4-RANKS OF CLASS GROUPS OF QUADRATIC EXTENSIONS OF CERTAIN QUADRATIC FUNCTION FIELDS

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ABSTRACT. We obtain some density results for the 4-ranks of class groups of quadratic extensions of quadratic function fields analogous to those results of F. Gerth in the classical case.

1. INTRODUCTION

In 1980's F. Gerth studied the structure of Sylow 2-subgroups of the class groups of quadratic number fields K ([3]), or quadratic extensions K of a certain quadratic number field F ([2],[5],[6]). More precisely, let R_K be the 4-class rank of the class group of K in the narrow sense when K is a quadratic number field, and in the usual sense when K is a quadratic extension of a certain quadratic number field. Then he presented results which specify how likely it is that $R_K = 0, 1, 2, \ldots$.

Let q be a power of an odd prime p. In ([1]), the analogous problem when K is a quadratic extension of $\mathbb{F}_q(T)$, which we call *quadratic function field*, is studied. In this article we study the problem in the case when K is a quadratic extension of a certain quadratic function field.

In the function field case the behavior of ∞ in an extension is much more diverse than number field case, where there are only two cases, real or imaginary. To be consistent with number field case, we only consider the extension fields K of $k = \mathbb{F}_q(T)$ such that every embedding of K into a fixed complete algebraically closed field C of $k_\infty = \mathbb{F}_q((\frac{1}{T}))$ over k is contained in $\mathbb{F}_q((\sqrt{\frac{-1}{T}}))$. For an infinite place ∞ of K, we say that ∞ is *real* (resp. *imaginary*) if $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ (resp. $K_\infty = \mathbb{F}_q((\sqrt{\frac{-1}{T}})))$.

The case when $q \equiv 1 \mod 4$ is much different from the classical case, because -1 is a square in this case. This makes getting the limit density much easier. The case when $q \equiv 3 \mod 4$ is very similar to the classical case. But we have to mention some interesting points compared to the classical case. In the classical case dyadic primes and the signs at archimedean places have effects to determine some Hilbert symbols. In the function field

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case there are no dyadic primes. However, the degrees and signs at infinite places would determine some Hilbert symbols and force to get the same result as in the classical case.

Notations:

$k := \mathbb{F}_q(T), \qquad \mathbb{A} := \mathbb{F}_q[T]$

γ:= a fixed generator of $\mathbb{F}_q^*$

∞:= the place of k associated to $(\frac{1}{T})$

Assume that for every extension field K of k, every embedding of K into C is contained in $\mathbb{F}_q((\sqrt{\frac{-1}{T}}))$.

F := a quadratic function field with odd class number h

$\mathbb{B}$:= the integral closure of $\mathbb{A}$ in F

$\mathcal{A}$:= a set of quadratic extensions of F

For $K \in \mathcal{A}$, let C_K be the 2-class group of the integer ring O_K.

D_K := the discriminant of K/F

r_K := 2-rank of C_K, $\qquad R_K$:= 4-rank of C_K.

$B_t = \{K \in \mathcal{A} :$ exactly t finite primes of F ramify in $K\}$

$B_{t;n} := \{K \in B_t : \deg D_K = n\}$

$B_{t,j;n} := \{K \in B_{t;n} : R_K = j\}$

$d_{t,j} = \lim_{n\to\infty} \frac{|B_{t,j;n}|}{|B_{t;n}|}$

$d_{\infty,j} = \lim_{t\to\infty} d_{t,j}$

2. F is real

In this section we assume that F is a real quadratic function field with odd class number h. Assume further that the fundamental unit ϵ of $\mathbb{B}$ satisfies $N_{F/k}(\epsilon \notin \mathbb{F}_q^{*2}$. Then $F = k(\sqrt{P})$ for some monic irreducible polynomial $P \in \mathbb{A}$ of even degree. Fix a sign function $sgn : k_\infty^* \longrightarrow \mathbb{F}_q^*$. Let ∞_1, ∞_2 be two infinite places of F. Let $sgn_{\infty_i} : F^* \to \mathbb{F}_q^*$ be the sign function at ∞_i, which is defined by

$$sgn_{\infty_i}(\alpha) = sgn(\sigma_i(\alpha)),$$

where σ_i is the embedding corresponding to ∞_i. An element $\alpha \in F^*$ is called *positive* at ∞_i if $sgn_{\infty_i}(\alpha) \in \mathbb{F}_q^2$. α is called *totally positive* if it is positive at every ∞_i.

For $0 \neq a \in \mathbb{B}$ and a prime ideal $\mathfrak{p}$ of $\mathbb{B}$, $(a, K/F)_\mathfrak{p}$ be the norm residue symbol for K/F. For $0 \neq b \in \mathbb{B}$, define the Hilbert symbol $(a,b)_\mathfrak{p} \in \{\pm 1\}$ by

$$(a, K/F)_\mathfrak{p}\sqrt{b} = (a,b)_\mathfrak{p}\sqrt{b},$$

where $K = F(\sqrt{b})$.

Since $N_{F/k}(\epsilon) \notin \mathbb{F}_q^2$, we may assume ϵ is positive at ∞_1 and negative at ∞_2.

For $a \in F^*$, $\deg_i a$ the order of pole of a at ∞_i.

For a totally positive $a \in \mathbb{B}$, a is said to be of type I (resp. II, III and IV) if $(\deg_1 a, \deg_2 a)$ is (even,odd) (resp. (odd,even), (odd,odd) and (even,even)). Put, for $q \equiv 3 \mod 4$,

$$\tilde{a} = \begin{cases} \epsilon a & \text{if } a \text{ is of type I} \\ -\epsilon a & \text{if } a \text{ is of type II} \\ -a & \text{if } a \text{ is of type III} \\ a & \text{if } a \text{ is of type IV,} \end{cases}$$

and for $q \equiv 1 \mod 4$, put $\tilde{a} = a$. Note that -1 is a square if and only if $q \equiv 1 \mod 4$. From the properties of norm residue symbols, we see easily that, for $a \in \mathbb{B}$ totally positive, $(\gamma, \tilde{a})_{\infty_i} = 1$ (resp. -1) if $\deg_i a$ is even (resp. odd), and $(\epsilon, \tilde{a})_{\infty_2} = 1$ (resp. -1) if $\deg_2 a$ is even (resp. odd). We always have $(\epsilon, \tilde{a})_{\infty_1} = 1$.

Lemma 2.1. *Let $\mathfrak{p}$ be a prime ideal and $\mathfrak{p}^{\mathfrak{h}} = (\mathfrak{a})$ for totally positive $a \in \mathbb{B}$. We have*

$$(\gamma, \tilde{a})_{\mathfrak{p}} = \begin{cases} 1 & \textit{if } a \textit{ is of type III or IV} \\ -1 & \textit{if } a \textit{ is of type I or II} \end{cases}$$

$$(\epsilon, \tilde{a})_{\mathfrak{p}} = \begin{cases} 1 & \textit{if } a \textit{ is of type II or IV} \\ -1 & \textit{if } a \textit{ is of type I or III} \end{cases}$$

proof From the product formula for Hilbert symbols, we have

$$(\gamma, \tilde{a})_{\mathfrak{p}} = (\gamma, \tilde{a})_{\infty_1} (\gamma, \tilde{a})_{\infty_2},$$

and

$$(\epsilon, \tilde{a})_{\mathfrak{p}} = (\epsilon, \tilde{a})_{\infty_1} (\epsilon, \tilde{a})_{\infty_2}.$$

The result follows from the above consideration. □

Let K be a quadratic extension of F, where exactly t finite primes of F are ramified in K, say, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Let a_i be a totally positive element in $\mathbb{B}$ with $\mathfrak{p}_i^{\mathfrak{h}} = (\mathfrak{a}_i)$. Let

$$\mu := a_1 \cdots a_t, \qquad \tilde{\mu} = \tilde{a}_i \cdots \tilde{a}_t.$$

Then $K = F(\sqrt{\tilde{\mu}})$. Let $M'_K = (m'_{ij})$ be the $(t+1) \times (t+2)$ matrix over $\mathbb{F}_2$ defined by;

$$(-1)^{m'_{ij}} = \begin{cases} (\gamma, \tilde{\mu})_{\infty_i} & \text{for } i = 1, 2,\ j = 1 \\ (\epsilon, \tilde{\mu})_{\infty_i} & \text{for } i = 1, 2,\ j = 2 \\ (a_{j-2}, \tilde{\mu})_{\infty_i} & \text{for } i = 1, 2,\ j = 3, \ldots, t+2 \\ (\gamma, \tilde{\mu})_{\mathfrak{p}_{i-2}} & \text{for } i = 3, \ldots, t+1,\ j = 1 \\ (\epsilon, \tilde{\mu})_{\mathfrak{p}_{i-2}} & \text{for } i = 3, \ldots, t+1,\ j = 2 \\ (a_{j-2}, \tilde{\mu})_{\mathfrak{p}_{i-2}} & \text{for } i = 3, \ldots, t+1,\ j = 3, \ldots, t+2 \end{cases}$$

Let M''_K be the $(t+1)\times 2$ matrix consisting of the first two columns of M'_K. Then from §1 of [7],

$$r_K = \begin{cases} t+1-\mathrm{rank} M''_K & \text{if both } \infty_1 \text{ and } \infty_2 \text{ are ramified, i.e. } \mu \text{ is of type III} \\ t-\mathrm{rank} M''_K & \text{if one of } \infty_1 \text{ and } \infty_2 \text{ is ramified, i.e. } \mu \text{ is of type I or II} \\ t-1-\mathrm{rank} M''_K & \text{if both } \infty_1 \text{ and } \infty_2 \text{ split, i.e. } \mu \text{ is of type IV,} \end{cases}$$

and

$$R_K = \begin{cases} t+1-\mathrm{rank} M'_K & \text{if both } \infty_1 \text{ and } \infty_2 \text{ are ramified, i.e. } \mu \text{ is of type III} \\ t-\mathrm{rank} M'_K & \text{if one of } \infty_1 \text{ and } \infty_2 \text{ is ramified, i.e. } \mu \text{ is of type I or II} \\ t-1-\mathrm{rank} M'_K & \text{if both } \infty_1 \text{ and } \infty_2 \text{ split, i.e. } \mu \text{ is of type IV.} \end{cases}$$

Suppose first that μ is of type III. In this case both ∞_1 and ∞_2 are ramified in K. It is easy to see that

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ & * \\ \vdots & \vdots \end{bmatrix}$$

Thus rank $M''_K = 2$ and $r_K = t-1$. Let M_K be the matrix obtained from M'_K by deleting the first two rows and first two columns. Since each a_j is totally positive, the first two rows of M'_K are given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then rank $M'_K = 2 + \mathrm{rank} M_K$ and so,

$$R_K = t-1-\mathrm{rank}\ M_K.$$

By renumbering, we may write $M_K = (m_{ij})$ with $1 \le i \le t-1$ and $1 \le j \le t$, where

$$(-1)^{m_{ij}} = (a_j, \tilde{\mu})_{\mathfrak{p}_i}.$$

Note that, for $q \equiv 3 \mod 4$,

$$(\mu, \tilde{\mu})_{\mathfrak{p}_i} = (-\tilde{\mu}, \tilde{\mu})_{\mathfrak{p}_i} = 1,$$

and, for $q \equiv 1 \mod 4$,

$$(\mu, \tilde{\mu})_{\mathfrak{p}_i} = (\mu, \mu)_{\mathfrak{p}_i} = (-1, \mu)_{\mathfrak{p}_i}(-\mu, \mu)_{\mathfrak{p}_i} = 1.$$

Thus the sum of each row of M_K is 0. We may discard any column of M_K without changing the rank, and thus we may consider M_K to be a $(t-1)\times(t-1)$ matrix.

Assume first that $q \equiv 1 \mod 4$. Then

$$(a_j, \tilde{\mu})_{\mathfrak{p}_i} = (a_j, a_i)_{\mathfrak{p}_i} = (a_i, a_j)_{\mathfrak{p}_j},$$

by the product formula for Hilbert symbols and the fact that $(a_j, a_i)_{\infty_k} = 1$, since a_j is totally positive. Therefore, M_K is symmetric if $q \equiv 1 \mod 4$. Now use [3], Proposition 3.7 to get

$$d_{\infty,j} = \frac{2^{-\frac{j(j+1)}{2}}}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=1}^{\infty}(1+2^{-i})}.$$

Now assume that $q \equiv 3 \mod 4$. Let $\epsilon_i := \tilde{a}_i/a_i$. Then using product formula and the multiplicativity of Hilbert symbols, we see easily that

$$\begin{aligned}(a_i, \tilde{\mu})_{\mathfrak{p}_j} &= (\epsilon_i, \tilde{a}_j)_{\mathfrak{p}_j}(\tilde{a}_i, \tilde{\mu})_{\mathfrak{p}_j} = (\epsilon_i, \tilde{a}_j)_{\mathfrak{p}_j}(\tilde{a}_i, \tilde{a}_j)_{\mathfrak{p}_j} \\ &= (\epsilon_i, \tilde{a}_j)_{\mathfrak{p}_j}(\tilde{a}_j, \tilde{a}_i)_{\mathfrak{p}_i}(\tilde{a}_j, \tilde{a}_i)_{\infty_1}(\tilde{a}_j, \tilde{a}_i)_{\infty_2} \\ &= (\epsilon_i, \tilde{a}_j)_{\mathfrak{p}_j}(\epsilon_j, \tilde{a}_i)_{\mathfrak{p}_i}(\epsilon_j, \tilde{a}_i)_{\infty_1}(\epsilon_j, \tilde{a}_i)_{\infty_2}(a_j, \tilde{\mu})_{\mathfrak{p}_i},\end{aligned}$$

since $(a_j, \tilde{a}_i)_{\infty_k} = 1$. Using Lemma 1.1, product formula for Hilbert symbols and noting that $(-1, \tilde{a})_{\mathfrak{p}} = (\gamma, \tilde{a})_{\mathfrak{p}}$, we see that

$$(\epsilon_i, \tilde{a}_j)_{\mathfrak{p}_j}(\epsilon_j, \tilde{a}_i)_{\mathfrak{p}_i}(\epsilon_j, \tilde{a}_i)_{\infty_1}(\epsilon_j, \tilde{a}_i)_{\infty_2} = -1$$

if and only if the pair (type of a_i, type of a_j) is $(I, I), (I, III), (II, II), (II, III), (III, I)$ or (III, II). We order $a_1, \ldots, a_{t-1}$ so that $a_1, \ldots, a_{k_1}$ are of type I; $a_{k_1+1}, \ldots, a_{k_1+k_2}$ are of type II; $a_{k_1+k_2+1}, \ldots, a_{k_1+k_2+k_3}$ are of type III; and $a_{k+k_2+k_3+1}, \ldots, a_{k_1+k_2+k_3+k_4}$ are of type IV. Then M_K is of the form

$$\begin{bmatrix} M_1 & | & M_2 & | & & | & \\ --- & | & --- & | & M_4 & | & M_5 \\ M_2^T & | & M_3 & | & & | & \\ --- & --- & --- & --- & --- & --- & --- \\ & M_4^T + J & & | & & & \\ --- & --- & --- & & & M_6 & \\ & M_5^T & & | & & & \end{bmatrix}$$

where M_1, M_3 are antisymmetric, M_6 is symmetric and J is the $k_3 \times (k_1 + k_2)$ matrix with each entry equal to 1, which is the same form as in [5], (3.11). Therefore

$$d_{\infty,j} = \frac{2^{-j^2}\prod_{i=1}^{\infty}(1-2^{-i})}{\prod_{i=1}^{j}(1-2^{-i})^2}.$$

It is not hard to see that for $q \equiv 3 \mod 4$, the case that μ is of type I or II is the same as the case (ii) of [5] and the case that μ is of type IV is the same as the case (iii) of [5].

Suppose that $q \equiv 1 \mod 4$ and μ is of type I or II. Then the associated matrix M_K, which is obtained from M'_K by deleting first two rows and first column, is of the form

$$M_K = \begin{bmatrix} H_1 & | & \\ -- & | & \\ 0_2 & | & \\ -- & | & M \\ H_3 & | & \\ -- & | & \\ 0_4 & | & \end{bmatrix}$$

where H_i (resp. 0_i) is a vector with each component 1(resp. 0), and M is symmetric. By changing the order, we get

$$M_K = \begin{bmatrix} H & | & \\ -- & | & M \\ 0 & | & \end{bmatrix}$$

which is the same matrix as in [6], (3.7). Note that the probability that $H = \phi$ tends to 0 as $t \to \infty$. Then following [6], we get

$$d_{\infty,j} = \frac{2^{-j(j+3)/2}}{\prod_{i=1}^{\infty}(1+2^{-i})\prod_{i=1}^{j}(1-2^{-i})}.$$

Now suppose that $q \equiv 1 \mod 4$ and μ is of type IV. Then the associated matrix M_K, which is obtained from M'_K by deleting first two rows, is of the form

$$M_K = \begin{bmatrix} H_1 & H_1 & | & \\ -- & -- & | & \\ H_2 & 0_2 & | & \\ -- & -- & | & M \\ 0_3 & H_3 & | & \\ -- & -- & | & \\ 0_4 & 0_4 & | & \end{bmatrix}$$

where H_i (resp. 0_i) is a vector with each component 1(resp. 0), and M is symmetric. Let J be $n \times 2$ matrix consisting the first two columns of M_K. Let

$$M_1 = \begin{bmatrix} J & M \end{bmatrix}$$

$$M_2 = \begin{bmatrix} J & M & V \\ \mathbf{u} & V^T & v \end{bmatrix},$$

where M is symmetric and $\mathbf{u} \in \mathbb{F}_{\nvDash}^{\nvDash}$. We get the following lemma, whose proof is almost the same as that of [6], Lemma 3.1.

Lemma 2.2. *Suppose that rank $M_1 = r$. Then of all possible M_2,*
i) $2^{n+1} - 2^{r+1}$ *have rank* $M_2 = r+2$
ii) $2^{r+1} - 2^{r-2}$ *have rank* $M_2 = r+1$
iii) 2^{r-2} *have rank* $M_2 = r$.

Then we have, using Lemma 1.5 of [4],

$$d_{\infty,j} = \frac{2^{-\frac{j(j+5)}{2}}}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=3}^{\infty}(1+2^{-i})}.$$

We summarize these in the following theorem.

Theorem 2.1. *Let F be a real quadratic function field with odd class number.*
i) If $\mathcal{A} = \{$quadratic extensions K of F in which both infinite places of F ramify$\}$, then

$$d_{\infty,j} = \begin{cases} \frac{2^{-j^2}\prod_{i=1}^{\infty}(1-2^{-i})}{\prod_{i=1}^{j}(1-2^{-i})^2} & \text{if } q \equiv 3 \mod 4 \\ \frac{2^{-\frac{j(j+1)}{2}}}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=1}^{\infty}(1+2^{-i})} & \text{if } q \equiv 1 \mod 4. \end{cases}$$

ii) If $\mathcal{A} = \{$quadratic extensions K of F in which exactly one infinite place of F ramify$\}$, then

$$d_{\infty,j} = \begin{cases} \frac{2^{-j(j+1)}\prod_{i=1}^{\infty}(1-2^{-i})}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=1}^{j+1}(1-2^{-i})} & \text{if } q \equiv 3 \mod 4 \\ \frac{2^{-j(j+3)/2}}{\prod_{i=2}^{\infty}(1+2^{-i})\prod_{i=1}^{j}(1-2^{-i})} & \text{if } q \equiv 1 \mod 4. \end{cases}$$

iii) If $\mathcal{A} = \{$quadratic extensions K of F in which no infinite place of F ramify$\}$, then

$$d_{\infty,j} = \begin{cases} \frac{2^{-j(j+2)}\prod_{i=1}^{\infty}(1-2^{-i})}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=1}^{j+2}(1-2^{-i})} & \text{if } q \equiv 3 \mod 4 \\ \frac{2^{-\frac{j(j+5)}{2}}}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=3}^{\infty}(1+2^{-i})} & \text{if } q \equiv 1 \mod 4. \end{cases}$$

Remark 2.1. *Let F be a real quadratic function field such that Sylow 2-subgroup of the class group C_F is elementary and $N\epsilon \notin \mathbb{F}_q^{*2}$, and K a quadratic extension of F where both infinite primes of F ramify as in* [2]. *Then exactly the same method used in this section works in this case too, as in* [2].

3. F is imaginary

In this section we assume that F is imaginary, that is, ∞ ramifies in F. Let K be a quadratic extension of F, where the unique infinite place ∞ of F splits in K. $\mathfrak{p}_j$, μ and a_j's are the same as in the section 1. For an ideal $\mathfrak{a}$ of $\mathbb{B}$, we define the degree $\deg_F \mathfrak{a}$ of $\mathfrak{a}$ with respect to F to be $\dim_{\mathbb{F}_q}(\mathbb{B}/\mathfrak{a})$. In fact, for $a \in \mathbb{A}$, $\deg_F(a) = 2\deg a$. Note that $\deg_F \mu$ is even and $K = F(\sqrt{\mu})$.

Lemma 3.1.

i) We have

$$(\gamma,\mu)_{\mathfrak{p}_j} = \begin{cases} 1 & \textit{if } \deg_F \mathfrak{p}_j \textit{ is even} \\ -1 & \textit{if } \deg_F \mathfrak{p}_j \textit{ is odd.} \end{cases}$$

ii) For $q \equiv 1 \mod 4$, we have

$$(-1,a_j)_\infty = (a_j,a_k)_\infty = 1.$$

iii) For $q \equiv 3 \mod 4$, we have

$$(-1,a_j)_\infty = \begin{cases} 1 & \textit{if } \deg_F \mathfrak{p}_j \textit{ is even} \\ -1 & \textit{if } \deg_F \mathfrak{p}_j \textit{ is odd} \end{cases}$$

$$(a_j,a_k)_\infty = \begin{cases} 1 & \textit{if } \deg_F \mathfrak{p}_j \textit{ or } \deg_F \mathfrak{p}_k \textit{ is even} \\ -1 & \textit{if both } \deg_F \mathfrak{p}_j \textit{ and } \deg_F \mathfrak{p}_k \textit{ are odd.} \end{cases}$$

Proof Note that $(\gamma,\mu)_{\mathfrak{p}_j} = (\gamma,a_j)_{\mathfrak{p}_j} = (\gamma,a_j)_\infty$. Then i) follows from the fact that $\deg_F \mathfrak{p}_j \equiv \deg_{\mathfrak{F}} \mathfrak{a}_j \mod 2$, since the class number h of F is odd.

ii) follows from the fact that -1 is a square in this case and that a_j is positive.

The first part of iii) is contained in i). Let $a_j^* := (-1)^{\deg_F a_j} a_j$ and $\epsilon_j = (-1)^{\deg_F a_j}$. Then

$$(a_j,a_k)_\infty = (a_j,\tilde{a_k})_\infty (a_j,\epsilon_k)_\infty = (a_j,\epsilon_k)_\infty,$$

since $(a_j,\tilde{a_k})_\infty = 1$. Since $F_\infty(\sqrt{\epsilon_k}) = \mathbb{F}_\swarrow((\Bbbk/\mathbb{T}))$ where $r = q$ or q^2 according to $\epsilon_k = 1$ or -1. Now the result follows.

Suppose that $\deg_F p_j$ is odd for $j = 1,\dots,\ell$ and $\deg_F p_j$ is even for $j = \ell+1,\dots,t$. Then, using Lemma 2.1 and the fact that

$$(a_i,a_j)_{\mathfrak{p}_j} = (a_j,a_i)_{\mathfrak{p}_i}(a_j,a_i)_\infty,$$

the associated matrix $M_K' = (m_{ij}')$ has the form

$$M_K' = \begin{bmatrix} H & | & \\ -- & | & M \\ 0 & | & \end{bmatrix},$$

where M is a symmetric $(t-1)\times(t-1)$ matrix if $q \equiv 1 \mod 4$ and

$$M = \begin{bmatrix} M_1 & | & M_2 \\ M_2^T & | & M_3 \end{bmatrix}$$

with M_1 anti-symmetric $\ell\times\ell$ matrix, M_2 arbitrary $\ell\times(t-1-\ell)$ matrix, and M_3 is symmetric $(t-1-\ell)\times(t-1-\ell)$ matrix if $q \equiv 3 \mod 4$.

Then following the ideas of Gerth, or §1, we have the following theorem.

Theorem 3.1. *Let F be an imaginary quadratic function field with odd class number. If $\mathcal{A} = \{$quadratic extensions K of Fin which the infinite place of F splits$\}$, then we have*
i) for $q \equiv 1 \mod 4$,

$$d_{\infty,j} = \frac{2^{-j(j+3)/2}}{\prod_{i=2}^{\infty}(1+2^{-i})\prod_{i=1}^{j}(1-2^{-k})}.$$

ii) for $q \equiv 3 \mod 4$,

$$d_{\infty,j} = \frac{2^{-j(j+1)}\prod_{i=1}^{\infty}(1-2^{-k})}{\prod_{i=1}^{j}(1-2^{-i})\prod_{i=1}^{j+1}(1-2^{-k})}.$$

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