# A GENERALIZATION OF CASTELNUOVO-MUMFORD REGULARITY FOR REPRESENTATIONS OF NONCOMMUTATIVE ALGEBRAS

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Abstract. We introduce and generalize the notion of Castelnuovo-Mumford regularity for representations of noncommutative algebras, effectively establishing a measure of *complexity* for such objects. The Gröbner-Shirshov basis theory for modules over noncommutative algebras is developed, by which a noncommutative analogue of Schreyer's Theorem is proved for computing syzygies. By a repeated application of this theorem, we construct free resolutions for representations of noncommutative algebras. Some interesting examples are included in which graded free resolutions and regularities are computed for representations of various algebras. In particular, using the Bernstein-Gelfand-Gelfand resolutions for integrable highest weight modules over Kac-Moody algebras, we compute the projective dimensions and regularities explicitly for the cases of finite type and affine type $A_n^{(1)}$.

## Introduction

One of the motivations for this paper is to introduce a measure of complexity for various representations of noncommutative algebras. In the commutative case, the well-established concept of *Castelnuovo-Mumford regularity* [5, 9] provides such a measure. In order to obtain a noncommutative analogue, one has to study how to compute the free resolutions of modules over noncommutative algebras, starting from the computation of the first syzygy module. While a fairly straightforward generalization based on such a free resolution produces an obvious noncommutative analogue, the resulting regularity often becomes infinite and does not produce a sensible notion of complexity.

Let $A$ be a noncommutative algebra. In order to study free resolutions for representations of $A$, we start by developing Gröbner-Shirshov basis theory for $A$-modules. The Gröbner-Shirshov basis theory provides a powerful tool for understanding the structure of associative algebras and their representations, especially in computational aspect. The main idea originates from Buchberger's algorithm [7] of computing Gröbner bases for commutative algebras and Shirshov's Composition Lemma [25] for Lie algebras. In [18], Kang and Lee developed the Gröbner-Shirshov basis theory for cyclic $A$-modules by introducing the notion of *Gröbner-Shirshov pair*. In this paper, we generalize their result to arbitrary finitely generated $A$-modules and prove an analogue of *Schreyer's Theorem* for computing syzygies for $A$-modules (Theorem 2.1). Using

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Theorem 2.1, we can find a set of generators in the module of syzygies for an $A$-module. Then we apply Theorem 2.1 inductively to produce a free resolution of a given module.

In [14], Green, Solberg and Zacharia constructed projective resolutions for $A$-modules when $A$ is a quotient of a path algebra, using a filtration of projective $A$-modules. On the other hand, in [21], Levandovskyy investigated the case of $G$-algebras; i.e., the algebras with Poincaré-Birkhoff-Witt bases, and he constructed free resolutions of finitely generated modules over $G$-algebras. Our approach is more general in that we deal with *all* noncommutative algebras and their representations defined by generators and relations. Even for the universal enveloping algebras of Lie algebras, which are $G$-algebras, our approach is different from [21]. For instance, let $U(\mathfrak{g})$ be the universal enveloping algebra of a finite dimensional simple Lie algebra $\mathfrak{g}$. While the elements of a basis of $\mathfrak{g}$ and the commutation relations among them play a crucial role in [21], we take the Chevalley generators and the Serre relations as the key ingredients so that our approach can be extended to Kac-Moody algebras and their highest weight modules.

A lot of important information on the complexity of a graded algebraic object can be derived from the minimal graded free resolutions if the uniqueness of such resolutions is established. For this purpose, using the graded version of Nakayama's Lemma, we introduce several equivalent conditions for the minimality of graded free resolutions of a given graded $A$-module. This allows one to prove that a graded $A$-module has a unique minimal graded free resolution up to isomorphism and that the length of its minimal resolution, called the *projective dimension*, is well-defined. Furthermore, the *Castelnuovo-Mumford regularity* is well-defined, although often infinite. When the regularity $r(M)$ of a graded $A$-module $M$ is finite, it roughly means that, past degree $r(M)$, nothing tricky happens in $M$. Therefore, in the case of finite regularity, it can be interpreted as a measure of complexity for graded $A$-modules.

The Hilbert syzygy theorem holds for PBW algebras [13, Proposition 4.1], and any left ideal in a PBW algebra has a finite Gröbner-Shirshov basis [12, Theorem III]. Thus, one concludes that finitely generated modules over PBW algebras have *finite* Castelnuovo-Mumford regularity. For the quantum polynomial algebras, the Castelnuovo-Mumford regularity of graded modules are defined in [15], using homological methods.

However, in general, the Castelnuovo-Mumford regularity of graded modules over graded noncommutative algebras can be infinite. So we need to modify and generalize the notion of the Castelnuovo-Mumford regularity to analyze such cases. For this purpose, we introduce the notions of the *exponent of growth* and the *rate of growth* of twistings in a minimal free resolution. The latter one is a generalization of the rate of growth introduced by Backelin [2]. We define the *regularity* of an $A$-module $M$ to be the pair $(e(M), r(M))$, where $e(M)$ is the exponent of growth of $M$ and $r(M)$ is the rate of growth of $M$. As expected, $e(M) = 0$ if and only if $M$ has a finite Castelnuovo-Mumford regularity, and in such a case, the rate of growth $r(M)$ coincides with the Castelnuovo-Mumford regularity in the usual sense.

As an application, we investigate the regularity of integrable highest weight modules over Kac-Moody algebras. Using the Bernstein-Gelfand-Gelfand resolution (see, for example, [11, 20, 24]), we compute the projective dimensions and regularities of integrable highest weight modules over Kac-Moody algebras of finite type and affine type $A_n^{(1)}$ explicitly.

The paper is organized as follows: In Section 1, we develop the Gröbner-Shirshov basis theory for $A$-modules over a noncommutative algebra. We also introduce a linear algebraic algorithm of computing Gröbner-Shirshov bases for $A$-modules, which is a representation-theoretic analogue of $F_4$-algorithm given in [10, 17]. Section 2 contains the proof of an analogue of Schreyer's Theorem for computing syzygies, which allows one to find free resolutions of $A$-modules. In Section 3, we study minimal graded free resolutions of $A$-modules and define the notion of regularity. We include several examples in which we compute the minimal graded free resolutions of modules over down-up algebras and exterior algebras. In Section 4, by using the Bernstein-Gelfand-Gelfand resolution, we compute the projective dimension and regularity of integrable highest weight modules over Kac-Moody algebras of finite type and affine type $A_n^{(1)}$ explicitly.

## 1. Gröbner-Shirshov basis theory for $A$-modules

Let $\mathcal{R} = \mathbb{F}\langle x_1, x_2, \ldots, x_n\rangle$ be the free associative algebra generated by a finite set $\{x_1, x_2, \ldots, x_n\}$ over a field $\mathbb{F}$ and $\mathfrak{m}(\mathcal{R})$ be the set of all monomials in $\mathcal{R}$. Let $A$ be a noncommutative algebra over $\mathbb{F}$, which is identified with the quotient algebra $\mathcal{R}/I$, where $I = \langle S\rangle$ is a two-sided ideal of $\mathcal{R}$ generated by a Gröbner-Shirshov basis $S$. In this section, the Gröbner-Shirshov basis theory for finitely generated left $A$-modules is developed, which is a generalization of [18].

### 1.1. **Basic theory.**

Let $F = \bigoplus_{i=1}^t Ae_i$ be a free left $A$-module with basis $\{e_1, \ldots, e_t\}$ and let $\widetilde{F} = \bigoplus_{i=1}^t \mathcal{R}\widetilde{e_i}$ be a free left $\mathcal{R}$-module with basis $\{\widetilde{e_1}, \ldots, \widetilde{e_t}\}$. A *monomial* in $\widetilde{F}$ is an element of the form $r\widetilde{e_i}$, where $r$ is a monomial in $\mathcal{R}$. Let $\mathfrak{m}(\widetilde{F})$ denote the set of all monomials in $\widetilde{F}$. The *degree* (or the *length*) of a monomial $m = r\widetilde{e_i} \in \widetilde{F}$ is defined to be the degree of $r \in \mathcal{R}$, and will be denoted by $\deg m$ (or $l(m)$). The degree of a polynomial $f \in \widetilde{F}$ is defined to be the maximal degree of monomials appearing in $f$. Once $\mathcal{R}$ has a monomial order $>_{\mathcal{R}}$, one can define a monomial order $>$ on $\widetilde{F}$ induced by $>_{\mathcal{R}}$ as follows:

**Definition 1.1.** A *monomial order* $>$ on $\widetilde{F}$ induced by $>_{\mathcal{R}}$ is a total order on $\widetilde{F}$ such that

(i) for all $m, n \in \mathfrak{m}(\widetilde{F})$ and $r, s \in \mathfrak{m}(\mathcal{R})$,

$$m > n \text{ implies } rm > rn; \qquad r >_{\mathcal{R}} s \text{ implies } rm > sm;$$

(ii) every nonempty subset of $\mathfrak{m}(\widetilde{F})$ has a minimal element.

*Remark.* If $\mathcal{R} = \mathbb{F}[x_1, \ldots, x_n]$, the commutative polynomial ring, then the condition (ii) follows from the Noetherian property of $\mathcal{R}$. However, when $\mathcal{R}$ is noncommutative, the Artinian condition (ii) is necessary for the division algorithm and the calculation of compositions.

For each $a \in A$, write $a = r + I$ for some $r \in \mathcal{R}$ and define $\widetilde{a}$ to be the normal form of $r$. Since $S$ is a Gröbner-Shirshov basis of $I$, $\widetilde{a}$ is well-defined. Thus we get a natural map $\psi_F : F \to \widetilde{F}$ given by $\psi_F(\sum_{i=1}^t a_i e_i) = \sum_{i=1}^t \widetilde{a_i}\widetilde{e_i}$. For $f \in F$, we denote by $\widetilde{f}$ the image of $f$ under $\psi_F$. Similarly, for a subset $T$ of $F$, we define $\widetilde{T} = \{\widetilde{f} \mid f \in T\}$. When there is no danger of confusion, we will simply write $f$ and $T$ for $\widetilde{f}$ and $\widetilde{T}$, respectively.

A monomial in $\mathcal{R}$ is called $S$-standard if it is not divisible by any leading monomial for all polynomials in $S$. A monomial $r\widetilde{e_i}$ in $\widetilde{F}$ is called *$S$-standard* if $r$ is $S$-standard. A monomial in $F$ is an element of the form $ae_i$, where $\widetilde{a}$ is an $S$-standard monomial in $\mathcal{R}$. Note that $\psi_F$ defines a bijection between the set of monomials in $F$ and the set of $S$-standard monomials in $\widetilde{F}$. We denote by $\mathfrak{m}(F)$ the set of all monomials in $F$.

Let $<$ be a monomial order on $\widetilde{F}$ induced by a monomial order on $\mathcal{R}$. Then $<$ induces a total order on $\mathfrak{m}(F)$ via the map $\psi_F$. For a nonzero element $f \in F$, the *leading monomial* $\overline{f}$ denotes the maximal monomial appearing in $f$ with respect to $<$. A nonzero element $f \in F$ is said to be *monic* if the leading coefficient $\mathrm{lc}(f)$ of $f$ is 1. For any subset $T$ of $F$, we denote by $\overline{T} = \{\overline{f} \mid f \in T\}$ the set of leading monomials of $T$.

For $m, n \in \mathfrak{m}(F)$, we say that $m$ is *divisible* by $n$, or $m$ is a *multiple* of $n$, denoted by $n|m$, if there exists $a \in \mathfrak{m}(A)$ such that $m = an$. A monomial $t \in \mathfrak{m}(F)$ is called a *common multiple* of $m$ and $n$ if $m|t$ and $n|t$. The *least common multiple* of $m$ and $n$ is defined to be the common multiple $t \in \mathfrak{m}(F)$ of $m$ and $n$ that divides any common multiple of $m$ and $n$.

**Definition 1.2.** For a subset $T$ of $F$, we denote by $\langle T\rangle$ the $A$-submodule of $F$ generated by $T$.

(a) For an $A$-submodule $M$ of $F$, the *initial submodule* $\mathrm{in}(M)$ of $M$ is the $A$-submodule of $F$ generated by $\overline{M}$. That is, $\mathrm{in}(M) = \langle\overline{M}\rangle$.
(b) We say that $T$ is a *Gröbner-Shirshov basis* of $M$ if $M = \langle T\rangle$ and $\mathrm{in}(M) = \langle\overline{T}\rangle$.
(c) A Gröbner-Shirshov basis $T$ of $M$ is called *minimal* if $\bar{t}$ does not divide $\overline{t'}$ for all distinct $t, t' \in T$.
(d) A Gröbner-Shirshov basis $T$ of $M$ is called *reduced* if no term in $t'$ is divisible by any $\bar{t}$ for all distinct $t, t' \in T$.

*Remark.* If $F = A$, then $\langle T\rangle$ is a left ideal of $A$. In this case, the pair $(S, T)$ is called a *Gröbner-Shirshov pair* (See [17, 18, 19]).

**Definition 1.3.**

(a) Let $p$ and $q$ be monic elements of $\widetilde{F}$. If there exists $a$ in $\mathfrak{m}(\mathcal{R})$ such that $\overline{p} = a\overline{q} = w$, then the *right-justified composition* is defined to be $(p, q)_w = p - aq$.
(b) Let $p \in \mathcal{R}$ and $q \in \widetilde{F}$ be monic elements.
    (i) If there exist $a \in \mathfrak{m}(\mathcal{R})$ and $b \in \mathfrak{m}(\widetilde{F})$ such that $\overline{p}b = a\overline{q} = w$ with $l(\overline{p}) > l(a)$, then the *composition of intersection* is defined to be $(p, q)_w = pb - aq$.
    (ii) If there exist $a \in \mathfrak{m}(\mathcal{R})$ and $b \in \mathfrak{m}(\widetilde{F})$ such that $a\overline{p}b = \overline{q} = w$, then the *composition of inclusion* is defined to be $(p, q)_{a,b} = apb - q$.

We denote by $\mathcal{C}^w_{a,b}(p, q)$ the composition of $p$ and $q$ determined by $a$, $b$ and $w$. If the composition is not defined, we simply put $\mathcal{C}^w_{a,b}(p, q) = 0$.

**Definition 1.4.**

(a) Let $\widetilde{T}$ be a subset of $\widetilde{F}$ and let $p, q \in \widetilde{F}$, $w \in \mathfrak{m}(\widetilde{F})$. We say that $p$ is *congruent* to $q$ modulo $\widetilde{T}$ and $w$, written as $p \equiv q \mod (\widetilde{T}; w)$, if there exist $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_i, c_j \in \mathfrak{m}(\mathcal{R})$, $s_i \in S$ and $t_j \in \widetilde{T}$ such that $p - q = \sum \alpha_i a_i s_i b_i e_i + \sum \beta_j c_j t_j$ with $\overline{c_j t_j} \precneqq w$ for all $i, j$.

(b) A subset $\widetilde{T}$ of monic elements in $\widetilde{F}$ is said to be *closed under composition* if $\mathcal{C}^w_{a,b}(p,q) \equiv 0 \mod (\widetilde{T}; w)$ for all $p \in S$ or $p \in \widetilde{T}$, $q \in \widetilde{T}$, $w \in \mathfrak{m}(\widetilde{F})$, $a \in \mathfrak{m}(\mathcal{R})$ and $b \in \mathfrak{m}(\widetilde{F})$.

A monomial $m$ in $\widetilde{F}$ is called $\widetilde{T}$*-standard* if it is $S$-standard and it is not divisible by $\overline{t}$ for all $t \in \widetilde{T}$. As in the case of cyclic $A$-modules, we obtain the following *division algorithm* [17, 18, 19].

**Proposition 1.5.** *Let $\widetilde{T}$ be a set of monic elements in $\widetilde{F}$. Then every element $f \in \widetilde{F}$ can be written as*

$$f = \sum \alpha_i a_i s_i b_i e_i + \sum \beta_j c_j t_j + \sum \gamma_k m_k \tag{1.1}$$

*where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$, $a_i, b_i, c_j \in \mathfrak{m}(\mathcal{R})$, $s_i \in S$, $t_j \in \widetilde{T}$, $m_k \in \mathfrak{m}(\widetilde{F})$, $\overline{a_i s_i b_i} \le \overline{f}$, $\overline{c_j t_j} \le \overline{f}$, $m_k \le \overline{f}$ and $m_k$ are $\widetilde{T}$-standard.*

The term $\sum \gamma_k m_k$ in the above expression (1.1) is called a *normal form* (or a remainder) of $f$ with respect to $\widetilde{T}$ (and with respect to the monomial order $<$). In general, a normal form of $f$ is not unique.

In the following theorem, we characterize the basic properties of Gröbner-Shirshov bases. The proof is standard (See [9, Theorem 15.8], [18, §3] or [19, Proposition 1.9]).

**Theorem 1.6.** *Let $T$ be a set of monic elements in $F$, let $M$ be the $A$-submodule of $F$ generated by $T$, and let $\mathcal{B}$ be the set of all $\widetilde{T}$-standard monomials in $\widetilde{F}$. Then the following are equivalent:*

(a) *$T$ is a Gröbner-Shirshov basis of $M$.*
(b) *Every $f \in \widetilde{F}$ has a unique normal form with respect to $\widetilde{T}$.*
(c) *$\mathcal{B}$ is an $\mathbb{F}$-linear basis of $F/M$.*
(d) *$\widetilde{T}$ is closed under composition.*

## 1.2. **Linear algebraic approach.**

The part (d) of Theorem 1.6 gives an analogue of *Buchberger's algorithm* of computing a Gröbner-Shirshov basis of a given $A$-submodule $M$. More precisely, let $T$ be a set of monic elements in $F$ and let $M = \langle T \rangle$ be the $A$-submodule of $F$ generated by $T$. We define

$$\begin{aligned}
\widetilde{T}^{(0)} &= \left\{ f/\mathrm{lc}(f) |\ f \in \widetilde{T} \right\}, \\
\widetilde{T}_{(i)} &= \left\{ (p,q)_w \not\equiv 0 \mod (\widetilde{T}^{(i)}; w) |\, p, q \in \widetilde{T}^{(i)}, (p,q)_w \text{ is right-justified} \right\}, \\
\widetilde{T}^{(i+1)} &= \widetilde{T}^{(i)} \cup \widetilde{T}_{(i)}.
\end{aligned}$$

Then the set $\widetilde{T}^c = \bigcup_{i \ge 0} \widetilde{T}^{(i)}$ is closed under the right-justified composition with respect to $S$. We now consider the compositions of elements in $S$ and $\widetilde{T}^c$. Let $X^{(0)} = \widetilde{T}^c$ and for $i \ge 0$, define

$$\begin{aligned}
X_{(i)} &= \left\{ (p,q)_w \not\equiv 0, (p,q)_{a,b} \not\equiv 0 \mod (X^{(i)}; w) |\, p \in S, q \in X^{(i)} \right\}, \\
X^{(i+1)} &= \left( X^{(i)} \cup X_{(i)} \right)^c.
\end{aligned}$$

Then, by construction, $\widetilde{\mathcal{T}} = \bigcup_{i \ge 0} X^{(i)}$ is closed under composition, and hence $\mathcal{T} = \psi_F^{-1}(\widetilde{\mathcal{T}})$ is a Gröbner-Shirshov basis of $M$.

However, in general, there is no guarantee that this algorithm would terminate in finitely many steps. To overcome this difficulty, a linear algebraic approach was introduced in [17] so that one can compute *truncated* Gröbner-Shirshov bases for cyclic $A$-modules whose elements are bounded by a fixed monomial. This method can be considered as a representation-theoretic analogue of $F_4$ algorithm [10]. In this subsection, we develop an analogous algorithm of computing Gröbner-Shirshov bases for finitely generated $A$-modules.

Let $A=(a_{ij})$ be an $s\times t$ matrix over $\mathbb{F}$ and $M_A=(m_1,\dots,m_t)$ be an ordered set of distinct monomials in $\widetilde{F}$. We define $\mathrm{Rows}(A,M_A)$ to be $\left\{\sum_{j=1}^t a_{ij}m_j | i=1,\dots,s\right\}\setminus\{0\}$, the set of polynomials given by $(A,M_A)$. Conversely, for a set $T$ of elements in $F$, we make $M(\widetilde{T})$ an ordered set of all monomials appearing in $\widetilde{T}$ with respect to the monomial order on $\widetilde{F}$. Then we obtain an $|\widetilde{T}|\times|M(\widetilde{T})|$ matrix $A_T$, the *Macaulay matrix* of $T$, whose $(i,j)$-entry is the coefficient of $j$th monomial in the $i$th element in $\widetilde{T}$. Let $\mathrm{RRE}\,(A_T)$ denote the unique reduced row echelon form of $A_T$, and we call the set $\mathrm{RRE}(T)=\mathrm{Rows}\left(\mathrm{RRE}\,(A_T)\,,M(\widetilde{T})\right)$ the *reduced row echelon form* of $T$.

We define the notion of *composition pairs from* $(S,\widetilde{T})$. For $p,q\in\widetilde{T}$, we define

$$Comp^0(p,q)=(p,aq) \quad \text{whenever the right-justified composition } (p,q)_w=p-aq \text{ is defined.}$$

For all $p\in S$, $q\in\widetilde{T}$, $a\in\mathfrak{m}(\mathcal{R})$, $b,w\in\mathfrak{m}(\widetilde{F})$, we define

$$Comp^1(p,q;w)=(pb,aq) \quad \text{whenever the composition of intersection } (p,q)_w=pb-aq \text{ is defined,}$$
$$Comp^2(p,q;a,b)=(apb,q) \quad \text{whenever the composition of inclusion } (p,q)_{a,b}=apb-q \text{ is defined.}$$

The set of all composition pairs from $(S,\widetilde{T})$ is denoted by $\mathscr{P}(S,\widetilde{T})$. We also define the set of all *composition data from* $(S,\widetilde{T})$ to be

$$\mathscr{D}(S,\widetilde{T})=\{f\in\widetilde{F}|\text{ there exists } g\in\widetilde{F} \text{ such that } (f,g)\in\mathscr{P}(S,\widetilde{T}) \text{ or } (g,f)\in\mathscr{P}(S,\widetilde{T})\}.$$

Recall that $S$ is a Gröbner-Shirshov basis in $\mathcal{R}$. Let $\widetilde{T}^{(0)}=\widetilde{T}$. Assume inductively that we have constructed $\widetilde{T}^{(i)}$ for $i\geq 0$. Set

$$D_i=\mathscr{D}(S,\widetilde{T}^{(i)}), \quad \text{and } P_i=D_i\setminus D_{i-1} \quad (D_{-1}=\emptyset).$$

Let $a_i=\min\{\deg f|f\in P_i\}$ and $b_i=\max\{\deg f|f\in P_i\}$, and let $P_i(d)\subseteq P_i$ be the subset consisting of elements of degree $d$, $a_i\leq d\leq b_i$. We define the sets $F_i(d)$ and $F_i(d)^+$ for each $d$ inductively from $a_i$ to $b_i$ as follows:

Let $P_i(d)^{(0)}=P_i(d)$, and assume, inductively on $k$, that $P_i(d)^{(k)}$ has been constructed for $k\geq 0$. For each $m\in\mathfrak{m}(P_i(d)^{(k)})\setminus\overline{P_i(d)^{(k)}}$ which is reducible mod $(S,\widetilde{T}^{(i)}\cup\bigcup_{j=a_i}^{d-1}F_i(j)^+)$ (mod $(S,\widetilde{T}^{(i)})$ if $d=a_i$), choose $f\in S$ and monomials $m'\in\mathfrak{m}(\mathcal{R})$ and $m''\in\mathfrak{m}(\widetilde{F})$ such that $m=m'\overline{f}m''$, or choose $f\in\widetilde{T}^{(i)}\cup\bigcup_{j=a_i}^{d-1}F_i(j)^+$ ($f\in\widetilde{T}^{(i)}$ if $d=a_i$) and $m'\in\mathfrak{m}(\mathcal{R})$ such that $m=m'\overline{f}$.

Let $P_i(d)^{(k+1)}$ be the set of all such $m'fm''$ or $m'f$, and define

$$F_i(d)=\bigcup_{k\geq 0}P_i(d)^{(k)}.$$

Let

$$F_i(d)^+ = \left\{ f \in \mathrm{RRE}\,(F_i(d)) \mid \overline{f} \notin \overline{F_i(d)} \right\},$$

where $\mathrm{RRE}\,(F_i(d))$ is the reduced row echelon form of $F_i(d)$.

Finally, define

$$\widetilde{T}^{(i+1)} = \widetilde{T}^{(i)} \cup \bigcup_{d=a_i}^{b_i} F_i(d)^+.$$

Then $\widetilde{\mathcal{T}} = \bigcup_{i\geq 0} \widetilde{T}^{(i)}$ is closed under composition and hence $\mathcal{T} = \psi_F^{-1}(\widetilde{\mathcal{T}})$ is a Gröbner-Shirshov basis of $M = \langle T \rangle$.

## 2. Syzygies for $A$-modules

Let $\widetilde{T} = \{t_1, \dots t_n\}$ be a set of monic elements of $\widetilde{F}$ and $M$ the submodule of $F$ generated by $T := \psi_F^{-1}(\widetilde{T})$. Consider a free $A$-module $F_1 = \bigoplus_{i=1}^n A\varepsilon_i$ and a natural surjective $A$-module homomorphism $\varphi : F_1 \to M$ given by $\varphi(\varepsilon_i) = \psi_F^{-1}(t_i)$ for $i = 1, \dots n$. Let $\widetilde{F_1} = \bigoplus_{i=1}^n \mathcal{R}\widetilde{\varepsilon_i}$ be a free $\mathcal{R}$-module and let

$$\sigma^w_{a,b}(p, t_j) = \begin{cases} \widetilde{\varepsilon_i} - a\widetilde{\varepsilon_j} & \text{if } \ p = t_i,\ \overline{t_i} = a\overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}), \\ -a\widetilde{\varepsilon_j} & \text{if } p \in S,\ \overline{p}b = a\overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}),\ b \in \mathfrak{m}(\widetilde{F}) \text{ with } l(\overline{p}) > l(a), \\ -\widetilde{\varepsilon_j} & \text{if } p \in S,\ a\overline{p}b = \overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}),\ b \in \mathfrak{m}(\widetilde{F}) \text{ with } a \neq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and consider the corresponding composition $\mathcal{C}^w_{a,b}(p, t_j)$ given by

$$\mathcal{C}^w_{a,b}(p, t_j) = \begin{cases} t_i - at_j & \text{if } \ p = t_i,\ \overline{t_i} = a\overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}), \\ pb - at_j & \text{if } p \in S,\ \overline{p}b = a\overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}),\ b \in \mathfrak{m}(\widetilde{F}) \text{ with } l(\overline{p}) > l(a), \\ apb - t_j & \text{if } p \in S,\ a\overline{p}b = \overline{t_j} = w \ \text{ for some } \ a \in \mathfrak{m}(\mathcal{R}),\ b \in \mathfrak{m}(\widetilde{F}) \text{ with } a \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Apply the division algorithm to obtain

$$\mathcal{C}^w_{a,b}(p, t_j) = \sum_k \alpha_k a_k s_k b_k + \sum_{r=1}^n \beta_r c_r t_r + h^w_{a,b},$$

where $\alpha_k, \beta_r \in \mathbb{F}$, $a_k, c_r \in \mathfrak{m}(\mathcal{R})$, $b_k \in \mathfrak{m}(\widetilde{F})$, $s_k \in S$, $\overline{c_r t_r} \lneq w$ and $c_r$ are $S$-standard for all $r$.

If $T$ is a Gröbner-Shirshov basis, then it is closed under composition, hence $h^w_{a,b} = 0$ for all $a \in \mathfrak{m}(\mathcal{R}),\ b, w \in \mathfrak{m}(\widetilde{F})$. We define the *syzygies* on $\widetilde{T}$ by

$$\tau^w_{a,b}(p, t_j) = \sigma^w_{a,b}(p, t_j) - \sum_{r=1}^n \beta_r c_r \widetilde{\varepsilon}_r$$

for all $p \in S$ or $p \in \widetilde{T}$, $a \in \mathfrak{m}(\mathcal{R}),\ b, w \in \mathfrak{m}(\widetilde{F})$ and $j = 1, \dots, n$.

The following is an $A$-module analogue of *Schreyer's Theorem* [9, Theorem 15.10].

**Theorem 2.1.** *Let $>$ be a monomial order on $\widetilde{F}$ and let $\widetilde{T} = \{t_1, \dots t_n\}$ be a set of monic elements of $\widetilde{F}$. Suppose that $T := \psi_F^{-1}(\widetilde{T})$ is a Gröbner-Shirshov basis of $\langle T \rangle$.*

(a) *For $a, b \in \mathfrak{m}(\mathcal{R})$, we define $a\widetilde{\varepsilon_i} \succ b\widetilde{\varepsilon_j}$ if and only if* (i) $\overline{at_i} > \overline{bt_j}$, *or* (ii) $\overline{at_i} = \overline{bt_j}$ *and $i < j$. Then $\succ$ is an monomial order on $\widetilde{F_1}$.*

(b) *Let*

$$K = \{\tau^w_{a,b}(p, q) |\ p \in S \text{ or } p \in \widetilde{T},\ q \in \widetilde{T},\ a \in \mathfrak{m}(\mathcal{R}) \text{ and } b, w \in \mathfrak{m}(\widetilde{F})\}$$

*be the set of syzygies induced from $\widetilde{T}$. Then $\psi_{F_1}^{-1}(K)$ is a Gröbner-Shirshov basis for* $\ker \varphi$.

*Proof.* (a) It is clear that the induced order $\succ$ is a total order on $\widetilde{F_1}$ and is preserved under the multiplication by a monomial from the left. The Artinian condition on $\widetilde{F_1}$ follows from that on $\widetilde{F}$.

(b) Let $\widetilde{\varphi} : \widetilde{F_1} \to \widetilde{F}$ be the lifting of $\varphi$ given by $\widetilde{\varphi}(\widetilde{\varepsilon_i}) = t_i$ for all $i$. Note that $\varphi = \psi_F^{-1} \circ \widetilde{\varphi} \circ \psi_{F_1}$. Let $f \in \ker \varphi$, then $\psi_{F_1}(f)$ can be written as $\sum_{i=1}^n \beta_i n_i \widetilde{\varepsilon_i}$ where $\beta_i \in \mathbb{F}$, $n_i \in \mathfrak{m}(\mathcal{R})$ and $n_i$ are $S$-standard. Without loss of generality, we may assume that $n_i\widetilde{\varepsilon_i} \succeq n_j\widetilde{\varepsilon_j}$ for all $i < j$. Let $r$ be the minimal positive integer such that $\beta_r \neq 0$. Then $\overline{\psi_{F_1}(f)} = n_r\widetilde{\varepsilon_r}$.

If $n_r\overline{t_r}$ is not $S$-standard, then $n_r\overline{t_r}$ is divisible by some element $s \in S$. Hence there exist a $S$-standard monomial $a \in \mathfrak{m}(\mathcal{R})$ and $b, w \in \mathfrak{m}(\widetilde{F})$ such that $\mathcal{C}^w_{a,b}(s, t_r) \neq 0$. It follows that $\tau^w_{a,b}(s, t_r) \neq 0$ and $\overline{\tau^w_{a,b}(s, t_r)}$ divides $\overline{\psi_{F_1}(f)}$. Therefore $\overline{\psi_{F_1}(f)} \in \langle \overline{K} \rangle$.

Assume that $n_r\overline{t_r}$ is $S$-standard. By definition, we have $\overline{\tau^w_{a,b}(p, q)} = \overline{\sigma^w_{a,b}(p, q)}$ for all $p \in S$ or $p \in T$, $q \in T$, $a \in \mathfrak{m}(\mathcal{R})$ and $b, w \in \mathfrak{m}(\widetilde{F})$. If $n_r\overline{t_r} \neq n_k\overline{t_k}$ for all $k > r$ then, since $n_r\widetilde{\varepsilon_r} \succ n_k\widetilde{\varepsilon_k}$, we have $n_r\overline{t_r} \gneq n_k\overline{t_k}$. This implies

$$\varphi(f) = \psi_F^{-1} \circ \widetilde{\varphi} \circ \psi_{F_1}(f) = \psi_F^{-1}(\beta_r n_r t_r + \sum_{k>r} \beta_k n_k t_k) \neq 0,$$

which is a contradiction. Therefore, we must have $n_r\overline{t_r} = n_k\overline{t_k}$ for some $k > r$. This implies that there exists $a, A \in \mathfrak{m}(\mathcal{R})$ and $b, w \in \mathfrak{m}(\widetilde{F})$ such that $n_r\widetilde{\varepsilon_r} - n_k\widetilde{\varepsilon_k} = A\overline{\sigma^w_{a,b}(t_r, t_k)}$. It follows that

$$\overline{\psi_{F_1}(f)} = n_r\widetilde{\varepsilon_r} = A\overline{\sigma^w_{a,b}(t_r, t_k)} = A\overline{\tau^w_{a,b}(t_r, t_k)} \in \langle \overline{K} \rangle.$$

Consequently,

$$\overline{\psi_{F_1}(\ker \varphi)} \subset \langle \overline{K} \rangle,$$

hence, since $\psi_{F_1}(\overline{f}) = \overline{\psi_{F_1}(f)}$ for $f \in F_1$,

$$\text{in}(\ker \varphi) = \langle \overline{\psi_{F_1}^{-1}(K)} \rangle.$$

From the fact that $\psi_{F_1}^{-1}(K) \subset \ker \psi$, $\psi_{F_1}^{-1}(K)$ is a Gröbner-Shirshov basis of $\ker \varphi$. □

Let $M$ be an $A$-module. Then there exists a free $A$-module $F_0$ and a canonical surjective $A$-module homomorphism $\varphi_0 : F_0 \to M$. Using the linear algebraic method described in Subsection 1.2, we find a Gröbner-Shirshov basis $T = \{t_1, \dots, t_n\}$ of $\ker \varphi_0$. Let $F_1 = \bigoplus_{i=1}^n A\varepsilon_i$ be the free $A$-module of rank $n$ and let $\rho_1 : F_1 \to \ker \varphi_0$ be the canonical surjective homomorphism defined by $\rho_1(\varepsilon_i) = t_i$ for $i = 1, \dots, n$. Set $\varphi_1 = \iota_0 \circ \psi_1 : F_1 \to F_0$, where $\iota_0 : \ker \varphi_0 \to F_0$ be the inclusion map. Then it is easy to see that $\text{im}\varphi_1 = \text{im}\rho_1 = \ker \varphi_0$.

Applying Theorem 2.1, find a Gröbner-Shirshov basis $K_1$ of $\ker \rho_1$ and let $F_2$ be the free $A$-module of rank $|K_1|$ with a canonical surjective homomorphism $\rho_2 : F_2 \to \ker \rho_1$. Set $\varphi_2 = \iota_1 \circ \rho_2 : F_2 \to F_1$, where $\iota_1 : \ker \rho_1 \to F_1$ is the inclusion map. By construction, we have $\mathrm{im}\varphi_2 = \ker \phi_1$. Repeating this procedure, we construct a free $A$-module resolution $(F_i, \varphi_i)_{i \geq 0}$ of $M$:

$$\cdots \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \longrightarrow 0.$$

## 3. Graded free resolutions for $A$-modules

For a monomial $m \in \mathfrak{m}(\mathcal{R})$, we define the *weight* of $m$ to be

$$\mathrm{wt}(m) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n,$$

where $\alpha_i$ is the number of $x_i$'s appearing in $m$, and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, the *$\alpha$-weight space* of $\mathcal{R}$ is defined to be

$$\mathcal{R}_\alpha = \mathrm{Span}_{\mathbb{F}}\{m \in \mathfrak{m}(\mathcal{R}) \mid \mathrm{wt}(m) = \alpha\}.$$

Then $\mathcal{R}$ is a $\mathbb{Z}^n$-graded algebra; i.e.,

$$\mathcal{R} = \bigoplus_{\alpha \in \mathbb{Z}^n} \mathcal{R}_\alpha, \quad \mathcal{R}_\alpha \mathcal{R}_\beta \subset \mathcal{R}_{\alpha+\beta}, \quad \text{and} \quad \dim \mathcal{R}_\alpha = \frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1! \cdots \alpha_n!}.$$

Let $I$ be a homogeneous ideal of $\mathcal{R}$ and set $A = \mathcal{R}/I$. Then $I$ can be written as $I = \bigoplus_{\alpha \in \mathbb{Z}^n} I_\alpha$, where $I_\alpha = I \cap \mathcal{R}_\alpha$. Moreover, if we define $A_\alpha = \mathcal{R}_\alpha / I_\alpha$, then we have $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ and $A$ becomes a $\mathbb{Z}^n$-graded algebra.

We say that an $A$-module $M$ is $\mathbb{Z}^n$-graded if $M$ has a decomposition

$$M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta \quad \text{such that} \quad A_\alpha M_\beta \subset M_{\alpha+\beta} \quad \text{for all} \quad \alpha, \beta \in \mathbb{Z}^n.$$

An element $m$ of $M$ is *homogeneous of weight* $\beta$ if $m \in M_\beta$ for some $\beta \in \mathbb{Z}^n$ and a submodule of $M$ is *homogeneous* if it is generated by a set of homogeneous elements.

**Definition 3.1.** Let $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ be a $\mathbb{Z}^n$-graded $A$-module with $\dim M_\beta < \infty$ for all $\beta \in \mathbb{Z}^n$ and $M_\beta = 0$ for $\beta \ll 0$. Then we define the *Hilbert series* $H_M(t_1, \ldots, t_n)$ *of* $M$ to be the formal Laurent series

$$H_M(t_1, \ldots, t_n) = \sum_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n} (\dim M_\beta) t_1^{\beta_1} \cdots t_n^{\beta_n}.$$

A homomorphism $\phi : M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha \to N = \bigoplus_{\alpha \in \mathbb{Z}^n} N_\alpha$ is called a *graded homomorphism of degree* $\beta$ if $\phi(M_\alpha) \subset N_{\alpha+\beta}$ for all $\alpha \in \mathbb{Z}^n$.

A free resolution $(F_i, \phi_i)_{i \geq 0}$ of $M$ is said to be *graded* if $\phi_i$ are graded homomorphisms of degree 0 for all $i \geq 0$. Once we are given a graded free resolution $(F_i, \phi_i)_{i \geq 0}$ of $M$, by the Euler-Poincaré principle, we have

$$H_M(t_1, \ldots, t_n) = \sum_{i \geq 0} (-1)^i H_{F_i}(t_1, \ldots, t_n).$$

Note that if $F = \bigoplus_{\varepsilon \in X} A\varepsilon$ is a free left $A$-module, then we have

$$H_F(t_1, \dots, t_n) = H_A(t_1, \dots, t_n) \sum_{\varepsilon \in X} t^{\mathrm{wt}(\varepsilon)}.$$

Basic properties on Hilbert series of noncommutative graded algebras are described briefly in [1].

Let $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ be a $\mathbb{Z}^n$-graded $A$-module with $M_\beta = 0$ for $\beta \ll 0$, and let $\mathfrak{p} = \bigoplus_{|\alpha|>0} A_\alpha$ the maximal homogeneous ideal of $A$. To define the notion of minimal graded free resolutions, we state the *graded* version of *Nakayama's Lemma* (See [9, Exercise 4.6] or [22, §II.8]).

**Lemma 3.2.** *Let $M = \bigoplus_{\beta \in \mathbb{Z}^n} M_\beta$ be a $\mathbb{Z}^n$-graded $A$-module with $M_\beta = 0$ for $\beta \ll 0$.*

(a) *If there exists a homogeneous ideal $I$ of $A$ contained in $\mathfrak{p}$ such that $IM = M$, then $M = 0$.*
(b) *If $N = \bigoplus_{\beta \in \mathbb{Z}^n} N_\beta$ is a $\mathbb{Z}^n$-graded submodule of $M$ such that $N + \mathfrak{p}M = M$, then $N = M$.*
(c) *Let $G = \{g_1, \dots, g_t\}$ be a set of homogeneous elements in $M$. If $G$ generates $M$ mod $\mathfrak{p}M$, then $G$ generates $M$.*

**Definition 3.3.** Let $(F_i, \phi_i)_{i \ge 0}$ be a complex of graded $A$-modules.

(a) A complex is *trivial* if it is isomorphic to

$$0 \longrightarrow A\varepsilon \xrightarrow{c} A\varepsilon \longrightarrow 0,$$

where $c$ is a nonzero scalar multiplication.
(b) We say that a complex can be *pruned* if it has a trivial subcomplex as a direct summand.
(c) A graded free resolution $(F_i, \phi_i)_{i \ge 0}$ of a graded module $M$ is called *minimal* if $(F_i, \phi_i)_{i \ge 0}$ cannot be pruned.

Let $\phi : \bigoplus_{j=1}^{t} A\varepsilon_j \to \bigoplus_{i=1}^{s} Ae_i$ be a grade homomorphism of degree 0 between free $A$-modules. For each $j = 1, \dots, t$, write

$$\phi(\varepsilon_j) = \sum_{i=1}^{s} a_{ij} e_i \quad \text{with} \quad a_{ij} \in A.$$

Then we obtain a matrix

$$[\phi] = (a_{ij})_{1 \le i \le s,\, 1 \le j \le t}\,,$$

which is called the *matrix presentation* of $\phi$.

The following proposition gives a characterization of minimal graded free resolutions. The proof is standard, as is given in [8, Ch.6, Proposition 3.10] or [9, Lemma 19.4].

**Proposition 3.4.** *Let $(F_i, \phi_i)_{i \ge 0}$ be a graded free resolution of a $\mathbb{Z}^n$-graded $A$-module $M$:*

$$\cdots \longrightarrow F_i \xrightarrow{\phi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0.$$

*Then the following are equivalent:*

(a) *The free resolution $(F_i, \phi_i)_{i \ge 0}$ is minimal.*
(b) *For each $i > 0$, $\mathrm{im}\phi_i$ is contained in $\mathfrak{p}F_{i-1}$.*
(c) *For each $i > 0$, if $[\phi_i] = (a_{pq})$, then we have $a_{pq} \notin \mathbb{F}^\times$ for all $1 \le p \le \operatorname{rank} F_{i-1}$, $1 \le q \le \operatorname{rank} F_i$.*

(d) *For each $i \geq 0$, the homomorphism $\phi_i$ takes an $A$-module basis of $F_i$ to a minimal set of generators of* $\mathrm{im}\phi_i$.

**Definition 3.5.** Let $M$ be a $\mathbb{Z}^n$-graded $A$-module. We say that two graded free resolutions $(F_i, \phi_i)_{i\geq 0}$ and $(G_i, \psi_i)_{i\geq 0}$ of $M$ are *isomorphic* if there exist graded isomorphisms $\alpha_i : F_i \to G_i$ of degree 0 for all $i \geq 0$ such that $\psi_0 \circ \alpha_0 = \phi_0$ and $\psi_i \circ \alpha_i = \alpha_{i-1} \circ \phi_i$.

In the next theorem, we obtain the uniqueness of a minimal graded free resolution of a given graded $A$-module. The proof is quite similar to that for the cases of finitely generated modules over graded commutative rings and local rings (See [8, Ch.6, Theorem 3.13] or [9, Exercise 4.11, Theorem 20.2]).

**Theorem 3.6.** *Let $M$ be a $\mathbb{Z}^n$-graded $A$-module.*

(a) *Any two minimal graded free resolutions of $M$ are isomorphic.*
(b) *Every graded free resolution of $M$ contains a minimal graded free resolution as a direct summand.*

**Definition 3.7.** Let $M$ be a $\mathbb{Z}^n$-graded $A$-module and let $(F_i, \phi_i)_{i\geq 0}$ be a minimal graded free resolution of $M$.

(a) The length of a minimal resolution $(F_i, \phi_i)_{i\geq 0}$ of $M$ is called the *projective dimension* of $M$ and is denoted by $\mathrm{pdim}_A M$.
(b) For each $i \geq 0$, write $F_i = \bigoplus_{j\geq 0} A\varepsilon_{ij}$, and set

$$T_i = \max\{|\mathrm{wt}(\varepsilon_{ij})| \mid j \geq 0\},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define the *exponent of growth* of $M$ to be

$$e(M) := \begin{cases} 0 & \text{if } \mathrm{pdim}_A M < \infty; \\ \limsup\limits_{i\to\infty} \dfrac{\log(T_i - i + 1)}{\log i} & \text{if } \mathrm{pdim}_A M = \infty. \end{cases}$$

(c) Assume that $e(M) < \infty$. We define the *rate of growth* of $M$ to be

$$r(M) := \begin{cases} \sup\{T_i - i \mid i \geq 0\} & \text{if } \ e(M) = 0; \\ \limsup\limits_{i\to\infty} \dfrac{T_i - i}{i^{e(M)}} & \text{if } \ e(M) \neq 0. \end{cases}$$

(d) We define the *regularity* of $M$ to be the pair

$$\mathrm{reg}_A M = (e(M), r(M)).$$

If $\mathrm{pdim}_A M < \infty$, then $e(M) = 0$, in which case the rate of growth of $M$ coincides with the Castelnuovo-Mumford regularity, $\sup\{T_i - i \mid i \geq 0\}$, in the usual sense. This detects the largest twisting in the resolution, and gives us a notion of complexity of $M$. In the commutative case, it often coincides with the degree complexity of $M$, i.e. the maximum degree $d(M)$ in the reduced Gröbner basis of $M$. One notes that the actual behavior in the commutative case depends on the term order used, although $d(M)$ is an upper bound for $r(M)$ in general, [3].

Suppose that $\mathrm{pdim}_A M = \infty$. Then, from the regularity $(e(M), r(M))$, one gets the following asymptotic behavior of the twistings:

$$T_i - i \quad \approx \quad r(M) i^{e(M)} \text{ for large } i.$$

Thus the regularity $(e(M), r(M))$ enables us to refine the notion of the Castelnuovo-Mumford regularity for the cases when it is infinite. In [2], Backelin introduced and studied the notion of the rate of growth when $e(M) = 1$. Our definition is a generalization to analyze the cases of the infinite Castelnuovo-Mumford regularity. An example with $e(M) = 2$ will be demonstrated in the next section. Let us close this section with some examples.

**Example 3.8.** We take the down-up algebra $A(\alpha, \beta, \gamma)$ introduced in [6]. Let $A = A(1, 1, 0) = \mathbb{C}\langle d, u\rangle/I$, where $I$ is the two-sided ideal generated by the homogeneous set

$$S = \left\{p := d^2u - dud - ud^2,\ q := du^2 - udu - u^2d\right\}.$$

Fix the degree-lexicographic order with $d > u$. Note that $S$ is a Gröbner-Shirshov basis for $A$, and the set of $S$-standard monomials is $\{u^i(du)^j d^k | i, j, k \geq 0\}$. Let

$$T_n = \{g_1 := du,\ g_2 := d,\ g_3 := u^n\},$$

and let $M_n$ be the quotient module of $A$ by the homogeneous left ideal generated by $T_n$.

Assume that $n \geq 2$. Then $T_n$ is a reduced Gröbner-Shirshov basis for the left $A$-module $M_n$. By considering all possible compositions between $S$ and $T_n$ and applying the method described in Theorem 2.1, we obtain the minimal graded free resolution of $M_n$ $(n \geq 2)$:

$$0 \to A\eta \to A\xi_1 \oplus A\xi_2 \oplus A\xi_3 \to A\varepsilon_1 \oplus A\varepsilon_2 \oplus A\varepsilon_3 \to Ae \to M_n \to 0,$$

where

$$\begin{aligned}
&\mathrm{wt}(e) = (0, 0), \quad \mathrm{wt}(\varepsilon_1) = (1, 1), \quad \mathrm{wt}(\varepsilon_2) = (1, 0), \quad \mathrm{wt}(\varepsilon_3) = (0, n),\\
&\mathrm{wt}(\xi_1) = (2, 1), \quad \mathrm{wt}(\xi_2) = (1, n), \quad \mathrm{wt}(\xi_3) = (1, n + 1), \quad \mathrm{wt}(\eta) = (2, n + 1).
\end{aligned}$$

It follows that

$$\mathrm{pdim}_A M_n = 3, \qquad \mathrm{reg}_A M_n = (0, n),$$

and

$$\begin{aligned}
H_{M_n}(d, u) &= \frac{1}{(1 - d)(1 - u)(1 - du)} \left(1 - (du + d + u^n) + (d^2u + du^n + du^{n+1}) - d^2u^{n+1}\right)\\
&= 1 + u + u^2 + \cdots + u^{n-1}.
\end{aligned}$$

**Example 3.9.** Let $A$ be the exterior algebra $\mathfrak{E}_3 = \mathbb{C}\langle x, y, z\rangle/I$, where $I$ is the two-sided ideal generated by

$$S = \left\{x^2, y^2, z^2, yx + xy, zx + xz, zy + yz\right\}.$$

Fix the degree-lexicographic order with $x < y < z$.

Let $N$ be the $A$-module defined by $(S,T)$, where $T=\{xyz\}$. By using Theorem 2.1 and Proposition 3.4, one can calculate the minimal graded free resolution of $N$:

$$\cdots \to \bigoplus_{a+b+c=i-1} A\varepsilon^i_{abc} \to \cdots \to \bigoplus_{a+b+c=1} A\varepsilon^2_{abc} \to A\varepsilon^1_{000} \to Ae \to N \to 0,$$

where

$$\mathrm{wt}(e)=(0,0,0),\quad \mathrm{wt}(\varepsilon^i_{abc})=(a+1,b+1,c+1),\quad \text{for } i\in\mathbb{Z}_{>0},\ a,b,c\in\mathbb{Z}_{\geq 0},\ a+b+c=i-1.$$

This implies

$$\mathrm{pdim}_A N=\infty,\qquad \mathrm{reg}_A N=(0,2),$$

and

$$\begin{aligned} H_N(x,y,z) &= (1+x)(1+y)(1+z)\left(1-xyz(1-h_1+h_2-h_3+\cdots)\right) \\ &= (1+e_1+e_2+e_3)\left(1-e_3+e_3h_1-e_3h_2+\cdots\right)=1+e_1+e_2, \end{aligned}$$

where $e_1:=x+y+z,\ e_2:=xy+yz+zx,\ e_3:=xyz$ are the elementary symmetric functions and

$$h_n:=\sum_{a,b,c\in\mathbb{Z}_{\geq 0},\ a+b+c=n} x^a y^b z^c \qquad (n\geq 1)$$

are the $n$th complete symmetric functions in the variables $x,y,z$, respectively.

## 4. Applications to Kac-Moody algebras

In this section, we concentrate on the regularity of integrable highest weight modules over Kac-Moody algebras. For Kac-Moody algebras of finite type, many results on Gröbner-Shirshov bases have been known (see, for example, [4, 19, 21]. However, for general Kac-Moody algebras, which are usually infinite dimensional, very little is known for Gröbner-Shirshov bases except for a few results on affine Kac-Moody algebras [23].

We will use the Bernstein-Gelfand-Gelfand resolutions [11, 20, 24] to investigate the regularity of integrable highest weight modules over Kac-Moody algebras. In particular, we will compute the regularity of integrable highest weight modules over Kac-Moody algebras of finite type and of affine type $A_n^{(1)}$.

Let us recall some of the basic facts on Kac-Moody algebras [16]. Let $I$ be a finite index set. An integral matrix $A=(a_{ij})_{i,j\in I}$ is called a *generalized Cartan matrix* if (i) $a_{ii}=2$ for all $i\in I$, (ii) $a_{ij}\leq 0$ for $i\neq j$, (iii) $a_{ij}=0$ if and only if $a_{ji}=0$. In this paper, it is assumed that $A$ is *symmetrizable*; i.e., there exists a diagonal matrix $D$ such that $DA$ is symmetric. A *Cartan datum* associated with $A$ is the quintuple $(A,\Pi,\Pi^\vee,P,P^\vee)$; i.e., (i) the *dual weight lattice* $P^\vee$ is a free ablelian group with a $\mathbb{Z}$-basis $\{h_i|i\in I\}\cup\{d_s|s=1,\ldots,|I|-\mathrm{rank}\,A\}$, (ii) the *weight lattice* is the set $P=\{\lambda\in\mathfrak{h}^*|\lambda(P^\vee)\subset\mathbb{Z}\}$, (iii) the set of *simple coroots* is $\Pi^\vee=\{h_i|i\in I\}$, (iv) the set of *simple roots* $\Pi$ is a linearly independent subset $\{\alpha_i|i\in I\}\subset\mathfrak{h}^*$ satisfying $\alpha_j(h_i)=a_{ij}$ and $\alpha_j(d_s)=0$ or 1 for $i,j\in I,\ s=1,\ldots,|I|-\mathrm{rank}\,A$.

The free abelian group $Q=\sum_{i\in I}\mathbb{Z}\alpha_i$ is called the *root lattice* and $Q_+=\sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ is called the *positive root lattice*. There is a partial order $\geq$ on $\mathfrak{h}^*$ by $\lambda\geq\mu$ if and only if $\lambda-\mu\in Q_+$ for $\lambda,\mu\in\mathfrak{h}^*$. For

$\alpha = \sum_{i\in I} k_i\alpha_i \in Q$, $\mathrm{ht}(\alpha) := \sum_{i=1}^n k_i$ is called the *height* of $\alpha$. We define

$$\begin{aligned} P_+ &:= \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0},\ i \in I\}, \\ P_{++} &:= \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}_{>0},\ i \in I\}. \end{aligned}$$

The elements of $P_+$ are called *dominant integral weights*. Choose an element $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = 1$ for all $i \in I$. The *Weyl group* $W$ is the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by *simple reflections* $\{r_i\}_{i\in I}$, where $r_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$ and $i \in I$. For each $k \in \mathbb{Z}_{\geq 0}$, we set

$$W(k) := \{w \in W \mid l(w) = k\},$$

where $l(w)$ is the length of a reduced expression of $w$.

**Definition 4.1.** The *Kac-Moody algebra* $\mathfrak{g}$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{h}$ and $e_i, f_i (i \in I)$ with the defining relations:

(a) $[h, h'] = 0$ for $h, h' \in P^\vee$,
(b) $[h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i$ for $h \in P^\vee$,
(c) $[e_i, f_j] = \delta_{ij}h_i$ for $i, j \in I$,
(d) $(\mathrm{ad}\ e_i)^{1-a_{ij}}(e_j) = 0,\ (\mathrm{ad}\ f_i)^{1-a_{ij}}(f_j) = 0$ for all $i \neq j$.

We have the *triangular decomposition*

$$\mathfrak{g} \quad = \quad \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+,$$

where $\mathfrak{g}_+$ (respectively, $\mathfrak{g}_-$) is the subalgebra of $\mathfrak{g}$ generated by the elements $e_i$ $(i \in I)$ (respectively, $f_i$ $(i \in I)$). For each $\alpha \in Q$, $\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$ is called the *root space* attached to $\alpha$, and $\alpha \in Q\backslash\{0\}$ such that $\mathfrak{g}_\alpha \neq \emptyset$ is called a *root* of $\mathfrak{g}$. The set of all roots is denoted by $\Delta$. The elements in $\Delta_+ := \Delta \cap Q_+$ (respectively, $\Delta_- := -\Delta_+$) are called *positive* (respectively, *negative*) roots. Then we have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha.$$

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{g}_+$. For $\lambda \in \mathfrak{h}^*$, $\mathbb{C}_\lambda := \mathbb{C}v_\lambda$ is the 1-dimensional $\mathfrak{b}$-module such that $hv_\lambda = \lambda(h)v_\lambda$ and $e_iv_\lambda = 0$ for $h \in \mathfrak{h}, i \in I$. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of a given Lie algebra $\mathfrak{g}$. Then $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ becomes a $U(\mathfrak{g})$-module under the natural left action, which is called the *Verma module*. For $\lambda \in P_+$, let $N(\lambda)$ be the submodule of $M(\lambda)$ generated by $f_i^{\lambda(h_i)+1}$ for $i \in I$. Then $V(\lambda) := M(\lambda)/N(\lambda)$ becomes the irreducible highest weight module with highest weight $\lambda$.

In [13], Garland and Lepowsky constructed a certain natural resolution of $V(\lambda)$ $(\lambda \in P_+)$, called the Berstein-Gelfand-Gelfand resolution for Kac-Moody algebras, which is given below:

Let $\mathfrak{g}$ be a Kac-Moody algebra and $V(\lambda)$ the integrable irreducible highest weight module with highest weight $\lambda \in P_+$. Then there is an exact sequence of $\mathfrak{g}$-module homomorphisms:

$$\cdots \to F_i(\lambda) \to \cdots \to F_1(\lambda) \to F_0(\lambda) \to V(\lambda) \to 0, \tag{4.1}$$

where $F_i(\lambda) = \bigoplus_{w\in W(i)} M(w(\lambda + \rho) - \rho)$. Modifying the proof of [20, (3.23)], one can prove the following proposition.

**Proposition 4.2.** *With the same notations as above, the exact sequence* (4.1) *is a minimal graded free resolution of* $V(\lambda)$ *as a* $U(\mathfrak{g}_-)$*-module.*

*Proof.* Let $I = \{1, \ldots, n\}$ and $w * \lambda := w(\lambda + \rho) - \rho$ for $w \in W$. Note that $\lambda - w * \lambda \in Q_+$. Since every highest weight $\mathfrak{g}$-module can be decomposed as root spaces, there is a natural $\mathbb{Z}^n$-graded structure on $M(w * \lambda)$. More precisely, given $w \in W$ and $m \in M(w * \lambda)_\alpha$, if we define

$$\mathrm{wt}(m) := (a_1, \ldots, a_n),$$

where $M(w * \lambda)_\alpha$ is the $\alpha$-weight space and $\lambda - \alpha = \sum_{i=1}^n a_i \alpha_i$, then this gives a $\mathbb{Z}^n$-graded structure on $M(w * \lambda)$. When there is no danger of confusion, we identity $Q$ with $\mathbb{Z}^n$. On the other hand, since every Verma module is isomorphic to $U(\mathfrak{g}_-)$ as a $U(\mathfrak{g}_-)$-module, $F_p(\lambda)$ is a graded free $U(\mathfrak{g}_-)$-module. Moreover, it follows from the fact that the maps appearing in (4.1) are $\mathfrak{g}$-module maps that they are graded $U(\mathfrak{g}_-)$-homomorphisms of degree 0. It remains to show that the image of each map in (4.1) is contained in $\mathfrak{p}F_*(\lambda)$, where $\mathfrak{p}$ is the subalgebra of $U(\mathfrak{g}_-)$ generated by $\{f_i\}_{i=1}^n$. For $w \in W$ with $l(w) = p$, let $\mathcal{V} = \{v \in W \mid v \preceq w,\ l(v) = p - 1\}$ where $\preceq$ is the *Bruhat-Chevalley partial ordering* on $W$, and let $f$ be the $p$-th map of (4.1) from $F_p(\lambda)$ to $F_{p-1}(\lambda)$. Then, from [20, (3.23)], we have

$$f|_{M(w*\lambda)} = \sum_{v \in \mathcal{V}} a_v \iota_v,$$

for some $a_v \in \mathbb{C}$ and some nonzero embedding maps $\iota_v \in \mathrm{Hom}_{\mathfrak{g}}(M(w * \lambda), M(v * \lambda))$. Since $W$ acts simply transitively on chambers [16, Proposition 3.12], $w * \lambda \neq v * \lambda$. Thus, $w * \lambda \lneq v * \lambda$, which implies that $\mathrm{im}(\iota_v) \subset \mathfrak{p}M(v * \lambda)$. Therefore, the image of each map in (4.1) is contained in $\mathfrak{p}F_*(\lambda)$. $\square$

By Proposition 4.2, the Bernstein-Gelfand-Gelfand resolution is minimal. To compute the regularity of integrable modules, we restate the resolution (4.1) in terms of free modules and bases:

$$\cdots \to \bigoplus_{w \in W(i)} U(\mathfrak{g}_-)\varepsilon_w^i \to \cdots \to \bigoplus_{w \in W(1)} U(\mathfrak{g}_-)\varepsilon_w^1 \to \bigoplus_{w \in W(0)} U(\mathfrak{g}_-)\varepsilon_w^0 \to V(\lambda) \to 0, \tag{4.2}$$

where $|\mathrm{wt}(\varepsilon_w^i)| = \mathrm{ht}(\lambda + \rho - w(\lambda + \rho))$ for $w \in W(i)$. Then we obtain the following theorem.

**Theorem 4.3.** *Let* $\mathfrak{g}$ *be a Kac-Moody algebra and* $\lambda$ *a dominant integral weight. Set* $A = U(\mathfrak{g}_-)$*.*

(a) *If* $\mathfrak{g}$ *is of finite type, then*

$$\mathrm{pdim}_A V(\lambda) = l(w_0), \quad \mathrm{reg}_A V(\lambda) = (0,\ \mathrm{ht}(\lambda + \rho - w_0(\lambda + \rho)) - l(w_0)),$$

*where* $w_0$ *is the longest element of* $W$*.*

(b) *If* $\mathfrak{g}$ *is not of finite type, then*

$$\mathrm{pdim}_A V(\lambda) = \infty.$$

*Proof.* Let $\nu = \lambda + \rho$. For each $i \in \mathbb{Z}_{\geq 0}$, if $W(i) \neq \emptyset$, choose $\varpi_i \in W(i)$ such that

$$\mathrm{ht}(\nu - \varpi_i \nu) \geq \max\{\mathrm{ht}(\nu - w\nu) \mid w \in W(i)\}.$$

Suppose $W(i) \neq \emptyset$. The aim is to show that

$$\mathrm{ht}(\nu - \varpi_i \nu) > \mathrm{ht}(\nu - \varpi_{i-1}\nu).$$

Since $\varpi_{i-1}$ is not the maximal element of $W$, there is a simple reflection $r_k$ such that $l(r_k\varpi_{i-1}) > l(\varpi_{i-1})$. It follows from $r_k\varpi_{i-1} \succ \varpi_{i-1}$ with respect to the *Bruhat-Chevalley partial order* on $W$ that

$$\mathrm{ht}(\nu - r_k\varpi_{i-1}\nu) > \mathrm{ht}(\nu - \varpi_{i-1}\nu),$$

which implies that $\mathrm{ht}(\nu - \varpi_i\nu) > \mathrm{ht}(\nu - \varpi_{i-1}\nu)$.
(a) Since $\mathfrak{g}$ is of finite type, $W$ has a unique maximal element $w_0$. Since $\{\mathrm{ht}(\nu - \varpi_i\nu)\}_{i=1}^{l(w_0)}$ is strictly increasing,

$$\mathrm{pdim}_A V(\lambda) = l(w_0), \quad \mathrm{reg}_A V(\lambda) = (0, \mathrm{ht}\,(\lambda + \rho - w_0(\lambda+\rho)) - l(w_0))\,.$$

(b) Assume $\mathfrak{g}$ is not of finite type. Since there is no maximal element in $W$, the length of a minimal graded free resolution (4.2) is infinite. □

*Remark.* Note that, if $\mathfrak{g}$ is of finite type, the projective dimension depends only on the Weyl group. On the other hand, the regularity depends also on the highest weight. If $\mathfrak{g}$ is not of finite type, the projective dimension and the Castelnuovo-Mumford regularity are infinite. To analyze this case, we need a generalization of Castelnuovo-Mumford regularity introduced in Definition 3.7.

As an illustration, let us compute the regularity of integrable highest weight modules over the affine Kac-Moody algebras of type $A_n^{(1)}$. Set $I = \{0, 1, \dots, n\}$.

**Theorem 4.4.** *Let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A_n^{(1)}$ and $\lambda$ a dominant integral weight. Set $A = U(\mathfrak{g}_-)$. Then*

$$\mathrm{reg}_A V(\lambda) = \left(2,\ \frac{l+n+1}{2n}\right),$$

*where $l := \sum_{i=0}^n \lambda(h_i)$ is the level of $\lambda$.*

Before starting the proof of Theorem 4.4, we need a few lemmas. From now on, we assume that $\mathfrak{g}$ is the affine Kac-Moody algebra of type $A_n^{(1)}$, $W$ is the Weyl group corresponding to $\mathfrak{g}$ and $V(\lambda)$ the integrable irreducible highest weight module with highest weight $\lambda \in P_+$.

**Lemma 4.5.** *Let $x = r_0 r_1 r_2 \dots r_{n-1} r_n r_{n-1} \dots r_1 \in W$ and $\nu \in P_{++}$. Then, for each $i \in \mathbb{Z}_{\geq 0}$,*

(a) *$l(x^i) = 2ni$.*

(b) *Let $l$ be the level of $\nu$. Then*

$$\mathrm{ht}(\nu - x^i\nu) = (n+1)li^2 + (l - (n+1)\nu(h_0))i.$$

*Proof.* Let $l = \sum_{i=0}^n \nu(h_i)$, $\delta = \sum_{i=0}^n \alpha_i$ and put $\alpha_{r,s} := \sum_{i=r}^s \alpha_i$ for $1 \leq r \leq s \leq n$. Then since $\mathfrak{g}$ is of type $A_n^{(1)}$,

$$\Delta_+ = \{k\delta \mid k \in \mathbb{Z}_{>0}\} \cup \{\alpha_{r,s} + k\delta \mid 1 \leq r \leq s \leq n,\ k \in \mathbb{Z}_{\geq 0}\} \tag{4.3}$$

$$\cup \{-\alpha_{r,s} + k\delta \mid 1 \le r \le s \le n,\ k \in \mathbb{Z}_{>0}\}.$$

On the other hand, it follows from [16, (6.5.1)] and $r_1 r_2 \cdots r_{n-1}(\alpha_n) = \alpha_1 + \cdots + \alpha_n$ that

$$x^i \nu = \nu - \left(li^2 - \nu(h_0)i\right)\delta - li\alpha_0. \tag{4.4}$$

(a) Let us consider

$$N(x^i) := \{\mu \in \Delta_+ \mid x^i\mu \in \Delta_-\}.$$

By computing from (4.3) and (4.4) directly, we obtain that

$$N(x^i) = \{\alpha_{1,n} + (i+j)\delta \mid 0 \le j < i\} \cup \{\alpha_{r,s} + j\delta \mid r = 1 \text{ or } s = n,\ \ 0 \le j < i\},$$

which implies that $l(x^i) = |N(x^i)| = 2ni$.
(b) It follows from (4.4). □

**Lemma 4.6.** *Let $w \in W$ with $l(w) > 0$ and $\nu \in P_{++}$. Put*

$$\nu - w\nu = \sum_{i=0}^{n} a_i \alpha_i$$

*for some $a_i \in \mathbb{Z}_{\ge 0}$. Then*

$$\frac{|a_i - a_{i\pm 1}|}{l(w)} \le \max\{\nu(h_i) \mid i = 0, 1, \ldots, n\},$$

*where $a_{-1} = a_n$ and $a_{n+1} = a_0$.*

*Proof.* (a) Let $\mathcal{M} = \max\{\nu(h_i) \mid i = 0, 1, \ldots, n\}$. We may write $w = r_k w'$ with $l(w) = l(w') + 1$ for some $w' \in W$ and some simple reflection $r_k$. Let $\nu - w'\nu = \sum_{i=0}^n a'_i \alpha_i$ for some $a'_i \in \mathbb{Z}_{\ge 0}$. For simplicity, we write $a'_{-1} = a'_n$ and $a'_{n+1} = a'_0$. Then

$$\begin{aligned} w\nu &= r_k w' \nu \\ &= r_k \left(\nu - \sum_{i=0}^{n} a'_i \alpha_i\right) \\ &= \nu - \sum_{i=0}^{n} a'_i \alpha_i - \nu(h_k)\alpha_k + \sum_{i=0}^{n} a'_i \alpha_i(h_k)\alpha_k. \\ &= w'\nu - \nu(h_k)\alpha_k + \sum_{i=0}^{n} a'_i \alpha_i(h_k)\alpha_k, \end{aligned}$$

which implies that

$$a_i = \begin{cases} a'_{k+1} + a'_{k-1} - a'_k + \nu(h_k) & \text{if } i = k; \\ a'_i & \text{if } i \ne k. \end{cases} \tag{4.5}$$

If $l(w) = 1$, then the statement is true clearly. Assume that $l(w) > 1$. Let us check the following two cases by using (4.5). The first is that, for $k \ne i, i \pm 1$,

$$\frac{|a_i - a_{i\pm 1}|}{l(w)} = \frac{|a'_i - a'_{i\pm 1}|}{l(w') + 1} \le \frac{|a'_i - a'_{i\pm 1}|}{l(w')} \le \mathcal{M}.$$

The second case is that

$$\frac{|a_k - a_{k\pm1}|}{l(w)} = \frac{|a'_{k\mp1} - a'_k + \nu(h_k)|}{l(w')+1} \le \frac{l(w')\mathcal{M} + \nu(h_k)}{l(w')+1} \le \mathcal{M}.$$

Using a standard induction argument on $l(w)$, the proof is complete. □

**Lemma 4.7.** *Let $l$ be the level of $\lambda$ and $\mathcal{M} = \max\{\lambda(h_i)|\ i = 0, 1, \dots, n\}$. Then*

$$e(V(\lambda)) = 2 \qquad \textit{and} \qquad \frac{(n+1)(l+n+1)}{4n^2} \le r(V(\lambda)) \le \mathcal{M} + 1.$$

*Proof.* Let $\nu = \lambda + \rho$. Note that $\nu \in P_{++}$. Consider $w \in W$ with $l(w) > 0$. Then there exist an element $w' \in W$ and a simple reflection $r_k$ such that $w = r_k w'$ and $l(w) = l(w') + 1$. Note that $w'\nu > w\nu$. Put

$$\nu - w'\nu = \sum_{i=0}^{n} a'_i \alpha_i.$$

For simplicity, we write $a'_{-1} = a'_n$ and $a'_{n+1} = a'_0$. Then, from (4.5),

$$\mathrm{ht}(\nu - w\nu) \quad = \quad \mathrm{ht}(\nu - w'\nu) + \nu(h_k) + a'_{k+1} + a'_{k-1} - 2a'_k,$$

which implies that, by Lemma 4.6,

$$|\mathrm{ht}(\nu - w\nu) - \mathrm{ht}(\nu - w'\nu)| \le \nu(h_k) + |a'_k - a'_{k+1}| + |a'_k - a'_{k-1}| \le \ 2(\mathcal{M}+1)l(w') + \mathcal{M} + 1.$$

Therefore, since $w$ is arbitrary,

$$e(V(\lambda)) \le 2, \quad r(V(\lambda)) \le \mathcal{M} + 1.$$

On the other hand, with the same notations as in Lemma 4.5,

$$\begin{aligned}\limsup_{i\to\infty} \frac{\mathrm{ht}(\nu - x^i\nu)}{l(x^i)^2} &= \lim_{i\to\infty} \frac{(n+1)(l+n+1)i^2 + (l+n+1-(n+1)\nu(h_0))i}{(2ni)^2} \\ &= \frac{(n+1)(l+n+1)}{4n^2}.\end{aligned}$$

Therefore, by definition,

$$2 \le e(V(\lambda)), \qquad \frac{(n+1)(l+n+1)}{4n^2} \le r(V(\lambda)).$$

□

By Lemma 4.7, $e(V(\lambda)) = 2$ and the rate of growth $r(V(\lambda))$ exists. To compute $r(V(\lambda))$ precisely, the following lemma is needed.

**Lemma 4.8.** *Let $x_0, x_1, \dots, x_n \in \mathbb{Z}$ such that $x_0 + \cdots + x_n = 0$ and $x_i \ne 0$ for some $i$. Then*

$$\frac{x_0^2 + \cdots + x_n^2}{\left(\sum_{0\le i<j\le n} |x_i - x_j|\ \right)^2} \ \le \ \frac{1}{n(n+1)}.$$

*The equality holds if and only if there is $k$ such that $x_i = x_j$ for $i, j \ne k$.*

*Proof.* It is enough to show that

$$\left(\sum_{0\le i<j\le n}|x_i-x_j|\right)^2-n(n+1)\left(x_0^2+\cdots+x_n^2\right)\ge 0.$$

Then

$$\begin{aligned}
&\left(\sum_{0\le i<j\le n}|x_i-x_j|\right)^2-n(n+1)(x_0^2+\cdots+x_n^2)\\
&=\sum_{\substack{0\le i<j\le n\\0\le k<l\le n\\(i,j)\ne(k,l)}}|x_i-x_j||x_k-x_l|+\sum_{0\le i<j\le n}|x_i-x_j|^2-n(n+1)(x_0^2+\cdots+x_n^2)\\
&=\sum_{\substack{0\le i<j\le n\\0\le k<l\le n\\(i,j)\ne(k,l)}}|x_i-x_j||x_k-x_l|-n^2(x_0^2+\cdots+x_n^2)-2\sum_{0\le i<j\le n}x_ix_j,\\
&=\sum_{\substack{0\le i<j\le n\\0\le k<l\le n\\(i,j)\ne(k,l)}}|x_i-x_j||x_k-x_l|-n(n-1)(x_0^2+\cdots+x_n^2)+2(n-1)\sum_{0\le i<j\le n}x_ix_j\\
&=\sum_{\substack{0\le i<j\le n\\0\le k<l\le n\\(i,j)\ne(k,l)}}|x_i-x_j||x_k-x_l|-(n-1)\sum_{0\le i<j\le n}|x_i-x_j|^2\\
&=2\sum_{1\le i<j\le n}|x_i-x_j|\left(\sum_{\substack{0\le k<l\le n\\(i,j)\ne(k,l)}}|x_k-x_l|\right)-(n-1)\sum_{1\le i<j\le n}|x_i-x_j|^2\\
&\quad+\ 2\sum_{1\le i<j\le n}|x_0-x_i||x_0-x_j|-(n-1)\sum_{1\le i\le n}|x_0-x_i|^2\\
&=2\sum_{1\le i<j\le n}|x_i-x_j|\left(\sum_{\substack{0\le k<l\le n\\(i,j)\ne(k,l)}}|x_k-x_l|\right)-(n-1)\sum_{1\le i<j\le n}|x_i-x_j|^2\\
&\quad-\sum_{1\le i<j\le n}||x_0-x_i|-|x_0-x_j||^2,\\
&=2\sum_{1\le i<j\le n}|X_i-X_j|\left(\sum_{\substack{1\le k<l\le n\\(i,j)\ne(k,l)}}|X_k-X_l|+\sum_{i=1}^n|X_i|\right)-(n-1)\sum_{1\le i<j\le n}|X_i-X_j|^2\\
&\quad-\sum_{1\le i<j\le n}||X_i|-|X_j||^2\qquad\text{if we let } X_i=x_i-x_0 \text{ for } i=1,\dots n,\\
&=\sum_{1\le i<j\le n}|X_i-X_j|\left(\sum_{1\le k<l\le n}|X_k-X_l|+\sum_{i=1}^n|X_i|-n|X_i-X_j|\right)
\end{aligned}$$

$$+\sum_{1\le i<j\le n}|X_i-X_j|\left(\sum_{\substack{1\le k<l\le n\\(i,j)\neq(k,l)}}|X_k-X_l|+\sum_{i=1}^{n}|X_i|\right)-\sum_{1\le i<j\le n}||X_i|-|X_j||^2$$

$$\geq 0,\qquad \text{since } |X_i-X_k|+|X_k-X_j|\geq|X_i-X_j| \text{ and } |X_i-X_j|\geq||X_i|-|X_j||.$$

Moreover, the equality holds if and only if $X_1=\cdots=X_n$ or there is $k$ such that $X_i=0$ for $i\neq k$. □

Finally we are ready to prove Theorem 4.4.

*Proof of Theorem 4.4.* Put $K:=\sum_{i=0}^n h_i$, $l:=\lambda(K)$ and $\delta:=\sum_{i=0}^n\alpha_i$. Let $T$ be the group of translations of $W$, $\mathring{W}$ the subgroup generated by $r_1,\dots,r_n$ and $\mathring{\Delta}_+$ the set of positive roots of $\mathring{W}$. Then

$$W=T\rtimes\mathring{W}.$$

For $w\in W$, denote by $\tilde{w}$ the translation part of $w$. Since $l(\mathring{w})\leq\frac{n(n+1)}{2}$ for $\mathring{w}\in\mathring{W}$,

$$l(\tilde{w})\leq l(w)+\frac{n(n+1)}{2}\quad\text{and}\quad l(w)\leq l(\tilde{w})+\frac{n(n+1)}{2}. \tag{4.6}$$

Note that $\mathring{w}(\lambda)(K)=\lambda(K)$ for $\mathring{w}\in\mathring{W}$. By Lemma 4.7, [16, (6.5.2)] and (4.6),

$$r(V(\lambda))=\limsup_{i\to\infty}\left(\max_{w\in W(i)}\frac{\mathrm{ht}(\nu-w\nu)}{(l(w))^2}\right)=\limsup_{i\to\infty}\left(\max_{w\in W(i)}\frac{\mathrm{ht}(\nu-\tilde{w}\nu)}{(l(\tilde{w}))^2}\right).$$

Let $\mathring{Q}:=\bigoplus_{i=1}^n\mathbb{Z}\alpha_i$. Choose a nonzero weight $\alpha=\sum_{i=1}^n k_i\alpha_i\in\mathring{Q}$ for $k_i\in\mathbb{Z}$ and denote by $t_\alpha$ the translation with $\alpha$. For simplicity, let $k_0=k_{n+1}=0$. From [16, (6.5.2)],

$$t_\alpha(\nu)=\nu+\nu(K)\alpha-\left(\nu(\alpha^\vee)+\frac{\nu(K)\alpha(\alpha^\vee)}{2}\right)\delta, \tag{4.7}$$

where $\alpha^\vee:=\sum_{i=1}^n k_ih_i$. Note that

$$\alpha(\alpha^\vee)=k_1^2+(k_1-k_2)^2+\cdots+(k_{n-1}-k_n)^2+k_n^2.$$

Then, using (4.3) and (4.7),

$$\begin{aligned}
l(t_\alpha)&=|\{\mu\in\Delta_+|\ t_\alpha\mu\in\Delta_-\}|\\
&=|\{\mathring{\mu}+i\delta\mid\mathring{\mu}+i\delta-\mathring{\mu}(\sum_{i=1}^n k_ih_i)\delta\in-\Delta_+,\ \mathring{\mu}\in\mathring{\Delta}_+,\ i\geq0\ \}|\\
&\quad+|\{-\mathring{\mu}+i\delta\mid\ -\mathring{\mu}+i\delta+\mathring{\mu}(\sum_{i=1}^n k_ih_i)\delta\in-\Delta_+,\ \mathring{\mu}\in\mathring{\Delta}_+,\ i>0\ \}|\\
&=\sum_{1\le i\le j\le n}|-k_{i-1}+k_i+k_j-k_{j+1}|.
\end{aligned}$$

Let $x_i=k_{i+1}-k_i$ for $i=0,\dots,n$. Note that $x_0+\cdots+x_n=0$. By Lemma 4.8,

$$r(V(\lambda))=\limsup_{i\to\infty}\left(\max_{w\in W(i)}\frac{\mathrm{ht}(\nu-\tilde{w}\nu)}{(l(\tilde{w}))^2}\right)$$

$$
\begin{aligned}
&= \limsup_{i\to\infty} \left( \max_{t_\alpha \in W(i), \alpha \in \mathring{Q}} \frac{\mathrm{ht}(\nu - t_\alpha \nu)}{(l(t_\alpha))^2} \right) \\
&\leq \sup_{0 \neq (k_1,\dots,k_n) \in \mathbb{Z}^n} \frac{(n+1)(l+n+1)\left(k_1^2 + (k_1-k_2)^2 + \cdots + (k_{n-1}-k_n)^2 + k_n^2\right)}{2\,\Big(\sum\limits_{1\leq i\leq j\leq n} |-k_{i-1}+k_i+k_j-k_{j+1}|\Big)^2} \\
&= \frac{(n+1)(l+n+1)}{2} \left( \sup_{\substack{0 \neq (x_0,\dots,x_n) \in \mathbb{Z}^{n+1} \\ x_0+\cdots+x_n=0}} \frac{x_0^2 + x_1^2 + \cdots + x_n^2}{\Big(\sum\limits_{0\leq i<j\leq n} |x_i - x_j|\Big)^2} \right) \\
&\leq \frac{l+n+1}{2n}.
\end{aligned}
$$

Moreover, if we let $k_i = ki$ for $i = 1, \dots, n$ and $k \in \mathbb{Z} \setminus \{0\}$, then the equality holds. Therefore,

$$r(V(\lambda)) = \frac{l+n+1}{2n}.$$

□

In the case of affine Kac-Moody algebras of type $A_n^{(1)}$, the projective dimension is infinite and $T_i$ appearing in Definition 3.7(c) goes to infinity as $i$ goes to infinity. It follow from Theorem 4.4 that

$$T_i \quad = \quad \frac{l+n+1}{2n}\, i^2 + o(i^2) \qquad \text{for large enough } i,$$

which means that the regularity introduced in this paper describes the asymptotic behavior of the twisting $T_i$ quite well. Moreover, since $r\,(V(\lambda))$ is proportional to the level of $\lambda$, it is compatible with our intuitive notion of complexity on representations. Further study is necessary in order to understand the regularity of representations of various algebras not considered in this paper. Especially, representations with negative levels and critical levels of Kac-Moody algebras should be carefully studied.

## References


[1] D. J. Anick, *Non-commutative graded algebras and their Hilbert series*, J. Algebra **78** (1982), 120–140.

[2] J. Backelin, *On the rates of growth of the homologies of Veronese subrings*, In: Algebra, algebraic topology and their interactions (J.-E. Roos ed.), Proc. the conference (Stockholm, 1983), Lect. Notes in Math. **1183**, Springer-Verlag, Berlin, 1986, pp. 79–100.

[3] D. Bayer and D. Mumford, *What can be computed in algebraic geometry*, In: Computational Algebraic Geometry and Commutative Algebra (D. Eisenbud and L. Robbiano eds.), Proc. the conference (Stockholm, 1983), Symposia Mathematica XXXIV., Cambridge University Press, 1993, pp. 1–48.

[4] L. A. Bokut, A. A. Klein, *Serre relations and Gröbner-Shirshov bases for simple Lie algebras I, II*, Internat. J. Algebra Comput. **6** (1996), 389–400, 401–412.

[5] D. Bayer, M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87** (1987), 1–11.

[6] G. Benkart, T. Roby, *Down-up algebras*, J. Algebra **209** (1998), 305–344.

[7] B. Buchberger, *An algorithm for finding a basis for the residue class ring of a zero-dimensional ideal*, Ph.D. thesis, University of Innsbruck, 1965 (in German). Translated in: J. Symbolic Comput. **41** (2006), 475–511.

[8] D. Cox, J. Little, D. O'Shea, *Using Algebraic Geometry*, 2nd ed., Grad. Texts in Math. **185**, Springer-Verlag, New York, 2005.

[9] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Grad. Texts in Math. **150**, Springer-Verlag, New York, 1995.

[10] J. C. Faugère, *A new efficient algorithm for computing Gröbner bases ($F_4$)*, J. Pure Appl. Algebra **139** (1999), 61–88.

[11] H. Garland, J. Lepowsky, *Lie algebra homology and the Macdonald-Kac formulas*, Invent. Math. **34** (1976), 37–76.

[12] T. Gateva-Ivanova, *On the noetherianity of some associative finitely presented algebras*, J. Algebra **138** (1991), 13–35.

[13] J. Gómez-Torrecillas, F. J. Lobillo, *Global homological dimension of multifiltered rings and quantized enveloping algebras*, J. Algebra **225** (2000), 522–533.

[14] E. L. Green, Ø. Solberg, D. Zacharia, *Minimal projective resolutions*, Trans. Amer. Math. Soc. **353** (2001), 2915–2939.

[15] P. Jørgensen, *Non-commutative Castelnuovo-Mumford regularity*, Math. Proc. Camb. Phil. Soc. **125** (1999), 203–221.

[16] V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.

[17] S.-J. Kang, D.-I. Lee, K.-H. Lee, H. Park, *Linear algebraic approach to Gröbner-Shirshov basis theory*, J. Algebra **313** (2007), 988–1004.

[18] S.-J. Kang, K.-H. Lee, *Gröbner-Shirshov bases for representation theory*, J. Korean Math. Soc. **37** (2000), 55–72.

[19] ———, *Gröbner-Shirshov bases for irreducible $sl_{n+1}$-modules*, J. Algebra **232** (2000), 1–20.

[20] S. Kumar, *Bernstein-Gelfand-Gelfand resolution for arbitrary Kac-Moody algebras*, Math. Ann. **286** (1990), 709–729.

[21] V. Levandovskyy, *Non-commutative Computer Algebra for polynomial algebras: Gröbner bases, applications and implementation*, Ph.D. thesis, Universität Kaiserslautern, 2005.

[22] C. Năstăsescu, F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, Lect. Notes in Math. **758**, Springer-Verlag, Berlin, 1979.

[23] E. N. Poroshenko, *Gröbner-Shirshov bases for the Kac-Moody algebras of the type $A_n^{(1)}$*, Comm. Algebra **30** (2002), 2617–2637.

[24] A. Rocha-Caridi, N. Wallach, *Projective modules over graded Lie algebras I*, Math. Z. **180**, (1982), 151–177.

[25] A. I. Shirshov, *Some algorithmic problems for Lie algebras*, Sibirk. Math. Z. **3** (1962), 292–296 (in Russian). Translated in: ACM SIGSAM Bull. Commun. Comput. Algebra **33** (1999) no. 3, 3–6.



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