ON FORMULAS FOR THE INDEX OF THE CIRCULAR DISTRIBUTIONS

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Abstract. A circular distribution is a Galois equivariant map ψ from the roots of unity μ_∞ to an algebraic closure of $\mathbb{Q}$ such that ψ satisfies product conditions, $\prod_{\zeta^d=\epsilon}\psi(\zeta)=\psi(\epsilon)$ for $\epsilon\in\mu_\infty$ and $d\in\mathbb{N}$, and congruence conditions for each prime number l and $s\in\mathbb{N}$ with $(l,s)=1$, $\psi(\epsilon\zeta)\equiv\psi(\zeta)$ modulo primes over l for all $\epsilon\in\mu_l,\zeta\in\mu_s$, where μ_l and μ_s denote respectively the sets of lth and sth roots of unity. For such ψ, let P_s^ψ be the group generated over $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\mu_s)/\mathbb{Q})]$ by $\psi(\zeta),\zeta\in\mu_s$ and let C_s^ψ be $P_s^\psi\bigcap U_s$, where U_s denotes the global units of $\mathbb{Q}(\mu_s)$. We give formulas for the indices $(P_s:P_s^\psi)$ and $(C_s:C_s^\psi)$ of P_s^ψ and C_s^ψ inside the circular numbers P_s and units C_s of Sinnott over $\mathbb{Q}(\mu_s)$.

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1. Introduction

Circular distributions of Coleman(cf. [2]) arise in various contexts in number theory. In particular, they give rise to Euler systems (cf. [5]) and higher special units over number fields(cf. [11]). They play a role in connecting the structure of the ideal class group with that of a certain quotient of special units of an abelian number field. Circular distributions can also be characterized using an Archimedean place(cf. [2]). One of the fundamental questions of this paper is computation of the indices between the circular units coming from various circular distributions. Let μ_s be the set of sth roots of unity and let ζ_s be a primitive sth root of unity in a fixed algebraic closure $\mathbb{Q}^{\mathrm{alg}}$ of $\mathbb{Q}$. Let $\mu_\infty=\bigcup_{s\in\mathbb{N}}\mu_s$ and $\mu_s^*=\mu_s\setminus\{1\}$, $\mu_\infty^*=\mu_\infty\setminus\{1\}$, where $\mathbb{N}$ is the set of positive integers. A circular distribution after Coleman is a Galois equivariant map f from μ_∞^* to $\mathbb{Q}^{\mathrm{alg}}$ such that f satisfies product conditions,

$$\prod_{\zeta^d=\epsilon} f(\zeta)=f(\epsilon),\ \text{for } \epsilon\in\mu_\infty^* \text{ and } d\in\mathbb{N}$$

and congruence conditions for each prime number l and $s\in\mathbb{N}$ with $(l,s)=1$,

$$f(\epsilon\zeta)\equiv f(\zeta) \text{ modulo primes over } l$$

for all $\epsilon\in\mu_l^*,\zeta\in\mu_s^*$. Let $\mathcal{F}$ denote the set of all circular distributions. Let $R_s=\mathbb{Z}[G(\mathbb{Q}(\mu_s)/\mathbb{Q})]$ be the group ring of the Galois group $G(\mathbb{Q}(\mu_s)/\mathbb{Q})$ and $R_\infty=\varprojlim R_s$ be the projective limit of R_s with respect to the natural restriction maps. Then R_∞ acts naturally on the group $\mathcal{F}$. In this note, l denotes a fixed odd prime. For an abelian group A, we denote by $A[l]$ the profinite l-completion $\varprojlim A/A^{l^n}$ of A. For any subgroup of the global units of a number field being finitely generated as a $\mathbb{Z}$-module, its profinite l-completion can be identified with the tensor product with $\mathbb{Z}_l$. Let $\mathcal{F}(s)$ be the group generated by $f(\zeta)$ for all f in $\mathcal{F}$ and all ζ in μ_s^* and let

$\mathcal{F}_s$ be the intersection of $\mathcal{F}(s)$ and the global units of $\mathbb{Q}(\mu_s)$. Then $\mathcal{F}(s)$ and $\mathcal{F}_s$ contain respectively the circular numbers P_s and the circular units C_s of $\mathbb{Q}(\mu_s)$ in the sense of Sinnott, as defined in [14] and [15]. In this paper, we let s_n denote l^{n+1}. Let $k = k_0 = \mathbb{Q}(\mu_{s_0})$ and $k_n = \mathbb{Q}(\mu_{s_n})$ be the intermediate field of the cyclotomic $\mathbb{Z}_l$-extension $k_\infty = \bigcup_n k_n$ of degree l^n over k_0. In fact, we have

$$\mathcal{F}(s_n)[l] = P_{s_n}[l], \;\; \mathcal{F}_{s_n}[l] = C_{s_n}[l]$$

for all $n \in \mathbb{N}\bigcup\{0\}$(cf. [10]), where P_{s_n} and C_{s_n} denote the circular numbers and the circular units respectively of k_n in the sense of Sinnott. For each ψ in $\mathcal{F}$, let $P^{\psi}_{s_n}$ be the group generated over R_{s_n} by $\psi(\zeta), \zeta \in \mu^*_{s_n}$, and let $C^{\psi}_{s_n} = P^{\psi}_{s_n} \bigcap U_{s_n}$, where U_{s_n} denotes the global units of k_n. Then from the above equation, we have

$$P^{\psi}_{s_n}[l] \subseteq P_{s_n}[l], \;\; C^{\psi}_{s_n} \subseteq C_{s_n}[l]$$

for all $n \in \mathbb{N}\bigcup\{0\}$. The group ring R_{s_n} acts naturally on the groups $P^{\psi}_{s_n}$ and $C^{\psi}_{s_n}$. Let ξ be the element of $\mathcal{F}$ defined by $\xi(\zeta) = 1-\zeta, \; \zeta \in \mu^*_\infty$. Notice that $P^{\xi}_{s_n}$ and $C^{\xi}_{s_n}$ are the circular numbers P_{s_n} and the circular units C_{s_n} of k_n. We compute formulas for the indices of the ψ-circular numbers $P^{\psi}_{s_n}$ and units $C^{\psi}_{s_n}$ coming from a single circular distribution ψ of $\mathcal{F}$ rather than the whole circular units $\mathcal{F}_{s_n}$ inside the circular numbers P_{s_n} and units C_{s_n}. Suppose that l is prime to $\phi(s)$, the Euler phi function. Let

$$h_{s_n} = [k_n : \mathbb{Q}(\psi(\zeta_{s_n}))]$$

be the degree of the field extension $k_n/\mathbb{Q}(\psi(\zeta_{s_n}))$. Notice that if $h_{s_n} > 1$ for infinitely many numbers n, then $\psi(\zeta_{s_n})$ is a root of unity for all n(cf. [12]). By Dirichlet's unit theorem, in order for the index $(C_{s_n} : C^{\psi}_{s_n})$ to be finite, ψ must satisfy the assumption that $\psi(\zeta_{s_n}) \notin \mu_{s_n}$, and $h_{s_n} = 1$ as well. Hence in this note, we will assume that $h_{s_n} = 1$. Let Ξ be the set of nontrivial Dirichlet characters of $\mathbb{Q}(\mu_{s_n})$, which are even, i.e., $\chi(-1) = 1$. If χ in Ξ is of conductor f, then for each $\psi \in \mathcal{F}$, we write

$$t^{\psi}(\chi) = -2^{-1} \sum_{(a,f)=1} \chi(a) \log |\psi(\zeta^a_f)|.$$

We write $t(\chi) = t^{\xi}(\chi)$. Let $\mathcal{O}_{k_n}$ denote the ring of integers of k_n. For each prime l, we define the sign,

$$\mathrm{sgn}_l(\psi) = v_{\mathfrak{l}_n}(\psi(\zeta_{l^n}))$$

of ψ at l to be the $\mathfrak{l}_n$-adic valuation $v_{\mathfrak{l}_n}(\psi(\zeta_{l^n}))$ of $\psi(\zeta_{l^n})$, where $\mathfrak{l}_n$ is the unique prime of $\mathbb{Q}(\mu_{l^n})$ lying over l and $v_{\mathfrak{l}_n}$ is the discrete valuation of k_n associated to $\mathfrak{l}_n$ defined as $\alpha\mathcal{O}_{k_n} = \mathfrak{l}_n^{v_{\mathfrak{l}_n}(\alpha)}\mathfrak{a}$ with $(\mathfrak{a}, \mathfrak{l}_n) = 1$. According to the product conditions of circular distributions, $\mathrm{sgn}_l(\psi)$ does not depend on n. It depends only on ψ and prime l. Let v_l denote the l-adic valuation defined as $v_l(l) = 1$ and let $|.|_l$ denote the l-adic absolute value normalized by $|l|_l = 1/l$. For a finite set S, we denote by $\#(S)$ the cardinality of S.

Theorem 1.1. *$P^{\psi}_{s_n}[l]$ and $C^{\psi}_{s_n}[l]$ are contained in $P_{s_n}[l]$ and $C_{s_n}[l]$, respectively. Suppose that $\mathrm{sgn}_l(\psi) \neq 0$, and $t^{\psi}(\chi) \neq 0$ for all $\chi \in \Xi$, then the ψ-numbers $P^{\psi}_{s_n}[l]$ and the ψ-units $C^{\psi}_{s_n}[l]$ have finite indices in $P_{s_n}[l]$ and $C_{s_n}[l]$, respectively. Conversely, if $P^{\psi}_{s_n}[l]$ and $C^{\psi}_{s_n}[l]$ have finite indices respectively in $P_{s_n}[l]$ and $C_{s_n}[l]$, then we have $\mathrm{sgn}_l(\psi) \neq 0$, and $t^{\psi}(\chi) \neq 0$ for all $\chi \in \Xi$. In this case, we have the* *following formulas,*

$$(P_{s_n}[l] : P^{\psi}_{s_m}[l]) = l^{v_l(N_{\mathbb{Q}}(\psi(\zeta_{s_m})))}|\theta_{s_n} \prod_{\chi\in\Xi} t(\chi)t^{\psi}(\chi)^{-1} |_l$$

$$(C_{s_n}[l] : C^{\psi}_{s_n}[l]) = |\theta_{s_n} \frac{\prod_{\chi\in\Xi} t(\chi)}{\prod_{\chi\in\Xi} t^{\psi}(\chi)} |_l = |\theta_{s_n} \prod_{\chi\in\Xi} t(\chi)t^{\psi}(\chi)^{-1} |_l$$

where $\theta_{s_n} = \#(\mu_{s_n}/\mu_{s_n} \bigcap C^{\psi}_{s_n})$.

In §2, we compute the cohomology groups of the circular numbers and circular units and relate it to the Λ-module structures of the inverse limits of circular numbers and units. We give the outline of the proof of Theorem 1.1.

In the appendix, we apply Sinnott's argument to compute the index of the ψ-circular numbers and units using the cohomology groups of ψ-circular numbers and units, and hence recover the proof of Theorem 1.1. Finally, we remark that if r is bigger than or equal to the rank $\mathrm{rank}_{\mathbb{Q}_l}\mathrm{Cl}_{s_n}[l]$ of $\mathbb{Q}_l$-vector space $\mathrm{Cl}_{s_n}[l] \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ of the l-primary part $\mathrm{Cl}_{s_n}[l]$ of the ideal class group Cl_{s_n} of $\mathbb{Q}(\mu_{s_n})$, then all the results in this note remain valid when the circular distributions are replaced by the truncated Euler systems $\mathcal{E}^r_{\mathbb{Q}}$ of depth r(cf. [13]). The notion of $\mathcal{E}^r_{\mathbb{Q}}$ will be briefly introduced at the end of the appendix.

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2. Cohomology of circular distributions

For a finitely generated $\mathbb{Z}$-module A, let $\overline{A}$ be the quotient of A by the torsion subgroup A_{tor}, $\overline{A} = A/A_{\mathrm{tor}}$. Let $k = k_0$ be the cyclotomic field $\mathbb{Q}(\mu_s)$ or $\mathbb{Q}(\mu_{sl})$ according as s is divisible by l or not. Let k_∞ be the cyclotomic $\mathbb{Z}_l$-extension of k with its unique subfield $k_n = \mathbb{Q}(\mu_{s_n})$, which is fixed by the Galois group Γ^{l^n} where $\Gamma = G(k_\infty/k)$. Let $\overline{C}_\infty[l]$ be the inverse limit of $\overline{C}_{s_n}[l]$ with respect to the norm maps, where C_{s_n} is the circular units of k_n. Then $\overline{C}_\infty[l]$ has a natural R_{l^∞}-module structure, where $R_{l^\infty} = \varprojlim \mathbb{Z}_l[G(k_n/\mathbb{Q})]$ is the completed group ring of $G(k_\infty/\mathbb{Q})$. The group $G(k_\infty/\mathbb{Q})$ has a direct decomposition $G(k_\infty/\mathbb{Q}) = G(k_\infty/k)\times G(k_0/\mathbb{Q})$ into the l-part $G(k_\infty/k)$ and the prime to l-part $G(k_0/\mathbb{Q})$. We have $R_{l^\infty} = \Lambda[G(k_0/\mathbb{Q})]$, where $\Lambda = \mathbb{Z}_l[[\Gamma]] = \varprojlim \mathbb{Z}_l[\Gamma/\Gamma_n]$, and $\Gamma_n = \Gamma^{l^n}$ is the unique subgroup of Γ of index l^n. We need the following theorem of Kuz'min.

Theorem 2.1 (=Theorem 1.1 of [7]). *The groups* $\overline{P}_\infty[l]$ *and* $\overline{C}_\infty[l]$ *are free* Λ-*modules of rank* $\frac{1}{2}[k_0 : \mathbb{Q}]$.

For a number field F, let U'_F be the l-units of F, i.e., the group of elements of F which are unit over the primes not dividing l. For any $\mathbb{Z}_l$-extension $F_\infty = \bigcup_n F_n$ of F, let U'_{s^n} be the l-units of F_n and let $\overline{U'}_\infty[l] = \varprojlim \overline{U'}_{s^n}[l]$ be the inverse limit of $\overline{U'}_{s^n}[l]$ with respect to the norm maps. The following theorem is due to Kuz'min.

Theorem 2.2 (=Theorem 7.2 of [6]). *Suppose that* F_∞/F *is cyclotomic. Then* $\overline{U'}_\infty[l]$ *is* Λ-*free of rank* $r_1 + r_2$, *where* r_1, r_2 *are the numbers of real places and complex places of* F, *respectively.*

Note that Greither(cf. [3]) proved a more general theorem on $\overline{U'}_\infty[l]$ when F_∞/F is cyclotomic or F does not contain a primitive lth root of unity. In this case, it is proved that $\overline{U'}_\infty[l]$ is Λ-free of rank $r_1 + r_2 + g$, where g is the number of primes lying over l, which split completely over F_∞/F.

Let $\overline{C^\psi_\infty}[l]$ be the inverse limit of $\overline{C^\psi_{s^n}}[l]$ with respect to the norm maps, where $C^\psi_{s^n}$ is the ψ-circular units of k_n. We need a proposition of Belliard(cf. [1]). In the proposition, G_n denotes $G(k_n/\mathbb{Q})$.

Proposition 2.3 (=Proposition 1.3 of [1]). *Let $M_n \subset L_n$ be $\mathbb{Z}_l[G_n]$-modules equipped with the natural norm map $N_{m,n} : L_m \to L_n$ and the extension map $i_{n,m} : L_n \to L_m$ satisfying the following properties:*
(i) *$L_\infty := \varprojlim L_n$ is Λ-free.*
(ii) *$i_{n,m} : L_n \to L_m^{G(k_m/k_n)}$ is injective.*
(iii) *Let M_n verify asymptotic Galois descent, i.e., $\exists N \in \mathbb{N}$ such that for all $m \geq n \geq N$, $M_m^{G(k_m/k_n)} = M_n$. Then $M_\infty := \varprojlim M_n$ is Λ-free.*

For a circular distribution ψ, the product condition $\prod_{\zeta^d=\epsilon} \psi(\zeta) = \psi(\epsilon)$ for $\epsilon \in \mu_\infty$ and $d \in \mathbb{N}$, is known to be equivalent to the following conditions (cf. [9]). For any prime number, l, and square free integer, r, with $(r, l) = 1$,

$$N_{\mathbb{Q}(\mu_{lr})/\mathbb{Q}(\mu_r)}\psi(\zeta_l\zeta_r) = \psi(\zeta_r)^{\ \mathrm{Fr}_l - 1}$$

and for $n - i \geq 1$,

$$N_{\mathbb{Q}(\mu_{l^n r})/\mathbb{Q}(\mu_{l^{n-1} r})}\psi(\zeta_{l^n}\zeta_r) = \psi(\zeta_{l^{n-1}}\zeta_r^l)$$

where Fr_l is the Frobenius at l. In the sections to follow, we often use these equivalent conditions rather than the product conditions. In the following proposition, we give a necessary and sufficient condition for the 0th Tate cohomology group of the ψ-circular units to be trivial. The proof follows from arguments similar to those used for Theorem 2.2 of Belliard(cf. [1]) and Lemma 4.2 of Nguyen Quang Do(cf. [8]).

Proposition 2.4. *Suppose $(\overline{C}_\infty[l]/\overline{C}^\psi_\infty[l])^{\Gamma_n} < \infty$, for all $n \in \mathbb{N}$. Then the following two statements are equivalent.*
(i) *There exists a positive integer N such that $\widehat{H}^0(G(k_m/k_n), \overline{C^\psi_{s_m}}[l])$ is trivial for all $m \geq n \geq N$.*
(ii) *$\overline{C}^\psi_\infty[l]$ is a Λ-free module.*

Proof. If $\widehat{H}^0(G(k_m/k_n), \overline{C^\psi_{s_m}}[l]) = 0$, then $(\overline{C^\psi_{s_m}}[l])^{G(k_m/k_n)} = N_{k_m/k_n}\overline{C^\psi_{s_m}}[l]$, which is equal to $\overline{C^\psi_{s_n}}[l]$ by the assumption on l. By Theorem 2.2 and Proposition 2.3, $\overline{C}^\psi_\infty[l]$ is a Λ-free module. Suppose now that $\overline{C}^\psi_\infty[l]$ is a Λ-free module. Let π be the natural projection from U_∞ into U_k. We denote the group $\pi(U_\infty)$ of norm coherent units by U_k^{coh}, and similarly for $U_{s_n}^{\mathrm{coh}}$ and

$$(C^\psi_{s_n})^{\mathrm{coh}} = C^\psi_{s_n} \bigcap U^{\mathrm{coh}}_{s_n}$$

for the coherent global units and coherent ψ-circular units of k_n, respectively. When $\psi = \xi$, the following lemma is the same as Lemma 2.5 of Belliard(cf. [1]). For a general ψ, it follows easily from the same argument. We leave the proof of the following lemma to the reader.

Lemma 2.5. *Let K be a number field and K_∞/K be a $\mathbb{Z}_l$-extension. Let K_n be the unique subfield of K_∞, which has degree l^n over K. Let $n \in \mathbb{N}$ and let I_n be the* *inertia field of K_n at l. Then we have*

$$\overline{C_n^\psi}[l] = \overline{C_{I_n}^\psi}[l](\overline{C_n^\psi})^{\mathrm{coh}}[l].$$

If we let I denote the inertia field of p in K_∞ then we have $I \subset K_n$:

$$\overline{C_n^\psi}[l] = \overline{C_I^\psi}[l](\overline{C_n^\psi})^{\mathrm{coh}}[l].$$

It follows from the above Lemma that if l is totally ramified in k then the ψ-circular units of k_n are norm coherent. It follows from the exact sequence $0 \to \overline{C_\infty^\psi}[l] \to \overline{U'}_\infty[l] \to \overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l] \to 0$ and Theorem 2.2 and the snake lemma that $0 \to (\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_n} \to (\overline{C_\infty^\psi}[l])_{\Gamma_n} \to (\overline{U'}_\infty[l])_{\Gamma_n}$. As Λ is noetherian, there is N such that $(\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_n} = (\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_N}$ for all $n \geq N$. Theorem 7.3 of Kuz'min in [6] tells us that the natural map $(\overline{U'}_\infty[l])_{\Gamma_n} \to \overline{U'}_{s^n}[l]$ is injective. Hence, we obtain an exact sequence,

$$0 \to (\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_N} \to (\overline{C_\infty^\psi}[l])_{\Gamma_n} \to (\overline{C_{s_n}^\psi})^{\mathrm{coh}}[l] \to 0.$$

As $\overline{C_\infty^\psi}[l]$ is Λ-free, $(\overline{C_\infty^\psi}[l])_{\Gamma_n}$ is $\mathbb{Z}_l[G(k_m/k_n)]$-free. Moreover, $G(k_m/k_n)$ acts trivially on $\mathbb{Z}_l$-free module $(\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_N}$. For all $n \geq N$, we have that the Tate cohomology $\widehat{H}^0(G(k_m/k_n), \overline{C_m^\psi}[l])$ is isomorphic to $\widehat{H}^0(G(k_m/k_n), (\overline{C_{s_m}^\psi})^{\mathrm{coh}}[l]) = \widehat{H}^1(G(k_m/k_n), (\overline{U'}_\infty[l]/\overline{C_\infty^\psi}[l])^{\Gamma_N}) = 0$. Let k_n^+ be the maximal real subfield of k_n. For every subgroup A of k_n, we write $A^+ := A \bigcap k_n^+$ for its intersection with k_n^+. Let $D_n := \langle\, \mathfrak{l} \mid \mathfrak{l}|l \,\rangle \bigcap (\mathrm{Cl}_n^+ \otimes \mathbb{Z}_l)$ be the subgroup of l-Sylow subgroup $\mathrm{Cl}_n^+ \otimes \mathbb{Z}_l$ of k_n^+ generated by the primes lying over l. Taking the inverse limits to the exact sequence $0 \to D_n \to \mathrm{Cl}_n^+ \otimes \mathbb{Z}_l \to \mathrm{Cl}_n^+ \otimes \mathbb{Z}_l/D_n \to 0$, we have an exact sequence, $0 \to D_\infty \to \mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l \to \mathrm{Cokernel} \to 0$. As l is totally ramified, Γ_n acts trivially on D_∞. By the snake lemma, we have $0 \to D_\infty \to (\mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l)^{\Gamma_n} \to (\mathrm{Cokernel})^{\Gamma_n}$. Leopoldt conjecture, which is true for our abelian case, tells us that the second term $(\mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l)^{\Gamma_n}$ is finite and hence so is D_∞. Write S_n for the set of primes of k_n^+ lying over l. From the exact sequence $0 \to \overline{(U'_{s_n})^+}[l]/\overline{U_{s_n}^+}[l] \to \mathbb{Z}_l[S_n] \to D_n \to 0$, we have

$$0 \to \overline{U_\infty^+}[l]/\overline{(C_\infty^\psi)^+}[l] \to \overline{(U'_\infty)^+}[l]/\overline{(C_\infty^\psi)^+})[l] \to \mathbb{Z}_l[S_\infty] \to D_\infty \to 0$$

where $\mathbb{Z}_l[S_\infty] = \varprojlim \mathbb{Z}_l[S_n] = \mathbb{Z}_l[S_0]$ by the assumption on l. Since D_∞ is finite, we have $0 \to \overline{U_\infty^+}[l]/\overline{(C_\infty^\psi)^+}[l] \to (\overline{U'_\infty})^+[l]/\overline{(C_\infty^\psi)^+})[l] \to \mathrm{Fr} \to 0$, where Fr is a free $\mathbb{Z}_l$-module with trivial Galois action. The snake lemma induces,

$$0 \to (\overline{U_\infty^+}[l]/\overline{(C_\infty^\psi)^+}[l])^{\Gamma_n} \to ((\overline{U'_\infty})^+[l]/\overline{(C_\infty^\psi)^+})[l])^{\Gamma_n} \to \mathrm{Fr}^{\Gamma_n}.$$

Thanks to the Iwasawa main conjecture, $(\overline{U_\infty^+}[l]/\overline{(C_\infty)^+}[l])$ and $\mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l$ have the same characteristic ideal. For the latter group, $\mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l$, its Galois coinvariant $(\mathrm{Cl}_\infty^+ \otimes \mathbb{Z}_l)^{\Gamma_n}$ is finite by the Leopoldt conjecture. Hence, from the assumption, we have

$$(\overline{U_\infty^+}[l]/\overline{(C_\infty^\psi)^+}[l])^{\Gamma_n} < \infty.$$

If j denotes the complex conjugation then $N_j = 1 + j$ is the norm map from k_n to k_n^+. If α lies in the kernel of N_j, then $\alpha^j = \alpha^{-1}$, which implies that all the conjugates of α have an absolute value of one. Because the kernel is a subset of the units, the kernel is the set $\mu(k_n)$ of all roots of unity in k_n. The exact sequence $0 \to \mu(k_n) \bigcap C^{\psi}_{s_n} \to C^{\psi}_{s_n} \stackrel{N_j}{\to} (C^{\psi}_{s_n})^+$ induces

$$1 \longrightarrow \overline{C^{\psi}_{s_n}}[l] \longrightarrow \overline{(C^{\psi}_{s_n})^+}[l] \longrightarrow \overline{(C^{\psi}_{s_n})^+}[l]/N\overline{C^{\psi}_{s_n}}[l] \longrightarrow 0$$

and $((C^{\psi}_{s_n})^+)^2 \subseteq N_j C^{\psi}_{s_n} \subseteq (C^{\psi}_{s_n})^+$. For an odd prime l, we have $\overline{(C^{\psi}_{s_n})^+}[l]/N_j\overline{C^{\psi}_{s_n}}[l] = 0$, $\overline{C^{\psi}_{s_n}}[l] \cong \overline{(C^{\psi}_{s_n})^+}[l]$ and $\overline{C^{\psi}_{\infty}}[l] \cong \overline{(C^{\psi}_{\infty})^+}[l]$. It follows that $\overline{(C^{\psi}_{\infty})^+}[l]$ is a free Λ-module under our assumption. Since $\overline{U^+_{\infty}}[l]$ is Λ-free(cf. [1]), $M := \overline{U^+_{\infty}}[l]/\overline{(C^{\psi}_{\infty})^+}[l]$ has a projective dimension less than or equal to one. Hence, the maximal finite submodule M_0 must be trivial. From the exact sequence above, it follows that $(\overline{(U'_{\infty})^+}[l]/\overline{(C^{\psi}_{\infty})^+}[l])^{\Gamma_n}$ is isomorphic to a $\mathbb{Z}_l$-free module with trivial Galois action. Hence, we can apply the same argument for the maximal real subfield to obtain the following isomorphisms. The Tate cohomology $\widehat{H}^0(G(k_m/k_n), \overline{(C^{\psi}_{s_m})^+}[l])$ is isomorphic to $\widehat{H}^0(G(k_m/k_n), \overline{((C^{\psi}_{s_m})^+)^{\mathrm{coh}}}[l])$, which is isomorphic to the first Tate group $\widehat{H}^1(G(k_m/k_n), (\overline{(U'_{\infty})^+}[l]/\overline{(C^{\psi}_{\infty})^+}[l])^{\Gamma_N})$, which is the trivial group. The isomorphism above shows that for all $m \geq n \geq N$, $\widehat{H}^0(G(k_m/k_n), \overline{C^{\psi}_{s_m}}[l]) \cong \widehat{H}^0(G(k_m/k_n), \overline{(C^{\psi}_{s_m})^+}[l]) = 0$. This completes the proof of Proposition 2.4. □

Let ψ be a circular distribution in $\mathcal{F}$. As was defined in the introduction, let $P^{\psi}_{s_n}$ be the subgroup of the multiplicative group $k_n^{\times}$ of k_n generated by the elements $\psi(\zeta)$ and μ_{s_n}, for $\zeta \in \mu_{s_n}, \zeta \neq 1$, and let $P^{\psi}_{\infty} = \varprojlim P^{\psi}_{s_n}$. If $\mathrm{sgn}_l(\psi) = 0$, then the index of circular numbers become infinity,

$$(P_{s_n}[l] : P^{\psi}_{s_n}[l]) = \infty.$$

Hence, we will consider the case $\psi(\zeta_{s_n})$ with $\mathrm{sgn}_l(\psi) \neq 0$. Notice that if $\psi(\zeta_{s_n})$ is $\mathbb{Z}$-torsion, then P^{ψ}_{∞} is either trivial or $\mathbb{Z}_l$. Let δ_m be the degree of field extension $[\mathbb{Q}(\psi(\zeta_{s_m})) : \mathbb{Q}]$ and $h_m = [k_m : \mathbb{Q}]/\delta_m$. Notice that if $h_m > 1$ for infinitely many numbers m, then $\psi(\zeta_{s_m})$ is a root of unity for all m. Hence, if $\psi(\zeta_{s_m})$ is $\mathbb{Z}$-torsion free then there is an integer N such that for all $m \geq N, h_m = 1$. If the l-part of the index $(C_{s_m}[l] : C^{\psi}_{s_m}[l])$ is finite, then ψ must satisfy the assumption that $\psi(\zeta_{s_m}) \notin \mu_{s_m}$ for an m and hence $\psi(\zeta_{s_a}) \notin \mu_{s_a}$ for all $a \geq m$ equivalently, and $h_m = 1$ as we assumed in the introduction. The latter condition is asymptotically implied by the former condition in the following sense(cf. [12]).

Lemma 2.6. *If $\psi(\zeta_{s_m}) \notin \mu_{s_m}$, then for all sufficiently large numbers m, $h_m = 1$.*

In order to apply the cohomology groups of circular distributions to the index formula in the next section, we determine the cohomology groups in the following proposition. When $\psi = \xi$, the cohomologies are well known(cf. [4]). For an arbitrary distribution ψ, we need more calculations. We give here a detailed proof in a self-contained way.

Proposition 2.7. *Suppose that the quotient $C_{s_m}[l]/C^{\psi}_{s_m}[l]$ is finite and $\mathrm{sgn}_l(\psi) \neq 0$. Then the Tate cohomology groups of ψ-circular units $C^{\psi}_{s_m}$ are*

$$\widehat{H}^0(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) = 0, \;\; \widehat{H}^1(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) = \mathbb{Z}/\phi(s_m)\mathbb{Z}$$

and the Tate cohomology groups of ψ-circular numbers $P^{\psi}_{s_m}$ are

$$\widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) = 0, \;\; \widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) = 0.$$

Proof. Let $G = G(k_m/\mathbb{Q})$, $H = G(k_m/\mathbb{Q}(\psi(\zeta_{s_m})))$ and $\Delta = G/H$. We have a natural surjection from the group ring $\mathbb{Z}[\Delta]$ to $P^{\psi}_{s_m}$ via $e \to \psi(\zeta_{s_m})$. If $I^{\psi}_{s_m}$ denotes the kernel of this surjection, then $\mathbb{Z}[\Delta]/I^{\psi}_{s_m} \cong P^{\psi}_{s_m}$. Since $I^{\psi}_{s_m}$ is the annihilator of $P^{\psi}_{s_m}$ inside $\mathbb{Z}[\Delta]$, each element $\sum_{\sigma\in\Delta} a_\sigma \sigma \in I^{\psi}_m$ must satisfy $\sum_{\sigma\in\Delta} a_\sigma = 0$ from $\mathrm{sgn}_l(\psi) \neq 0$. It follows that $(I^{\psi}_m)^G = 0$ and $(I^{\psi}_m)_{s(G)} = I^{\psi}_m$, and hence $\widehat{H}^0(G(k_m/\mathbb{Q}), I^{\psi}_{s_m}) = 0$. We have an exact hexagon

$$\to \widehat{H}^0(G(k_m/\mathbb{Q}), I^{\psi}_m) \to \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z}[\Delta]) \to \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \to$$
$$\leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), \mathbb{Z}[\Delta]) \leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), I^{\psi}_m) \hookleftarrow$$

induced from the exact sequence

$$0 \longrightarrow I^{\psi}_m \longrightarrow \mathbb{Z}[\Delta] \longrightarrow P^{\psi}_{s_m} \longrightarrow 0.$$

It follows from the exact hexagon above that

$$\widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) = 0$$

because the first cohomology group $\widehat{H}^1(G(k_m/\mathbb{Q}), \mathbb{Z}[\Delta])$ is trivial. From the assumption of $\mathrm{sgn}_l(\psi) \neq 0$, the absolute norm map $N_{\mathbb{Q}} = N_{k_m/\mathbb{Q}}$ induces an exact sequence,

$$0 \longrightarrow C^{\psi}_{s_m} \longrightarrow P^{\psi}_{s_m} \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the last map is $N_{\mathbb{Q}}$, and $\mathbb{Z}$ is the cyclic group generated by $N_{\mathbb{Q}}(\psi(\zeta_{s_m}))$. The exact sequence induces the exact hexagon.

$$\to \widehat{H}^0(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \to \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \to \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z}) \to$$
$$\leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), \mathbb{Z}) \leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \leftarrow \widehat{H}^1(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \hookleftarrow$$

From the remark above, the exact hexagon reduces to

$$0 \longrightarrow \widehat{H}^0(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \longrightarrow \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \xrightarrow{N_{\mathbb{Q}}} \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z}) \longrightarrow$$
$$\longrightarrow \widehat{H}^1(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \longrightarrow 0.$$

First, we show that the map $N_{\mathbb{Q}}$ is injective. Let $\alpha \in (P^{\psi}_{s_m})^{G(k_m/\mathbb{Q})}$. Suppose that α belongs to the kernel of $N_{\mathbb{Q}}$. Then there is an integer r such that

$$\alpha^{\phi(s_m)} = N_{\mathbb{Q}}(\alpha) = l^{n_l r\phi(s_m)} = N_{\mathbb{Q}}(\psi(\zeta_{s_m}))^{r\phi(s_m)}$$

where n_l is the integer with $v_l(N_{\mathbb{Q}}(\psi(\zeta_{s_m}))) = n_l$. Hence $\alpha = \pm N_{\mathbb{Q}}(\psi(\zeta_{s_m}))^r$ because both lie in the rational field and the roots of unity in the rational field are ± 1. Hence, the map is injective modulo ± 1. We can ignore this part because we will consider for the l-parts, an odd prime l. Hence, we have

$$\widehat{H}^0(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) = 0.$$

The exact hexagon reduces to the following short exact sequence,

$$0 \longrightarrow \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \xrightarrow{N_{\mathbb{Q}}} \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z}) \longrightarrow \widehat{H}^1(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \longrightarrow 0.$$

We now claim that $\widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m})$ is trivial, and hence $N_{\mathbb{Q}}$ is a trivial map. In order to confirm this, we first consider the short exact sequence above with ψ replaced by ξ. Then $n_l = 1$ and $\mathbb{Z}$ is generated by l, and $(P^{\xi}_{s_m})^{G(k_m/\mathbb{Q})} = P^{G(k_m/\mathbb{Q})}_{s_m} = l^{\mathbb{Z}}$, as $P^{G(k_m/\mathbb{Q})}_{s_m}$ are l-units in the rational field containing $l^{\mathbb{Z}}$. The absolute norm map $N_{\mathbb{Q}}$ is then a trivial map from $P^{G(k_m/\mathbb{Q})}_{s_m} = l^{\mathbb{Z}}$ to $\mathbb{Z}/\phi(s_m)\mathbb{Z} = \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z})$ in the short exact sequence above. Hence, the first cohomology

$\widehat{H}^0(G(k_m/\mathbb{Q}), P_{s_m})$ is a trivial group from the exact hexagon above. From this, we compute the first cohomology $\widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m})$ using the Herbrand quotient. Since $l \nmid \phi(s)$, $\mathcal{F}(s_m)/P^{\xi}(s_m) = \mathcal{F}(s_m)/P(s_m)$ is of order prime to l from Theorem A of [10] and hence $\mathcal{F}(s_m) \otimes \mathbb{Z}_l = P(s_m) \otimes \mathbb{Z}_l$. From this, a natural inclusion $P^{\psi}_{s_m}[l] \hookrightarrow P_{s_m}[l]$ follows. Then the cokernel is either infinite or finite. By applying the snake lemma to the exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{\psi}_{s_m}[l] & \longrightarrow & P^{\psi}_{s_m}[l] & \xrightarrow{N_{\mathbb{Q}}} & \mathbb{Z}_l & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle N_{\mathbb{Q}}(\psi(\zeta_{s_m}))} & & \\ 0 & \longrightarrow & C_{s_m}[l] & \longrightarrow & P_{s_m}[l] & \xrightarrow{N_{\mathbb{Q}}} & \mathbb{Z}_l & \longrightarrow & 0 \end{array}$$

we have that $P_{s_m}[l]/P^{\psi}_{s_m}[l]$ is finite if and only if $C_{s_m}[l]/C^{\psi}_{s_m}[l]$ is finite and the top absolute norm map $N_{\mathbb{Q}}$ is non trivial, i.e., $\mathrm{sgn}_l(\psi) \neq 0$. In this case the indices of circular numbers and units are given as follows,

$$(P_{s_m}[l] : P^{\psi}_{s_m}[l]) = l^{v_l(N_{\mathbb{Q}}(\psi(\zeta_{s_m})))}(C_{s_m}[l] : C^{\psi}_{s_m}[l]).$$

If it is infinite then the index $(C_{s_m}[l] : C^{\psi}_{s_m}[l])$ is infinite and hence we exclude this case. By applying the Herbrand quotient $h_{G(k_m/\mathbb{Q})}(-)$ to the short exact sequence

$$0 \longrightarrow P^{\psi}_{s_m}[l] \longrightarrow P_{s_m}[l] \longrightarrow P_{s_m}[l]/P^{\psi}_{s_m}[l] \longrightarrow 0,$$

we have

$$h_{G(k_m/\mathbb{Q})}(P_{s_m}[l]) = h_{G(k_m/\mathbb{Q})}(P^{\psi}_{s_m}[l])$$

since the quotient $P_{s_m}[l]/P^{\psi}_{s_m}[l]$ is finite. It follows from the triviality of the cohomologies $\widehat{H}^1(G(k_m/\mathbb{Q}), P^{\psi}_{s_m})$, $\widehat{H}^0(G(k_m/\mathbb{Q}), P_{s_m})$, $\widehat{H}^1(G(k_m/\mathbb{Q}), P_{s_m})$, and the equality $h_{G(k_m/\mathbb{Q})}(P_{s_m}) = h_{G(k_m/\mathbb{Q})}(P^{\psi}_{s_m})$ that

$$\widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}[l]) = 0.$$

Let $G_0 = G(k_m/k_0)$ be the l-Sylow subgroup of $G(k_m/\mathbb{Q})$ and $G_1 = G(k_m/\mathbb{Q})/G_0$. As G_0 is an l-group, we have $\widehat{H}^0(G_0, P^{\psi}_{s_m}) = \widehat{H}^0(G_0, P^{\psi}_{s_m}[l])$. It follows that $\widehat{H}^0(G_1, (P^{\psi}_{s_m})^{G_0}) = 0$ because the order of G_1 is prime to l and $\widehat{H}^0(G_1, (P^{\psi}_{s_m})^{G_0})$ is of order l-power. From the following equality,

$$1 = \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}[l]) = (P^{\psi}_{s_m}[l])^{G(k_m/\mathbb{Q})}/N_{k_m/\mathbb{Q}}P^{\psi}_{s_m}[l]$$

which is equal to $((P^{\psi}_{s_m}[l])^{G_1})^{G_0}/N_{G_0}(N_{G_1}P^{\psi}_{s_m}[l])$ and by applying the triviality of $\widehat{H}^0(G_1, P^{\psi}_{s_m}) = \widehat{H}^0(G_1, P^{\psi}_{s_m}[l])$, we have

$$1 = ((P^{\psi}_{s_m}[l])^{G_1})^{G_0}/N_{G_0}(P^{\psi}_{s_m}[l])^{G_1} = \widehat{H}^0(G_0, P^{\psi}_{s_m}[l]^{G_1}) = \widehat{H}^0(G_0, (P^{\psi}_{s_m})^{G_1}[l]).$$

Since G_0 is an l-group and $\widehat{H}^0(G_1, P^{\psi}_{s_m})$ is a trivial group, it reduces to

$$\widehat{H}^0(G_0, (P^{\psi}_{s_m})^{G_1}) = \widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}).$$

Hence we obtain

$$\widehat{H}^0(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) = 1.$$

It follows from the reduced exact hexagon that $\widehat{H}^0(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) = 0$, and

$$\widehat{H}^1(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \cong \widehat{H}^0(G(k_m/\mathbb{Q}), \mathbb{Z}) = \mathbb{Z}/\phi(s_m)\mathbb{Z}.$$

This completes the proof of Proposition 2.7. □

Notice that the structure of cohomology groups in Proposition 2.7 is not valid without the finiteness condition on $C_{s_m}[l]/C^{\psi}_{s_m}[l]$. Write $s(G(\mathbb{Q}(\mu_l)/\mathbb{Q}))$ for the sum in $\mathbb{Z}[G(\mathbb{Q}(\mu_l)/\mathbb{Q})]$ of elements of $G(\mathbb{Q}(\mu_l)/\mathbb{Q})$. Let $\psi(\zeta) = (1-\zeta)^{\tilde{N}_l}$, where $\tilde{N}_l$ is a fixed preimage $\pi^{-1}(s(G(\mathbb{Q}(\mu_l)/\mathbb{Q})))$ of $s(G(\mathbb{Q}(\mu_l)/\mathbb{Q}))$ under the natural projection map π from the Galois group $G(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$ of the maximal abelian extension $\mathbb{Q}(\mu_\infty)$ to the Galois group $G(\mathbb{Q}(\mu_l)/\mathbb{Q})$. In this case, $\widehat{H}^0(G(k_0/\mathbb{Q}), P^{\psi}_{s_0}) = \mathbb{Z}/(l-1)\mathbb{Z}$. Hence, in general, the cohomology groups vary depending on ψ. As l is an odd prime, it follows that for $i = 1, 2$ and any subgroup μ of μ_{s_m}, the cohomology group $\widehat{H}^i(G(k_m/\mathbb{Q}), \mu)$ is trivial. From this and the short exact sequences $0 \to \mu_{s_m} \bigcap C^{\psi}_{s_m} \to C^{\psi}_{s_m} \to \overline{C^{\psi}_{s_m}} \to 0$, and $0 \to \mu_{s_m} \bigcap P^{\psi}_{s_m} \to P^{\psi}_{s_m} \to \overline{P^{\psi}_{s_m}} \to 0$ induce the following isomorphisms

$$\widehat{H}^i(G(k_m/\mathbb{Q}), C^{\psi}_{s_m}) \cong \widehat{H}^i(G(k_m/\mathbb{Q}), \overline{C^{\psi}_{s_m}}), \widehat{H}^i(G(k_m/\mathbb{Q}), P^{\psi}_{s_m}) \cong \widehat{H}^i(G(k_m/\mathbb{Q}), \overline{P^{\psi}_{s_m}}).$$

Hence, Proposition 2.7 leads to the following corollary.

Corollary 2.8. *Under the assumptions of Proposition 2.7, we have*

$$\widehat{H}^0(G(k_m/\mathbb{Q}), \overline{C^{\psi}_{s_m}}) = 0, \quad \widehat{H}^1(G(k_m/\mathbb{Q}), \overline{C^{\psi}_{s_m}}) = \mathbb{Z}/\phi(s_m)\mathbb{Z}$$
$$\widehat{H}^0(G(k_m/\mathbb{Q}), \overline{P^{\psi}_{s_m}}) = 0, \quad \widehat{H}^1(G(k_m/\mathbb{Q}), \overline{P^{\psi}_{s_m}}) = 0.$$

In the next section, we will find certain conditions on $\mathrm{sgn}_l(\psi)$, h_m and characters where the finiteness of indices is satisfied. As a direct consequence of Propositions 2.4, 2.7, we infer that if the quotient $C_{s_m}[l]/C^{\psi}_{s_m}[l]$ is finite then $\overline{C}^{\psi}_{\infty}[l]$ is Λ-free. From the assumption of l and Corollary 2.8 we have that $(\overline{P^{\psi}_{s_m}}[l])^{G(k_m/k_n)} = N_{k_m/k_n}\overline{P^{\psi}_{s_m}}[l]$, which is equal to $\overline{P^{\psi}_{s_n}}[l]$. By Theorem 2.2 and Proposition 2.3, $\overline{P}^{\psi}_{\infty}[l]$ is a Λ-free module. We derive the following corollary, which asserts that $\overline{P}^{\psi}_{\infty}[l]$ and $\overline{C}^{\psi}_{\infty}[l]$ are Λ-free.

Corollary 2.9. *Under the assumptions of Proposition 2.7, we have that* $\overline{P}^{\psi}_{\infty}[l]$ *and* $\overline{C}^{\psi}_{\infty}[l]$ *are* Λ*-free.*

From this freeness result, we can outline the proof of Theorem 1.1.

Sketch of the proof of Theorem 1.1. With simplified notation, let $\overline{C}_{\infty}[l]$ be generated over R_∞ by ξ_∞, and $\overline{C}^{\psi}_{\infty}[l]$ by ψ_∞. Then there is a unique $\alpha \in R_\infty$ with $\psi_\infty = \xi_\infty \alpha$. By applying an even character χ plus the logarithmic embedding explained in the appendix to this we see that $\chi(\alpha)$ is exactly $t^{\psi}(\chi)/t(\chi)$. On the other hand, $\overline{C}_{\infty}[l]/\overline{C}^{\psi}_{\infty}[l]$ is R_∞-isomorphic to $R_\infty/\alpha R_\infty$. By codescent to finite level for each character χ, the result follows. □

In the appendix, we will give a detailed version of the proof by applying Sinnott's argument for its own interest. If $\psi = \xi$, then $\mathrm{sgn}_l(\psi) = 1$ and $h_m = 1$, and hence the corollary above tells us that $\overline{P}_{\infty}[l]$ and $\overline{C}_{\infty}[l]$ are Λ-free, which is known. Theorem A of [10] leads us to the exact sequence, $0 \to \overline{C^{\psi}_{s_n}}[l] \to \overline{C}_{s_n}[l] \to \overline{C}_{s_n}[l]/\overline{C^{\psi}_{s_n}}[l] \to 0$, and because the functor of the inverse limit is left exact we have the short exact sequence,

$$0 \longrightarrow \overline{C^{\psi}_{\infty}}[l] \longrightarrow \overline{C}_{\infty}[l] \longrightarrow \overline{C}_{\infty}[l]/\overline{C^{\psi}_{\infty}}[l] \longrightarrow 0.$$

We now suppose that the quotient $\overline{C}_\infty[l]/\overline{C^\psi_\infty}[l]$ is finite. For a finitely generated Λ-module A, the rank of A is the vector space dimension of $A \otimes_\Lambda Q(\Lambda)$ over $Q(\Lambda)$, where $Q(\Lambda)$ is the quotient field of Λ. We write $A_{\mathrm{tor}} := \{a \in A \mid \exists \lambda \in \Lambda, \lambda a = 0, \lambda \neq 0\}$. A is said to be a Λ-torsion free module if $A_{\mathrm{tor}} = 0$. We need the following proposition of Kuz'min.

Proposition 2.10 (=Proposition 1.1 of [6])**.** *Let A be a Λ-torsion free module of rank r. Then A is isomorphic to a submodule of finite index of some Λ-free module of rank r. This index is equal to one if and only if A is free.*

According to Corollary 2.9, Proposition 2.10 and the short exact sequence above show that when $\overline{C}_\infty[l]/\overline{C^\psi_\infty}[l]$ is finite $\overline{C}^\psi_\infty[l]$ must be equal to $\overline{C}_\infty[l]$, which is a Λ-free module.

Corollary 2.11. *Under the assumptions of Proposition 2.7, we have that $\overline{C}_\infty[l]/\overline{C^\psi_\infty}[l]$ is finite if and only if $\overline{C}^\psi_\infty[l] = \overline{C}_\infty[l]$.*

3. Appendix

In this appendix, we apply Sinnott's argument of [14] to our circular distributions. Let ψ be a circular distribution in $\mathcal{F}$. Let $\widetilde{P}^\psi_{s_n}$ be the subgroup of the multiplicative group $k_n^\times$ of k_n generated by the elements $\psi(\zeta)$ and μ_{s_n}, for $\zeta \in \mu_{s_n}, \zeta \neq 1$, and let $\widetilde{C}^\psi_{s_n} = \widetilde{P}^\psi_{s_n} \bigcap U_{s_n}$. Hence, the ψ-circular units $C^\psi_{s_n}$ of k_n satisfy $\widetilde{C}^\psi_{s_n} = C^\psi_{s_n}\mu_{s_n}$. Write θ_{s_n} for the cardinality $\#(\mu_{s_n}/\mu_{s_n} \bigcap C^\psi_{s_n})$. If $\psi(\zeta) = \xi(\zeta) = 1 - \zeta$, then $\widetilde{C}^\psi_{s_n} = C^\psi_{s_n}$. In this section, we will compute the index ψ-circular units of k_n inside the circular units of Sinnott. In order for the index to be finite we need to assume that $\psi(\zeta_{s_n}) \notin \mu_{s_n}$ and $h_n = 1$, as in the introduction. By Lemma 2.6, the former condition implies the latter condition for all sufficiently large numbers n. Let $U^+_{s_n}$ be the global units of the maximal subfield k_n^+ of k_n and let $(C^\psi_{s_n})^+ = \widetilde{C}^\psi_{s_n} \bigcap U^+_{s_n}$. By the assumption on l, we have that $U_{s_n} = U^+_{s_n}\mu_{s_n} = U^+_{s_n}\widetilde{C}^\psi_{s_n}$ and an isomorphism, $U_{s_n}/\widetilde{C}^\psi_{s_n} \cong U^+_{s_n}/(C^\psi_{s_n})^+$. Write G_{s_n} for the absolute Galois group $G(k_n/\mathbb{Q})$ of k_n and write $s(G_{s_n})$ for the sum in $\mathbb{Z}[G_{s_n}]$ of elements of G_{s_n}.

Lemma 3.1. *Let $\alpha = \psi(\zeta) \in \widetilde{P}^\psi_{s_n}$. Then $\alpha^{s(G_{s_n})} = 1$ if and only if $\alpha \in \widetilde{C}^\psi_{s_n}$, and $\alpha^{1+j} = 1$ if and only if $\alpha \in \mu_{s_n}$.*

Proof. If $\alpha \in \widetilde{C}^\psi_{s_n}$, then its absolute norm $\alpha^{s(G_{s_n})}$ is one because our k_n is imaginary. If $\alpha^{s(G_{s_n})} = 1$, then since $\widetilde{C}^\psi_{s_n}$ contains $\psi(\zeta)^{1-\sigma}$ for $\sigma \in G_{s_n}$, $1 = \alpha^{s(G_{s_n})} \equiv \alpha^{\#(G_{s_n})}$ modulo $\widetilde{C}^\psi_{s_n}$. This shows that α must be a unit and hence an element inside $\widetilde{C}^\psi_{s_n}$. If $\alpha^{1+j} = 1$, then $\alpha^{s(G_n)} = 1$ and hence α must be a unit whose conjugates have an absolute value of one. This shows that α is a root of unity. □

Following Sinnott, we now define the map t from $k_n^\times \to \mathbb{R}[G_{s_n}]$ to be

$$t(\alpha) = \sum_{\sigma \in G_{s_n}} -2^{-1} \log |\alpha^\sigma| \sigma^{-1}.$$

If $t(\alpha) = 0$, then $\alpha^{1+j} = 1$. By Lemma 3.1, α must be a root of unity in k_n. Hence, the map t induces an isomorphism $U_{s_n}/\widetilde{C}^\psi_{s_n} \cong t(U_{s_n})/t(\widetilde{C}^\psi_{s_n})$. For an R_{s_n}-module

A, write $A_{G_{s_n}}$ for the set of elements of A, which is annihilated by $s(G_{s_n})$. Using Lemma 3.1 and the argument of Lemma 4.2 of Sinnott in [14], we obtain

Lemma 3.2. *Let* $T^{\psi}_{s_n} = t(P^{\psi}_{s_n})$. *Then* $t(\widetilde{C}^{\psi}_{s_n}) = (T^{\psi}_{s_n})_{G_{s_n}}$.

Let $e_1 = |G_{s_n}|^{-1}\sum_{\sigma\in G_{s_n}}\sigma$ be the idempotent associated to the trivial character of G_{s_n}. We prove the following lemma, which will be used in the main theorem.

Lemma 3.3. $(T^{\psi}_{s_n})_{G_{s_n}} = T^{\psi}_{s_n}\bigcap(1-e_1)T^{\psi}_{s_n}$. *If g denotes the number of primes dividing s_n, then the index $((1-e_1)T^{\psi}_{s_n} : (T^{\psi}_{s_n})_{G_{s_n}})$ is finite and it is given by* $((1-e_1)T^{\psi}_{s_n} : (T^{\psi}_{s_n})_{G_{s_n}}) = 2^{-g}\phi(s_n)$.

Proof. As $s(G_{s_n})s(G_{s_n}) = \#(G_{s_n})s(G_{s_n})$, $(T^{\psi}_{s_n})_{G_{s_n}} = T^{\psi}_{s_n}\bigcap(1-e_1)T^{\psi}_{s_n}$ follows immediately from the definitions. $((1-e_1)T^{\psi}_{s_n}/(T^{\psi}_{s_n})_{G_{s_n}} \cong ((1-e_1)T^{\psi}_{s_n}/T^{\psi}_{s_n}\bigcap(1-e_1)T^{\psi}_{s_n})$, which is isomorphic to $T^{\psi}_{s_n} + (1-e_1)T^{\psi}_{s_n}/T^{\psi}_{s_n} \cong (T^{\psi}_{s_n} + e_1T^{\psi}_{s_n})/T^{\psi}_{s_n} \cong T^{\psi}_{s_n}/(T^{\psi}_{s_n}\bigcap e_1T^{\psi}_{s_n})$. As $T^{\psi}_{s_n}\bigcap e_1T^{\psi}_{s_n} \cong (T^{\psi}_{s_n})^{G_{s_n}}$, there results an isomorphism

$$((1-e_1)T^{\psi}_{s_n}/(T^{\psi}_{s_n})_{G_{s_n}} \cong e_1T^{\psi}_{s_n}/(T^{\psi}_{s_n})^{G_{s_n}}.$$

First, we compute the numerator,

$$e_1T^{\psi}_{s_n} = \phi(s_n)^{-1}s(G_{s_n})t(P^{\psi}_{s_n}) = \phi^{-1}(s_n)t((P^{\psi}_{s_n})^{s(G_{s_n})}).$$

We need the following lemma of Coleman. We include its proof for the reader's convenience.

Lemma 3.4 (Coleman)**.** *Let f be a circular distribution in $\mathcal{F}$. If s_n is divisible by at least two different primes, then $f(\zeta_{s_n})$ is a unit of $\mathbb{Q}(\mu_{s_n})$. If s_n is a prime power l^{n+1}, then $f(\zeta_{s_n})$ is an l-unit of $\mathbb{Q}(\mu_{s_n})$.*

Proof. By the norm coherent property of circular distribution f, we have the equality, $N_{\mathbb{Q}(\mu_{l^{n+r}})/\mathbb{Q}(\mu_{l^n})}f(\zeta_{l^{n+r}}) = f(\zeta_{l^n})$, for any $r \geq 0$. Let $\mathfrak{l}^u$ denote the prime ideal of $\mathbb{Q}(\mu_{l^u})$ lying over l. Let $\mathfrak{p}$ be a prime ideal dividing the principal ideal $(f(\zeta_{l^n}))$, which is prime to l. Let p be the rational prime, which is divisible by $\mathfrak{p}$. The decomposition group D_p of p must be $l^a\mathbb{Z}_l$, for some $a \geq 0$. The inertia group I_p of p is either zero or $l^b\mathbb{Z}_l$, for some $b \geq 0$. By the local class field theory, the inertia group I_p comes from the local units of the complete local field at p. Hence if I_p is $l^b\mathbb{Z}_l$, then we have a surjection from $\mathbb{Z}_p$ to $l^b\mathbb{Z}_l$ which is impossible. Hence, the inertia group I_p of p must be trivial. The decomposition D_p has an infinite residue class degree, which means $(f(\zeta_{l^n}))$ is infinitely divisible by the prime $\mathfrak{p}$. This contradiction implies that $f(\zeta_{l^n})$ is an l-unit. If s_n is divisible by at least two different primes l, v, then $f(\zeta_{s_n})$ is a unit outside both l and v. Hence, $f(\zeta_{s_n})$ must be a global unit. This completes the proof of Lemma 3.4. □

If $1 \neq \zeta \in \mu_{s_n}$ and the order of ζ is not a prime power, then we have $\psi(\zeta)^{s(G_{s_n})} = 1$ by Lemma 3.4. If the order of ζ is a prime power l^a with $a > 0$, then by Lemma 3.4 and under the assumption $\mathrm{sgn}_l(\psi) \neq 0$, there exists a positive rational $n_l(\psi)$ such that

$$\psi(\zeta)^{s(G_{s_n})} = l^{n_l(\psi)\phi(s_n)/\phi(l^a)}$$

and hence $(\widetilde{P}^{\psi}_{s_n})^{s(G_{s_n})} = \prod_{l|s_n} l^{\phi(s_n)/\phi(l^a)\mathbb{Z}}$. It follows that

$$e_1T^{\psi}_{s_n} = \phi(s_n)^{-1}t((\widetilde{P}^{\psi}_{s_n})^{s(G_{s_n})}) = \sum_{l|m}\frac{n_l(\psi)}{2\phi(l^{n+1})}\log l \cdot s(G_{s_n})\mathbb{Z}.$$

We now compute $(T_{s_n}^{\psi})^{G_{s_n}}$. For an element $\alpha \in \widetilde{P}_{s_n}^{\psi}$, $t(\alpha) \in (T_{s_n}^{\psi})^{G_{s_n}}$ if and only if $(\sigma-1)t(\alpha) = t(\alpha^{\sigma-1}) = 0$, for all $\sigma \in G_{s_n}$. As the kernel of t in $\widetilde{P}_{s_n}^{\psi}$ is the group of roots of unity, $t(\alpha) \in (T_{s_n}^{\psi})^{G_{s_n}}$ if and only if $1 = (\alpha^{\sigma-1})^{1+j} = (\alpha^{1+j})^{\sigma-1}$, for all $\sigma \in G_{s_n}$, if and only if $\alpha^{1+j} \in \mathbb{Q}^{\times}$. Let $(\widetilde{P}_{s_n}^{\psi})_1$ be the set of elements $\alpha \in \widetilde{P}_{s_n}^{\psi}$ satisfying $\alpha^{1+j} \in \mathbb{Q}^{\times}$. Then we have $t((\widetilde{P}_{s_n}^{\psi})_1) = (T_{s_n}^{\psi})^{G_{s_n}}$. Write $e^+ = (1+j)/2$. By applying e^+ we have $(T_{s_n}^{\psi})^{G_{s_n}} = 2^{-1}t((\widetilde{P}_{s_n}^{\psi})_1^{1+j})$. Let L be the imaginary quadratic field in $\mathbb{Q}(\mu_l)$. Write $\alpha_l = N_{\mathbb{Q}(\mu_l)/L}\psi(\zeta_l)$. The integer $n_l(\psi)$ defined above satisfies $\alpha_l^{1+j} = N_{\mathbb{Q}}\psi(\zeta_l) = l^{n_l(\psi)}$. Hence, α_l is contained in $(\widetilde{P}_{s_n}^{\psi})_1$ and $(\widetilde{P}_{s_n}^{\psi})_1^{1+j}$ contains the subgroup $H^{\psi} = \langle l^{n_l(\psi)}\rangle$ of $\mathbb{Q}^{\times}$ generated by the prime power $l^{n_l(\psi)}$. As stated in the introduction, for the index to be finite, it is assumed that $h_{s_n} = 1$. It follows that

$$\langle l^{n_l(\psi)}\rangle = H^{\psi} \subseteq (\widetilde{P}_{s_n}^{\psi})_1^{1+j} \subseteq (\widetilde{P}_{s_n}^{\psi})^{G_{s_n}}$$

and by Proposition 2.7, $(\widetilde{P}_{s_n}^{\psi})^{G_{s_n}}$ is contained in $N_{G_{s_n}}\widetilde{P}_{s_n}^{\psi} \subseteq \langle N_{G_s}\psi(\zeta_l)\rangle = \langle l^{n_l(\psi)}\rangle$. Hence, $H^{\psi} = (\widetilde{P}_{s_n}^{\psi})_1^{1+j}$ and $(T_{s_n}^{\psi})^{G_{s_n}} = 2^{-1}(2^{-1}n_l(\psi)\log l)\cdot s(G_{s_n})\mathbb{Z}$. By comparing the formulas $e_1 T_{s_n}^{\psi}$ and $(T_{s_n}^{\psi})^{G_{s_n}}$, we obtain the index formula of Lemma 3.3. □

Here, we define notions that will be used in the proof of Theorem 1.1. Write $u^{\psi}(r) = -2^{-1}\log|\psi(\zeta_r)|$, for $r \in \mathbb{Q}\backslash 1$. Then for each even nontrivial Dirichlet character χ associated with k, $u^{\psi}(\chi) = \sum_{(a,f)=1}\chi(a)u^{\psi}(a/f) = -2^{-1}\sum_{(a,f)=1}\chi(a)\log|\psi(\zeta_f^a)| = t^{\psi}(\chi)$. Let $\overline{\chi}$ be the complex conjugate of the primitive Dirichlet character associated with χ. Write $w^{\psi} = \sum_{\chi\neq 1}u^{\psi}(\overline{\chi})e_{\chi}$, with the sum taken over the nontrivial characters of G_{s_n}. Let e_{χ} denote the idempotent $|G_{s_n}|^{-1}\sum_{\sigma\in G_{s_n}}\chi(\sigma)\sigma^{-1}$ associated with χ. Let $\overline{\sigma}_l = \sum_{\chi}\overline{\chi}(l)e_{\chi}$, and let H_f denote the subgroup of G_{s_n} consisting of the elements σ_t, with $(t,m) = 1$ and $t \equiv 1 \bmod f$. By Proposition 2.1 of Sinnott in [14], $T_{s_n}^{\psi}$ satisfies $(1-e_1)T_{s_n}^{\psi} = w^{\psi}W$, where W denotes the $\mathbb{Z}[G_{s_n}]$-module generated in $\mathbb{C}[G_{s_n}]$ by the elements $s(H_f)\prod_{l|f}(1-\overline{\sigma}_l), 1 \le f \le s_n, f|s_n$. As in the introduction, let v_l denote the l-adic valuation defined as $v_l(l) = 1$ and let $|.|_l$ denote the l-adic absolute value normalized by $|l|_l = 1/l$. For a finite set S, $\#(S)$ denotes the cardinality of S. From $U_{s_n}/\widetilde{C}_{s_n}^{\psi} \cong U_{s_n}^+/(C_{s_n}^{\psi})^+ \cong t(U_{s_n})/t(\widetilde{C}_{s_n}^{\psi})$ and Theorem 4.1 of [14] and Lemma 3.3, we have that the index $(t(U_{s_n}) : t(\widetilde{C}_{s_n}^{\psi}))$ is equal to $(t(U_{s_n}) : e^+\mathbb{Z}[G_{s_n}]_{G_{s_n}})(e^+\mathbb{Z}[G_{s_n}]_{G_{s_n}} : e^+W_{G_{s_n}})(e^+W_{G_{s_n}} : (1-e_1)T_{s_n}^{\psi})((1-e_1)T_{s_n}^{\psi} : (T_{s_n}^{\psi})_{G_{s_n}})$. It is known that the first two indices are finite and independent of our ψ. The last index is Lemma 3.3, and hence it remains to compute the index $(e^+W_{G_n} : (1-e_1)T_{s_n}^{\psi})$. Note that $(1-e_1)T_{s_n}^{\psi} \subset X := (1-e_1)e^+\mathbb{R}[G_{s_n}]$. It follows from $(1-e_1)T_{s_n}^{\psi} = w^{\psi}W$ and $(1-e_1)e^+x = x$ for any $x \in X$ that $(1-e_1)T_{s_n}^{\psi} = w^{\psi}(1-e_1)T_{s_n}^{\psi}$. It also follows from *ibid* that $(1-e_1)W = W_{G_n}$. The linear transformation A^{ψ} on X induced by $A^{\psi}(x) = w^{\psi}x$ and hence $A^{\psi}(e^+W_{G_n}) = (1-e_1)T_{s_n}^{\psi}$. Hence, the index is finite if and only if $\det(A^{\psi}) = \prod_{\chi\neq 1}^{+}t^{\psi}(\chi) \neq 0$. In this case, the index is equal to $\det(A^{\psi}) = \prod_{\chi\neq 1}^{+}t^{\psi}(\chi)$. As in the introduction, $\mathcal{F}(s_n)$ denotes the group generated by $f(\zeta)$ for all f in $\mathcal{F}$ and all ζ in $\mu_{s_n}^*$. Then $\mathcal{F}(s_n)$ contains the circular numbers P_{s_n} of $\mathbb{Q}(\mu_{s_n})$. As $l \nmid \phi(s)$, the index $(\mathcal{F}(s_n) : P_{s_n})$ is finite by Theorem A of [10]. Hence, there is an integer q prime to $\phi(s)$ such that $\widetilde{C}^{\psi}(s_n)^q \subseteq P_{s_n}$. If $t^{\psi}(\chi) \neq 0$ for all $\chi \in \Xi$, then $\widetilde{C}^{\psi}(s_n)^q$ has also a finite index in the global units

U_{s_n} and hence $(\widetilde{C}^{\psi}(s_n)^q : P_{s_n})$ is finite. The formula above shows that we can compute the indices $(U'_{s_n} : \widetilde{C}^{\psi}(s_n)^q) = (U_{s_n} : (\widetilde{C}^{\psi}_{s_n})^q) = (t(U_{s_n}) : t((\widetilde{C}^{\psi}_{s_n})^q))$ and $(U'_{s_n} : P_{s_n}) = (U_{s_n} : C_{s_n}) = (t(U_{s_n}) : t(C_{s_n}))$. The l-primary part of $P_{s_n}/\widetilde{C}^{\psi}(s_n)^q$ is trivial as q is prime to l. Hence $(P_{s_n}[l] : \widetilde{C}^{\psi}(s_n)^q[l]) = (P_{s_n}[l] : \widetilde{C}^{\psi}(s_n)[l])$. We have the following indices,

$$(C_{s_n}[l] : \widetilde{C}^{\psi}_{s_n}[l]) = (t(C_{s_n}[l]) : t(\widetilde{C}^{\psi}_{s_n}[l])) = \frac{(t(U_{s_n}[l]) : t(\widetilde{C}^{\psi}_{s_n}[l]))}{(t(U_{s_n}[l]) : t(C_{s_n}[l]))} = \mid \frac{\det(A^{\psi})}{\det(A^{\xi})} \mid_l$$

which, by $\det(A^{\psi}) = \prod^{+}_{\chi\neq 1} t^{\psi}(\chi)$, is equal to,

$$\mid \frac{\prod_{\chi\in\Xi} t(\chi)}{\prod_{\chi\in\Xi} t^{\psi}(\chi)} \mid_l = \mid \prod_{\chi\in\Xi} t(\chi) t^{\psi}(\chi)^{-1} \mid_l .$$

From the isomorphism $\widetilde{C}^{\psi}_{s_n}/C^{\psi}_{s_n} = C^{\psi}_{s_n}\mu_{s_n}/C^{\psi}_{s_n} \cong \mu_{s_n}/\mu_{s_n}\bigcap C^{\psi}_{s_n}$, we prove the second index formula,

$$(C_{s_n}[l] : C^{\psi}_{s_n}[l]) = l^{v_l(\theta_{s_n})}(t(C_{s_n}[l]) : t(\widetilde{C}^{\psi}_{s_n}[l])) = l^{v_l(\theta_{s_n})}\frac{(t(U_{s_n}[l]) : t(\widetilde{C}^{\psi}_{s_n}[l]))}{(t(U_{s_n}[l]) : t(C_{s_n}[l]))}$$

where θ_{s_n} denotes the cardinality $\#(\mu_{s_n}/\mu_{s_n}\bigcap C^{\psi}_{s_n})$. Finally from the equality that $(P_{s_m}[l] : P^{\psi}_{s_m}[l]) = l^{v_l(N_{\mathbb{Q}}(\psi(\zeta_{s_m})))}(C_{s_m}[l] : C^{\psi}_{s_m}[l])$, we have

$$(P_{s_m}[l] : P^{\psi}_{s_m}[l]) = l^{v_l(N_{\mathbb{Q}}(\psi(\zeta_{s_m})))}|\theta_{s_n}\frac{\prod_{\chi\in\Xi} t(\chi)}{\prod_{\chi\in\Xi} t^{\psi}(\chi)}|_l$$

which proves the first index formula. This recovers the proof of Theorem 1.1. □

Remark. As we mentioned in the introduction, our results in this study are valid when the circular distributions are replaced by the truncated Euler systems of a certain depth. We briefly recall truncated Euler systems of a fixed depth r(cf. [13]). Let K be an abelian extension of F containing the Hilbert class field of F. Let $\mathrm{I}^r_{K/F}$ be the set of square free integral fractional ideals $\mathfrak{a}$ of F, such that each prime $\mathfrak{p}$ dividing $\mathfrak{a}$ has an absolute degree one and splits completely in K, and the number of primes dividing $\mathfrak{a}$ is less than or equal to r. For each prime ideal $\mathfrak{p}$ of F, let $F(\mathfrak{p})$ denote the ray class field of F modulo $\mathfrak{p}$, and for an integral fractional ideal $\mathfrak{a}$, let $F(\mathfrak{a})$ denote the composite field of $F(\mathfrak{p})$ for all prime divisors $\mathfrak{p}$ of $\mathfrak{a}$. Let $K(\mathfrak{a})$ be the composite of K and $F(\mathfrak{a})$. We define truncated Euler systems $\mathcal{E}^r_{K/F}$ of depth r to be the set of maps ψ from $\mathrm{I}^r_{K/F}$ to a fixed algebraic closure $\mathbb{Q}^{\mathrm{alg}}$, such that for each $\mathfrak{m}, \mathfrak{n} \in \mathrm{I}^r_{K/F}$ with $\mathfrak{n}|\mathfrak{m}$, $\psi(\mathfrak{m}) \in K(\mathfrak{m})$, $N_{K(\mathfrak{m})/K(\mathfrak{n})}\psi(\mathfrak{m})$ is equal to $\psi(\mathfrak{n})^{\prod_{\mathfrak{p}|\mathfrak{m},\mathfrak{p}\nmid\mathfrak{n}}(1-\mathrm{Frob}^{-1}_{\mathfrak{p}})}$, and $\psi(\mathfrak{n}\mathfrak{p})$ is congruent to $\psi(\mathfrak{n})$ modulo primes over $\mathfrak{p}$, whenever $\mathfrak{n}$ is prime to $\mathfrak{p}$. Write $\mathcal{E}^r_{\mathbb{Q}}$ for $\mathcal{E}^r_{K/F}$ when $F = K = \mathbb{Q}$. Let Cl_{s_m} denote the ideal class group of k_m and let $\mathrm{rank}_{\mathbb{Q}_l}\mathrm{Cl}_{s_m}[l]$ denote the $\mathbb{Q}_l$-rank of $\mathrm{Cl}_{s_m}[l] \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Based on these definitions, if $r \geq \mathrm{rank}_{\mathbb{Q}_l}\mathrm{Cl}_{s_m}[l]$, then all the results in this note are valid when the circular distributions are replaced by the truncated Euler systems $\mathcal{E}^r_{\mathbb{Q}}$ of depth r. This follows from the fact(cf. *ibid*) that under the assumption $r \geq \mathrm{rank}_{\mathbb{Q}_l}\mathrm{Cl}[\mathrm{l}]$ and $l \nmid \phi(s)$, we have the following equality,

$$\mathcal{E}^r_{\mathbb{Q}}(s_m) \otimes \mathbb{Z}_l = P_{s_m} \otimes \mathbb{Z}_l,$$

where $\mathcal{E}^r_{\mathbb{Q}}(s_m)$ denotes the set $\{\ \psi(\zeta) \mid \psi \in \mathcal{E}^r_{\mathbb{Q}},\ \zeta \in \mu_{s_m}\}$ of all special numbers coming from truncated Euler systems $\mathcal{E}^r_{\mathbb{Q}}$ of depth r.

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