PARTITIONS WEIGHTED BY THE PARITY OF THE CRANK

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Abstract. The 'crank' is a partition statistic which originally arose to give combinatorial interpretations for Ramanujan's famous partition congruences. In this paper, we establish an asymptotic formula and a family of Ramanujan type congruences satisfied by the number of partitions of n with even crank $M_e(n)$ minus the number of partitions of n with odd crank $M_o(n)$. We also discuss the combinatorial implications of q-series identities involving $M_e(n) - M_o(n)$. Finally, we determine the exact values of $M_e(n) - M_o(n)$ in the case of partitions into distinct parts. These values are at most two, and zero for infinitely many n.

1. Introduction

A partition λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Then λ_1 is the largest part and k is the number of parts of λ . In 1944, Dyson [15] defined the *rank of a partition* λ by

$$\text{rank}(\lambda) := \lambda_1 - k.$$

Let $p(n)$ denote the number of partitions of n. Dyson observed that the rank appeared to give combinatorial interpretations for Ramanujan's famous congruences $p(5n + 4) \equiv 0 \pmod 5$ and $p(7n + 5) \equiv 0 \pmod 7$ by splitting the partitions of $5n + 4$ (resp. $7n + 5$) into 5 (resp. 7) equal classes when sorted according to the rank modulo 5 (resp. 7). Dyson also conjectured the existence of another partition statistic, which he named the 'crank', that would similarly explain Ramanujan's third partition congruence $p(11n + 6) \equiv 0 \pmod{11}$. His observations on the rank were first proved by Atkin and Swinnerton-Dyer [7] in 1954 and the crank was found by Andrews and Garvan [5] in 1988. This Andrews-Garvan crank is defined by

$$\text{crank}(\lambda) := \begin{cases} \lambda_1, & \text{if } \mu(\lambda) = 0; \\ \nu(\lambda) - \mu(\lambda), & \text{if } \mu(\lambda) > 0, \end{cases}$$

where $\mu(\lambda)$ denotes the number of ones in λ and $\nu(\lambda)$ denotes the number of parts of λ that are strictly larger than $\mu(\lambda)$. In fact this crank turned out to give combinatorial interpretations for all three of Ramanujan's congruences.

Studying the number of partitions with even rank minus the number with odd rank has led to some rather intriguing mathematics. For example, if we let $N_e(n)$ (resp. $N_o(n)$) denote the

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number of partitions of n with even (resp. odd) rank, then we have

$$\sum_{n=0}^{\infty}(N_e(n)-N_o(n))q^n=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=:f(q), \tag{1.1}$$

where

$$(z;q)_n:=\prod_{j=0}^{n-1}(1-zq^j).$$

This q-series is one of Ramanujan's third order mock theta functions and has been the subject of a number of works (e.g. [2, 6, 9, 10, 14, 19, 23]). Most recently, Bringmann and Ono [9, 10] have shown that $f(q)$ belongs to the theory of *weak Maass forms*. This has many implications, including an infinitude of congruences for the number of partitions restricted by their ranks modulo t and the truth of a longstanding conjecture of Andrews and Dragonette giving an exact formula for $N_e(n)-N_o(n)$.

For another example, let $N_e(\mathcal{D},n)$ (resp. $N_o(\mathcal{D},n)$) denote the number of partitions into distinct parts with even (resp. odd) rank. Then

$$\sum_{n=0}^{\infty}(N_e(\mathcal{D},n)-N_o(\mathcal{D},n))q^n=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n}=:R(q).$$

In [4], Andrews, Dyson, and Hickerson showed that the coefficients of $R(q)$ have multiplicative properties determined by a certain Hecke character associated to the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{6})$, and Cohen [13] subsequently showed that $R(q)$ belongs to the theory of *Maass waveforms*. The main theorem of [4] gives an exact formula for $N_e(\mathcal{D},n)-N_o(\mathcal{D},n)$, which has numerous interesting consequences, such as the fact that $N_e(\mathcal{D},n)-N_o(\mathcal{D},n)$ is almost always zero and assumes every integer infinitely often.

In this paper we pass from the rank to the crank, studying the number of partitions with even crank minus the number with odd crank. In doing so, we leave the world of weak Maass forms and Maass waveforms for the world of classical modular forms. Specifically, if we define, for $n>1$, $M(m,n)$ to be the number of partitions of n with crank m, and define $M(0,1)=-1$, $M(-1,1)=M(1,1)=1$ and $M(m,1)=0$ otherwise, then the generating function for $M(m,n)$ is given in [5] by the infinite product

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m,n)a^mq^n=\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}, \tag{1.2}$$

where

$$(z;q)_\infty:=\lim_{n\to\infty}(z;q)_n.$$

Letting $M_e(n)$ (resp. $M_o(n)$) denote the number of partitions of n with even (resp. odd) crank, then by setting $a=-1$ in (1.2) we have

$$\sum_{n=0}^{\infty}(M_e(n)-M_o(n))q^n=\frac{(q;q)_\infty}{(-q;q)_\infty^2}:=g(q). \tag{1.3}$$

Apparently the only study of $g(q)$ was done by Andrews and Lewis [6], who proved that $M_e(n) > M_o(n)$ if n is even and $M_e(n) < M_o(n)$ if n is odd (by showing all the coefficients of $g(-q)$ are positive). The purpose of this paper is to give a thorough study on $M_e(n) - M_o(n)$ by examining several other aspects of $g(q)$ such as congruences, asymptotics, and q-series identities.

We begin with congruence properties satisfied by $M_e(n) - M_o(n)$. From work of Treneer [22], we know immediately that $M_e(n) - M_o(n)$ has infinitely many congruences in arithmetic progressions modulo any prime coprime to 6. The obvious question, then, is whether any of these congruences are as simple and elegant as those of Ramanujan for the partition function. Work of Garvan [16] implies that $M_e(5n+4) - M_o(5n+4) \equiv 0 \pmod 5$. This turns out to be the first case of an infinite family of congruences for the crank difference $M_e(n) - M_o(n)$ modulo powers of 5.

Theorem 1.1. *For all $\alpha \geq 0$, we have*

$$M_e(n) - M_o(n) \equiv 0 \pmod{5^{\alpha+1}} \quad \text{if} \quad 24n \equiv 1 \pmod{5^{2\alpha+1}}.$$

Following from the proof of Theorem 1.1 will be a nice generating function for $M_e(5n+4) - M_o(5n+4)$:

Theorem 1.2.

$$\sum_{n=0}^{\infty}(M_e(5n+4) - M_o(5n+4))q^n = \frac{5(q;q^2)_\infty^2(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2}.$$

Next we turn to the asymptotics of $g(q)$. We apply O. Chan's modification [11] of the Hardy-Ramanujan circle method to obtain an asymptotic formula for $M_e(n) - M_o(n)$:

Theorem 1.3. *If n is a positive integer, then*

$$M_e(n) - M_o(n) = \frac{1}{\sqrt{n-1/24}} \sum_{0<k<\sqrt{\pi(n-1/24)/2}} \frac{B_k(n)}{\sqrt{k}} \cosh\left(\frac{\pi}{k}\sqrt{\frac{n-1/24}{6}}\right) + E_n,$$

where

$$B_k(n) := \sum_{(h,2k)=1} e^{\pi i(2s(h,k)-3s(h,2k))} e^{-2\pi i n h/2k},$$

for the Dedekind sum $s(h,k)$, and where $|E_n| < 95(n-1/24)^{1/4}$.

To finish our study of the function $g(q)$, we discuss a weighted partition identity involving $M_e(n) - M_o(n)$. To state this we use the notion of the "initial run" of a partition, by which we mean the largest increasing sequence of part sizes starting with 1. For example, the partition $(7,7,5,3,3,3,3,2,1,1)$ has initial run $(1,2,3)$, while the partition $(6,6,5,2,2,2,2)$ has no initial run at all.

Theorem 1.4. *For a nonempty partition λ, define the weight $\omega(\lambda)$ to be*

$$\omega(\lambda) := 1 + 4\sum_j (-1)^j,$$

where the sum is over those j in the initial run which occur an odd number of times in λ. Then

$$M_e(n) - M_o(n) = \sum_\lambda \omega(\lambda),$$

where the sum is over all partitions λ of n.

For example, take $n = 4$. The partition 4 has weight 1, $(3,1)$ has weight -3, $(2,2)$ has weight 1, $(2,1,1)$ has weight 5, and $(1,1,1,1)$ has weight 1. Summing these weights gives $1-3+1+5+1=5$, which is, as expected, $M_e(4)-M_o(4)$.

Finally, inspired by the work of Andrews, Dyson and Hickerson in [4], we look at what happens if we restrict our crank difference to partitions into distinct parts. In this case, the definition of the crank simplifies considerably, and we are able to use basic manipulations of q-series to prove an exact formula, which is an analogue to Euler's pentagonal number theorem.

Let $M_e(\mathcal{D},n)$ (resp. $M_o(\mathcal{D},n)$) denote the number of partitions into distinct parts with even (resp. odd) crank. For partitions into distinct parts the crank is either the largest part, if there is no one appearing, or the number of parts minus 2 if there is a one. Let $\mathcal{P}$ denote the set of pentagonal numbers, i.e., numbers of the form $m(3m+1)/2$ for m an integer. If $n = m(3m+1)/2$, we write $R(n) = m$. Finally, we use the notations $\lfloor n\rfloor_p$ and $\lceil n\rceil_p$, to denote the pentagonal floor and ceiling of n, i.e., the largest (resp. smallest) pentagonal number $\leq$ (resp. $\geq$) n.

Theorem 1.5. *For positive integers n we have*

$$M_e(\mathcal{D},n)-M_o(\mathcal{D},n) = \begin{cases} 1, & \text{if } n\in\mathcal{P} \text{ and } R(n) \text{ is odd and positive},\\ -1, & \text{if } n\in\mathcal{P} \text{ and } R(n) \text{ is not as above},\\ 2, & \text{if } n\notin\mathcal{P},\ R(\lfloor n\rfloor_p) \text{ is odd and positive, and } n\equiv\lfloor n\rfloor_p \pmod 2,\\ -2, & \text{if } n\notin\mathcal{P},\ R(\lfloor n\rfloor_p) \text{ is even and positive, and } n\equiv\lfloor n\rfloor_p \pmod 2,\\ -2(-1)^{n-\lfloor n\rfloor_p}, & \text{if } n\notin\mathcal{P} \text{ and } R(\lfloor n\rfloor_p) \text{ is even and negative},\\ 0, & \text{otherwise}. \end{cases}$$

Corollary 1.6. *The quantities $M_e(\mathcal{D},n)$ and $M_o(\mathcal{D},n)$ differ by at most 2 and are equal for infinitely many n.*

To illustrate the above theorem, take $n = 6$. Then n is not a pentagonal number and $R(\lfloor 6\rfloor_p) = R(5) = -2$. Hence by the penultimate case we expect $M_e(\mathcal{D},6)-M_o(\mathcal{D},6)$ to be $-2(-1)^{6-5} = 2$. Indeed, the four partitions of 6 into distinct parts are (6), $(5,1)$, $(4,2)$, and $(3,2,1)$, each of which has even crank except for $(3,2,1)$.

The paper is organized as follows. In the next section, we prove the family of congruences in Theorem 1.1 and the generating function in Theorem 1.2. In Section 3, we establish the asymptotic formula in Theorem 1.3. In Section 4, we prove the weighted identities in Theorem 1.4 and discuss similar identities. Finally, in Section 5 we treat the exact formula in Theorem 1.5.

2. Proof of Theorems 1.1 and 1.2

Here we follow the exposition in Gordon and Hughes' [17] rediscovery of some congruences of Rødseth [21]. The reader should have some familiarity with the preliminaries in [17]. Define

$$F(\tau) := \frac{\eta(\tau)^3}{\eta(2\tau)^2}\frac{\eta(50\tau)^2}{\eta(25\tau)^3}.$$

Here $\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n)$ and $q := e^{2\pi i\tau}$. Applying [17, Theorem 2, Theorem 3], we have $F(\tau) \in M_0^!(\Gamma_0(50))$, the space of weight 0 weakly holomorphic modular functions (functions that may have poles at the cusps) on $\Gamma_0(50)$. Then $F(\tau)$ is holomorphic on the upper half plane $\mathbb{H}$ and its orders at the cusps ν/δ are as given below by [17, Theorem 3].

δ	1	2	5	10	25	50
$ord_{\nu/\delta}F$	4	-1	0	0	-4	1

Next, recall the U_d-operator, which acts on power series by

$$\sum_{n=-n_0}^{\infty} a(n)q^n \big| U_d := \sum_{n=-n_0}^{\infty} a(dn)q^n$$

for an integer n_0. Note that if f and g are power series in q, we have

$$\left(f(q^d)g(q)\right) \big| U_d = f(q) \left(g(q)\big| U_d\right).$$

By [17, Theorem 5] we have that $F(\tau)|U_5 \in M_0(\Gamma_0(10))$, with the following lower bounds for the orders of $F|U_5$ at the cusps.

δ	1	2	5	10
$ord_{\nu/\delta}F\vert U_5 \geq$	0	-1	0	1

Now consider the function

$$G(\tau) := \frac{\eta(\tau)^2}{\eta(2\tau)^4}\frac{\eta(10\tau)^4}{\eta(5\tau)^2}.$$

Applying [17, Theorem 2, Theorem 3], we find that $G(\tau) \in M_0^!(\Gamma_0(10))$ and that its orders at the cusps are as follows.

δ	1	2	5	10
$ord_{\nu/\delta}G$	0	-1	0	1

Since the only holomorphic modular functions of weight 0 are the constant functions, comparing the last two tables and the Fourier series expansions of $F|U_5$ and G gives $F|U_5 = 5G$. Now, if we consider $G(\tau)$ as a function in $M_0^!(\Gamma_0(50))$, rather than the subfield $M_0^!(\Gamma_0(10))$, we find from [17, Theorem 3] that its orders at the cusps are as follows.

δ	1	2	5	10	25	50
$ord_{\nu/\delta}G$	0	-5	0	1	0	1

Hence by [17, Theorem 5] applied to G^i and FG^i, $G^i|U_5$ and $FG^i|U_5$ are on $\Gamma_0(10)$ and we have the following lower bounds for the orders of these functions at the cusps.

δ	1	2	5	10
$ord_{\nu/\delta}G^i\vert U_5 \geq$	0	$-5i$	0	$i/5$
$ord_{\nu/\delta}FG^i\vert U_5 \geq$	0	$-5i-1$	0	$(i+1)/5$

If $i \geq 0$, this implies that $G^i|U_5$ and $FG^i|U_5$ are polynomials in G of degrees at most $5i$ and $5i+1$ respectively. Hence

$$G^i|U_5 = \sum_{j\geq 0} a_{ij}G^j \quad \text{and} \quad FG^i|U_5 = \sum_{j\geq 0} b_{ij}G^j \tag{2.1}$$

for complex coefficients a_{ij} and b_{ij}.

Let S be the vector space of all polynomials $P = \sum_{j\geq 0} c_jG^j$ and T be the subspace of such polynomials with 0 as constant terms. By considering our lower bounds for the orders of $G^i|U_5$ and $FG^i|U_5$ and (2.1), we see that U_5 maps S to itself as well as T to itself. In addition, the linear transformation $V : P \to (FP)|U_5$ maps S into T. With respect to the basis G, G^2, G^3, $\dots$ of T the matrices of U_5 and V restricted to T are respectively

$$U_5 = (A := (a_{ij})) \quad \text{and} \quad V = (B := (b_{ij})),$$

for $1 \leq i, j < \infty$.

If we define a sequence of functions L_ν ($\nu \geq 0$) inductively by putting for $\alpha \geq 0$,

$$L_0 = 1,\ L_{2\alpha+1} = FL_{2\alpha}|U_5,\ \text{and}\ L_{2\alpha+2} = L_{2\alpha+1}|U_5, \tag{2.2}$$

then

$$L_1 = 5G = (5, 0, 0, \dots),$$
$$L_{2\alpha+1} = (5, 0, 0, \dots)(AB)^\alpha,$$

and

$$L_{2\alpha+2} = (5, 0, 0, \dots)(AB)^\alpha A,$$

where the matrices A and B act on the right. On the other hand, it follows from induction on α that

$$L_{2\alpha+1} = \frac{(q^{10}; q^{10})_\infty^2}{(q^5; q^5)_\infty^3} \sum_{n=1}^{\infty} (M_e(m) - M_o(m))q^n, \tag{2.3}$$

where $m = 5^{2\alpha+1}n - 1 - 5^2 - \cdots - 5^{2\alpha}$. Theorem 1.1 will then follow from the following theorem.

Theorem 2.1. *If we set $L_{2\alpha+1} = (l_1(2\alpha+1), l_2(2\alpha+1), \dots)$, then $l_i(2\alpha+1)$ are integers divisible by $5^{\alpha+1}$.*

Proof. Following Gordon and Hughes' approach, we would first show that if $\pi(n)$ denotes the 5-adic order of n, then for all i, j, we have $a_{ij},\ b_{ij} \in \mathbb{Z}$ and

$$\pi(a_{ij}) \geq [\frac{5j - i - 1}{6}],$$

and

$$\pi(b_{ij}) \geq [\frac{5j - i - 1}{6}] \quad \text{and} \quad \pi(b_{ij}) \geq 1 \quad \text{if} \quad i \equiv 1 \pmod 5.$$

But as the proofs for these results are almost identical with those for [17, Lemma 7 and Lemma 9] except that the key function we use for our case is

$$\phi(\tau) := \frac{\eta(\tau)}{\eta(2\tau)^2} \frac{\eta(50\tau)^2}{\eta(25\tau)} \in M_0^!(\Gamma_0(50)),$$

we omit the proofs. Using the fact that $F|U_5 = 5G$ together with the lower bounds for $\pi(a_{ij})$ and $\pi(b_{ij})$ above as in the proof of [17, Theorem 10], we obtain

$$\pi(l_j(2\alpha + 1)) \geq \alpha + 1 + [\frac{j - 1}{2}],$$

$$\pi(l_j(2\alpha + 2)) \geq \alpha + 1 + [\frac{j}{2}]$$

for all $\alpha \geq 0$ and $j \geq 1$. This implies that $l_j(2\alpha + 1) \equiv 0 \pmod{5^{\alpha+1}}$. This then completes the proofs of both Theorems 2.1 and 1.1. $\square$

For Theorem 1.2, recall that we have shown that $F|U_5 = 5G$. Hence we have

$$\begin{aligned}
\frac{5\eta^3(\tau)\eta(5\tau)\eta^2(10\tau)}{\eta^2(2\tau)} &= \frac{\eta^3(\tau)\eta(5\tau)\eta^2(10\tau)}{\eta^2(2\tau)}\Big|U_5 \\
&= \left((q^5;q^5)_\infty(q^{10};q^{10})_\infty^2\sum_{n=0}^{\infty}\left(M_e(n)-M_o(n)\right)q^{n+1}\right)\Big|U_5 \\
&= (q)_\infty(q^2;q^2)_\infty^2\sum_{n=0}^{\infty}\left(M_e(5n+4)-M_o(5n+4)\right)q^{n+1}.
\end{aligned}$$

Multiplying the first and last terms in the above string of equations by

$$\frac{1}{q\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n})^2}$$

gives Theorem 1.2. □

3. Proof of Theorem 1.3

As Chan's method in [11] can be applied to find asymptotics for general eta-quotients, we only summarize the computation and refer to his results whenever it is necessary. By Cauchy's integral formula, for a circle C of radius $r = e^{-2\pi\rho} = e^{-2\pi/N^2}$ for a positive N to be determined, we have

$$\begin{aligned}
M_e(n)-M_o(n) &= \frac{1}{2\pi i}\int_C g(q)q^{-n-1}dq \\
&= \int_0^1 g\left(\exp(-2\pi\rho+2\pi i\theta)\right)e^{2\pi n\rho-2\pi in\theta}d\theta \\
&= \sum_{\substack{0\le h\le k\le N\\ (h,k)=1}} e^{-\frac{2\pi inh}{k}}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} g\left(\exp\left\{\frac{2\pi ih}{k}-2\pi(\rho-i\phi)\right\}\right)e^{2\pi n(\rho-i\phi)}d\phi, \qquad (3.1)
\end{aligned}$$

where the last equality follows from Andrews' dissection of the circle of integration in [3, Ch. 5]. As each of $\theta'_{h,k}$ and $\theta''_{h,k}$ is the mediant of the Farey number h/k and the adjacent Farey numbers, they satisfy $1/(2kN) \le \theta \le 1/(kN)$.

If we set $z = k(\rho - i\phi)$ and $\tau = (h+iz)/k$ in (3.1), we obtain

$$M_e(n)-M_o(n) = \sum_{\substack{0\le h\le k\le N\\ (h,k)=1}} e^{-\frac{2\pi inh}{k}}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} g(e^{2\pi i\tau})e^{2\pi n(\rho-i\phi)}d\phi. \qquad (3.2)$$

Recall from [11, (3.13)] that if $H(q) := 1/(q;q)_\infty$ and $(n,k)=1$, then

$$H(e^{2\pi in\tau}) = e^{\frac{\pi}{12k}\left(\frac{1}{nz}-nz\right)}e^{\pi is(h_n,k)}\sqrt{nz}\,H(e^{2\pi i\gamma_{(n,k)}(n\tau)}), \qquad (3.3)$$

where $k>0$, $(h,k)=1$, $h_n = nh$ and $\gamma_{(n,k)} := \begin{pmatrix} * & * \\ k & -h_n \end{pmatrix} \in SL_2(\mathbb{Z})$. Since

$$g(q) = (q;q)_\infty(q;q^2)_\infty^2 = \frac{(q;q)_\infty^3}{(q^2;q^2)_\infty^2} = \frac{H(q^2)^2}{H(q)^3}, \qquad (3.4)$$

it follows from (3.3) that for k odd,

$$g(e^{2\pi i\tau}) = 2e^{\frac{\pi}{12k}(-\frac{2}{z}-z)}e^{\pi i(2s(2h,k)-3s(h,k))}z^{-1/2}\frac{H^2(e^{2\pi i\gamma_{(2,k)}(2\tau)})}{H^3(e^{2\pi i\gamma_{(1,k)}(\tau)})}, \tag{3.5}$$

and for $k = 2\ell$,

$$g(e^{2\pi i\tau}) = e^{\frac{\pi}{12(2\ell)}(\frac{1}{z}-z)}e^{\pi i(2s(h,\ell)-3s(h,2\ell))}z^{-1/2}\frac{H^2(e^{2\pi i\gamma_{(1,\ell)}(2\tau)})}{H^3(e^{2\pi i\gamma_{(1,2\ell)}(\tau)})}. \tag{3.6}$$

Now we set

$$A_{h,k} := e^{-\frac{2\pi inh}{k}}g(e^{2\pi i\tau})e^{2\pi n(\rho-i\phi)} \tag{3.7}$$

and write

$$M_e(n) - M_o(n) = \sum_{\substack{k\ \text{odd}\\ 0\le h\le k\le N\\ (h,k)=1}}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} A_{h,k}d\phi + \sum_{\substack{k\ \text{even}\\ 0\le h\le k\le N\\ (h,k)=1}}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} A_{h,k}d\phi := \Sigma_1 + \Sigma_2. \tag{3.8}$$

Applying the value of $\mathrm{Im}(\gamma((n,k)(n\tau))$ and the bounds on z , $\mathrm{Re}(z)$, and $H(e^{2\pi i\tau})$ computed in [11, Section 3] into (3.5), we find that, for odd k,

$$|A_{h,k}| \le 2\exp\left(\frac{2e^{-\pi/2}}{(1-e^{-\pi/2})^2}\right)\exp\left(\frac{3e^{-\pi}}{(1-e^{-\pi})^2}\right)e^{-\pi/12}e^{2\pi(n-1/24)\rho}k^{-1/2}N.$$

This implies that

$$|\Sigma_1| \le 3.4404\sum_{\substack{k\ \text{odd}\\ 1\le k\le N}}\sum_{1\le h\le k}\int_{-\theta'_{h,k}}^{\theta''_{h,k}} e^{2\pi(n-1/24)\rho}k^{-1/2}Nd\phi.$$

As the length of the integral is at most $2/kN$, we obtain

$$|\Sigma_1| \le 3.4404e^{2\pi(n-1/24)\rho}\sum_{1\le k\le N}2k^{-1/2} \le 13.7616e^{2\pi(n-1/24)\rho}N^{1/2}, \tag{3.9}$$

where the last inequality follows from $\sum_{k=1}^{N}k^{-1/2} \le 2N^{1/2}$.

For Σ_2, we utilize (3.6) with $\tau' = \gamma_{(1,2\ell)}(\tau)$ and $\omega_{h,\ell} = e^{\pi i(2s(h,\ell)-3s(h,2\ell))}$ so that

$$\begin{aligned}\Sigma_2 \quad &= \sum_{1\le 2\ell\le N}\sum_{\substack{0\le h\le 2\ell\\ (h,2\ell)=1}} e^{-2\pi inh/2\ell}\omega_{h,\ell}\left[\int_{-\theta'_{h,2\ell}}^{\theta''_{h,2\ell}} e^{\frac{\pi}{12(2\ell)}(\frac{1}{z}-z)}z^{-1/2}e^{2\pi n\rho}e^{-2\pi in\phi}\Psi(e^{2\pi i\tau'})d\phi + I_{h,2\ell}\right]\\ &:= \Sigma_{21} + \Sigma_{22},\end{aligned} \tag{3.10}$$

where

$$\Psi(e^{2\pi i\tau'}) := \frac{H^2(e^{2\pi i(2\tau')})}{H^3(e^{2\pi i(\tau')})} - 1 \quad \text{and} \quad I_{h,k} := \int_{-\theta'_{h,k}}^{\theta''_{h,k}} e^{\frac{\pi}{12k}(\frac{1}{z}-z)}z^{-1/2}e^{2\pi n\rho}e^{-2\pi in\phi}d\phi. \tag{3.11}$$

Since $|M_e(n)-M_o(n)|\le p(n)$,

$$|g(x)-1|\le |H(x)-1|,\ \text{and hence}\ \left|\frac{\Psi(x)}{x^{1/24}}\right|\le \frac{H(|x|)-1}{|x^{1/24}|}.$$

According to the computation of $\dfrac{H(|x|)-1}{|x^{1/24}|}$ in [11, p. 126],

$$\left|e^{\frac{\pi}{12k}\left(\frac{1}{z}\right)}\Psi(e^{2\pi i\tau'})\right|\le 0.0551.$$

Thus

$$\begin{aligned} |\Sigma_{21}| &< 0.0551\sum_{1\le 2\ell\le N}\sum_{\substack{0\le h\le 2\ell\\(h,2\ell)=1}}\int_{-\theta'_{h,2\ell}}^{\theta''_{h,2\ell}} e^{-\frac{\pi}{12(2\ell)}\mathrm{Re}(z)}|z|^{-1/2}e^{2\pi n\rho}d\phi \\ &\le 0.2204e^{2\pi(n-1/24)\rho}N^{1/2}, \end{aligned}\tag{3.12}$$

again by the bounds on z and $\mathrm{Re}(z)$ in [11].

Lastly, it follows from [11, Lemma 3.2] that

$$\Sigma_{22}=\sum_{1\le 2\ell\le N}B_\ell(n)\sqrt{\frac{2}{2\ell(n-1/24)}}\cosh\left(\frac{\pi}{2\ell}\sqrt{\frac{2}{3}(n-1/24)}\right)+\sum_{1\le 2\ell\le N}\sum_{\substack{0\le h\le 2\ell\\(h,2\ell)=1}}E(I),\tag{3.13}$$

where $B_\ell(n)$ is as defined in Theorem 1.3 and

$$|E(I)|\le 1.2828\frac{e^{2\pi(n-1/24)\rho}N^{1/2}}{n-1/24}.\tag{3.14}$$

Therefore, by (3.8), (3.9), (3.10), (3.12), (3.13) and (3.14), we obtain

$$M_e(n)-M_o(n)=\sum_{1\le 2\ell\le N}B_\ell(n)\sqrt{\frac{2}{2\ell(n-1/24)}}\cosh\left(\frac{\pi}{2\ell}\sqrt{\frac{2}{3}(n-1/24)}\right)+E_n,$$

with

$$|E_n|\le e^{2\pi(n-1/24)\rho}N^{1/2}(13.7616+0.2204+1.2828\frac{N}{n-1/24}).$$

We choose the value $N=\sqrt{2\pi(n-1/24)}$ for the minimization of the error and complete the proof of Theorem 1.3.

4. Weighted identities

In this section we prove Theorem 1.4 and discuss another weighted identity of similar type.

Proof of Theorem 1.4. Using the identity [8]

$$g(q)=\frac{1}{(q;q)_\infty}\left(1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n}\right),\tag{4.1}$$

we have that

$$\sum_{n\geq 0}(M_e(n)-M_o(n))q^n = \frac{1}{(q;q)_\infty} + 4\sum_{n\geq 1}\frac{(-1)^n q^{n(n+1)/2}}{(q;q)_{n-1}(1-q^{2n})(q^{n+1};q)_\infty}. \tag{4.2}$$

We shall interpret the right hand side of the identity as the weighted count of partitions in Theorem 1.4. The term $1/(q)_\infty$ initializes the weight of each partition λ to 1. If there are no ones in λ, then it is not counted at all by the sum on the right hand side, and so the weight just remains 1.

Otherwise, we look at the "initial run" of λ. A partition λ will be counted by the sum on the right hand side for each j in the initial run that occurs an odd number of times. For each such j, we add $4(-1)^j$ to the weight. (Those that occur an even number of times contribute nothing.) Then, summing over all partitions of n, each counted according to its weight, gives $M_e(n) - M_o(n)$. □

As an application, we give a combinatorial proof of the following q-series identity:

Corollary 4.1.

$$\frac{1}{(q;q)_\infty} + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q;q)_{n-1}(1-q^{2n})(q^{n+1};q)_\infty} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(1-q^{n+1})}{(q;q)_n(1+q^{n+1})(q^{n+2};q)_\infty}. \tag{4.3}$$

Of course, this identity may also be established analytically. For example, take $a = b = 0$, $c = d = q$, and $z = -1$ in the following identity of S. H. Chan [12, Eq. (3.1)], valid for $|a/c| < 1$ and $|bq/d| < 1$,

$$\frac{(az, b/z, q)_\infty}{(cz, d/z)_\infty} = \frac{(a/c, bc)_\infty}{(cd)_\infty}\sum_{n=0}^{\infty}\frac{(cq/a, cd)_n (a/c)^n}{(q, bc)_n(1-czq^n)} + \frac{d(ad, b/d)_\infty}{(cd)_\infty}\sum_{n=0}^{\infty}\frac{(dq/b, cd)_n (bq/d)^n}{(q, ad)_n(z-dq^n)},$$

where $(z_1, z_2, \ldots, z_k)_n = \prod_{i=1}^{k}(z_i;q)_n$. The result shows that the right hand side of (4.3) is indeed the generating function for $M_e(n) - M_o(n)$. This with (4.2) implies Corollary 4.1.

Proof. We define a second weight, ω_1 by

$$\omega_1(\lambda) := (-1)^{\text{length of initial run}} - 2\sum_j (-1)^j(-1)^{\#\text{ occurrences of } j},$$

where the sum is over those j in the initial run of λ.

We shall argue by induction on the length of the initial run that for any partition λ, we have $\omega(\lambda) = \omega_1(\lambda)$. First, if the initial run is empty, then $\omega(\lambda) = 1 = \omega_1(\lambda)$. Now, suppose λ has an initial run of length $n+1$. Let $\lambda - \overline{n+1}$ denote the partition λ with all of the parts of size $n+1$ removed. Then the induction hypothesis says that $\omega(\lambda - \overline{n+1}) = \omega_1(\lambda - \overline{n+1})$.

Let us now determine the effect on the weights of adding back in the parts of size $n+1$. First, consider the effect on ω. If $n+1$ occurs an even number of times, the weight is unchanged. If it occurs an odd number of times, then 4 is added to the weight if n is odd and -4 is added if n is even.

Now, consider the effect on ω_1. If n is even, the length of the initial run changes from even to odd, giving us a -2. If $n+1$ occurs an even number of times, then we get a $+2$ and the net change is 0. If $n+1$ occurs an odd number of times, we get a -2 and the net change is -4. This matches the change to ω in these cases. A similar argument in the case where n is odd shows that the changes to ω and ω_1 are always the same. Hence, we have $\omega(\lambda) = \omega_1(\lambda)$ for all partitions λ.

To complete the proof, it suffices to argue as in the proof of Theorem 1.4 that the right hand side of (4.3) is the generating function for partitions λ counted with weight $\omega_1(\lambda)$. The details are very similar to the case of (4.2), so we omit them. $\square$

Before continuing, we note that similar weighted identities can be found for $M_e(n)$ and $M_o(n)$ separately as well as for the rank difference $f(q)$. These rely on the equations

$$\sum M_e(n)q^n = \frac{1}{(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n}\right)$$

and

$$\sum M_o(n)q^n = \frac{2}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1+q^n},$$

easily deduced from (4.1), and Watson's equation [23]

$$f(q) = \frac{1}{(q;q)_\infty}\left(1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}\right). \tag{4.4}$$

5. Proof of Theorem 1.5

In this section we prove Theorem 1.5. We begin by deducing a key generating function for $M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n)$. It is easily seen using the definition of the crank that

$$\sum_{n=1}^{\infty}(M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n))q^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+3)/2}}{(-q;q)_n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_{n-1}}.$$

But this generating function does not tell us much about $M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n)$. However, we can prove a more useful form.

Theorem 5.1.

$$\sum_{n=1}^{\infty}(M_e(\mathcal{D}, n) - M_o(\mathcal{D}, n))q^n = \frac{1}{1+q}\sum_{n=1}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) - q(q^2;q)_\infty. \tag{5.1}$$

Proof. We shall demonstrate that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+3)/2}}{(-q;q)_n} = \frac{1}{1+q}\sum_{n=1}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}) \tag{5.2}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_{n-1}} = -q(q^2;q)_\infty. \tag{5.3}$$

Shifting the summation variable by 1 on both sides of (5.2), we obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+5)/2}}{(-q^2;q)_n} = \sum_{n=0}^{\infty} q^{(3n^2+7n)/2}(1-q^{2n+3}). \tag{5.4}$$

But this is precisely the specialization $(a,b,c,d,e) \to (q^3, -q^2, \infty, \infty, q)$ of the following limiting case of the Watson-Whipple transformation:

(5.5)
$$\sum_{n=0}^{\infty} \frac{(aq/bc,d,e)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c)_n} = \frac{(aq/d,aq/e)_\infty}{(aq,aq/de)_\infty} \sum_{n=0}^{\infty} \frac{(a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e)_n(aq)^{2n}(-1)^n q^{n(n-1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e)_n(bcde)^n}.$$

For (5.3), shifting the summation variable as before leaves us with the task of proving that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q;q)_n} = (q^2;q)_\infty. \tag{5.6}$$

This follows immediately from the case $z = -q$ of

$$\sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)/2}}{(q;q)_n} = (-zq;q)_\infty. \tag{5.7}$$

This completes the proof Theorem 5.1. □

We may now use this theorem to deduce the formula for $M_e(\mathcal{D},n) - M_o(\mathcal{D},n)$. First we record a formula for each of the two series in (5.1).

Lemma 5.2. *If $a(n)$ is defined by*

$$\sum_{n=1}^{\infty} a(n)q^n = \frac{1}{1+q}\sum_{n=1}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}), \tag{5.8}$$

then

$$a(n) = \begin{cases} -(-1)^{n-\lfloor n \rfloor_p}, & \text{if } R(\lfloor n \rfloor_p) \text{ is even and positive,} \\ 0, & \text{if } R(\lfloor n \rfloor_p) \text{ is odd and negative,} \\ (-1)^{n-\lfloor n \rfloor_p}, & \text{if } R(\lfloor n \rfloor_p) \text{ is odd and positive,} \\ -2(-1)^{n-\lfloor n \rfloor_p}, & \text{if } R(\lfloor n \rfloor_p) \text{ is even and negative.} \end{cases} \tag{5.9}$$

Proof. The proof is elementary. One simply expands $1/(1+q)$ as $\sum(-1)^n q^n$ in (5.8), multiplies the series together, and verifies that the result is (5.9). □

Lemma 5.3. *If $b(n)$ is defined by*

$$\sum_{n=1}^{\infty} b(n)q^n = -q(q^2;q)_\infty, \tag{5.10}$$

then

$$b(n) = \begin{cases} 0, & \text{if } R(\lceil n \rceil_p) \text{ is positive,} \\ -1, & \text{if } R(\lceil n \rceil_p) \text{ is odd and negative,} \\ 1, & \text{if } R(\lceil n \rceil_p) \text{ is even and negative.} \end{cases} \tag{5.11}$$

Proof. This is another elementary calculation, using the fact that

$$(q^2;q)_\infty = \frac{1}{1-q}\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/2}.$$

□

Theorem 1.5 now follows by combining the above two lemmas and checking all of the different cases. □

In closing this section, we mention that it is undoubtedly possible to give a purely combinatorial proof of Theorem 1.5 by adapting Franklin's involution [3, pp.10-11].

6. Concluding Remarks

We wish to end by offering two suggestions for future research. First, there do not seem to be any simple congruences of the form $M_e(pn+a) - M_o(pn+a) \equiv 0 \pmod{p}$ for p prime except when $p = 5$. Can the work of Ahlgren-Boylan [1] and Kiming-Olsson [18] be adapted to prove that this is indeed the case? Second, can the ideas of [20] be applied to extend the congruences modulo 5^a to congruences modulo 5^{a+1} within certain arithmetic progressions?

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