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CALIBER NUMBER OF REAL QUADRATIC FIELDS

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Abstract. We obtain lower bound of caliber number of real quadratic field $K = \mathbb{Q}(\sqrt{d})$ using splitting primes in K. We find all real quadratic fields of caliber number 1 and find all real quadratic fields of caliber number 2 if d is not 5 modulo 8. In both cases, we don't rely on the assumption on $\zeta_K(1/2)$.

Contents

1. Introduction

In [5], Gauss had conjectured that there exist exactly nine imaginary quadratic fields of class number 1. Later, this was solved after diverse works of Stark, Heegner and Baker.

Further in this direction Goldfeld found an explicit lower bound of the class number of a given discriminant assuming existence of an elliptic curve on each imaginary quadratic field $K = \mathbb{Q}(\sqrt{d})$ whose Hasse-Weil L-function have order of vanishing 3 at $s = 1$(cf. [4]). Together with Gross-Zagier's formula for $L'_K(E, 1)$, Goldfeld's bound yields an explicit upper bound of a discriminant $|d|$ with $h(d) = h_0$, where $h(d)$ is a class number of $K = \mathbb{Q}(\sqrt{d})$. Finally, this gives an effective way of finding all imaginary quadratic fields with a given class number h_0.

Contrary to imaginary quadratic case, in real quadratic field case, the same question remains still unanswered. It is believed that there are infinitely many real quadratic fields of class number 1. As the first step has not been answered, at this moment, it does not make much

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sense to ask a similar generalization as above due to Stark, Heegner, Baker, Goldfeld, Gross and Zagier et al.

If we replace class number with caliber number, there is a room for a parallel generalization for real quadratic fields as in imaginary quadratic fields. Let d be a positive square free integer and D be a discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$. Denote by $[A, B, C]$ a binary quadratic form $Q(X, Y) = AX^2 + BXY + CY^2 \in \mathbb{Z}[X, Y]$. Then $GL_2(\mathbb{Z})$ acts on the set $\mathfrak{Q}(D)$ of primitive binary quadratic forms $[A, B, C]$ of discriminant $D = B^2 - 4AC$ by $S \circ Q(X, Y) = Q(aX + bY, cX + dY)$ for $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ and $Q(X, Y) \in \mathfrak{Q}(D)$. Let $H(D)$ be the set of equivalent classes $\mathfrak{Q}(D)/GL_2(\mathbb{Z})$. The cardinality of $H(D)$ is the class number $h(D) = h(d)$ of $\mathbb{Q}(\sqrt{d})$.

A quadratic form $[A, B, C]$ of discriminant D is called *reduced* if the coefficients satisfy the following inequalities:

$$A > 0,\ B < 0,\ C < 0,\ |B| < \sqrt{D},\ \sqrt{D} - |B| < 2A < \sqrt{D} + |B|. \tag{1}$$

Let $\mathfrak{Q}_{red}(D)$ be the set of reduced forms. The caliber number $\kappa(D) = \kappa(d)$ of $\mathbb{Q}(\sqrt{d})$ is the cardinality of $\mathfrak{Q}_{red}(D)$.

For a real quadratic irrationality w in K, its caliber $m(w)$ is simply the length of the periodic part in the continued fraction expansion. Let ω_Q be a root of $Q(X, 1)$, then $m([Q])$ is actually a class invariant in such a way that $m([Q]) = m(w_{Q_1}) = m(w_{Q_2})$ if Q_1 and Q_2 are in the same class $[Q] \in H(D)$. And it is well-known that $m([Q])$ is the number of reduced forms in the class $[Q]$. Thus the caliber number $\kappa(D)$ is rewritten as follows:

$$\kappa(D) = \sum_{[Q] \in H[D]} m([Q])$$

In [7], Lachaud obtained an effective lower bound of $\kappa(d)$ assuming $\zeta_K(\frac{1}{2}) \leq 0$:

$$\kappa(d) > \frac{1}{8.46} \log(d - 3). \tag{2}$$

This is a real analogue of Goldfeld's work. Moreover, Lachaud determined all real quadratic fields with caliber number 1 with assumption of $\zeta_K(\frac{1}{2}) \leq 0$.

We explain the content of this article.

In Section 2, we recall the definition of a set $S_D(A)$ and its cardinality $\rho_D(A)$ for a positive integer A. We write a lower bound and an upper bound of $\kappa(d)$ in terms $\rho_D(A)$. Some further properties of $\rho_D(-)$ are studied to give a lower bound of the caliber number of a real quadratic field in terms of D and a rational prime that splits above. The other results of this paper relies on this estimate:

Theorem 2.6. *Let d be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D. Suppose a rational prime p splits in K. Then*

$$\kappa(d) > 2\left[\frac{\log \frac{\sqrt{D}}{2}}{\log p}\right].$$

In Section 3, we investigate the caliber number problem of real quadratic fields without the assumption on $\zeta_K(1/2)$.

Theorem 3.3. ($\kappa(d) = 1$) *Suppose d is a positive square free integer. Then $\kappa(d) = 1$ if and only if d is one of the following:* $2, 13, 29, 53, 173, 293$.

Since we have related the lower bound of caliber number with a splitting prime and the values of $\rho_D(-)$, we further obtain an existence of splitting prime smaller than $\sqrt{D}$ in case of $\kappa(d) \neq 1$. We further study the $\kappa(d) = 2$ problem for some cases. We apply some results on class number problems of Richaud-Degert type due to Biró , Byeon and the second named author (cf. [1], [2], [3], [8]), we list all real quadratic fields of caliber number one and all real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ when $d \not\equiv 5$ modulo 8 with caliber number two.

Theorem 3.9. ($\kappa(d) = 2$ with $d \not\equiv 5$ (8))*Real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with $d \not\equiv 5$ modulo 8 with caliber number 2 are the followings:*

$$3, 6, 11, 38, 83, 227.$$

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2. Lower bound of caliber number

Throughout this article, $D > 0$ denotes the discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{d})$.

For each positive integer A, we associate a set

$$S_D(A) := \{[B] \in \mathbb{Z}/2A\mathbb{Z} | B^2 \equiv D \pmod{4A}\}$$

and let $\rho_D(A)$ be the cardinality of $S_D(A)$.

Lemma 2.1. *Suppose $A < \frac{\sqrt{D}}{2}$. Then for a given $[B] \in S_D(A)$, there exists a unique pair of integers (B, C) such that $B \in [B]$ and $[A, B, C]$ is reduced.*

Proof. Let B_0 be any integer representative of $[B]$. Then for any integer k one can find uniquely an integer $C(k)$ satisfying

$$D = (B_0 + 2Ak)^2 - 4AC(k).$$

Moreover, we have for a unique integer k_0,

$$-\sqrt{D} < B_0 + 2Ak_0 < 2A - \sqrt{D}.$$

If we set $B = B_0 + 2Ak_0$ and $C = C(k_0)$, from $A < \sqrt{D}/2$, one can check easily $B < 0$ and $2A < \sqrt{D} - B$. □

Theorem 2.2. *The caliber number $\kappa(D)$ of $\mathbb{Q}(\sqrt{D})$ satisfies*

$$\sum_{A<\frac{\sqrt{D}}{2}} \rho_D(A) \leq \kappa(D) \leq \sum_{A<\sqrt{D}} \rho_D(A).$$

Proof. The lower bound is immediate from Lemma 2.1. If a primitive quadratic form $[A, B, C]$ is reduced then for $B < \sqrt{D}$,

$$2A < B + \sqrt{D}.$$

Thus we obtain

$$A < \sqrt{D}.$$

This yields the upper bound of $\kappa(d)$. □

Lemma 2.3. *Let d be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$ with discriminant D. Set*

$$\omega_D = \begin{cases} \frac{\sqrt{D}}{2} & D \equiv 0 \pmod 4 \\ \frac{1+\sqrt{D}}{2} & D \equiv 1 \pmod 4. \end{cases}$$

Then an integral ideal is of the form $[a, b + c\omega_D]$ for some positive integers a, b, c such that $c|b, c|a$ and $ac|N(b + c\omega_D)$.

Proof. See Thorem 1.2.1 and Definition 1.2.1 in [9]. □

We say that an integral ideal $[A, \frac{B+\sqrt{D}}{2}]$ of K is *primitive* if the integers A, B satisfy

$$B^2 \equiv D \pmod{4A}.$$

(See Theorem 1.2.1 and Definition 1.2.1 in [9].)

Lemma 2.4. *Let d be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$ with discriminant D.*
1) $\rho_D(A)$ equals the number of primitive ideals of K with norm A.
2) Any integral ideal can be written as $[fA, \frac{fB+f\sqrt{D}}{2}]$ for a positive integer f and a primitive ideal $[A, \frac{B+\sqrt{D}}{2}]$.

Proof. 1) Note that

$$[A, \frac{B'+\sqrt{D}}{2}] = [A, \frac{B+\sqrt{D}}{2}]$$

if and only if

$$B' \equiv B \pmod{2A}.$$

Thus for a primitive ideal I of K, there exists exactly a unique pair of integers (A, B) with $[B] \in S_D(A)$ and $I = [A, \frac{B+\sqrt{D}}{2}]$. If $I = [A, \frac{B+\sqrt{D}}{2}]$ is a primitive ideal,

$$N(I) = A.$$

This completes the proof.
2) It is an immediate consequence of Lemma 2.3.

□

Proposition 2.5. *1) If $(n, m) = 1$, $\rho_D(nm) = \rho_D(n)\rho_D(m)$.*
2) For $p \not| D$,

$$\rho_D(p^\alpha) = 1 + \chi_D(p),$$

where χ_D be the Kronecker character(ie. $\chi_D(\cdot) = (\frac{D}{\cdot})$). For $p|D$,

$$\rho_D(p^\alpha) = \begin{cases} 0, & \alpha > 1 \\ 1, & \alpha = 1 \end{cases}$$

Proof. 1) It is clear from Lemma 3.2 of pp. 48 in [6].
2) From Lemma 2.4, we find that

$$\sum_n \rho_D(n)n^{-s} = \zeta(2s)^{-1}\zeta_K(s).$$

The Euler factor at p of $\sum_n \rho_D(n)n^{-s}$ is

$$1 + \sum_{n=1}^{\infty} \frac{\rho_D(p^n)}{p^{ns}}.$$

If $\chi_D(p) = 1$ (resp. $\chi_D(p) = -1$, $\chi_D(p) = 0$) then the Euler factor at p of $\zeta(2s)^{-1}\zeta_K(s)$ is

$$1 + \sum_{n=1}^{\infty} \frac{2}{p^{ns}} \quad (\text{resp. } 1, 1 + p^{-s}).$$

By comparing the Euler factors of $\sum_A \rho_D(A)A^{-s}$ and $\zeta(2s)^{-1}\zeta_K(s)$, we can prove Proposition. $\square$

Theorem 2.6. *Let d be a positive square free integer and $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D. Suppose a rational prime p splits in K. Then*

$$\kappa(d) > 2\Big[\frac{\log\frac{\sqrt{D}}{2}}{\log p}\Big].$$

Proof. It suffices to show the theorem for the smallest splitting prime. Let p_1 be the smallest prime that splits in K.

From Theorem 2.2, we have:

$$\sum_{p_1^\alpha < \frac{\sqrt{D}}{2}} \rho_D(p_1^\alpha) \leq \sum_{A < \frac{\sqrt{D}}{2}} \rho_D(A) < \kappa(d).$$

Lemma 2.5 implies that $\rho_D(p_1^\alpha) = 0$ for any α.

Therefore,

$$\begin{aligned}\sum_{p_1^\alpha < \frac{\sqrt{D}}{2}} \rho_D(p_1^\alpha) &= 2 \cdot (\text{the number of } \alpha\text{'s:} p_1^\alpha < \sqrt{D}/2\ ) \\ &= 2\Big[\frac{\log\frac{\sqrt{D}}{2}}{\log p_1}\Big].\end{aligned}$$

This completes the proof. $\square$

Corollary 2.7. *Suppose $d \equiv 1 \pmod 8$ be a positive square free integer. Then*

$$2^{\kappa(d)+4} > d.$$

Remark 2.8. *The result of this section is comparable to Section 22.5 of* [10].

For an imaginary quadratic field of discriminant $D < 0$, one has

$$\sum_{A \leq \sqrt{\frac{|D|}{4}}} \rho_D(A) \leq h(D) \leq \sum_{A \leq \sqrt{\frac{|D|}{3}}} \rho_D(A). \tag{3}$$

As $h(D) = \kappa(D)$ for $D < 0$, the above extends Proposition 2.5 to negative discriminant case. The inequality (3) *turns out to a lower bound of $h(D)$ in terms of* $\log p_r$

$$\log p_r \geq \frac{\log\frac{|D|}{4}}{\sqrt[r]{2h(D)r!}}$$

where $p_1 < p_2 < \cdots < p_r$ are the first r splitting primes.

3. Determination of real quadratic fields with small caliber numbers

In this section, we determine all the real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with caliber 1 and $\mathbb{Q}(\sqrt{d})$ with caliber number 2 when $d \not\equiv 5$ modulo 8. For the determination, we don't assume $\zeta_K(1/2) \leq 0$.

For both cases, we need some ideas on continued fractions of quadratic irrationalities. For general and precise idea on continued fractions we refer the readers to [9], [12], [13], etc.

Consider a real quadratic irrationality x. The caliber $m(x)$ is simply the length of the periodic part in the continued fraction expansion. x is said to be reduced if $x > 1$ and $-1 < x' < 0$. It is well known that the reduced elements x has purely periodic continued fraction expansion.

Proposition 3.1. *Let d be a positive square free integer. Then $K = \mathbb{Q}(\sqrt{d})$ is of caliber one only if it has class number one and*

$$d = n^2 + 4 \text{ or } n^2 + 1.$$

Proof. Suppose $K = \mathbb{Q}(\sqrt{d})$ is of caliber 1. Since $h(d) \leq \kappa(d)$, clearly $h(d) = 1$.

Suppose now x is reduced with period 1 then for a positive integer r, x satisfies

$$x = r + \frac{1}{x}.$$

Solving the above equality, we get

$$x = \frac{r + \sqrt{r^2 + 4}}{2}.$$

Suppose $d \equiv 1 \pmod 4$. Since $\kappa(d) = h(d) = 1$, $[1, \frac{1+\sqrt{d}}{2}]$ is principal and $m(\frac{1+\sqrt{d}}{2}) = 1$. As $(\frac{1+\sqrt{d}}{2} - [\frac{1+\sqrt{d}}{2}])^{-1}$ is reduced, we have for a positive integer r

$$(\frac{1+\sqrt{d}}{2} - \Big[\frac{1+\sqrt{d}}{2}\Big])^{-1} = \frac{r + \sqrt{r^2 + 4}}{2}.$$

Thus

$$\frac{1+\sqrt{d}}{2} - \Big[\frac{1+\sqrt{d}}{2}\Big] = \frac{-r + \sqrt{r^2 + 4}}{2}.$$

From above equation, we have

$$d = \Big(2\Big[\frac{1+\sqrt{d}}{2}\Big] - 1\Big)^2 + 4.$$

If $K = \mathbb{Q}(\sqrt{d})$ with $d \equiv 2, 3$ modulo 4 of caliber number 1, similarly we can conclude that d is of the form $n^2 + 1$ for an integer n. $\square$

In the above case of d (i.e.. $d = n^2 + 4$ or $n^2 + 1$), Biró found the full list of d with $h(d) = 1$.

Proposition 3.2 (Biró [1], [2]).
I. Let $d = n^2 + 4$ be a square free integer. $h(d) = 1$ if and only if

$$d = 13, 19, 53, 173, 293. \tag{4}$$

II. Let $d = n^2 + 1$ be a square free integer. $h(d) = 1$ if and only if

$$d = 2, 17, 37, 101, 197, 677. \tag{5}$$

Combining Proposition 3.1 and Proposition 3.2, we can list all d with $\kappa(d) = 1$:

Theorem 3.3 ($\kappa(d) = 1$). *Suppose d is a positive square free integer. Then $\kappa(d) = 1$ if and only if d is one of the following:* $2, 13, 29, 53, 173, 293$.

Corollary 3.4. *Suppose that d is a rational prime that is not in*

$$S := \{2, 13, 29, 53, 173, 293\}$$

. Let D be the discriminant of $\mathbb{Q}(\sqrt{d})$. Then there exists a rational prime that splits in $\mathbb{Q}(\sqrt{d})$ and smaller than or equal to $\sqrt{D}$.

Proof. After Theorem 3.3, we know that $\kappa(d) \geq 2$ for $d \notin S$.

Let $A \neq 1$ be a positive integer smaller than $\sqrt{D}$. If we suppose conversely that there no prime $\leq \sqrt{D}$ splits above, then $\rho_D(A) = 0$ or 1. If the multiplicity of a prime factor p of A is greater than 2, $\rho_D(A) = 0$. And if a prime factor p of A inerts in $\mathbb{Q}(\sqrt{d})$, $\rho_D(A) = 0$ from the multiplicative property of ρ_D as in Proposition 2.5.

Since $S_D(1) \subseteq \mathbb{Z}/2\mathbb{Z}$ cannot contain 0, $\rho_D(1) < 2$. As D is either d or $4d$, $d > \sqrt{D}$.

Therefore, we have

$$\kappa(d) \leq \sum_{A \leq \sqrt{D}} \rho_D(A) \leq \rho_D(1) \leq 1.$$

This contradicts to our assumption. □

Now we move to the case of $\kappa(d) = 2$.

A positive square free integer $d = n^2 + r$ with $r|4n$ is said to be of Richaud-Degert type. Lemma 3.5 and Proposition 3.6 imply that if $\kappa(d) = 2$ then d is necessarily of Richaud-Degert type:

Lemma 3.5. *Let $d = n^2 + 1$ be a square free integer. If $d \equiv 2 \pmod 4$ then the ideal $[2, \sqrt{d}]$ is not principal ideal except $d = 2$ and if $d \equiv 3$ modulo 4 then the ideal $[2, 1+\sqrt{d}]$ is not principal ideal if $d \equiv 1$ modulo 8 then the ideal $[2, \frac{1+\sqrt{d}}{2}]$ is not principal ideal except $d = 17$.*

Proof. See the proof of Theorem 2.6 in [3]. □

Proposition 3.6. *Let $d \not\equiv 5$ modulo 8 be a positive square free integer. The field $\mathbb{Q}(\sqrt{d})$ is of caliber number 2 then d is a Richaud-Degert type with class number one.*

Proof. Suppose $K = \mathbb{Q}(\sqrt{d})$ has caliber number 2 and class number 2. Then the principal ideal has caliber 1. Thus from the proof of Proposition 3.1, we know that d is either n^2+1 or n^2+4. As we assumed that $d \not\equiv 5 \pmod 8$, we can exclude the case $d = n^2+4$. From Lemma 3.5, if $n^2+1 \equiv 2 \pmod 4$, $[1, \sqrt{n^2+1}]$ and $[2, \sqrt{n^2+1}]$ represent two distinct ideal classes of caliber 1. Thus $[2, \sqrt{n^2+1}] \sim [1, \sqrt{n^2+1}/2] \sim [1, x]$, where

$$x^{-1} = \frac{\sqrt{n^2+1}}{2} - \left[\frac{\sqrt{n^2+1}}{2}\right] = \frac{-r+\sqrt{r^2+4}}{2}$$

for a positive integer r. In the above, comparing the rational parts, one can see that

$$r = 2\left[\tfrac{\sqrt{n^2+1}}{2}\right],$$
$$n^2+1 = r^2+4.$$

This contradict to the assumption that $d = n^2+1$ is square free. Similarly, for the rest cases $d \equiv 3 \pmod 4$ or $d \equiv 1 \pmod 8$, we obtain contradiction.

Therefore, if $d \not\equiv 5 \pmod 8$ and $\kappa(d) = 2$, then $h(d) = 1$.

Suppose now x is a reduced quadratic irrationality of caliber 2. Then

$$x = a + \frac{1}{b + \frac{1}{x}}. \tag{6}$$

for two distinct positive integers a and b. Solving the above equation, we obtain

$$x = \frac{ab + \sqrt{a^2b^2+4ab}}{2b}. \tag{7}$$

Consider the case $d \equiv 1 \pmod 8$. Since $\kappa(d) = 2$ implies $h(d) = 1$, $[1, \frac{1+\sqrt{d}}{2}]$ is principal and $m(\frac{1+\sqrt{d}}{2}) = 2$. Thus from the equation (7), we have

$$x^{-1} = \frac{1+\sqrt{d}}{2} - \left[\frac{1+\sqrt{d}}{2}\right] = \frac{-ab+\sqrt{a^2b^2+4ab}}{2a}. \tag{8}$$

And (3.9) implies that

$$b = 2\left[\tfrac{1+\sqrt{d}}{2}\right] - 1,$$
$$d = b^2 + 4\tfrac{b}{a}.$$

Thus we find that d is of the form $n^2 + r$ with $r|4n$.

Similarly, for a square free integer $d \equiv 2, 3$ modulo 4, we conclude that if $\kappa(d) = 2$, then d is of the form $n^2 + r$ with $r|2n$. $\square$

For Richaud-Degert types of $d \not\equiv 5 \pmod 8$, we recall a class number 1 criterion by Byeon and Kim (cf. [3]):

Proposition 3.7 (Byeon-Kim)**.** *Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of R-D type and $h(d)$ be the class number of K. Then*

I. $d = n^2 + r \equiv 2, 3$ *(mod 4)*
  (i) $|r| \neq 1, 4$, $h(d) > 1$ *except* $r = \pm 2$
  (ii) $|r| = 1$, $h(d) > 1$ *except* $d = 2, 3$

II. $d = n^2 + r \equiv 1$ *(mod 8)*
  (i) $|r| \neq 1, 4$ $h(d) > 1$ *except* $d = 33$
  (ii) $|r| = 1$ *(hence $r = 1$ and n even)* $h(d) > 1$ *except* $d = 17$.

After the Proposition 3.7, if d is a Richaud-Degert type of $h(d) = 1$, then $d = n^2 \pm 2, 2, 3, 33$ or 17.

For Richaud-Degert type of $n^2 \pm 2$, the second named author found the whole list of d with $h(d) = 1$ in [8]:

Proposition 3.8 (Lee)**.** *Let $d = n^2 \pm 2$ be a square free integer. Then $h(d) = 1$ if and only if*

$$d = 3, 6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398.$$

Finally, we obtain the following list of $d(\not\equiv 5 \pmod 8)$ with $\kappa(d) = 2$:

Theorem 3.9 ($\kappa(d) = 2$ with $d \not\equiv 5$ (8))**.** *Real quadratic fields $K = \mathbb{Q}(\sqrt{d})$ with $d \not\equiv 5$ modulo 8 with caliber number 2 are the followings:*

$$3, 6, 11, 38, 83, 227.$$

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