilon_1^0 \to V(\lambda) \to 0,$$

where $|\mathrm{wt}(\varepsilon_w^i)| = \mathrm{ht}(\lambda + \rho - w(\lambda + \rho))$ for $w \in W(i)$.

Let a_i, $a_i^\vee$ ($i = 0, 1, \ldots, n$) be the numerical labels given in [5, Ch.4]. Then we have $\delta = \sum_{i=0}^n a_i \alpha_i$ and $K = \sum_{i=0}^n a_i^\vee h_i$, where K is the *canonical central element* of $\mathfrak{g}$. The *Coxeter number* and the *dual Coxeter number* are defined to be

$$h = \sum_{i=0}^n a_i, \qquad h^\vee = \sum_{i=0}^n a_i^\vee,$$

respectively.

Let $\overset{\circ}{\mathfrak{g}}$ be the finite dimensional simple Lie algebra associated with the Cartan matrix $\overset{\circ}{A} := (a_{ij})_{i,j=1}^n$ and let $\overset{\circ}{W}$ be the Weyl group of $\overset{\circ}{\mathfrak{g}}$ generated by r_i ($i = 1, \ldots, n$). We denote by $\overset{\circ}{\Delta}$, $\overset{\circ}{\Delta}_+$, $\overset{\circ}{\Delta}_S$, $\overset{\circ}{\Delta}_L$, $\overset{\circ}{\Delta}_{+,S}$, and $\overset{\circ}{\Delta}_{+,L}$ the set of roots, positive roots, short roots, long roots, positive short roots, and positive long roots of $\overset{\circ}{\mathfrak{g}}$, respectively. We also denote by $\overset{\circ}{\rho}$ the element of $\mathfrak{h}^*$ such that $\overset{\circ}{\rho}(h_i) = 1$ for $i = 1, \ldots, n$. Thus we have $\overset{\circ}{\rho} = \frac{1}{2}\sum_{\alpha \in \overset{\circ}{\Delta}_+} \alpha$. Set $\overset{\circ}{\rho}_S = \frac{1}{2}\sum_{\alpha \in \overset{\circ}{\Delta}_{+,S}} \alpha$ and $\overset{\circ}{\rho}_L = \frac{1}{2}\sum_{\alpha \in \overset{\circ}{\Delta}_{+,L}} \alpha$.

Let $\mathfrak{h}_{\mathbb{R}}^* := \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_n\}$ (respectively, $\mathfrak{h}^* := \mathrm{Span}_{\mathbb{C}}\{\alpha_1, \ldots, \alpha_n\}$) and we denote the closure of the fundamental Weyl chamber relative to $\overset{\circ}{\Pi} = \{\alpha_1, \ldots, \alpha_n\}$ by

$$\mathcal{C} := \{\alpha \in \mathfrak{h}_{\mathbb{R}}^* \mid (\alpha|\alpha_i) \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Note that $\mathcal{C}$ is the fundamental domain for the action of $\overset{\circ}{W}$ on $\mathfrak{h}_{\mathbb{R}}^*$. For $\alpha \in \mathfrak{h}^*$, we define $t_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ by

$$t_\alpha(\lambda) = \lambda + \langle \lambda, K \rangle \alpha - \left((\lambda|\alpha) + \frac{1}{2}\|\alpha\|^2 \langle \lambda, K \rangle \right)\delta,$$

where $(.|.)$ is the normalized bilinear form on $\mathfrak{h}^*$ and $\|\alpha\|^2 = (\alpha|\alpha)$. Note that $t_\alpha(\lambda) = \lambda - (\lambda|\alpha)\delta$ if $\langle \lambda, K \rangle = 0$. Let

$$M = \begin{cases} \overset{\circ}{Q} & \text{if } A \text{ is symmetric or } r > a_0, \\ \mathbb{Z}\overset{\circ}{\Delta}_L & \text{if } A \text{ is not symmetric and } r = 1, \\ \dfrac{1}{2}\mathbb{Z}\overset{\circ}{\Delta}_L & \text{if } \mathfrak{g} = A_{2n}^{(2)}. \end{cases}$$

The free abelian group $T = \langle t_\alpha \mid \alpha \in M \rangle$ is called the *group of translations*. It is known that

$$W = T \rtimes \overset{\circ}{W}$$

(see, for example, [5, Ch.6]).

3. The exponent of growth

In this section, we show that the exponents of growth are always 2 for all integrable highest weight modules over affine Kac-Moody algebras. We first prove:

Lemma 3.1. *For a translation $t_\alpha \in T$, we have*

$$l(t_\alpha) = \begin{cases} \displaystyle\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|\alpha)| & \text{if } r = 1 \text{ or } X_N^{(r)} = A_{2n}^{(2)}, \\ \displaystyle\sum_{\mu \in \overset{\circ}{\Delta}_{+,S}} |(\mu|\alpha)| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \left| \left\lceil \frac{(\mu|\alpha)}{r} \right\rceil \right| & \text{otherwise,} \end{cases}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof. Since $l(w) = |\{\mu \in \Delta_+ | \ w(\mu) \in \Delta_-\}|$, we have

$$l(t_\alpha) = |\{\mu \in \Delta_+ | \ t_\alpha(\mu) \in \Delta_-\}|$$
$$= |\{\mu \in \Delta_+^{re} | \ \mu - (\mu|\alpha)\delta \in \Delta_-^{re}\}|.$$

If $r = 1$, it follows from [5, Proposition 6.3] that

$$l(t_\alpha) = |\{\mu + n\delta \mid \mu + n\delta - (\mu|\alpha)\delta \in \Delta_-^{re}, \ n \in \mathbb{Z}_{\geq 0}, \ \mu \in \overset{\circ}{\Delta}_+\}|$$

$$+ |\{-\mu + n\delta \mid -\mu + n\delta + (\mu|\alpha)\delta \in \Delta_-^{re}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_+\}|$$

$$= \sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|\alpha)|.$$

If $\mathfrak{g} = A_{2n}^{(2)}$, we have

$$l(t_\alpha) = |\{\mu + n\delta \mid \mu + n\delta - (\mu|\alpha)\delta \in \Delta_-^{re}, \ n \in \mathbb{Z}_{\geq 0}, \ \mu \in \overset{\circ}{\Delta}_{+,S}\}|$$

$$+ |\{-\mu + n\delta \mid -\mu + n\delta + (\mu|\alpha)\delta \in \Delta_-^{re}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,S}\}|$$

$$+ |\{\tfrac{1}{2}(\mu + (2n-1)\delta) \mid \tfrac{1}{2}(\mu + (2n-1)\delta - (\mu|\alpha)\delta) \in \Delta_-^{re}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$+ |\{\tfrac{1}{2}(-\mu + (2n-1)\delta) \mid \tfrac{1}{2}(-\mu + (2n-1)\delta + (\mu|\alpha)\delta) \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$+ |\{\mu + 2n\delta \mid \mu + 2n\delta - (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{\geq 0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$+ |\{-\mu + 2n\delta \mid -\mu + 2n\delta + (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$= \sum_{\mu \in \overset{\circ}{\Delta}_{+,S}} |(\mu|\alpha)| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \left| \left\lfloor \frac{(\mu|\alpha)}{2} \right\rfloor \right| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \left| \left\lceil \frac{(\mu|\alpha)}{2} \right\rceil \right|$$

$$= \sum_{\mu \in \overset{\circ}{\Delta}_{+}} |(\mu|\alpha)|.$$

Here, $\lfloor x \rfloor$ is the largest integer less than or equal to x.

If $r = 2, 3$ and $\mathfrak{g} \neq A^{(2)}_{2n}$, by a similar argument, we obtain

$$l(t_\alpha) = |\{\mu + n\delta \mid \mu + n\delta - (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{\geq 0}, \ \mu \in \overset{\circ}{\Delta}_{+,S}\}|$$

$$+ |\{-\mu + n\delta \mid -\mu + n\delta + (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,S}\}|$$

$$+ |\{\mu + rn\delta \mid \mu + rn\delta - (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{\geq 0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$+ |\{-\mu + rn\delta \mid -\mu + rn\delta + (\mu|\alpha)\delta \in \Delta^{re}_{-}, \ n \in \mathbb{Z}_{>0}, \ \mu \in \overset{\circ}{\Delta}_{+,L}\}|$$

$$= \sum_{\mu \in \overset{\circ}{\Delta}_{+,S}} |(\mu|\alpha)| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \left| \left\lceil \frac{(\mu|\alpha)}{r} \right\rceil \right|.$$

$\square$

Using Lemma 3.1, we can compute the exponent of growth for the integrable highest weight module $V(\lambda)$.

Theorem 3.2. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. Then for any dominant integral weight $\lambda \in P_+$, we have*

$$e(V(\lambda)) = 2.$$

Proof. Let $w = t_\alpha \overset{\circ}{w} \in W$ for $\alpha \in M$ and $\overset{\circ}{w} \in \overset{\circ}{W}$. Set

$$HT(w) := \mathrm{ht}\,(\lambda + \rho - w(\lambda + \rho))$$

and let $\overset{\circ}{\omega} := \overset{\circ}{w}(\lambda + \rho) - \lambda - \rho \in \overset{\circ}{Q}_- := -\sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\alpha_i$. Then we have

$$HT(w) = \mathrm{ht}\left(\lambda + \rho - t_\alpha(\lambda + \rho + \overset{\circ}{\omega})\right)$$

$$= \mathrm{ht}\left(\lambda + \rho - (\lambda + \rho + \overset{\circ}{\omega} + \langle \lambda + \rho + \overset{\circ}{\omega}, K\rangle\alpha - ((\lambda + \rho + \overset{\circ}{\omega}|\alpha) + \tfrac{1}{2}\|\alpha\|^2\langle \lambda + \rho + \overset{\circ}{\omega}, K\rangle)\delta\,)\right).$$

Since $|\overset{\circ}{W}| < \infty$, we obtain

$$HT(w) - \frac{1}{2}\langle \lambda + \rho, K\rangle\,\mathrm{ht}(\delta)\,\|\alpha\|^2 = o(\|\alpha\|^2).$$

On the other hand, since $|\overset{\circ}{\Delta}| < \infty$, Lemma 3.1 implies there exist $A \geq B > 0$ such that

$$B\|\alpha\| \leq l(w) \leq A\|\alpha\|.$$

Therefore, Definition 1.1 and the Bernstein-Gelfand-Gelfand resolution (2.2) yield

$$e(V(\lambda)) = \limsup_{\substack{w \in W \\ l(w) \to \infty}} \frac{\log(HT(w) - l(w) + 1)}{\log(l(w))} = 2.$$

$\square$

4. The rate of growth

The remaining task is to compute the rate of growth of $V(\lambda)$. For this purpose, we need a couple of lemmas.

Lemma 4.1. *For a dominant integral weight $\lambda \in P_+$, we have*

$$r(V(\lambda)) \;=\; \frac{1}{2}\langle \lambda + \rho, K \rangle \; \mathrm{ht}(\delta) \; \limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\|\alpha\|^2}{l(t_\alpha)^2}.$$

Proof. With the same notations as in the proof of Theorem 3.2, we get

$$\frac{\mathrm{ht}\,(\lambda + \rho - w(\lambda + \rho))}{l(w)^2} = \frac{\mathrm{ht}\left(\lambda + \rho - t_\alpha(\lambda + \rho + \overset{\circ}{\omega})\right)}{l(t_\alpha \overset{\circ}{w})^2}$$

$$= \frac{l(t_\alpha)^2}{l(t_\alpha \overset{\circ}{w})^2} \; \frac{\langle \lambda + \rho, K \rangle \; \mathrm{ht}(\delta) \; \|\alpha\|^2 + f(\|\alpha\|)}{2l(t_\alpha)^2}$$

for some $f(\|\alpha\|) = o(\|\alpha\|^2)$. Then, using the fact that $|l(t_\alpha \overset{\circ}{w}) - l(t_\alpha)| \leq |\overset{\circ}{\Delta}_+|$ and $\|\alpha\| \to \infty$ if and only if $l(t_\alpha) \to \infty$, we obtain

$$r(V(\lambda)) = \limsup_{\substack{w \in W \\ l(w) \to \infty}} \frac{\mathrm{ht}\,(\lambda + \rho - w(\lambda + \rho))}{l(w)^2}$$

$$= \limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{1}{2}\langle \lambda + \rho, K \rangle \; \mathrm{ht}(\delta) \; \frac{\|\alpha\|^2}{l(t_\alpha)^2}.$$

$\square$

Lemma 4.2.

$$\limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\|\alpha\|^2}{l(t_\alpha)^2} = \begin{cases} \displaystyle\limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{(2\overset{\circ}{\rho} \mid \alpha)^2} & \text{if } r = 1 \text{ or } \mathfrak{g} = A_{2n}^{(2)}, \\[2em] \displaystyle\limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{(2\overset{\circ}{\rho}_S + \frac{2}{r}\overset{\circ}{\rho}_L \mid \alpha)^2} & \text{otherwise.} \end{cases}$$

Proof. First, assume that $r = 1$ or $\mathfrak{g} = A_{2n}^{(2)}$. For any $\alpha \in M$ and $c \in \mathbb{Z}$, we have

$$\frac{\|\alpha\|^2}{l(t_\alpha)^2} = \frac{\|\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|\alpha)|\right)^2} = \frac{\|c\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|c\alpha)|\right)^2}.$$

Note that, for any $\alpha \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{Q}}$, there exists $c \in \mathbb{Z}$ such that $c\alpha \in M$. Since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, we get

$$\limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\|\alpha\|^2}{l(t_\alpha)^2} = \limsup_{\alpha \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}} \frac{\|\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|\alpha)|\right)^2}.$$

On the other hand, for any simple reflection r_i $(i = 1, \ldots, n)$ and $\alpha \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$, we have

$$\frac{(\alpha|\alpha)}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\alpha|\mu)|\right)^2} = \frac{(r_i\alpha|r_i\alpha)}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(r_i\alpha|r_i\mu)|\right)^2} = \frac{(r_i\alpha|r_i\alpha)}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(r_i\alpha|\mu)|\right)^2}.$$

Note that $(.|.)$ is invariant under W and $r_i \overset{\circ}{\Delta}_+ = \left(\overset{\circ}{\Delta}_+ \setminus \{\alpha_i\}\right) \cup \{-\alpha_i\}$ for $i = 1, \ldots, n$. Thus, since the closure $\mathcal{C}$ of the fundamental Weyl chamber is the fundamental domain for the action of $\overset{\circ}{W}$ on $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$, we obtain

$$\limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\|\alpha\|^2}{l(t_\alpha)^2} = \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} |(\mu|\alpha)|\right)^2} = \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_+} (\mu|\alpha)\right)^2}$$

$$= \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{(\sum_{\mu \in \overset{\circ}{\Delta}_+} \mu|\alpha)^2} = \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{(2\overset{\circ}{\rho}|\alpha)^2}.$$

Next, assume that $r = 2, 3$ and $\mathfrak{g} \neq A_{2n}^{(2)}$. Let

$$\tilde{l}(t_\alpha) = \sum_{\mu \in \overset{\circ}{\Delta}_{+,S}} |(\mu|\alpha)| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \frac{|(\mu|\alpha)|}{r}$$

for $\alpha \in M$. Note that $|l(t_\alpha) - \tilde{l}(t_\alpha)| \leq |\overset{\circ}{\Delta}_{+,L}|$. Thus by a similar argument given above, we can show that

$$\limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\|\alpha\|^2}{l(t_\alpha)^2} = \limsup_{\substack{\alpha \in M \\ \|\alpha\| \to \infty}} \frac{\tilde{l}(t_\alpha)^2}{l(t_\alpha)^2} \frac{\|\alpha\|^2}{\tilde{l}(t_\alpha)^2}$$

$$= \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{\left(\sum_{\mu \in \overset{\circ}{\Delta}_{+,S}} |(\mu|\alpha)| + \sum_{\mu \in \overset{\circ}{\Delta}_{+,L}} \frac{|(\mu|\alpha)|}{r} \right)^2}$$

$$= \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{\left((2\overset{\circ}{\rho}_S|\alpha) + \frac{(2\overset{\circ}{\rho}_L|\alpha)}{r} \right)^2}$$

$$= \limsup_{\alpha \in \mathcal{C}} \frac{\|\alpha\|^2}{(2\overset{\circ}{\rho}_S + \frac{2}{r}\overset{\circ}{\rho}_L \mid \alpha)^2}.$$

$\square$

Now we are ready to state and prove the main result of this paper.

Theorem 4.3. *Let λ be a dominant integral weight of level $\ell = \lambda(K)$. Then we have*

$$(4.1) \qquad r(V(\lambda)) = \begin{cases} \dfrac{h(\ell + h^\vee)}{2} \displaystyle\max_{i \in I \setminus \{0\}} \dfrac{\|\Lambda_i\|^2}{(2\overset{\circ}{\rho} \mid \Lambda_i)^2} & \text{if } r = 1 \text{ or } \mathfrak{g} = A_{2n}^{(2)} , \\[3mm] \dfrac{h(\ell + h^\vee)}{2} \displaystyle\max_{i \in I \setminus \{0\}} \dfrac{\|\Lambda_i\|^2}{(2\overset{\circ}{\rho}_S + \frac{2}{r}\overset{\circ}{\rho}_L \mid \Lambda_i)^2} & \text{otherwise.} \end{cases}$$

Proof. Let

$$\xi := \begin{cases} 2\overset{\circ}{\rho}, & \text{if } r = 1 \text{ or } \mathfrak{g} = A_{2n}^{(2)} , \\[2mm] 2\overset{\circ}{\rho}_S + \dfrac{2}{r} \overset{\circ}{\rho}_L, & \text{otherwise.} \end{cases}$$

and

$$l_i := \bigcap_{j \in I \setminus \{0,i\}} H_j,$$

where $H_j = \{\alpha \in \overset{\circ}{\mathfrak{h}}_{\mathbb{R}}^* \mid (\alpha|\alpha_j) = 0\}$ $(j \in I \setminus \{0\})$. Note that $\xi \in \mathcal{C}$ and $l_i = \mathbb{R}\Lambda_i$. Since $(.|.)$ is a symmetric positive-definite bilinear form on $\overset{\circ}{\mathfrak{h}}_{\mathbb{R}}^*$, if $\|\alpha\|$ is fixed, $(\xi|\alpha)$ achieves its minimum value at the boundary of $\mathcal{C}$. Therefore

$$\limsup_{\alpha \in \mathcal{C}} \frac{(\alpha|\alpha)}{(\xi|\alpha)^2} = \max_{i \in I \setminus \{0\}} \limsup_{\alpha \in l_i} \frac{(\alpha|\alpha)}{(\xi|\alpha)^2}$$

$$= \max_{i \in I \setminus \{0\}} \frac{(\Lambda_i|\Lambda_i)}{(\xi|\Lambda_i)^2}.$$

Since $\langle \lambda + \rho, K \rangle = \ell + h^\vee$ and $\mathrm{ht}(\delta) = h$, the formula (4.1) follows from Lemma 4.1 and Lemma 4.2. $\square$

Corollary 4.4. *Let λ be a dominant integral weight of level $\ell = \lambda(K)$ and write $\xi = \sum_{i \in I \setminus \{0\}} k_i \alpha_i$ with $k_i \in \mathbb{Z}$. Then we have*

$$r(V(\lambda)) = \frac{h(\ell + h^\vee)}{2} \max_{i \in I \setminus \{0\}} \left(\frac{2a'_{ii}}{\|\alpha_i\|^2 \, k_i^2} \right),$$

where $(a'_{ij})_{i,j \in I \setminus \{0\}}$ is the inverse matrix of $\overset{\circ}{A}$.

Proof. Since $2(\alpha_i|\alpha) = (\alpha_i|\alpha_i)\alpha(h_i)$ and $\Lambda_i = \sum_{j \in I \setminus \{0\}} a'_{ji}\alpha_j$ for $\alpha \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$, $i = 1, \ldots, n$, we have

$$(\Lambda_i|\Lambda_i) = \left(\sum_{j \in I \setminus \{0\}} a'_{ji}\alpha_j \mid \Lambda_i \right) = \frac{(\alpha_i|\alpha_i)a'_{ii}}{2}$$

and

$$(\xi|\Lambda_i) = \left(\sum_{j \in I \setminus \{0\}} k_j \alpha_j \mid \Lambda_i \right) = \frac{(\alpha_i|\alpha_i)k_i}{2}.$$

Our assertion follows immediately from Theorem 4.3. $\square$

Combining Theorem 3.2 and Corollary 4.4, we can compute explicitly the generalized Castelnuovo-Mumford regularities for integrable highest weight representations of all affine Kac-Moody algebras. Note that the rates of growth depend only on the type and rank of algebras and the level of representations.

Corollary 4.5. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $X_N^{(r)}$ and let $V(\lambda)$ be an integrable highest weight $\mathfrak{g}$-module with highest weight $\lambda \in P_+$. Then we have*

$$\mathrm{reg}_{U(\mathfrak{g}_-)} V(\lambda) = (2, \, r(V(\lambda))),$$

where $r(V(\lambda))$ is given in the following table.

$X_N^{(r)}$	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$
$r(V(\lambda))$	$\frac{\ell+n+1}{2n}$	$\frac{n(\ell+2n-1)}{(2n-1)^2}$	$\frac{\ell+n+1}{2n}$	$\frac{\ell+2n-2}{4(n-1)}$	$\frac{\ell+12}{32}$
$X_N^{(r)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$A_2^{(2)}$
$r(V(\lambda))$	$\frac{\ell+18}{54}$	$\frac{15(\ell+30)}{1682}$	$\frac{6(\ell+9)}{121}$	$\frac{9(\ell+4)}{50}$	$\frac{3(\ell+3)}{2}$
$X_N^{(r)}$	$A_{2n}^{(2)}$	$A_{2n-1}^{(2)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
$r(V(\lambda))$	$\frac{(2n+1)(\ell+2n+1)}{4n^2}$	$\frac{\ell+2n}{2n-1}$	$\frac{(n+1)(\ell+2n)}{2n^2}$	$\frac{9(\ell+12)}{121}$	$\frac{9(\ell+6)}{25}$

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