MOVING LEMMA FOR ADDITIVE CHOW GROUPS AND APPLICATIONS

AMALENDU KRISHNA, JINHYUN PARK

Abstract. We study additive higher Chow groups with several modulus conditions. Apart from exhibiting the validity of all known results for the additive Chow groups with these modulus conditions, we prove the moving lemma for them: for a smooth projective variety X and a finite collection $\mathcal{W}$ of its locally closed algebraic subsets, every additive higher Chow cycle is congruent to an admissible cycle intersecting properly all members of $\mathcal{W}$ times faces. This is the additive analogue of the moving lemma for the higher Chow groups studied by S. Bloch and M. Levine.

As applications, we show that any map from a quasi-projective variety to a smooth projective variety induces a pull-back map of additive higher Chow groups. Using the moving lemma, we also establish the structure of graded-commutative differential graded algebra (CDGA) on the additive higher Chow groups.

Contents

2000 *Mathematics Subject Classification.* Primary 14C25; Secondary 19E15.
Key words and phrases. Chow group, algebraic cycle, moving lemma, CDGA.

1. Introduction

Working with algebraic cycles, formal finite sums of closed subvarieties of a variety, often requires some forms of moving results, as differential geometry often requires Sard's lemma. A classical example is Chow's moving lemma in [6] that moves algebraic cycles under rational equivalence. A modern one for higher Chow groups shows in [2, 16] that, for a smooth quasi-projective variety X and a finite set of locally closed subvarieties of X, one can move (modulo boundaries) admissible cycles to other admissible cycles that intersect a given finite set of subvarieties in the right codimensions. Any such results on moving of cycles is generally referred to as moving lemmas. Such moving results have played a very crucial role in the development and applications of the theory of higher Chow groups. The primary goal of this paper is to prove this latter kind of moving lemma for the additive higher Chow groups of a smooth and projective variety and to study some very important applications on the structural properties of additive higher Chow groups.

The additive Chow groups of zero cycles on a field were first introduced by Bloch and Esnault in [4] in an attempt to describe the K-theory and motivic cohomology of the ring of dual numbers via algebraic cycles. Bloch and Esnault [5] later defined these groups by putting a *modulus* condition on the additive Chow cycles in the hope of describing the K-groups of any given truncated polynomial ring over a field. The additive higher Chow groups of any given variety were defined in the most general form by Park in [18] and were later studied in more detail in [14], where many of the expected properties of these groups were also established.

The most crucial part of the existing definition(s) of the additive higher Chow groups which makes them distinct from the higher Chow groups, is the *modulus* condition on the admissible additive cycles. This condition also brings the extra subtlety which does not persist with the higher Chow groups. As conjectured in [14, 18], the additive higher Chow groups are expected to complement higher Chow groups for non-reduced schemes so as to obtain the right motivic cohomology groups. In particular, for a smooth projective variety X, one expects a Atiyah-Hirzebruch spectral sequence

$$\mathrm{TCH}^{-q}(X, -p-q; m) \Rightarrow K^{\mathrm{nil}}_{-p-q}(X), \tag{1.1}$$

where K^{nil} is the homotopy fiber of the restriction map $K(X \times \mathrm{Spec}(k[t])) \to K(X \times \mathrm{Spec}(k[t]/t^{m+1}))$.

Since these beliefs are still conjectural, it is not clear if the modulus conditions used to study the additive higher Chow groups of varieties in the literature are the right ones which would give the correct motivic cohomology, *e.g.*, the one which would satisfy (1.1). One aspect of this paper is to exhibit that the modulus condition (which we call M_{sup} in this paper) used in [14] may not be the best possible one, if one expects the theory of additive higher Chow groups to yield the correct motivic cohomology and the motives of non-reduced schemes.

We study the theory of additive Chow groups based on two other modulus conditions in this paper: $M = M_{sum}$ is based on the modulus condition used by Bloch-Esnault-Rülling in [5, 21], and $M = M_{ssup}$ is a new modulus condition introduced in this paper. Although this new modulus condition M_{ssup} may appear mildly stronger than that used in [14, 18], it turns out that the resulting additive Chow groups have all the properties known for the additive Chow groups of [5], [14], and [18]. In addition, we prove many other crucial structural properties of the additive higher Chow groups based on the modulus conditions M_{sum} and M_{ssup}. Although it may seem surprising, the techniques used in proving the results of this paper make one believe that such results may not be possible for the additive higher Chow groups based on the modulus condition M_{sup} of [14, 18], if Conjecture 2.9 turns out to be false.

We now outline the structure of this paper and elaborate on our main results. We define our basic objects, the additive higher Chow groups with various modulus conditions, in Section 2. We also prove some preliminary results which are used repeatedly in the paper. In Section 3, we prove the basic properties of these additive Chow groups. In particular, we demonstrate all those results for the additive higher Chow groups based on the modulus condition M_{ssup}, which are known for the additive higher Chow groups of [5], [14] and [18]. Section 4 gives the proofs of further preliminary results needed to prove our moving lemma for the additive higher Chow groups.

The subsequent Sections 5 and 6 are devoted to our first main result, the moving lemma for additive higher Chow groups. As in the case of higher Chow groups, any theory of additive motivic cohomology which would compute the K-theory as in (1.1) is expected to have a form of moving lemma to make them more amenable to deeper study. This was one of the primary motivation for working on this paper. We show in Theorem 4.1 that for a smooth projective variety X and a finite collection $\mathcal{W}$ of its locally closed algebraic subsets, every additive higher Chow cycle is congruent to an admissible cycle intersecting properly all members of $\mathcal{W}$ times faces. This is the additive analogue of the moving lemma for the higher Chow groups studied by S. Bloch and M. Levine.

While lack of this result for general quasi-projective varieties may seem disappointing, one would rather not expect this to be the case. For instance, $\mathbb{A}^1$-homotopy invariance and localization sequences fail for additive higher Chow groups, but these are indirectly implied if the moving lemma is assumed for all quasi-projective varieties such as $X \times \mathbb{A}^1$. A concrete quasi-projective example, where the standard arguments fail, is given in Example 12.2.

Our proof of the above result is broadly speaking based on the techniques of [1] and [16] that prove the analogous result for the higher Chow groups. The main difficulty with the techniques of higher Chow groups which does not immediately allow them to be adapted into the additive world is that these arguments are mostly intersection theoretic and are not equipped to handle the most delicate modulus condition of additive Chow cycles. We show one such phenomenon in the last section of this paper. However, in the case of projective varieties, we carefully modify the arguments at every step so that we can keep track of the modulus condition whenever we encounter new cycles in the process, especially in the construction of the chain homotopy variety. This is achieved using our new *containment* sort of argument and some results of [14]. This essentially proves the above theorem. On the log-additive higher Chow groups, one can prove the moving lemma for any general smooth quasi-projective varieties using our main theorem.

In Section 7, we give our first application of the moving lemma. We establish the contravariant functoriality property of the additive higher Chow groups in the most general form by showing in Theorem 7.1 that for a morphism $f : X \to Y$ of quasi-projective varieties over a field k, where Y is smooth and projective, there is a pull-back map $f^* : \mathrm{TCH}^q(Y, n; m) \to \mathrm{TCH}^q(X, n; m)$, and this satisfies the expected composition law.

If X is also smooth and projective, the pull-back map on the additive Chow groups was constructed in [14] using the action of higher Chow groups on the additive ones. However, the contravariance functoriality in this general form as above (even if Y is smooth) is new and is based on a crucial use of Theorem 4.1 as is the case of general pull-back maps of higher Chow groups (*cf.* [1, Theorem 4.1]), and another use of our containment argument. Even in the special case of X being smooth and projective, our proof is different and more direct than the one in [14].

Our final set of main results of this paper are motivated by the question of what are the important and necessary properties one would expect the additive higher Chow groups to satisfy, if they are the right motivic cohomology to compute the nil K-theory of the infinitesimal deformations of smooth schemes. In particular, one could ask what are the necessary implications on the structural properties of the additive higher Chow groups if there is indeed a spectral sequence as in (1.1). The reader would recall in this context that the K-theory of the infinitesimal deformations of the base field k is expressed in terms of the modules of absolute Kähler differentials and the absolute de Rham-Witt complex of Hesselholt-Madsen, as shown for example in [12]. For general smooth projective varieties over k, one expects these K-groups to be given by the cohomology of the absolute de Rham-Witt complex. It is well known that these de Rham-Witt complexes have the structure of a graded-commutative differential graded algebra (CDGA) and they are initial objects in the category of so called Witt complexes over a base scheme (*cf.* [13, 21]). This makes it imperative that the additive higher Chow groups posses such structures. Our next set of results together implies that this is indeed the case.

In Section 8, we show in Theorem 8.16 that the additive higher Chow groups indeed have a wedge product which makes the direct sum of all additive higher Chow groups a graded-commutative algebra. This product structure has all the functoriality properties and satisfies the projection formula. Furthermore, this is compatible with the module structure on the additive higher Chow groups over the Chow ring of the variety. We show in Theorem 9.7 that these are equipped with a differential operator, too.

For the modulus condition M_{sum}, we further show in Theorem 11.13 that these differential operators and the wedge product turn the resulting additive higher Chow groups into a CDGA. As a very important ingredient needed to achieve this, we introduce the normalized version of additive cycle complex and additive higher Chow groups in Section 11. We show that this normalized additive cycle complex is in fact quasi-isomorphic to the additive cycle complex if one uses the modulus condition M_{sum}. This is an additive analog of a similar result of S. Bloch [3] for higher Chow groups. This is the last result we need to complete the program of CDGA structure on additive higher Chow groups.

Although we do not go into this in order not to further increase the length of this seemingly long paper, the reader can easily see using our techniques that the push-forward and pull-back maps given by the power map $a \mapsto a^r$ on $\mathbb{G}_m$ (which is finite and flat) induce on the additive higher Chow groups of smooth projective varieties two operators, so called the *Frobenius* and the *Verschiebung* operators. Together with the above CDGA structure, this turns them into a Witt complex.

One hopes that this very general abstract structure of a Witt complex will help us in making a significant progress towards the eventual goal of showing that the additive higher Chow groups are the right motivic cohomology of the infinitesimal deformations of smooth schemes. This goal was in some sense the starting point of the theory of additive higher groups.

In the last section, we append some calculations of the additive higher Chow groups the authors found in the process of working on the problem. This suggests some kind of "pseudo"-$\mathbb{A}^1$-homotopy properties of additive higher Chow groups.

We finally remark that the only reason for not including the modulus condition M_{ssup} in our Theorem 11.13 is the lack of an affirmative answer of Question 11.8 in this case. We strongly believe the answer to be indeed positive and hope that a proof will be available soon.

Throughout this paper, a *k-scheme*, or a *scheme over k*, is always a separated scheme of finite type over a perfect field k. A *k-variety* is an integral k-scheme. The ground field k will be fixed throughout this paper.

2. Additive higher Chow groups

In this section, we define additive higher Chow groups from a unifying perspective than those in the literature by Bloch-Esnault, Rülling, Krishna-Levine, and Park, treating the modulus conditions as "variables". We also prove some elementary results that are needed to study and compare the additive Chow groups based on various modulus conditions.

We begin by fixing some notations which will be used throughout this paper. We write $\mathbf{Sch}/k$, $\mathbf{Sm}/k$ and $\mathbf{SmProj}/k$ for the categories of k-schemes, smooth quasi-projective varieties, and smooth projective varieties, respectively. $D^-(\mathbf{Ab})$ is the derived category of bounded above complexes of abelian groups. Recall from [14, 18] that for a normal variety X over k, and a finite set of Weil divisors $\{Y_1, \cdots, Y_s\}$ on X, the supremum of these divisors, denoted by $\sup_{1 \le i \le s} Y_i$, is the Weil divisor defined to be

$$\sup_{1 \le i \le s} Y_i = \sum_{Y \in Pdiv(X)} (\max_{1 \le i \le s} \mathrm{ord}_Y(Y_i))[Y], \tag{2.1}$$

where $Pdiv(X)$ is the set of all prime Weil divisors of X. One observes that the set of all Cartier divisors on a normal scheme X is contained in the set of all Weil divisors, and the supremum of a collection of Cartier divisors may not remain a

Cartier divisor in general, unless X is factorial. We shall need some elementary results about Cartier and Weil divisors on normal varieties.

Here are some basic facts about divisors on normal varieties:

Lemma 2.1. *Let X be a normal variety and let D_1 and D_2 be effective Cartier divisors on X such that $D_1 \geq D_2$ as Weil divisors. Let $Y \subset X$ be a closed subset which intersects D_1 and D_2 properly. Let $f : Y^N \to X$ be the composite of the inclusion and the normalization of $Y_{\rm red}$. Then $f^*(D_1) \geq f^*(D_2)$.*

Proof. For any effective Cartier divisor D on X, let $\mathcal{I}_D$ denote the sheaf of ideals defining D as a locally principal closed subscheme of X. We first claim that $D_1 \geq D_2$ if and only if $\mathcal{I}_{D_1} \subset \mathcal{I}_{D_2}$. We only need to show the only if part as the other implication is obvious. Now, $D_1 \geq D_2$ implies that $D = D_1 - D_2$ is effective as a Cartier divisor since the group of Cartier divisors forms a subgroup of Weil divisors on a normal scheme. Since $\mathcal{I}_{D_1} \subset \mathcal{I}_{D_2}$ is a local question, we can assume that $X = \mathrm{Spec}(A)$ is local normal integral scheme and $\mathcal{I}_{D_i} = (a_i)$. Put $a = a_1/a_2$ as an element of the function field of X. We need to show that $a \in A$. Since A is normal, it suffices to show that $a \in A_{\mathfrak{p}}$ for every height one prime ideal $\mathfrak{p}$ of A. But this is precisely the meaning of $D_1 \geq D_2$. This proves the claim.

Since D_i intersect Y properly, we see that $f^*(D_i)$ is a locally principal closed subscheme of Y^N for $i = 1, 2$. The lemma now follows directly from the above claim. □

The following is a refinement of [14, Lemma 3.2]:

Lemma 2.2. *Let $f : Y \to X$ be a surjective map of normal integral k-schemes. Let D be a Cartier divisor on X such that $f^*(D) \geq 0$ on Y. Then $D \geq 0$ on X.*

Proof. As is implicit in the proof of the Lemma 2.1, we can localize at the generic points of $\mathrm{Supp}(D)$ and assume that $X = \mathrm{Spec}(A)$, where A is a dvr which is essentially of finite type over k. The divisor D is then given by a rational function $a \in K$, where K is the field of fractions of A. Choosing a uniformizing parameter π of A, we can write a uniquely as $a = u\pi^n$, where $u \in A^\times$ and $n \in \mathbb{Z}$.

Since f is surjective, there is a closed point $y \in Y$ such that $f(y)$ is the closed point of X. Since Y is integral, the surjectivity of f also implies that the generic point of Y (which is also the generic point of $\mathrm{Spec}(\mathcal{O}_{Y,y})$) must go to the generic point of X under f. Hence the map $\mathrm{Spec}(\mathcal{O}_{Y,y}) \to X$ is surjective. This implies in particular that the image of π in $\mathcal{O}_{Y,y}$ is a non-zero element of the maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{Y,y}$. On the other hand, $f^*(D) \geq 0$ implies that as a rational function on Y, a actually lies in $\mathcal{O}_{Y,y}$. Since $u \in \mathcal{O}_{Y,y}^\times$ and $\pi \in \mathfrak{m}$, this can happen only when $n \geq 0$. That is, D is effective. □

We assume a k-scheme X is equi-dimensional in this paper to define the additive Chow groups although one can easily remove this condition by writing the additive Chow cycles in terms of their dimensions rather than their codimensions. Throughout this paper, for any such scheme X, we shall denote the normalization of $X_{\rm red}$ by X^N. Thus X^N is the disjoint union of the normalizations of all the irreducible components of $X_{\rm red}$.

Set $\mathbb{A}^1 := \mathrm{Spec}\, k[t]$, $\mathbb{G}_m := \mathrm{Spec}\, k[t, t^{-1}]$, $\mathbb{P}^1 := \mathrm{Proj}\, k[Y_0, Y_1]$ and let $y := Y_1/Y_0$ be the standard coordinate function on $\mathbb{P}^1$. We set $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$. For $n \geq 1$, let $B_n = \mathbb{G}_m \times \square^{n-1}$, $\widetilde{B}_n = \mathbb{A}^1 \times \square^{n-1}$, $\overline{B}_n = \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \supset \widetilde{B}_n$ and $\widehat{B}_n =$

$\mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \supset \overline{B}_n$. We use the coordinate system $(t, y_1, \cdots, y_{n-1})$ on $\widehat{B}_n$, with $y_i := y \circ q_i$, where $q_i : \widehat{B}_n \to \mathbb{P}^1$ is the projection onto the i-th $\mathbb{P}^1$.

Let $F^1_{n,i}$, for $i = 1, \dots, n-1$, be the Cartier divisor on $\widehat{B}_n$ defined by $\{y_i = 1\}$ and $F_{n,0} \subset \widehat{B}_n$ the Cartier divisor defined by $\{t = 0\}$. Notice that the divisor $F_{n,0}$ is in fact contained in $\overline{B}_n \subset \widehat{B}_n$. Let F^1_n denote the Cartier divisor $\sum_{i=1}^{n-1} F^1_{n,i}$ on $\widehat{B}_n$.

A *face* of B_n is a subscheme F defined by equations of the form

$$y_{i_1} = \epsilon_1, \dots, y_{i_s} = \epsilon_s;\ \epsilon_j \in \{0, \infty\}.$$

For $\epsilon = 0, \infty$, and $i = 1, \cdots, n-1$, let

$$\iota_{n,i,\epsilon} : B_{n-1} \to B_n$$

be the inclusion

$$\iota_{n,i,\epsilon}(t, y_1, \dots, y_{n-2}) = (t, y_1, \dots, y_{i-1}, \epsilon, y_i, \dots, y_{n-2}). \tag{2.2}$$

We now define the modulus conditions that we shall consider for defining our additive higher Chow groups.

2.1. Modulus conditions.

Definition 2.3. Let X be a k-scheme as above and let V be an integral closed subscheme of $X \times B_n$. Let $\overline{V}$ denote the closure of V in $X \times \widehat{B}_n$ and let $\nu : \overline{V}^N \to X \times \widehat{B}_n$ denote the induced map from the normalization of $\overline{V}$. We fix an integer $m \geq 1$.

(1) We say that V satisfies the modulus m condition M_{sum} (or the *sum*-modulus condition) on $X \times B_n$ if as Weil divisors on $\overline{V}^N$,

$$(m+1)[\nu^*(F_{n,0})] \leq [\nu^*(F^1_n)].$$

This condition was used by Bloch-Esnault and Rülling in [1, 21] to study additive Chow groups of zero cycles on fields.

(2) We say that V satisfies the modulus m condition M_{sup} (or the *sup*-modulus condition) on $X \times B_n$ if as Weil divisors on $\overline{V}^N$,

$$(m+1)[\nu^*(F_{n,0})] \leq \sup_{1 \leq i \leq n-1} [\nu^*(F^1_{n,i})].$$

This condition was used by Park and Krishna-Levine in [14, 18] to define their additive higher Chow groups.

(3) We say that V satisfies the modulus m condition M_{ssup} (or the *strong sup*-modulus condition) on $X \times B_n$ if there exists an integer $1 \leq i \leq n-1$ such that

$$(m+1)[\nu^*(F_{n,0})] \leq [\nu^*(F^1_{n,i})]$$

as Weil divisors on $\overline{V}^N$.

Since the modulus conditions are defined for a given fixed integer m, we shall often simply say that V satisfies a modulus condition M without mentioning the integer m. Notice that since V is contained in $X \times B_n$, its closure $\overline{V}$ intersects all the Cartier divisors $F_{n,0}$ and $F^1_{n,i}$ $(1 \le i \le n-1)$ properly in $X \times \widehat{B}_n$. In particular, their pull-backs of $F_{n,0}$ and $F^1_{n,i}$ are all effective Cartier divisors on $\overline{V}^N$. Notice also that

$$M_{ssup} \Rightarrow M_{sup} \Rightarrow M_{sum}. \tag{2.3}$$

The following restriction property of the modulus conditions M_{sum} and M_{ssup} will be used repeatedly in this paper.

Proposition 2.4 (Containment lemma)**.** *Let X be a k-scheme and let $Y \subset X \times B_n$ be a closed subvariety such that its closure $\overline{Y} \subset X \times \widehat{B}_n$ intersects the Cartier divisors $X \times F_{n,0}$ and $X \times F^1_n$ properly. Let V be an irreducible closed subvariety of $X \times B_n$ such that V satisfies the modulus condition M_{sum} or M_{ssup} on $X \times B_n$. Let $\overline{V}$ be its closure in $X \times \widehat{B}_n$. Let V_Y be an irreducible component of $V \cap Y$ and let $\widehat{V}^N_Y$ denote the normalization of the closure of V_Y in $\overline{Y}$. Let $\nu_Y : \widehat{V}^N_Y \to X \times \widehat{B}_n$ denote the natural map.*
(1) If V satisfies the modulus condition M_{ssup}, then there is an $1 \le i \le n-1$ such that $(m+1)[\nu^*_Y(F_{n,0})] \le [\nu^*_Y(F^1_{n,i})]$, that is, V_Y also satisfies M_{ssup}.
(2) If V satisfies the modulus condition M_{sum}, then $(m+1)[\nu^*_Y(F_{n,0})] \le [\nu^*_Y(F^1_n)]$, that is, V_Y also satisfies M_{sum}.

Proof. If $V \cap Y = \emptyset$, then there is nothing to prove. Hence, we assume that $V \cap Y$ is nonempty, so there is at least one nonempty irreducible component V_Y. We consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{V}^N_Y \arrow[r, hook] \arrow[d, "f_Y"'] & Z^N_1 \arrow[r, "\overline{g}"] \arrow[d, "f^N"'] & Z_1 \arrow[r, "\overline{p}"] \arrow[d, "f"'] & \overline{V}^N \arrow[d] \arrow[dd, bend left, "\nu"] \\
\widehat{V}^N_Y \arrow[r, hook] \arrow[drr, "\nu_Y"'] & Z^N \arrow[r, "g"] \arrow[dr, "h"] & Z \arrow[r, "p"] \arrow[d] & \overline{V} \arrow[d, hook] \\
 & & \overline{Y} \arrow[r, hook, "j"'] & X \times \widehat{B}_n.
\end{tikzcd} \tag{2.4}$$

Here Z and Z_1 are defined so that both the upper and the lower squares on the right are Cartesian. It is then easy to check that $Z \cap Y = V \cap Y$ and hence $\overline{V \cap Y}$ is a union of irreducible components of Z. In particular, $\widehat{V}^N_Y$ is one of the disjoint components of Z^N. Since f^N is finite and surjective, there is a component $\widetilde{V}^N_Y$ of Z^N_1 lying over $\widehat{V}^N_Y$, and the restriction f_Y of f^N also is a finite and surjective map. Since $V \cap F_{n,0} = \emptyset$ and $V_Y \neq \emptyset$, we see that $F_{n,0}$ and $F^1_{n,i}$ all intersect Z properly. Now if we use M_{ssup}, then the modulus condition for V and Lemma 2.1 imply that there is an integer $1 \le i \le n-1$ such that $\overline{g}^* \circ \overline{p}^*[\nu^*(F^1_{n,i} - (m+1)F_{n,0})] \ge 0$ on Z^N_1 and hence on $\widetilde{V}_Y^N$. In particular, by commutativity, we get ${f_Y}^*[{\nu_Y}^*(F_{n,i}^1 - (m+1)F_{n,0})] \geq 0$ on $\widetilde{V}_Y^N$. Since f_Y is finite and surjective map of normal varieties, the proposition now follows from Lemma 2.2 for the modulus condition M_{ssup}. The case of M_{sum} follows exactly the same way using F_n^1 instead of $F_{n,i}^1$, and the fact that F_n^1 is also an effective Cartier divisor. □

As one can see from the above proposition, although the modulus condition M_{sup} lies between the other two modulus conditions M_{sum} and M_{ssup}, it turns out that the additive higher Chow groups based on the latter modulus conditions have better structural properties.

In this paper, we study the additive higher Chow groups based on the modulus conditions M_{sum} and M_{ssup}. We shall show in the next section that the additive Chow groups based on our new modulus condition M_{ssup} satisfy all the properties known to be satisfied by the additive higher Chow groups of Krishna-Levine, Park, Bloch-Esnault and Rülling.

The following lemma is not used in the paper, but we decided to keep it for it might be useful for some follow-up works.

Lemma 2.5. *Let X be a normal variety and let D_1 and D_2 be effective Cartier divisors on X such that $D_1 \geq D_2$ as Weil divisors. Let X' denote the normalization of the blow-up $Bl_X(Z)$ of X along a closed subscheme $Z \subset X$ of codimension at least two. Let $f : X' \to X$ be the natural map. Then $f^*(D_1) \geq f^*(D_2)$.*

Proof. Let $D_i = \sum n_{ij} V_{ij}$ for $i = 1, 2$, where V_{ij} are prime divisors on X. Since Z is of codimension at least two, we see that for each i, $f^*(D_i) = \sum n_{ij} V'_{ij} + \sum n_l E_l$, where E_l are the components of the exceptional divisor, V'_{ij} is the proper transform of V_{ij} and $n_l \geq 0$. The lemma now immediately follows. □

2.2. **Additive cycle complex.** We define the additive cycle complex based on the above modulus conditions.

Definition 2.6. Let M be the modulus condition M_{sum} or M_{ssup}. Let X be a k-scheme, and let r, m be integers with $m \geq 1$.

(0) $\underline{\mathrm{Tz}}_r(X, 1; m)_M$ is the free abelian group on integral closed subschemes Z of $X \times \mathbb{G}_m$ of dimension r.

For $n > 1$, $\underline{\mathrm{Tz}}_r(X, n; m)_M$ is the free abelian group on integral closed subschemes Z of $X \times B_n$ of dimension $r + n - 1$ such that:

(1) (Good position) For each face F of B_n, Z intersects $X \times F$ properly:

$$\dim(Z \cap (X \times F)) \leq r + \dim(F) - 1, \text{and}$$

(2) (Modulus condition) Z satisfies the modulus m condition M on $X \times B_n$.

As our scheme X is equi-dimensional of dimension d over k, we write for $q \geq 0$

$$\underline{\mathrm{Tz}}^q(X, n; m)_M = \underline{\mathrm{Tz}}_{d+1-q}(X, n; m)_M.$$

We now observe that the good position condition on Z implies that the cycle $(\mathrm{id}_X \times \iota_{n,i,\epsilon})^*(Z)$ is well-defined and each component satisfies the good position condition. Moreover, letting $Y = X \times F$ for $F = \iota_{n,i,\epsilon}(B_{n-1})$ in Proposition 2.4, we first of all see that $\overline{Y}$ intersects $X \times F_{n,0}$ and $X \times F_n^1$ properly in $X \times \widehat{B}_n$, and each component of $(\mathrm{id}_X \times \iota_{n,i,\epsilon})^*(Z)$ satisfies the modulus condition M on $X \times B_{n-1}$. We thus conclude that if $Z \subset X \times B_n$ satisfies the above conditions (1) and (2), then every component of ${\iota_{n,i,\epsilon}}^*(Z)$ also satisfies these conditions on $X \times B_{n-1}$. In particular, we have the cubical abelian group $\underline{n} \mapsto \underline{\mathrm{Tz}}^q(X, n; m)_M$.

Definition 2.7. The *additive cycle complex* $\mathrm{Tz}^q(X, \bullet; m)_M$ of X in codimension q and with modulus m condition M is the non-degenerate complex associated to the cubical abelian group $n \mapsto \underline{\mathrm{Tz}}^q(X, n; m)_M$, i.e.,

$$\mathrm{Tz}^q(X, n; m)_M := \frac{\underline{\mathrm{Tz}}^q(X, n; m)_M}{\underline{\mathrm{Tz}}^q(X, n; m)_{M,\mathrm{degn}}}.$$

The boundary map of this complex at level n is given by $\partial = \sum_{i=1}^{n-1}(-1)^i(\partial_i^\infty - \partial_i^0)$, which satisfies $\partial^2 = 0$. The homology

$$\mathrm{TCH}^q(X, n; m)_M := H_n(\mathrm{Tz}^q(X, \bullet; m)_M);\ n \geq 1$$

is the *additive higher Chow group* of X with modulus m condition M.

From now on, we shall drop the subscript M from the notations and it will be understood that the additive cycle complex or the additive higher Chow group in question is based on the modulus condition M, where M could be either M_{sum} or M_{ssup}. The reader should however always bear in mind that these two are different objects.

There are a few comments in order. We could also have defined our additive cycle complex by taking $\underline{\mathrm{Tz}}_r(X, n; m)$ to be the free abelian group generated by integral closed subschemes of $X \times \widetilde{B}_n$ which have good intersection property with respect to the faces of $\widetilde{B}_n$, and which satisfy the modulus condition on $X \times \overline{B}_n$ (*cf.* [14, 18]). However, the following easy consequence of the modulus condition shows that this does not change the cycle complex.

Lemma 2.8. *Let M be a modulus condition in Definition 2.3. Then, there is a canonical bijective correspondence between the set of irreducible closed subvarieties $V \subset X \times B_n$ satisfying the modulus m condition M and the set of irreducible closed subvarieties $W \subset X \times \widetilde{B}_n$, whose Zariski closure in $X \times \overline{B}_n$ satisfies the modulus m condition M. Here, the correspondence is actually given by the identity map.*

Proof. First of all, since for any integral closed subscheme V of $X \times \widehat{B}_n$, the pull-back $\nu^*(F_{n,0})$ on V^N is contained in the open subset $\nu^{-1}(X \times \overline{B}_n)$, we can replace $\widehat{B}_n$ by $\overline{B}_n$ in the definition of the modulus conditions.

Now, if Σ and $\widetilde{\Sigma}$ are the two sets in the statement, then the modulus condition forces that if $V \in \Sigma$, then V is same as its closure in $X \times \widetilde{B}_n$. Conversely, if $V \in \widetilde{\Sigma}$, then the modulus condition again forces V to be contained in $X \times B_n$. $\square$

Let $\mathrm{Tz}^q(X, \bullet; m)_{sup}$ be the additive cycle complex as defined in [14, 18]. This complex is based on the modulus condition M_{sup} above. It follows from (2.3) that there are natural inclusions of cycle complexes

$$\mathrm{Tz}^q(X, \bullet; m)_{ssup} \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)_{sup} \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)_{sum} \tag{2.5}$$

and hence there are natural maps

$$\mathrm{TCH}^q(X, \bullet; m)_{ssup} \to \mathrm{TCH}^q(X, \bullet; m)_{sup} \to \mathrm{TCH}^q(X, \bullet; m)_{sum}. \tag{2.6}$$

One drawback of the cycle complex based on M_{sup} is that the underlying modulus condition for a cycle is not necessarily preserved when it is restricted to a face of B_n. This forces one to put an extra *induction* condition in the definition of $\mathrm{Tz}^q(X, \bullet; m)_{sup}$ that requires for cycles to be admissible, not only the cycles themselves to satisfy M_{sup} on $X \times B_n$, but also all their intersections with various faces to satisfy M_{sup}. In particular, as n gets large, this condition gets more serious, and it might be a very tedious job to find admissible cycles. On the other hand, the definition of our cycle complexes shows that this extra *induction* condition is superfluous for the cycle complexes based on M_{sum} or M_{ssup}. Based on this discussion and all the results of this paper, one is led to believe that even though the modulus condition M_{ssup} may appear mildly stronger and M_{sum} weaker than the modulus condition M_{sup}, the following conjecture should be true.

Conjecture 2.9. *For a smooth projective variety X over k, the natural inclusions of cycle complexes $\mathrm{Tz}^q(X, \bullet; m)_{ssup} \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)_{sup} \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)_{sum}$ are quasi-isomorphisms.*

3. Basic properties of $\mathrm{TCH}^q(X, \bullet; m)$

In this section, our aim is to demonstrate that the additive higher Chow groups defined above for M_{sum} and M_{ssup} have all the properties (except Theorem 3.7 which we do not know for M_{sum}) which are known to be true for the additive Chow groups for M_{sup} of [14, 18]. Since most of the arguments in the proofs can be given either by quoting these references verbatim or by straight-forward modifications of the same, we only give the sketches of the proofs with minimal explanations whenever deemed necessary. We begin with the following structural properties of our additive Chow groups.

Theorem 3.1. *Let $f : Y \to X$ be a morphism of k-schemes.*

(1) *If f is projective, there is a natural map of cycle complexes $f_* : \mathrm{Tz}_r(Y, \bullet; m) \to \mathrm{Tz}_r(X, \bullet; m)$ which induces the analogous push-forward map on the homology.*
(2) *If f is flat, there is a natural map of cycle complexes $f^* : \mathrm{Tz}_r(X, \bullet; m) \to \mathrm{Tz}_r(Y, \bullet; m)$ which induces the analogous pull-back map on the homology. These pull-back and push-forward maps satisfy the obvious functorial properties.*
(3) *If X is smooth and projective, there is a product*
$$\cap_X : \mathrm{CCH}^r(X, p) \otimes \mathrm{TCH}_s(X, q; m) \to \mathrm{TCH}_{s-r}(X, p+q; m),$$
that is natural with respect to flat pull-back, and that satisfies the projection formula
$$f_*(f^*(a) \cap_X b) = a \cap_Y f_*(b)$$
for $f : X \to Y$ a morphism of smooth projective varieties. If f is flat in addition, we have an additional projection formula
$$f_*(a \cap_X f^*(b)) = f_*(a) \cap_Y b.$$

(4) *If X is smooth and quasi-projective, there is a product*

$$\cap_X : \mathrm{CCH}^r(X) \otimes \mathrm{TCH}_s(X, q; m) \to \mathrm{TCH}_{s-r}(X, q; m),$$

that is natural with respect to flat pull-back, and that satisfies the projection formula

$$f_*(f^*(a) \cap_X b) = a \cap_Y f_*(b)$$

for $f : X \to Y$ a projective morphism of smooth quasi-projective varieties. If f is flat in addition, we have an additional projection formula

$$f_*(a \cap_X f^*(b)) = f_*(a) \cap_Y b.$$

Furthermore, all products are associative.

Proof. (*cf.* [14]) Granting the flat pull-back and the projective push-forward, the theorem is a direct consequence of [14, Lemmas 4.7, 4.9] whose proofs are independent of the choice of the modulus conditions of Definition 2.3, as the interested reader may verify. The proofs of the flat pull-back and projective push-forward maps on the level of cycle complexes also follow in the same way as in *loc. cit.* using our Lemma 2.2. □

Theorem 3.2 (Projective Bundle and Blow-up formulae). *Let X be a smooth quasi-projective variety and let E be a vector bundle on X of rank $r + 1$. Let $p : \mathbb{P}(E) \to X$ be the associated projective bundle over X. Let $\eta \in CH^1(\mathbb{P}(E))$ be the class of the tautological line bundle $\mathcal{O}(1)$. Then for any $q, n \geq 1$ and $m \geq 2$, the map*

$$\theta : \bigoplus_{i=0}^{r} \mathrm{TCH}^{q-i}(X, n; m) \to \mathrm{TCH}^q(\mathbb{P}(E), n; m)$$

given by

$$(a_0, \cdots, a_r) \mapsto \sum_{i=0}^{r} \eta^i \cap_{\mathbb{P}(E)} p^*(a_i)$$

is an isomorphism.

If $i : Z \to X$ is a closed immersion of smooth projective varieties and $\mu : X_Z \to X$ is the blow-up of X along Z with $i_E : E \to X_Z$ the exceptional divisor with morphism $q : E \to Z$. Then the sequence

$$0 \to \mathrm{TCH}^s(X, n; m) \xrightarrow{(i^*, \mu^*)} \mathrm{TCH}^s(Z, n; m) \oplus \mathrm{TCH}^s(X_Z, n; m) \xrightarrow{q^* - i_E^*} \mathrm{TCH}^s(E, n; m) \to 0$$

is split exact.

Proof. (*cf.* [14]) The proof of both the formulae is a consequence of the corresponding decomposition of motives in the triangulated category of Chow motives Mot_k together with the fact that the additive Chow groups can be defined as a functor of graded abelian groups on Mot_k. But this functoriality is a direct consequence of Theorem 3.1. □

Theorem 3.3. *Assume that k admits resolution of singularities. Then the functor* $\mathrm{Tz}_r(-;m) : \mathbf{SmProj}/k \to D^-(\mathbf{Ab})$ *extends to a functor*

$$\mathrm{Tz}_r^{\log}(-;m) : \mathbf{Sch}/k \to D^-(\mathbf{Ab})$$

together with a natural transformation of functors $\mathrm{Tz}_r^{\log}(-;m) \to \mathrm{Tz}_r(-;m)$ *satisfying:*

(1) Let $\mu : Y \to X$ be a proper morphism in $\mathbf{Sch}/k$, $i : Z \to X$ a closed immersion. Suppose that $\mu : \mu^{-1}(X \setminus Z) \to X \setminus Z$ is an isomorphism. Set $E := \mu^{-1}(X \setminus Z)$ with maps $i_E : E \to Y$, $q : E \to Z$. There is a canonical extension of the sequence in $D^-(\mathbf{Ab})$

$$\mathrm{Tz}_r^{\log}(E;m) \xrightarrow{(i_{E*}, -q_*)} \mathrm{Tz}_r^{\log}(Y;m) \oplus \mathrm{Tz}_r^{\log}(Z;m) \xrightarrow{\mu_* + i_*} \mathrm{Tz}_r^{\log}(X;m)$$

to a distinguished triangle in $D^-(\mathbf{Ab})$.

(2) Let $i : Z \to X$ be a closed immersion in $\mathbf{Sch}/k$, $j : U \to X$ the open complement. Then there is a canonical distinguished triangle in $D^-(\mathbf{Ab})$:

$$\mathrm{Tz}_r^{\log}(Z;m) \xrightarrow{i_*} \mathrm{Tz}_r^{\log}(X;m) \xrightarrow{j^*} \mathrm{Tz}_r^{\log}(U;m) \to \mathrm{Tz}_r^{\log}(Z;m)[1],$$

which is natural with respect to proper morphisms of pairs $(X,U) \to (X',U')$.

(3) For any $X \in \mathbf{Sch}/k$, the natural map $\mathrm{TCH}_r^{\log}(X,n;m) \to \mathrm{TCH}_{r+p}^{\log}(X \times \mathbb{A}^p, n; m)$ is an isomorphism.

Proof. (cf. [14]) This follows directly from Theorem 3.1 and Theorem 3.2 above together with the main results of Guillén and Navarro Aznar [10]. We refer [14, Section 6] for details. The natural transformation of functors is an immediate consequence of the constructions of Guillén and Navarro Aznar using the proper hyper cubical resolutions, and the proper push-forward property of additive cycle complex. ☐

Next we study the question of the existence of the regulator maps from our additive higher Chow groups to the modules of absolute Kähler differentials. First we prove the following result of Bloch-Esnault [5] and Rülling [21] on 0-cycles for the modulus condition M_{ssup}.

Theorem 3.4. *Assume that $char(k) \neq 2$ and let $\mathbb{W}_m\Omega_k^\bullet$ denote the generalized de Rham-Witt complex of Hesselholt-Madsen (cf. [21]). Then there is a natural isomorphism*

$$R_{0,m}^n : \mathrm{TCH}^n(k,n;m) \to \mathbb{W}_m\Omega_k^{n-1}.$$

Proof. This is already known for M_{sum}. For the modulus condition M_{ssup}, we first note that the map $R_{0,m}^n$ is the composite map

$$\mathrm{TCH}^n(k,n;m)_{ssup} \to \mathrm{TCH}^n(k,n;m)_{sum} \xrightarrow{\theta} \mathbb{W}_m\Omega_k^{n-1},$$

where θ is constructed in [21] and this coincides with the regulator map of Bloch-Esnault for $m = 1$. Furthermore for $m = 1$, Bloch-Esnault define the inverse map $\Omega_k^{n-1} \to \mathrm{TCH}^n(k,n;1)_{sum}$ using a presentation of Ω_k^{n-1}. The reader can easily check from the proof of [5, Proposition 6.3] that the inverse map is actually defined from Ω_k^{n-1} to $\mathrm{TCH}^n(k,n;1)_{ssup}$. This completes the proof when $m=1$.

For $m \geq 2$, the proof of K. Rülling for $\mathrm{TCH}^n(k,n;1)_{sum}$ has three main steps, namely:

(1) The existence of map $R^n_{0,m}$,

(2) The isomorphism of $R^1_{0,m}$, and

(3) The existence of transfer maps on the additive higher Chow groups for finite extensions of fields.

(4) Showing that pro-group $\{\mathrm{TCH}^n(k,n;m)\}_{n,m\geq 1}$ is an example of a restricted Witt complex (see *loc. cit.*, Remark 4.22).

We have already shown (1) for our $\mathrm{TCH}^n(k,n;m)_{ssup}$. The proof of (3) is a simple consequence of Theorem 3.1. The surjectivity part of (2) follows from the result of Rülling and the isomorphism $\underline{\mathrm{Tz}}^n(k,n;m)_{ssup} = \underline{\mathrm{Tz}}^n(k,n;m)_{sum}$. To prove injectivity, we follow the proof of Corollary 4.6.1 of *loc. cit.* and observe that if there is a cycle $\zeta \in \underline{\mathrm{Tz}}^1(k,1;m)$ such that $R^1_{0,m}(\zeta)=0$, then ζ is the boundary of a curve C which is an admissible cycle with the modulus condition M_{sum}. But then C is admissible cycle also with the modulus condition M_{ssup} since one has $M_{ssup} = M_{sup} = M_{sum}$ when $n=2$ by definition. This proves (2). Note that this does not need any assumption on the characteristic of the ground field.

We now sketch the proof of (4) to complete the proof of the theorem. We have seen in Remark 11.15 that $\left(\{\mathrm{TCH}^n(k,n;m)\}_{n,m\geq 1}, \wedge, \delta_{alt}\right)$ forms a graded-commutative differential graded algebra. It is also easy to see that the push-forward and pull-back maps for the finite and flat map $a \mapsto a^r$ on $\mathbb{G}_m$ induces the Frobenius and Verschiebung operators F_r and V_r on these additive higher Chow groups, and they satisfy $\delta_{alt}F_r = rF_r\delta_{alt}$ and $r\delta_{alt}V_r = V_r\delta_{alt}$. Moreover, the same proof as in [21, Lemma 4.17] shows that if $\mathrm{char}(k)\neq 2$, then $F_r\delta_{alt}V_r = \delta_{alt}$. This proves (3).

As shown in *loc. cit.*, the above four ingredients and the universality of the de Rham-Witt complex imply that there is a map $\mathbb{W}_m\Omega_k^{n-1} \xrightarrow{S^n_{0,m}} \mathrm{TCH}^n(k,n;m)$ which is surjective. On the other hand, one checks from the construction of the map $R^n_{0.m}$ in *loc. cit.* that $R^n_{0.m}\circ S^n_{0.m}$ is identity. □

Remark 3.5. One would like to have the assumption $\mathrm{char}(k)\neq 2$ removed from the statement of Theorem 3.4. In this context, we remark that the only place we used this assumption was to show the identity $F_r\delta_{alt}V_r = \delta_{alt}$. This is an imrovement over the result of Rülling who needs this assumption even to get a CDGA structure on the additive Chow groups. It is possible that the identity $F_r\delta_{alt}V_r = \delta_{alt}$ holds in the additive higher Chow groups for our choice of derivation even if $\mathrm{char}(k)=2$. But we have not checked this.

The following is an immediate consequence of the results of Rülling and Theorem 3.4. This gives an evidence of Conjecture 2.9.

Corollary 3.6. *For every $n,m\geq 1$, the natural maps*

$$\mathrm{TCH}^n(k,n;m)_{ssup} \to \mathrm{TCH}^n(k,n;m)_{sup} \to \mathrm{TCH}^n(k,n;m)_{sum}$$

are isomorphisms.

□

We finally turn to the regulator maps for 1-cycles as considered in [18].

Theorem 3.7. *Suppose that k is of characteristic zero and assume the modulus condition to be M_{ssup}. Then there is a natural non-trivial regulator map*

$$R^n_{1,m} : \mathrm{TCH}^{n-1}(k,n;m)_{ssup} \to \Omega^{n-3}_k. \tag{3.1}$$

This map is surjective if k is moreover algebraically closed.

Proof. The map $R^n_{1,m}$ is the composite map

$$\mathrm{TCH}^n(k,n;m)_{ssup} \to \mathrm{TCH}^n(k,n;m)_{sup} \xrightarrow{\theta} \Omega^{n-3}_k,$$

where θ is constructed in [18]. For the non-triviality of $R^n_{1,m}$, J. Park constructs a 1-cycle Γ (see [19, Proposition 1.9], [14, 7.11]) and shows (see [19, Lemmas 1.7, 1.9]) that each component of Γ in fact satisfies the modulus condition M_{ssup}. Hence $R^n_{1,m}$ is non-trivial. If $k = \overline{k}$, then the proof of the surjectivity in [14, Section 7] follows from the following:
(1) An action of $k^\times$ on $\mathrm{TCH}^n(k,n;m)$,
(2) Suitable $k^\times$-equivariance of $R^3_{1,m}$ up to a scalar,
(3) The surjectivity of $R^3_{1,m}$,
(4) The cap product $CH^n(k,n)\otimes_{\mathbb{Z}}\mathrm{TCH}^2(k,3;m) \to \mathrm{TCH}^{n+2}(k,n+3;m)$.
The action of $k^\times$ on our additive higher Chow groups is given exactly as in [14] by

$$a * (x, t_1, \cdots, t_{n-1}) = (x/a, t_1, \cdots, t_{n-1}). \tag{3.2}$$

This action extends to an action of $k^\times$ on $\widehat{B}_n$. The proof of (2) now follows from the $k^\times$-equivariance of the natural map $\mathrm{Tz}_r(k,n;m)_{ssup} \to \mathrm{Tz}_r(k,n;m)_{sup}$ and the results of [14]. The proof of (3) is a direct consequence of (1), (2) and the fact that k is algebraically closed field of characteristic zero. Finally, (4) is already shown in Theorem 3.1. □

4. Preliminaries for Moving lemma

The underlying additive cycle complexes and additive higher Chow groups in all the results in the rest of this paper will be based on the modulus condition M_{sum} or M_{ssup}, unless one of these is specifically mentioned. Our next three sections will be devoted to proving our first main result of this paper:

Theorem 4.1. *Let X be a smooth projective variety over a perfect field k. Let $\mathcal{W}$ be a finite collection of locally closed subsets of X. Then, the inclusion of additive higher Chow cycle complexes (see below for definitions)*

$$\mathrm{Tz}^q_{\mathcal{W}}(X,\bullet;m) \hookrightarrow \mathrm{Tz}^q(X,\bullet;m)$$

is a quasi-isomorphism. In other words, every admissible additive higher Chow cycle is congruent to another admissible cycle intersecting properly all given finitely many locally closed subsets of X times faces.

In this section, we set up our notations and machinery that are needed to prove this theorem, and prove some preliminary steps. Let X be a smooth projective variety over k and we fix an integer $m \geq 1$. Let $\mathcal{W}$ be a finite collection of locally closed algebraic subsets of X. If a member of $\mathcal{W}$ is not irreducible, we always replace it by all of its irreducible components so that we assume all members of $\mathcal{W}$ are irreducible. For a locally closed subset $Y \subset X$, recall that the codimension $\mathrm{codim}_X Y$ is defined to be the minimum of $\mathrm{codim}_X Z$ for all irreducible components Z of Y.

Definition 4.2. We define $\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,n;m)$ to be the subgroup of $\underline{\mathrm{Tz}}^q(X,n;m)$ generated by integral closed subschemes $Z \subset X \times B_n$ such that
(1) Z is in $\underline{\mathrm{Tz}}^q(X,n;m)$ and
(2) $\mathrm{codim}_{W\times F}(Z \cap (W \times F)) \geq q$ for all $W \in \mathcal{W}$ and all faces F of B_n.

It is easy to see that $\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,\bullet;m)$ forms a cubical subgroup of $\underline{\mathrm{Tz}}^q(X,\bullet;m)$, giving us the subcomplex

$$\mathrm{Tz}^q_{\mathcal{W}}(X,\bullet;m) = \frac{\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,\bullet;m)}{\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,\bullet;m)_{\mathrm{degn}}} \subset \mathrm{Tz}^q(X,\bullet;m).$$

Let $\mathrm{TCH}^q_{\mathcal{W}}(X,\bullet;m)$ denote the homology of the complex $\mathrm{Tz}^q_{\mathcal{W}}(X,\bullet;m)$. Then the above inclusion induces a natural map of homology

$$\mathrm{TCH}^q_{\mathcal{W}}(X,\bullet;m) \to \mathrm{TCH}^q(X,\bullet;m). \tag{4.1}$$

More generally, if $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ is a set-theoretic function, then one can define subcomplexes $\underline{\mathrm{Tz}}^q_{\mathcal{W},e}(X,\bullet;m)$ replacing the condition (2) above by
($2e$) $\mathrm{codim}_{W\times F}(Z \cap (W \times F)) \geq q - e(W)$.
In this generality, the subcomplex $\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,\bullet;m)$ is same as $\underline{\mathrm{Tz}}^q_{\mathcal{W},0}(X,\bullet;m)$.

Remark 4.3. Let Φ be the set of all set-theoretic functions $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$. Give a partial ordering on Φ by declaring $e' \geq e$ if $e'(W) \geq e(W)$ for all $W \in \mathcal{W}$. If two functions $e, e' \in \Phi$ satisfy $e' \geq e$, then for any irreducible admissible variety $Z \in Tz^q_{\mathcal{W},e}(X,n;m)$, we have

$$\mathrm{codim}_{W\times F}(Z \cap (W \times F)) \geq q - e(W) \geq q - e'(W) \tag{4.2}$$

for all $W \in \mathcal{W}$ and all faces $F \subset B_n$. Thus, we have

$$\mathrm{Tz}^q_{\mathcal{W},e}(X,n;m) \subset \mathrm{Tz}^q_{\mathcal{W},e'}(X,n;m) \quad \text{for } e \leq e'. \tag{4.3}$$

Note that if $e \in \Phi$ satisfies $e \geq q$ where q is considered as a constant function in Φ, then automatically

$$\mathrm{Tz}^q_{\mathcal{W},q}(X,n;m) = \mathrm{Tz}^q_{\mathcal{W},e}(X,n;m) = \mathrm{Tz}^q(X,n;m). \tag{4.4}$$

Since $0 \leq e$ for all $e \in \Phi$, for each triple e, e', e'' such that $e \leq e' \leq q \leq e''$, we have

$$\begin{aligned} \mathrm{Tz}^q_{\mathcal{W}}(X,n;m) &\subset \mathrm{Tz}^q_{\mathcal{W},e}(X,n;m) \subset \mathrm{Tz}^q_{\mathcal{W},e'}(X,n;m) \\ &\subset \mathrm{Tz}^q_{\mathcal{W},q}(X,n;m) = \mathrm{Tz}^q_{\mathcal{W},e''}(X,n;m) = \mathrm{Tz}^q(X,n;m). \end{aligned}$$

All these (in)equalities are equivariant with respect to the boundary maps.

Remark 4.4. The main theorem is equivalent to that the inclusion induces an isomorphism $\mathrm{TCH}^q_{\mathcal{W}}(X,n;m) \simeq \mathrm{TCH}^q(X,n;m)$ for the given modulus conditions M. This is equivalent to that for each pair $e, e' \in \Phi$ with $e \leq e'$ the inclusion induces an isomorphism

$$\mathrm{TCH}^q_{\mathcal{W},e}(X,n;m) \simeq \mathrm{TCH}^q_{\mathcal{W},e'}(X,n;m). \tag{4.5}$$

Our remaining objective in this section is to prove the following additive analogue of the spreading argument of Bloch-Levine. We begin with the following results.

Lemma 4.5. *Let $f : X \to Y$ be a projective and dominant map of integral normal varieties and let η denote the generic point of Y. Consider the fiber diagram*

$$\begin{tikzcd} X_\eta \arrow[r, "j_\eta"] \arrow[d] & X \arrow[d, "f"] \\ \{\eta\} \arrow[r] & Y. \end{tikzcd} \tag{4.6}$$

Let D be a Weil divisor on X such that $j_\eta^*(D)$ is effective. Then there is a nonempty open subset $U \subset Y$ such that if $j : f^{-1}(U) \to X$ is the open inclusion, then $j^*(D)$ is also effective.

Proof. Let $D = \sum n_i D_i$. Then $j_\eta^*(D)$ is effective if and only if for every i with $n_i < 0$, one has $D_i \cap X_\eta = \emptyset$. Since D is a finite sum, it suffices to show that if D is a prime divisor on X such that $D \cap X_\eta = \emptyset$, then there is a nonempty open subset $U \subset Y$ such that $D \cap f^{-1}(U) = \emptyset$.

Since f is projective map, it is in particular closed. Hence $f(D)$ is closed in Y. Moreover, our hypothesis implies that $f(D)$ is a proper closed subset of Y. Thus $U = Y \backslash f(D)$ is the desired open subset of Y. □

Lemma 4.6. *Let X be a quasi-projective k-variety and let $\mathcal{W}$ be a finite collection of locally closed subsets of X. Let K be a finite field extension of k. Let X_K be the base extension $X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$, and let $\mathcal{W}_K$ be the set of the base extensions of sets in $\mathcal{W}$. Then there are natural maps*

$$p^* : \frac{\mathrm{Tz}^q(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m)} \to \frac{\mathrm{Tz}^q(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(X_K, \bullet; m)}$$

$$p_* : \frac{\mathrm{Tz}^q(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(X_K, \bullet; m)} \to \frac{\mathrm{Tz}^q(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m)}$$

such that $p_* \circ p^* = [K : k] \cdot id$.

Proof. By Theorem 3.1, one as well has the flat pull-back and finite push-forward maps $\mathrm{Tz}^q_{\mathcal{W}'}(X, \bullet; m) \to \mathrm{Tz}^q_{\mathcal{W}'_K}(X_K, \bullet; m)$ and $\mathrm{Tz}^q_{\mathcal{W}'_K}(X_K, \bullet; m) \to \mathrm{Tz}^q_{\mathcal{W}'}(X, \bullet; m)$ for any $\mathcal{W}'$. Taking for $\mathcal{W}'$, the collection $\{X\}$ and also $\mathcal{W}$, and then taking the quotients of the two, we get the desired maps. The last property of the composite map is obvious from the construction of the pull-back and the push-forward maps on the additive cycle complexes (*cf.* [14]). □

Proposition 4.7 (Spreading lemma)**.** *Let $k \subset K$ be a purely transcendental extension. For a smooth projective variety X over k and any finite collection $\mathcal{W}$ of locally closed algebraic subsets of X, let X_K and $\mathcal{W}_K$ be the base extensions as before. Let $p_K : X_K \to X_k$ be the natural map. Then, the pull-back map*

$$p_K^* : \frac{\mathrm{Tz}^q(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m)} \to \frac{\mathrm{Tz}^q(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(X_K, \bullet; m)}$$

is injective on homology.

Proof. If k is a finite field, then for each prime l different from the characteristic of k, there are infinite pro-l algebraic extensions of k. Combining this with Lemma 4.6, we can assume that k is infinite. Furthermore, since the additive Chow groups of X_K is a projective limit of the additive Chow groups of X_L, where $L(\subset K)$ is a purely transcendental extension of k of finite transcendence degree over k, we can assume that the transcendence degree of K over k is finite.

Now let $Z \in \mathrm{Tz}^q(X, n; m)$ be a cycle such that $\partial Z \in \mathrm{Tz}^q_{\mathcal{W}}(X, n; m)$ where there are admissible cycles $B_K \in \mathrm{Tz}^q(X_K, n+1; m)$ and $V_K \in \mathrm{Tz}^q_{\mathcal{W}_K}(X_K, n; m)$ satisfying $Z_K = \partial(B_K) + V_K$.

We first consider the natural inclusion of complexes $\mathrm{Tz}^q(X, \bullet; m) \hookrightarrow z^q(X \times \mathbb{A}^1_k, \bullet - 1)$. Since K is the function field of some affine space $\mathbb{A}^r_k$, we can use the specialization argument for Bloch's cycle complexes (*cf.* [1, Lemma 2.3]) to find an open subset $Y \subset \mathbb{A}^r_k$ and cycles

$$B_Y \in z^q(X \times Y \times \mathbb{A}^1_k, n), \ V_Y \in z^q_{\mathcal{W}\times Y \times \mathbb{A}^1_k}(X \times Y \times \mathbb{A}^1_k, n-1)$$

such that B_K and V_K are the restrictions of B_Y and V_Y respectively to the generic point of Y and $Z \times Y = \partial(B_Y) + V_Y$. In particular, all components of B_Y and V_Y intersect all faces of $X \times Y \times B_{n+1}$ and $X \times Y \times B_n$ properly. To make B_Y and V_Y admissible additive cycles, we modify them using our Lemma 4.5.

To check the modulus condition for our cycles, let η denote the generic point $\mathrm{Spec}(K)$ of Y. Let $\widehat{B}^N_Y$ and $\widehat{V}^N_Y$ denote the normalizations of the closures of B_Y and V_Y in $X \times Y \times \widehat{B}_{n+1}$ and $X \times Y \times \widehat{B}_n$ respectively.

We first prove the admissibility under the modulus condition M_{ssup} which is a priori more difficult than M_{sum}. The admissibility of B_K and V_K implies that there are integers $1 \le i \le n$ and $1 \le i' \le n-1$ such that in the Diagram (4.6), the Weil divisors $j^*_\eta(F^1_{n+1,i} - (m+1)F_{n+1,0})$ and $j^*_\eta(F^1_{n,i'} - (m+1)F_{n,0})$ are effective on $\widehat{B}^N_{Y,\eta}$ and $\widehat{V}^N_{Y,\eta}$ respectively. Since X and $\widehat{B}_n$ are projective, the maps $\widehat{B}^N_Y, \widehat{V}^N_Y \to Y$ are projective. These maps are dominant since B_K and V_K are non-zero cycles. Thus we can apply Lemma 4.5 to find an open subset $U \subset Y$ such that $j^*_U(F^1_{n+1,i} - (m+1)F_{n+1,0})$ and $j^*_U(F^1_{n,i'} - (m+1)F_{n,0})$ are also effective. The same argument applies for the modulus condition M_{sum} as well. We just have to replace the Cartier divisors $F^1_{n+1,i}$ and $F^1_{n,i'}$ by F^1_{n+1} and F^1_n respectively. Lemma 4.5 applies in this case, too.

Replacing Y by U, we see that

$$B_U \in \mathrm{Tz}^q(X \times U, n+1; m), \ V_U \in \mathrm{Tz}^q_{\mathcal{W}\times U}(X \times U, n; m), \ \ Z \times U = \partial(B_U) + V_U. \tag{4.7}$$

Next, (4.7) implies that for a k-rational point $u \in U(k)$ such that the restrictions of B_U and V_U to $X \times \{u\}$ give well-defined cycles in $z^q(X \times \mathbb{A}^1, n)$ and $z^q_{\mathcal{W}\times \mathbb{A}^1_k}(X \times \mathbb{A}^1_k, n-1)$, one has $Z = \partial\left(i^*_u(B_U)\right) + i^*_u(V_U)$, where $i_u : X \times \{u\} \to X \times U$ is the closed immersion. We can assume that $i^*_u(B_U)$ and $i^*_u(V_U)$ are not zero. We now only need to show that $i^*_u(B_U)$ and $i^*_u(V_U)$ satisfy the modulus condition on $X \times \{u\}$. But this follows directly from (4.7) and the containment lemma, Proposition 2.4. This completes the proof of the proposition. $\square$

5. Moving lemma for projective spaces

We follow the strategy of S. Bloch to prove the moving lemma for the additive higher Chow groups. This involves proving the moving lemma first for the projective spaces and then deducing the same for general smooth projective varieties using the techniques of linear projections. This section is devoted to the proof of the moving lemma for the projective spaces. We use the following technique a few times to prove the proper-intersection properties of moved cycles with the prescribed algebraic sets.

Lemma 5.1 (*cf.* [1, Lemma 1.1])**.** *Let X be an algebraic k-scheme, and G a connected algebraic k-group acting on X. Let $A, B \subset X$ be closed subsets, and assume that the fibers of the map*

$$G \times A \to X \quad (g, a) \mapsto g \cdot a$$

all have the same dimension, and that this map is dominant. Then, there exists a non-empty open subset $U \subset G$ such that for all $g \in U$, the intersection $g(A) \cap B$ is proper in X.

Proof. Consider the fiber square

$$\begin{tikzcd} C \arrow[r] \arrow[d] & G \times A \arrow[d] \\ B \arrow[r] & X, \end{tikzcd}$$

and take

$$U = \{g \in G | \text{the fiber of } C \to G \times A \to G \text{ over } g \text{ has the smallest dimension.}\}.$$

For such $g \in U$, we have the desired property. $\square$

Proposition 5.2 (Admissibility of projective image)**.** *Let $f : X \to Y$ be a projective morphism of quasi-projective varieties over a field k. Let $Z \in \underline{\mathrm{Tz}}^r(X, n; m)$ be an irreducible admissible cycle and let $V = f(Z)$. Then $V \in \underline{\mathrm{Tz}}^s(Y, n; m)$, where s is the codimension of V in $Y \times B_n$.*

Proof. We prove it in several steps.
Claim (1): *V intersects all codimension one faces F of B_n properly in B_n.*

Consider $F = F^{\epsilon}_{n,i} = \iota_{n,i,\epsilon}(B_{n-1})$ for some $i \in \{1, 2, \cdots, n-1\}$, $\epsilon \in \{0, \infty\}$, and consider the diagram

$$\begin{tikzcd} X \times B_{n-1} \arrow[r, "\iota_{n,i,\epsilon}"] \arrow[d, "f_{n-1}"] & X \times B_n \arrow[d, "f_n"] \\ Y \times B_{n-1} \arrow[r, "\iota_{n,i,\epsilon}"] & Y \times B_n. \end{tikzcd}$$

Since F is a divisor in B_n, that V intersects $Y \times F$ properly is equivalent to that $Y \times F \not\supset V$. Towards contradiction, suppose that $V \subset Y \times F$. Then,

$$Z \subset f_n^{-1}(f_n(Z)) = f_n^{-1}(V) \subset f_n^{-1}(Y \times F) = \iota_{n,i,\epsilon}(f_{n-1}^{-1}(Y \times B_{n-1}) = X \times F.$$

By assumption, Z intersects $X \times F$ properly so that we must have $Z \not\subset X \times F$. This is a contradiction. This proves Claim (1).

Claim (2): *V intersects all lower dimensional faces of B_n properly.*

By the admissibility assumption, all cycles $\partial_i^\epsilon(Z) = Z \cap (X \times F_{n,i}^\epsilon)$ are admissible. Moreover, it is easy to see that $\partial_i^\epsilon(V) = f_{n-1}(\partial_i^\epsilon(Z))$. Thus we can replace Z by $\partial_i^\epsilon(Z)$ and apply the same argument as above; inductively we see that V has good intersection property.

Claim (3): *For each face F of B_n, including the case $F = B_n$, the cycle $V \cap (Y \times F)$ has the modulus condition.*

For any face $F = \iota(B_i) \subset B_n$, where $\iota : B_i \hookrightarrow B_n$ is a face map, and for the projections $f_i : X \times B_i \to Y \times B_i$, note that $V \cap (Y \times F) = f_n(Z \cap (X \times F)) = f_i(Z|_{X \times F})$. But the admissibility of Z implies that $Z|_{X \times F}$ is also admissible (*cf.* Proposition 2.4). Hence, replacing $Z|_{X \times F}$ by Z, we only need to prove it for $F = B_n$, that is, we just need to show that V satisfies the modulus condition. Consider the diagram

$$\begin{tikzcd} X \times B_n \arrow[r] \arrow[d, "f_n = f"] & X \times \widehat{B}_n \arrow[d, "\bar{f}_n = \bar{f}"] \\ Y \times B_n \arrow[r] & Y \times \widehat{B}_n \end{tikzcd}$$

Subclaim: Let $\overline{V}$ be the closure of V in $Y \times \widehat{B}_n$ and let $\overline{Z}$ be the closure of Z in $X \times \widehat{B}_n$. Then $\overline{V} = \bar{f}(\overline{Z})$.

Since $Z \subset f^{-1}(V) \subset \bar{f}^{-1}(\overline{V})$ and V is closed, we have $\overline{Z} \subset \overline{f}^{-1}(\overline{V})$. Hence, $\bar{f}(\overline{Z}) \subset \overline{V}$. For the other inclusion, note that $W = f(Z) \subset \bar{f}(\overline{Z})$ and $\bar{f}(\overline{Z})$ is closed because $\bar{f}$ is projective. Hence $\overline{W} \subset \bar{f}(\overline{Z})$. This proves this subclaim.

To prove the modulus condition for V, we take the normalizations $\nu_{\overline{Z}} : \overline{Z}^N \to \overline{Z}$ and $\nu_{\overline{V}} : \overline{V}^N \to \overline{V}$ of $\overline{Z}$ and $\overline{V}$, and consider the following diagram

$$\begin{tikzcd} \overline{Z}^N \arrow[r, "\nu_{\overline{Z}}"] \arrow[d, "f_Z^N"] & \overline{Z} \arrow[r, "\iota_1"] \arrow[d, "f_Z = \bar{f}|_{\overline{Z}}"] & X \times \widehat{B}_n \arrow[d, "\bar{f}"] \\ \overline{V}^N \arrow[r, "\nu_{\overline{V}}"] & \overline{V} \arrow[r, "\iota_2"] & Y \times \widehat{B}_n, \end{tikzcd}$$

where ι_1, ι_2 are the inclusions, and f_Z^N is given by the universal property of the normalization $\nu_{\overline{V}}$ for dominant morphisms. Note that f_Z^N is automatically projective and surjective because f_Z is so. Let $q_{\overline{Z}} := \iota_1 \circ \nu_{\overline{Z}}$ and $q_{\overline{V}} = \iota_2 \circ \nu_{\overline{V}}$.

Suppose Z satisfies the modulus condition M_{ssup} and consider on $\widehat{B}_n$ the Cartier divisors $D_i := F_{n,i}^1 - (m+1)F_{n,0}$ for $1 \le i \le n-1$. That the cycle Z has the modulus condition means that $[q_{\overline{Z}}^* \circ \bar{f}^*(D_i)] \ge 0$ for an index i. By the commutativity of the above diagram, this means that the Cartier divisor $f_Z^{N*}[q_{\overline{V}}^*(D_i)] \ge 0$. By Lemma 2.2, this implies that $[q_{\overline{V}}^*(D_i)] \ge 0$, which is the modulus condition for V.

If Z satisfies the modulus condition M_{sum}, we use the same argument by replacing $F^1_{n,i}$ with F^1_n. This finishes the proof of the proposition. □

Remark 5.3. In Proposition 5.2, if X is projective, $Y = \mathrm{Spec}(k)$ and $n = 1$, then V is always a single point. To see this, let $Z \subset X \times B_1 = X \times \mathbb{G}_m$ be an admissible irreducible closed subvariety. Let $V = p(Z)$, where $p : X \times \mathbb{G}_m \to \mathbb{G}_m$ is the projection.

Since X is complete, p is a closed map. Hence, $V = p(Z)$ is an irreducible closed subvariety of $\mathbb{G}_m$. But the only closed subvarieties of $\mathbb{G}_m$ are finite subsets or all of $\mathbb{G}_m$. On the other hand, if $\overline{Z}$ is the closure of Z in $X \times \mathbb{A}^1$, then the modulus condition implies that $\overline{Z} \cap (X \times \{t = 0\}) = \emptyset$. This implies that V must be a proper subset and hence a finite subset. Since V is irreducible, consequently V must be a non-zero single point.

Hence $Z = W \times \{*\}$ for a closed subvariety $W \subset X$, and a closed point $\{*\} \in \mathbb{G}_m$. Conversely, any such variety is admissible. This classifies all admissible cycles Z when X is projective and $n = 1$.

For $n > 1$, all we can say is that Z is contained in $X \times V$, where V is admissible in $\mathrm{Tz}_s(k, n; m)$ for a suitable s.

5.1. **Homotopy variety.** Now we want to construct the "homotopy variety". First, we need the following simple result:

Lemma 5.4. *Let $SL_{r+1,k}$ be the $(r+1) \times (r+1)$ special linear group over k, and let η be the generic point of the k-variety $SL_{r+1,k}$. Let K be its function field (this is a purely transcendental extension of k). Let $SL_{r+1,K} := SL_{r+1,k} \otimes_k K$ be the base change. Then, there is a morphism of K-varieties $\phi : \square^1_K \to SL_{r+1,K}$ such that $\phi(0)$ is the identity element, and $\phi(\infty)$ is the generic point η considered as a K-rational point.*

Proof. By a general result on the special linear groups, every element of $SL_{r+1,K}$ is generated by the transvections $E_{ij}(a)$, $i \neq j$, $a \in K$, that are $(r+1) \times (r+1)$ matrices whose diagonal entries are 1, the (i,j)-entry is a, and all other entries are zero.

For each pair (i,j), the collection $\{E_{ij}(a) | a \in K\}$ forms a one-parameter subgroup of $SL_{r+1,K}$ isomorphic to $\mathbb{G}_{a,K}$. Thus, for each fixed $b \in K$, define $\phi^b_{ij} : \mathbb{A}^1_K \to SL_{r+1,K}$ by $\phi^b_{ij}(y) := E_{ij}(by)$.

Express the K-rational point η of $SL_{r+1,K}$ as the (ordered) product

$$\eta = \prod_{l=1}^{p} E_{i_l j_l}(a_l), \qquad \text{for some } i_l, j_l \in \{1, 2, \cdots, r+1\},\ a_l \in K,$$

and define $\phi' : \mathbb{A}^1_K \to SL_{r+1,K}$ by $\phi' = \prod_{l=1}^p \phi^{a_l}_{i_l j_l}$. By definition, we have $\phi'(0) = \mathrm{Id}$ and $\phi'(1) = \eta$. Composing with the automorphism $\sigma : \mathbb{P}^1_K \to \mathbb{P}^1_K$ given by $y \mapsto y/(y-1)$, that isomorphically maps $\square^1_K$ to $\mathbb{A}^1_K$, we obtain $\phi = \phi' \circ \sigma| : \square^1_K \to SL_{r+1,K}$. This ϕ satisfies the desired properties. □

Recall that one consequence of Lemma 2.8 is that the additive cycle complex with modulus m can also be defined as a complex whose level n term is the free abelian group of integral closed subschemes $Z \subset X \times \widetilde{B}_n$ which have good intersection property with all faces, and which satisfy the appropriate modulus condition on $X \times \widehat{B}_n$. The following lemma uses this particular definition of the additive cycle complex.

Lemma 5.5. *Let K be the function field of $SL_{r+1,k}$, and $\phi : \square^1_K \to SL_{r+1,K}$ be as in the previous lemma. Let $SL_{r+1,K}$ act on $\mathbb{P}^r_K$ naturally. Consider the composition $H_n = p_{K/k} \circ pr'_K \circ \mu_\phi$ of morphisms*

$$\mathbb{P}^r \times \mathbb{A}^1 \times \square^n_K \xrightarrow{\mu_\phi} \mathbb{P}^r \times \mathbb{A}^1 \times \square^n_K \xrightarrow{\mathrm{pr_K}'} \mathbb{P}^r \times \mathbb{A}^1 \times \square^{n-1}_K \xrightarrow{p_{K/k}} \mathbb{P}^r \times \mathbb{A}^1 \times \square^{n-1}_k$$

where

$$\begin{cases} \mu_\phi(x, t, y_1, \cdots, y_n) := (\phi(y_1)x, t, y_1, \cdots, y_n), \\ \mathrm{pr}'_K(x, t, y_1, \cdots, y_{n-1}) := (x, t, y_2, \cdots, y_{n-1}), \\ p_{K/k} : \text{ the base change.} \end{cases}$$

Then for any $Z \in \mathrm{Tz}^q(\mathbb{P}^r_k, n; m)$, the cycle $H^*_n(Z) = \mu^*_\phi \circ \mathrm{pr}'^*(Z_K)$ is admissible, hence it is in $\mathrm{Tz}^q(\mathbb{P}^r_K, n+1; m)$. Similarly, H^*_n carries $\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_k, n; m)$ to $\mathrm{Tz}^q_{\mathcal{W}_K}(\mathbb{P}^r_K, n+1; m)$.

Proof. It is enough to prove the second assertion that for any irreducible admissible Z in $\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r, n; m)$, the variety $Z' := H^*_n(Z)$, that we informally call the "homotopy variety" of Z, satisfies the admissibility conditions of Definition 2.6.

Claim (1): *The variety Z' intersects $W \times F_K$ properly for all $W \in \mathcal{W}$ and for each face F of B_{n+1}.*

This follows from the arguments of S. Bloch and M. Levine in [1, 16] without modification. We provide its proof for sake of completeness. We use Lemma 5.1 for this purpose. We may assume that $\mathcal{W}$ contains only one non-empty algebraic set W. There are cases to consider.
Case 1. Suppose F_K is of the form $F = \mathbb{A}^1 \times \{0\} \times F'_K$ for some face $F'_K \subset \square^{n-1}_K$. In this case, $Z' \cap (W \times F_K)$ is nothing but $Z_K \cap (W \times \mathbb{A}^1 \times F'_K)$ because $\phi(0) = \mathrm{Id} \in SL_{r+1,K}$. So, proper-intersection is obvious in this case.
Case 2. Suppose F_K is any other form. It comes from some $F \subset B_{n+1}$. We apply Lemma 5.1 with $G = SL_{r+1,k}$, $X = \mathbb{P}^r \times F$, $A = W \times F$, $B = \mathrm{pr_k}'^*(Z) \cap (\mathbb{P}^r \times F)$, where G acts on X by acting trivially on F and acting naturally on $\mathbb{P}^r$. By Lemma 5.1, there is a non-empty open subset $U \subset SL_{r+1}$ such that for all $g \in U$, the intersection $g(A) \cap B$ is proper. By shrinking U if necessary, we may assume that U is invariant under taking the multiplicative inverses. Take $g = \eta^{-1} \in U$, the inverse of the generic point. Thus, after base extension to K, the intersection of $\eta^{-1}(W_K \times F_K)$ with $\mathrm{pr}'^*(Z_K) \cap (\mathbb{P}^r \times F_K)$ is proper, which means $\eta(\mathrm{pr}'^*(Z_K) \cap (\mathbb{P}^r \times F_K))$ intersects properly with $W_K \times F_K$. But the intersection $\mathrm{pr}'^*(Z_K) \cap (\mathbb{P}^r \times F_K)$ is proper, as Z was admissible. Hence, $\eta(\mathrm{pr}'^*(Z_K))$ intersects with $W_K \times F_K$ properly and consequently Z' intersects with $W_K \times F_K$ properly. This proves the claim and hence Z' has good intersection property. Thus we only need to show the modulus condition for Z' to complete the proof of the lemma.

Claim (2): *Z' satisfies the modulus condition on $\mathbb{P}^r \times \widetilde{B}_{n+1,K}$.*

We prove this using our containment lemma. In the following, we casually drop the automorphism $\tau : \mathbb{P}^r \times \mathbb{A}^1 \times \square^n \to \mathbb{P}^r \times \mathbb{A}^1 \times \square^n$ that maps $(x, t, y_1, \cdots, y_n)$ to $(x, t, y_2, \cdots, y_n, y_1)$ from our notations for simplicity.

Take $V = p(Z)$, where $p : \mathbb{P}^r \times \widetilde{B}_n \to \widetilde{B}_n$ is the projection. Because $Z \subset p^{-1}(p(Z)) = \mathbb{P}^r \times V$, we have

$$Z' = \mu_\phi^*(Z \times \square^1_K) \subset \mu_\phi^*(\mathbb{P}^r \times V \times \square^1_K) = \mathbb{P}^r \times V \times \square^1_K =: Z_1, \text{say.} \tag{5.1}$$

Now, Proposition 5.2 implies that V is an irreducible admissible closed subvariety of $\widetilde{B}_n$. The flat pull-back property in turn implies that $p^*([V]) = \mathbb{P}^r \times V$ is an irreducible admissible closed subvariety of $\mathbb{P}^r \times \widetilde{B}_n$. In particular, the modulus condition holds for $\mathbb{P}^r \times V$. If $\overline{V}$ is the closure of V in $\widehat{B}_n$, then commutativity of the diagram

$$\begin{tikzcd}
\overline{Z}_1^N = \mathbb{P}^r \times \overline{V}^N \times \mathbb{P}^1_K \arrow[r] \arrow[d] & \mathbb{P}^r \times \widehat{B}_{n+1,K} \arrow[r] \arrow[d] & \widehat{B}_{n+1,K} \arrow[d] \\
\mathbb{P}^r \times \overline{V}^N \arrow[r] & \mathbb{P}^r \times \widehat{B}_n \arrow[r] & \widehat{B}_n
\end{tikzcd}$$

now implies that Z_1 satisfies the modulus condition on $\mathbb{P}^r \times \widetilde{B}_{n+1,K}$ even though it is a degenerate additive cycle. Furthermore, the admissibility of Z and the fact that μ_ϕ is an automorphism, imply that $\overline{Z}'$ intersects the Cartier divisors F^1_{n+1} and $F_{n+1,0}$ properly. Thus we can use (5.1) and apply Proposition 2.4 (with "X" $= \mathbb{P}^r_K$, "Y" $= Z'$ and "V" $= Z_1$) to conclude that Z' satisfies the modulus condition. This completes the proof of the lemma. $\square$

Lemma 5.6. *The collection $H^*_\bullet : \mathrm{Tz}^q(\mathbb{P}^r_k, \bullet; m) \to \mathrm{Tz}^q(\mathbb{P}^r_K, \bullet + 1; m)$ is a chain homotopy satisfying $\partial H^* + H^* \partial = p^*_{K/k}(Z) - \eta(Z_K)$. The same is true for $\mathrm{Tz}^q_{\mathcal{W}}$.*

Proof. It is enough to prove the second assertion. This is straightforward: let $Z \in \mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_k, n; m)$. Then

$$\begin{aligned}
H^*\partial(Z) &= H^*\left(\sum_{i=1}^{n}(-1)^i(\partial_i^\infty - \partial_i^0)(Z)\right) \\
&= \sum_{i=1}^{n}(-1)^i(\mu_\phi^* \mathrm{pr}'^* p_{K/k}^*)(\partial_i^\infty - \partial_i^0)(Z) \\
&= \sum_{i=1}^{n}(-1)^i(\partial_{i+1}^\infty - \partial_{i+1}^0)(\mu_\phi^* \mathrm{pr}'^* p_{K/k}^*(Z)) \\
&= -\sum_{i=2}^{n+1}(-1)^i(\partial_i^\infty - \partial_i^0)(H^*(Z)), \\
\partial H^*(Z) &= \sum_{i=1}^{n+1}(-1)^i(\partial_i^\infty - \partial_i^0)H^*(Z) \\
&= \sum_{i=1}^{n+1}(-1)^i(\partial_i^\infty - \partial_i^0)(H^*(Z)) \\
&= (-1)(\partial_1^\infty - \partial_1^0)(H^*(Z)) + \sum_{i=2}^{n+1}(-1)^i(\partial_i^\infty - \partial_i^0)(H^*(Z)).
\end{aligned}$$

Hence,

$$(\partial H^* + H^*\partial)(Z) = (\partial_1^0 - \partial_1^\infty)(H^*(Z)) = p_{K/k}^*(Z) - \eta(Z_K).$$

This proves the lemma. □

5.2. **Proof of the moving lemma for projective spaces.** We are now ready to finish the proof of Theorem 4.1 for $\mathbb{P}^r$.

By the Lemma 5.6, the base extension

$$p_{K/k}^* : \frac{\mathrm{Tz}^q(\mathbb{P}^r_k, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_k, \bullet; m)} \to \frac{\mathrm{Tz}^q(\mathbb{P}^r_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(\mathbb{P}^r_K, \bullet; m)}$$

is homotopic to the map $\eta p_{K/k}^*$. Note for each admissible cycle $Z \in \mathrm{Tz}^q(\mathbb{P}^r_k, n; m)$, the cycle $\eta(Z_K)$ lies in $\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_K, n; m)$. We can prove it as before.

We may assume that $\mathcal{W}$ has only one nonempty algebraic set, say W. In the Lemma 5.1, take $G = SL_{r+1}$, $X = \mathbb{P}^r \times B_n$ where G acts on $\mathbb{P}^r_K$ naturally and B_n trivially. Let F be a face B_n. Let $A = W \times F$, $B = Z \cap (\mathbb{P}^r \times F)$. Since SL_{r+1} acts transitively on $\mathbb{P}^r$, the map $G \times A \to X$ is surjective. Hence, by the Lemma 5.1, there is a non-empty open subset $U \subset G$ such that for all $g \in U$, the intersection $g(A) \cap B$ is proper in X. By shrinking U further, we may assume that U is closed under taking multiplicative inverse of η. Taking $g = \eta^{-1}$, the inverse of the generic point, we see that after base extension to K, the intersection of $\eta^{-1}(W \times F)$ with $Z_K \cap (\mathbb{P}^r \times F_K)$ is proper, which means $\eta(Z_K \cap (\mathbb{P}^r \times F_K))$ intersects $W_K \times F_K$

properly. Since Z_K intersects with $\mathbb{P}^r \times F_K$ properly by the assumption, we conclude that $\eta(Z_K)$ intersects $W_K \times F_K$ properly. Thus, $\eta(Z_K) \in \mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_K, n; m)$. Hence, the induced map on the quotient

$$\eta p^*_{K/k} : \frac{\mathrm{Tz}^q(\mathbb{P}^r_k, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_k, \bullet; m)} \to \frac{\mathrm{Tz}^q(\mathbb{P}^r_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(\mathbb{P}^r_K, \bullet; m)}$$

is zero. Hence the base extension $p^*_{K/k}$ induces a zero map on homology since it is homotopic to the zero map.

On the other hand, by the spreading lemma, Proposition 4.7, the chain map $p^*_{K/k}$ is injective on homology. Hence the quotient complex $\mathrm{Tz}^q(\mathbb{P}^r_k, \bullet; m)/\mathrm{Tz}^q_{\mathcal{W}}(\mathbb{P}^r_k, \bullet; m)$ must be acyclic. This proves Theorem 4.1 for the projective spaces. □

6. Generic projections and moving lemma for projective varieties

6.1. **Generic projections.** This section begins with a review of some facts about linear projections. In combination with the moving lemma for $\mathbb{P}^r$, that we saw in the previous section, we prove the moving lemma for general smooth projective varieties.

Lemma 6.1. *Consider two integers $N > r > 0$. Then for each linear subvariety $L \subset \mathbb{P}^N$ of dimension $N-r-1$, there exists a canonical linear projection morphism $\pi_L : \mathbb{P}^N \backslash L \to \mathbb{P}^r$.*

Proof. Fix the coordinates $x = (x_0; \cdots; x_N)$ of $\mathbb{P}^N$. A linear subvariety L is given by $(r+1)$ homogeneous linear equations in x whose corresponding $(N+1)\times(r+1)$ matrix A has the full rank $r+1$. Take the reduced row echelon form of A whose rows are the linear homogeneous functions $P_0(x), \cdots, P_r(x)$ in x.

For $x \in \mathbb{P}^N \backslash L$, define $\pi_L(x) := (P_0(x); \cdots; P_r(x))$. Since $x \notin L$, we have some $P_i(x) \neq 0$ so that the map π_L is well-defined. By elementary facts about reduced row echelon forms and row equivalences, the subvariety L uniquely decides this map π_L in this process. □

Let X be a smooth projective k-variety. Let $r = \dim X$. Suppose that we have an embedding $X \hookrightarrow \mathbb{P}^N$ for some $N > r$. Consider $\pi_L : \mathbb{P}^N \backslash L \to \mathbb{P}^r$. Whenever $L \cap X = \emptyset$, we have a finite morphism $\pi_{L,X} := \pi_L|_X : X \to \mathbb{P}^r$. Such L's form a non-empty open subset $Gr(N-r-1, N)_X$ of the Grassmannian $Gr(N-r-1, N)$. Note that such a map π_L is automatically flat since X is smooth (*cf.* [11, Ex. III-10.9, p. 276]). In particular, the pull-back $\pi^*_{L,X}$ and push-forward $\pi_{L,X*}$ are defined by Theorem 3.1.

For any closed integral admissible cycle Z on $X \times B_n$, define $\widetilde{L}(Z)$ to be

$$\widetilde{L}(Z) := \pi^*_{L,X}(\pi_{L,X_*}([Z])) - [Z].$$

Extending this map linearly, this defines a morphism of complexes

$$\widetilde{L} : \mathrm{Tz}^q(X, \bullet; m) \to \mathrm{Tz}^q(X, \bullet; m).$$

6.2. **Chow's moving lemma.** Recall that for two locally closed subsets A, B of pure codimension a and b, the *excess* of A, B is defined to be

$$e(A, B) := \max\{a + b - \mathrm{codim}_X(A \cap B), 0\}.$$

That the intersection $A \cap B$ is proper means $e(A, B) = 0$. If A, B are cycles, then we define $e(A, B) := e(\mathrm{Supp}(A), \mathrm{Supp}(B))$. The excess measures how far an intersection is from being proper.

Lemma 6.2 (cf. [14, Lemma 1.12])**.** *Let $X \subset \mathbb{P}^N$ be a smooth closed projective k-subvariety of dimension r. Let Z, W be cycles on X. Then there is a non-empty open subscheme $U_{Z,W} \subset Gr(N - r - 1, N)_X$ such that for each field extension $K \supset k$ and each K-point L of $U_{Z,W}$, we have*

$$e(\widetilde{L}(Z), W) \leq \max\{e(Z, W) - 1, 0\}.$$

For its proof, see J. Roberts [20, Main Lemma, p. 93], or [16, Lemma 3.5.4, p. 96] for a slightly different but equivalent version. The point of the projection business is the following lemma:

Lemma 6.3. *Let X be a smooth projective k-variety, and let $\mathcal{W}$ be a finite set of locally closed algebraic subsets of X. Let $m, n \geq 1$, $q \geq 0$ be integers. Let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set-theoretic function. Define $e - 1 : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ by*

$$(e - 1)(W) := \max\{e(W) - 1, 0\}.$$

Let K be the function field of $Gr(N-r-1, N)$, and let $L_{gen} \in Gr(N-r-1, N)_X(K)$ be the generic point. Then, the map

$$\widetilde{L}_{gen} : Tz^q(X, \bullet; m) \to Tz^q(X_K, \bullet; m)$$

maps $\mathrm{Tz}^q_{\mathcal{W},e}(X, \bullet; m)$ to $\mathrm{Tz}^q_{\mathcal{W}_K,e-1}(X_K, \bullet; m)$.

Proof. The arguments of [14, Lemma 1.13, p. 84] or [16, §3.5.6, p. 97] work in this additive context without change. The central idea is to use a variation of Chow's moving lemma as in Lemma 6.2. $\square$

6.3. **Proof of the moving lemma.**

Proof of Theorem 4.1. Let L_{gen} be the generic point of the Grassmannian $Gr(N - r - 1, N)$ as in Lemma 6.3. Then, for each function $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$, the morphism

$$\widetilde{L}_{gen} = \pi^*_{L_{gen}} \circ \pi_{L_{gen}*} - p^*_{K/k} : \frac{\mathrm{Tz}^q_{\mathcal{W},e}(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W},e-1}(X, \bullet; m)} \to \frac{\mathrm{Tz}^q_{\mathcal{W}_K,e}(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K,e-1}(X_K, \bullet; m)}$$

is zero. Hence $\pi^* L_{gen} \circ \pi_{L_{gen}*}$ is equal to the base extension morphism $p^*_{K/k}$ on the quotient complex.

On the other hand, $\pi^* L_{gen} \circ \pi_{L_{gen}*}$ factors as

$$\frac{\mathrm{Tz}^q_{\mathcal{W},e}(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W},e-1}(X, \bullet; m)} \xrightarrow{\pi_{L_{gen}*}} \frac{\mathrm{Tz}^q_{\mathcal{W}',e'}(\mathbb{P}^r_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}',e'-1}(\mathbb{P}^r_K, \bullet; m)} \xrightarrow{\pi^*_{L_{gen}}} \frac{\mathrm{Tz}^q_{\mathcal{W}_K,e}(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K,e-1}(X_K, \bullet; m)},$$

where $\mathcal{W}'$ and e' are defined as follows: for each $W \in \mathcal{W}$, the constructible subset $\pi_{L_{gen}}(W)$ can be written as

$$\pi_{L_{gen}}(W) = W'_1 \cup \cdots \cup W'_{i_W}$$

for some $i_W \in \mathbb{N}$ and locally closed irreducible sets W'_j in $\mathbb{P}^r_K$. Let $d_j = \operatorname{codim}_{\mathbb{P}^n_K}(W'_j) - \operatorname{codim}_X(C)$. Let $\mathcal{W}' = \{W'_j | W \in \mathcal{W}\}$. Define $e' : \mathcal{W}' \to \mathbb{Z}_{\geq 0}$ by the rule $e'(W'_j) := e(W) + d_j$. We have already shown in Section 5.2 that the moving lemma is true for all projective spaces. In particular, for all functions $e' : \mathcal{W}' \to \mathbb{Z}_{\geq 0}$, the complex in the middle

$$\frac{\mathrm{Tz}^q_{\mathcal{W}'_K, e'}(\mathbb{P}^r_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}'_K, e'-1}(\mathbb{P}^r_K, \bullet; m)}$$

is acyclic (see Remark 4.4). Hence, the base extension map

$$p^*_{K/k} : \frac{\mathrm{Tz}^q_{\mathcal{W}, e}(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}, e-1}(X, \bullet; m)} \to \frac{\mathrm{Tz}^q_{\mathcal{W}_K, e}(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K, e-1}(X_K, \bullet; m)}$$

is zero on homology. Consequently, by induction, the base extension map

$$p^*_{K/k} : \frac{\mathrm{Tz}^q(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m)} \to \frac{\mathrm{Tz}^q(X_K, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}_K}(X_K, \bullet; m)}$$

is zero on homology. On the other hand, this map is also injective on homology by Proposition 4.7. This happens only when

$$\frac{\mathrm{Tz}^q(X, \bullet; m)}{\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m)}$$

is acyclic, that is, the inclusion

$$\mathrm{Tz}^q_{\mathcal{W}}(X, \bullet; m) \to \mathrm{Tz}^q(X, \bullet; m)$$

is a quasi-isomorphism. This finishes the proof of Theorem 4.1. □

7. Application to contravariant functoriality

In this section, we prove the following general contravariance property of the additive higher Chow groups as an application of the moving lemma.

Theorem 7.1. *Let $f : X \to Y$ be a morphism of quasi-projective varieties over k, where Y is smooth and projective. Then there is a pull-back map*

$$f^* : \mathrm{TCH}^q(Y, n; m) \to \mathrm{TCH}^q(X, n; m)$$

such that for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ with Y and Z smooth and projective, we have

$$(g \circ f)^* = f^* \circ g^* : \mathrm{TCH}^q(Z, n; m) \to \mathrm{TCH}^q(X, n; m).$$

Before proving this functoriality, we mention one more consequence of our containment lemma (Proposition 2.4).

Corollary 7.2. *Let $X \xrightarrow{i} Y$ be a regular closed embedding of quasi-projective but not necessarily smooth varieties over k. Then there is a Gysin chain map of additive cycle complexes*

$$i^* : \mathrm{Tz}^q_{\{X\}}(Y, \bullet; m) \to \mathrm{Tz}^q(X, \bullet; m).$$

Proof. Let $\iota : Z \subset Y \times B_n$ be a closed irreducible admissible subvariety in $\mathrm{Tz}^q_{\{X\}}(Y, n; m)$. By assumption, Z intersects all faces $X \times F$ properly. Hence the abstract intersection product of cycles $(X \times B_n) \cdot Z = [\iota^*(X \times B_n)] \in z^q(X \times B_n)$ is well defined, and the intersection formula for the regular embedding implies that this intersection product commutes with the boundary maps ([8, §2.3, §6.3]). We want this cycle to be $i^*(Z)$. Thus we only need to show that each component of $Z \cap (X \times B_n)$ satisfies the modulus condition in order for i^* to be a map of additive cycle complexes. Since $X \times \widehat{B}_n$ clearly intersects F^1_n and $F_{n,0}$ properly on $Y \times \widehat{B}_n$, this modulus condition follows directly from Proposition 2.4, for Z has the modulus condition. ☐

Proof of Theorem 7.1. We do this by imitating the proof of [1, Theorem 4.1]. So, let $f : X \to Y$ be a map as in Theorem 7.1. Such a morphism can be factored as the composition $X \xrightarrow{gr_f} X \times Y \xrightarrow{pr_2} Y$, where gr_f is the graph of f and pr_2 is the projection. Notice that pr_2 is a flat map and moreover, the smoothness of Y implies that gr_f is a regular closed embedding. Let $\Gamma_f \subset X \times Y$ denote the image of gr_f which is necessarily closed.

For $0 \le i \le \dim Y$, let Y_i be the Zariski closure of the collection of all points $y \in Y$ such that $\dim f^{-1}(y) \ge i$. We use the convention that $\dim \emptyset = -1$. Let $\mathcal{W}$ be the collection of the irreducible components of all Y_i. Then $\mathcal{W}$ is a finite collection.

Claim : *Let $Z \in \mathrm{Tz}^q_{\mathcal{W}}(Y, n; m)$ be an irreducible admissible closed subvariety of $Y \times B_n$. Then $(pr_2 \times \mathrm{Id}_{B_n})^{-1}(Z) = X \times Z$ in $X \times Y \times B_n$ is an admissible closed subset that intersects $\Gamma_f \times F$ properly in $X \times Y \times B_n$ for all faces $F \subset B_n$. This gives a chain map*

$$pr_2{}^* : \mathrm{Tz}^q_{\mathcal{W}}(Y, \bullet; m) \to \mathrm{Tz}^q_{\{\Gamma_f\}}(X \times Y, \bullet, m).$$

That $(pr_2 \times \mathrm{Id}_{B_n})^{-1}(Z) = X \times Z$ is admissible is obvious by [14, §3.4]. Since Z intersects $W \times F$ properly for all $W \in \mathcal{W}$ and faces $F \subset B_n$, we have

$$\dim \widetilde{Z}_i \le \dim Y_i + \dim F - q, \quad \text{where } \widetilde{Z}_i := Z \cap (Y_i \times F).$$

Now, $(X \times Z) \cap (\Gamma_f \times F) = \cup_i X \times \widetilde{Z}_i$, and for each i we have

$$\begin{aligned} \dim(X \times \widetilde{Z}_i) &= \dim X + \dim \widetilde{Z}_i \\ &\le \dim X + \dim F - q = \dim(\Gamma_f \times F) - q. \end{aligned}$$

Hence $\mathrm{codim}_{\Gamma_f \times F}(X \times Z) \cap (\Gamma_f \times F) \ge q$ and we have the desired map $pr_2{}^* : \mathrm{Tz}^q_{\mathcal{W}}(Y, n; m) \to \mathrm{Tz}^q_{\{\Gamma_f\}}(X \times Y, n; m)$ for each $n \ge 1$. That this gives a chain map is obvious since f^* clearly commutes with the boundary maps. This proves the Claim.

The pull-back map f^* is now given by composing pr_2^* with the Gysin map gr^* of Corollary 7.2 and then using the moving lemma, Theorem 4.1. The composition law can be checked directly from the construction of f^*. This completes the proof of Theorem 7.1. $\square$

8. Algebra structure on additive higher Chow groups

In this section, we consider an algebra structure on the additive higher Chow groups of smooth projective varieties. This algebra structure corresponds to the exterior product on the cohomology of the sheaves of absolute Kähler differentials on the smooth projective varieties. We show that this algebra structure on the additive higher Chow groups is compatible with the module structure on these groups for the ordinary Chow ring of the variety. We draw some important consequences of this towards the end of this section. We shall deduce our algebra structure on the additive higher Chow groups as a consequence of the following general result whose proof will occupy most of this section.

Proposition 8.1. *Let X and Y be smooth projective varieties over a field k. Then there exists an external wedge product on the additive higher Chow groups*

$$\overline{\wedge} : \mathrm{TCH}^{q_1}(X, n_1; m) \otimes_{\mathbb{Z}} \mathrm{TCH}^{q_2}(Y, n_2; m) \to \mathrm{TCH}^{q}(X \times Y, n; m), \tag{8.1}$$

where $q = q_1 + q_2 - 1$, $n = n_1 + n_2 - 1$, and $q_i, n_i, m \geq 1$ for $i = 1, 2$. In the case of $X = Y$, one has

$$\xi \overline{\wedge} \eta = (-1)^{(n_1-1)(n_2-1)} \eta \overline{\wedge} \xi \tag{8.2}$$

for all classes $\xi \in \mathrm{TCH}^{q_1}(X, n_1; m)$ and $\eta \in \mathrm{TCH}^{q_2}(X, n_2; m)$.

8.1. **External wedge product.** The external wedge product is based on the product map $\mu : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ which clearly extends to the product map

$$\mu : \mathbb{G}_m \times \mathbb{P}^1 \to \mathbb{P}^1. \tag{8.3}$$

Note that this product defines a $\mathbb{G}_m$-action on $\mathbb{P}^1$ and hence is a smooth map.

We define the external product at the level of cycle complexes in the following way.

$$\begin{tikzcd} X \times \mathbb{G}_m \times \square^{n_1-1} \times Y \times \mathbb{G}_m \times \square^{n_2-1} \arrow[r, "\tau"] \arrow[dr, "\mu"'] & X \times Y \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n-1} \arrow[d, "1\times 1\times\mu\times 1"] \\ & X \times Y \times \mathbb{G}_m \times \square^{n-1}, \end{tikzcd} \tag{8.4}$$

where τ is the transposition map $(x, t, y, x', t', y') \mapsto (x, x', t, t', y, y')$. We denote the composite map also by μ.

Let $V_1 \in \mathrm{Tz}^{q_1}(X, n_1; m)$ and $V_2 \in \mathrm{Tz}^{q_2}(Y, n_2; m)$ be two irreducible admissible cycles. Define $\mu_*(V_1 \times V_2)$ to be the Zariski closure of $\mu(V_1 \times V_2)$ in $X \times Y \times B_n$. We first claim that $\mathrm{codim}_{X \times Y \times B_n}(\mu_*(V_1 \times V_2)) = q$, or, equivalently, $\dim(\mu(V_1 \times V_2)) = \dim(V_1) + \dim(V_2)$.

This is obvious if one of the V_i's lie in a fiber of the projection map to $\mathbb{G}_m$. Otherwise, the modulus condition implies that none of these can be of the form $W \times \mathbb{G}_m$. Thus, the set of points of W such that the fiber of V_i is $\mathbb{G}_m$ must be nowhere dense. In particular, there is a dense subset of closed points of W such that the fiber of V_i over this subset must be nowhere dense in $\mathbb{G}_m$. But then this fiber must be finite. Hence for any $c \in \mathbb{G}_m$ in a dense open subset, there are points $(a, t_1) \in \mathbb{G}_m \times X \times \square^{n_1-1}$ such that $(a', t_1) \notin \mathbb{G}_m \times X \times \square^{n_1-1}$ and $aa' \neq c$. Also, we have then $(ca^{-1}, t_2) \in \mathbb{G}_m \times Y \times \square^{n_1-1}$ as V_i map dominantly onto an open subset of $\mathbb{G}_m$. But then we see that $(a, ca^{-1}, t_1, t_2) \in V_1 \times V_2$ but $(a', ca'^{-1}, t_1, t_2) \notin V_1 \times V_2$. On the other hand, we have $\mu(a, ca^{-1}, t_1, t_2) = \mu(a', ca'^{-1}, t_1, t_2)$. This implies that $V_1 \times V_2 \neq \mu^{-1}(\mu(V_1 \times V_2))$. Since μ is flat of relative dimension one, this implies that $\dim(\mu(V_1 \times V_2)) = \dim(V_1) + \dim(V_2)$, proving the claim.

Thus we have shown that if $V_1 \in \mathrm{Tz}^{q_1}(X, n_1; m)$ and $V_2 \in \mathrm{Tz}^{q_1}(Y, n_2; m)$ are two irreducible admissible additive cycles, then $\mu_*(V_1 \times V_2)$ is a closed subvariety of $X \times Y \times B_n$ of codimension q. Our aim is to show the admissibility of $\mu_*(V_1 \times V_2)$ as an additive cycle which we do in several steps. Let us denote $\mu_*(V_1 \times V_2)$ by Z.

Lemma 8.2. *The cycle Z has the proper intersection property with all faces of B_n.*

Proof. To show the good intersection property, it is enough to intersect the Zariski-dense open subset $\mu(Z_1 \times Z_2)$ with $X \times Y \times F$ for any face F of B_n. Write $F = \mathbb{G}_m \times F_1 \times F_2$ for some faces $F_1 \subset \square^{n_1-1}$ and $F_2 \subset \square^{n_2-1}$.

Since the multiplication μ is equivariant with respect to all face maps ∂_i^ϵ given by the intersection with a codimension 1-face of $\square^{n_i-1}$, and since the faces F_i are obtained by intersecting a multiple number of those codimension 1-faces, we immediately see that $\mu(V_1 \times V_2)$ intersects $X \times Y \times F$ properly if V_1 intersects $X \times \mathbb{G}_m \times F_1$ properly and V_2 intersects $Y \times \mathbb{G}_m \times F_2$ properly. But this is indeed the case. $\square$

Proposition 8.3. *The cycle Z satisfies the modulus condition in $X \times Y \times B_n$.*

Proof. For the structure morphisms $p : X \to \mathrm{Spec}(k)$ and $p' : Y \to \mathrm{Spec}(k)$, consider $W_1 = (p \times \mathrm{Id}_{B_{n_1}})(V_1)$ and $W_2 = (p' \times \mathrm{Id}_{B_{n_2}})(V_2)$. By the admissibility of the projective images, Proposition 5.2, W_1 and W_2 are admissible in B_{n_1} and B_{n_2} respectively. In particular, $X \times W_1$ and $Y \times W_2$ are admissible cycles by the flat pull-back (*cf.* Theorem 3.1). Now, we have

$$V_1 \subset X \times W_1, \quad V_2 \subset Y \times W_2,$$

which implies that

$$Z = \mu_*(V_1 \times V_2) \subset X \times Y \times \mu_*(W_1 \times W_2). \tag{8.5}$$

Since $\overline{Z}$ intersects F_n^1 and $F_{n,0}$ properly in $X \times Y \times \widehat{B}_n$, we can use Proposition 2.4 to conclude that Z has the modulus condition if $\mu_*(W_1 \times W_2)$ has. That is, we reduce to the case when $X = Y = \mathrm{Spec}(k)$.

We first dispose of the case of the modulus condition M_{sum} as it is relatively straightforward. Let $\overline{V_1} \subset \overline{B}_{n_1}, \overline{V_2} \subset \overline{B}_{n_2}$, and $\overline{Z} \subset \overline{B}_n$ be the Zariski closures of V_1, V_2, and Z respectively. Take their normalizations $\nu_{\overline{V}_1} : \overline{V}_1^N \to \overline{V}_1$, $\nu_{\overline{V}_2} : \overline{V}_2^N \to \overline{Z}_2$, and $\nu_{\overline{Z}} : \overline{Z}^N \to \overline{Z}$. By [14, Lemma 3.1], the product of two reduced normal finite type k-schemes is again normal over perfect fields. Thus, the morphism

$$\nu := \nu_{\overline{V}_1} \times \nu_{\overline{V}_2} : \overline{V}_1^N \times \overline{V}_2^N \to \overline{V}_1 \times \overline{V}_2 = \overline{V_1 \times V_2}$$

is a normalization, under the identification $\mathbb{A}^2 \times \square^{n-1} = \mathbb{A}^1 \times \square^{n_1-1} \times \mathbb{A}^1 \times \square^{n_2-1}$. Thus, we can regard $\overline{V}_1^N \times \overline{V}_2^N$ as $\overline{V_1 \times V_2}^N$. This gives the following diagram:

$$\begin{tikzcd} \overline{V}_1^N \times \overline{V}_2^N \arrow[r, "\nu"] \arrow[d, "\overline{\mu}^N"] & \overline{V}_1 \times \overline{V}_2 \arrow[r, "\iota_1 \times \iota_2"] \arrow[d, "\overline{\mu}"] & \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1} \arrow[d, "\mu"] \\ \overline{Z}^N \arrow[r, "\nu_{\overline{Z}}"] & \overline{Z} \arrow[r, "\iota"] & \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}, \end{tikzcd}$$

where $\overline{\mu}$ is the restriction $\mu|_{\overline{V}_1 \times \overline{V}_2}$, and $\overline{\mu}^N$ is given by the universal property of the normalization $\nu_{\overline{Z}}$. Note that the restriction $\overline{\mu} : \overline{V}_1 \times \overline{V}_2 \to \overline{Z}$ is a surjective morphism. Hence, $\overline{\mu}^N$ is also surjective.

Let $(t_1, t_2, y_1, \cdots, y_{n-1}) \in \mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}$ and $(w, y_1, \cdots, y_{n-1}) \in \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}$ be the coordinates.

Consider the Cartier divisor $D := \sum_{i=1}^{n-1} \{t_i = 1\} - (m+1)\{w = 0\}$ on $\mathbb{A}^1 \times (\mathbb{P}^1)^{n-1}$. Then, as a Cartier divisor on $\mathbb{A}^2 \times (\mathbb{P}^1)^{n-1}$, we have

$$\begin{aligned} \mu^* D &= \sum_{i=1}^{n-1} \{y_i = 1\} - (m+1)\{t_1 = 0\} - (m+1)\{t_2 = 0\} \\ &= \left(\sum_{i=1}^{n_1-1} \{y_i = 1\} - (m+1)\{t_1 = 0\} \right) + \\ &\quad \left(\sum_{i=n_1}^{n-1} \{y_i = 1\} - (m+1)\{t_2 = 0\} \right) \\ &=: D^1 + D^2, \end{aligned}$$

where we note that $n - 1 = (n_1 - 1) + (n_2 - 1)$. Note that by pulling back along $\nu = \nu_{\overline{V}_1} \times \nu_{\overline{V}_2}$, we see that

$$((\iota_1 \times \iota_2) \circ \nu)^* (\mu^* D) \geq 0 \tag{8.6}$$

since we have

$$\begin{aligned} ((\iota_1 \times 1) \circ (\nu_{\overline{V}_1} \times 1))^* D^1 \geq 0, \\ ((1 \times \iota_2) \circ (1 \times \nu_{\overline{V}_2}))^* D^2 \geq 0, \end{aligned}$$

by the modulus condition M_{sum} of $V_1 \times B_{n_2}$ and $B_{n_1} \times V_2$, regarding $V_1 \times B_{n_2}$ as a cycle in $\mathrm{Tz}_{p_1+n_2}(B_{n_2}, n_1; m)$, and similarly for $B_{n_1} \times V_2$.

This inequality (8.6) is equivalent to $(\overline{\mu}^N)^* \left((\iota \circ \nu_{\overline{Z}})^* D \right) \geq 0$ by the commutativity of the diagram. Then, by Lemma 2.2 applied to the surjective morphism $\overline{\mu}^N$, we get $(\iota \circ \nu_{\overline{Z}})^* D \geq 0$. This is the modulus m condition M_{sum} for $Z = \mu_*(V_1 \times V_2)$.

Now we prove the M_{ssup} condition for Z if V_i's satisfy this modulus condition. This is a much more delicate case and we prove it in several steps. First suppose that V_1 or V_2 is contained in the locus of $\{t = a\}$ for a closed point $a \in \mathbb{G}_m$.

By symmetry, we may then assume that $V_1 = \{a\} \times W_1$ for a closed irreducible subvariety $W_1 \subset \square^{n_1-1}$ intersecting all faces of $\square^{n_1-1}$ properly. Then,

$$\mu(V_1 \times V_2) = a^{-1} * (\tau(W_1 \times V_2)),$$

where τ is the transposition

$$\square^{n_1-1} \times \mathbb{G}_m \times \square^{n_2-1} \to \mathbb{G}_m \times \square^{n_1-1} \times \square^{n_2-1},$$

and $*$ is the action of $\mathbb{G}_m$ as in (3.2). This is already closed in B_{n-1}, so we have

$$\mu_*(V_1 \times V_2) = \mu(V_1 \times V_2) = a^{-1} * (\tau(W_1 \times V_2)).$$

Furthermore, the modulus condition for V_2 implies the modulus condition for $a^{-1} * (\tau(W_1 \times V_2))$. Hence, $\mu_*(V_1 \times V_2)$ has the modulus condition M_{ssup}.

Hence, for the rest of the proof, we assume that neither V_1 nor V_2 lies in the loci of $\{t = a\}$ for some $a \in \mathbb{G}_m$. In particular, the images of V_1 and V_2 are open dense subsets of $\mathbb{G}_m$.

Let $V_1' = \mathbb{G}_m \times \square^{n_1-1}$ and let $\widetilde{Z} = \overline{\mu(V_1' \times V_2)}$, the closure of $\mu(V_1' \times V_2)$ in $\mathbb{G}_m \times B_n$. Note that V_1' is just a closed subvariety and not an admissible cycle. Note further that Z is a closed subvariety of $\widetilde{Z}$ and moreover $\overline{Z}$ intersects the divisors F_n^1 and $F_{n,0}$ properly in $\mathbb{P}^1 \times \widehat{B}_n$ since our V_1 and V_2 have this property. Hence by Proposition 2.4, to prove the modulus condition for Z, it suffices to prove the modulus condition for the closed subvariety $\widetilde{Z}$. So, from now on, we shall replace V_1 by V_1' and Z by $\widetilde{Z}$, while we call the new ones as still V_1 and Z, respectively.

We set $B := (\mathbb{P}^1)^{n-1}$, and let $\overline{V}_1$ and $\overline{V}_2$ be the closures of V_1 and V_2 in $\mathbb{G}_m \times (\mathbb{P}^1)^{n_1-1}$ and $\widehat{B}_{n_2}$, respectively.

Let $p : \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^{n-1}$ and $q : \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \to \mathbb{P}^1$ be the projections. Let $W = \mu(V_1 \times V_2)$ and $W' = \overline{\mu}(\overline{V}_1 \times \overline{V}_2)$, where $\overline{\mu} = \mu \times \mathrm{Id}_B : \mathbb{G}_m \times \mathbb{P}^1 \times B \to \mathbb{P}^1 \times B$ is the extension of μ as defined in (8.3). Then we get a commutative diagram

(8.7)
$$\begin{tikzcd}
V_1 \times V_2 \arrow[r] \arrow[d, "\mu"] & \overline{V}_1 \times \overline{V}_2 \arrow[r] \arrow[d, "\overline{\mu}"] & \mathbb{G}_m \times \mathbb{P}^1 \times B \arrow[ddd, "\overline{\mu}"] \\
W \arrow[r] \arrow[d] & W' \arrow[d] \arrow[ddr, "\phi"] & \\
Z \arrow[r] \arrow[d] & Z' \arrow[dr, "\phi'"] & \\
\mathbb{G}_m \times \square^{n-1} \arrow[rr] & & \mathbb{P}^1 \times B,
\end{tikzcd}$$

where Z' is the closure of W' in $\mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$, and ϕ, ϕ' are the inclusions. Note that all the arrows except μ and $\overline{\mu}$ are injective, and μ and $\overline{\mu}$ are surjective.

Notice that by the given assumptions, the composition under the first projection $\overline{V}_2 \subset \widehat{B}_{n_2} \to \mathbb{P}^1$ is surjective since this map is projective and hence a closed map, while V_2 is not in the locus of $\{t = a\}$ for some $a \in \mathbb{G}_m$ so that the map is dominant.

First note that W is open in W'. This is because $W = W' \cap (B_n)$ and B_n is open in $\widehat{B}_n$. Likewise, W' is open in Z'. This is because $W' = Z' \cap \mathrm{Image}(\overline{\mu})$, where $\mathrm{Image}(\overline{\mu})$ is open because $\overline{\mu}$ is flat. Hence, the Zariski closure of W in $\widehat{B}_n$ is equal to the closure of W', which is Z' by definition. Since Z is the Zariski closure of W in B_n, which is open in $\widehat{B}_n$, the Zariski closure $\overline{Z}$ of Z in $\widehat{B}_n$ is consequently Z'. In other words, the Zariski closures of W, Z, W' in $\widehat{B}_n$ are all equal to Z'.

Let $\overline{V}'_2 = \tau\left((\mathbb{P}^1)^{n_1-1} \times \overline{V}_2\right) \subset \widehat{B}_n$ for the transposition

$$\tau : (\mathbb{P}^1)^{n_1-1} \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n_2-1} \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1}.$$

Notice that W' is the orbit $\mathbb{G}_m \cdot \overline{V}'_2$ in $\widehat{B}_n$, where the action of $\mathbb{G}_m$ on $(\mathbb{P}^1)^{n-1}$ is by the identity. In particular, Z' is the orbit closure of $\overline{V}'_2$. Let $W_p := p(W')$.

Claim (1): *$W_p = p(W') = p(V'_2)$ and it is closed in $B = (\mathbb{P}^1)^{n-1}$.*

Consider the following diagram.

$$\begin{tikzcd}
\mathbb{G}_m \times \mathbb{P}^1 \times B \arrow[r, "\overline{\mu}"] \arrow[d, "p'"] & \mathbb{P}^1 \times B \arrow[d, "p"] \\
\mathbb{G}_m \times B \arrow[r, "r"'] & B
\end{tikzcd}$$

Note that the lower horizontal arrow and the two vertical arrows are the projection maps. Since p is projective, $p(\overline{V}'_2)$ is closed in B. Next we have,

$$p(W') = p \circ \overline{\mu}(\mathbb{G}_m \times \overline{V}'_2) = r \circ p'(\mathbb{G}_m \times \overline{V}'_2) = r(\mathbb{G}_m \times p(\overline{V}'_2)) = p(\overline{V}'_2)$$

since p' is identity on $\mathbb{G}_m$ and r is the projection. This proves Claim (1).

Claim (2): *There is a non-empty open subset $U \subset W_p$ such that $\mathbb{G}_m \times U \subset W'$ as an open subset.*

From Claim (1), we have a surjection $\overline{V}'_2 \to W_p$. Since $\overline{V}'_2$ is irreducible, so is W_p. Now recall the following well-known generic flatness theorem. For its proof, see [7, Theorem 5.12, p.123]:

Theorem 8.4. *Let $f : X \to Y$ be morphism of noetherian schemes of finite type over k, where Y is integral. Then there exists a non-empty open subset $U' \subset Y$ such that $f^{-1}(U') \to U'$ is flat.*

Using this theorem and the openness of a flat map, we see that the image of the open set $\overline{V}'_2 \cap q^{-1}(\mathbb{G}_m) \cap p^{-1}(U') \to U'$ is open in U' and hence in W_p. Notice that since the map $\overline{V}_2 \to \mathbb{P}^1$ is surjective, $\overline{V}'_2 \cap q^{-1}(\mathbb{G}_m)$ in not empty. Let U be this image in W_p.

Now, by the choice of U, we see that for each $u \in U$, the fibre $p^{-1}(u)$ meets $\overline{V}'_2 \cap q^{-1}(\mathbb{G}_m)$ non-trivially. This implies that the orbit of $\overline{V}'_2 \cap p^{-1}(U)$ contains at least $\mathbb{G}_m \times U$ (it might also contain some points of $\{0, \infty\} \times U$). In particular, we conclude that $\mathbb{G}_m \times U \subset W'$. Since $\mathbb{G}_m \times U$ is open in $p^{-1}(U) = \mathbb{P}^1 \times U$, it must be open in W', too. This proves Claim (2).

Claim (1) implies that $Z' \subset \mathbb{P}^1 \times W_p$ and Claim (2) implies that there is a non-empty open subset $\mathbb{G}_m \times U \subset Z' \subset \mathbb{P}^1 \times W'$. Since Z' is closed in $\mathbb{P}^1 \times W_p$ and irreducible, and since $\mathbb{G}_m \times U$ is open dense in $\mathbb{P}^1 \times W_p$ which is also irreducible, we conclude that $Z' = \mathbb{P}^1 \times W_p$.

Claim (3): *Let $S := Z' \backslash W'$. Then $Z' \cap q^*(\{t = 0\})$ is irreducible and* $\mathrm{codim}_{Z'}(S \cap q^*\{t = 0\}) \geq 2$, *where t is the coordinate of $\mathbb{P}^1$.*

Since we have just seen that $Z' = \mathbb{P}^1 \times W_p$, the closed subscheme $Z'_0 = Z' \cap q^*(\{t = 0\})$ is in fact $W_p \times \{0\}$ and hence irreducible as W_p is so. This also implies that $\dim(Z'_0) = \dim(W_p) = \dim(Z') - 1$. Now we recall that the map $\overline{V}_2 \to \mathbb{P}^1$ is surjective as a consequence of our assumption. Hence, $\overline{V}'_2 \to \mathbb{P}^1$ is also surjective. This implies in particular that $W' \to \mathbb{P}^1$ is surjective too. This in turn shows that $W' \cap q^*(\{t = 0\}) = W' \cap (Z' \cap q^*(\{t = 0\})) = W' \cap Z'_0$ is a non-empty open subset of Z'_0. Since we have just shown that Z'_0 is irreducible, this implies that $\dim_k(S \cap Z'_0) = \dim_k(Z'_0 \backslash (W' \cap Z'_0)) \leq \dim_k(Z'_0) - 1$. Thus we get

$$\dim_k(S \cap Z'_0) \leq \dim_k(Z'_0) - 1 = \dim_k(Z') - 2.$$

This proves Claim (3).

Claim (4): *For the composition*

$$\nu_{W'} : {W'}^N \to W' \xrightarrow{\phi} \widehat{B}_{n-1},$$

where the first arrow is the normalization, there exists an index $i \in \{1, \cdots, n_2 - 1\}$ for which we have on ${W'}^N$

$$\nu^*_{W'} \left[\{y_i = 1\} - (m+1)\{t = 0\}\right] \geq 0.$$

Consider the following normalization diagram:

(8.8)

$$\begin{tikzcd}
\overline{V}^N_1 \times \overline{V}^N_2 \arrow[r] \arrow[d, "\overline{\mu}^N"'] & \overline{V}_1 \times \overline{V}_2 \arrow[r] \arrow[d, "\overline{\mu}"] & \mathbb{G}_m \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \arrow[r, "pr"] \arrow[d, "\overline{\mu}"] & \mathbb{G}_m \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n_2-1} \arrow[d, "\overline{\mu}"] \\
{W'}^N \arrow[r] & W' \arrow[r, "\phi"] & \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \arrow[r, "pr"] & \mathbb{P}^1 \times (\mathbb{P}^1)^{n_2-1}.
\end{tikzcd}$$

Here the horizontal arrows in the last square are the obvious projection maps. We also note that $\overline{V}^N_1 \times \overline{V}^N_2$ is the normalization of $\overline{V}_1 \times \overline{V}_2$ by [14, Lemma 3.1]. We have seen that $\overline{\mu}$ is a surjective map of irreducible varieties and hence dominant. This gives the map of the corresponding normalizations, which must also be surjective. Since the modulus condition M_{ssup} holds for V_2, there is an $1 \leq i \leq n_2 - 1$ such that the Cartier divisor $\nu^*_2[\{y_i = 1\} - (m+1)\{t = 0\}] \geq 0$ on $\overline{V}^N_2$. This implies that in the diagram

$$\begin{array}{ccc} \overline{V}_1^N \times \overline{V}_2^N & \xrightarrow{\nu_{1,2}} & \mathbb{G}_m \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle s} \\ \overline{V}_2^N & \xrightarrow{\nu_2} & \mathbb{P}^1 \times (\mathbb{P}^1)^{n_2-1}, \end{array}$$

we have

$$\nu_{1,2}^* \circ s^*[\{y_i = 1\} - (m+1)\{t=0\}] \geq 0 \text{ on } \overline{V}_1^N \times \overline{V}_2^N, \tag{8.9}$$

where v, s are the obvious projections, and $\nu_{1,2}$ is the composition of the first two arrows of the upper part of the Diagram (8.8).

Next we observe that as $\mathbb{G}_m$ acts trivially on $(\mathbb{P}^1)^{n-1}$, one has $\overline{\mu}^*(\{y_i = 1\}) = \mathbb{G}_m \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-2} \times (\{y_i = 1\}) = s^*(\{y_i = 1\})$. We also observe that $\overline{\mu}^*(\{t = 0\}) = \mathbb{G}_m \times (\{t = 0\}) \times (\mathbb{P}^1)^{n-1} = s^*(\{t = 0\})$. In particular, we have for $1 \leq i \leq n_2 - 1$,

$$\begin{aligned} s^*[\{y_i = 1\} - (m+1)\{t=0\}] &= \overline{\mu}^* \circ pr^*[\{y_i = 1\} - (m+1)\{t=0\}] \\ &= \overline{\mu}^*[\{y_i = 1\} - (m+1)\{t=0\}]. \end{aligned}$$

Combining this with (8.9), we conclude that $\nu_{1,2}^* \circ \overline{\mu}^*[\{y_i = 1\} - (m+1)\{t = 0\}] \geq 0$ and hence $\overline{\mu}^{N*} \circ \nu_{W'}^*[\{y_i = 1\} - (m+1)\{t = 0\}] \geq 0$. We now apply Lemma 2.2 to the surjective morphism $\overline{\mu}^N$ of normal integral k-varieties to conclude that $\nu_{W'}^*[\{y_i = 1\} - (m+1)\{t = 0\}] \geq 0$ on ${W'}^N$. This proves Claim (4).

Now, we have the final statement of this lengthy proposition:

Claim (5): *The modulus condition M_{ssup} holds for Z.*

Since we have shown that the Zariski closure of Z in $\widehat{B}_n$ is Z', we need to show that the modulus condition holds on ${Z'}^N$. Consider the following commutative diagram

$$\begin{array}{ccccc} {W'}^N = f^{-1}(W') & \longrightarrow & {Z'}^N & \longleftarrow & f^{-1}(S) \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ W' & \longrightarrow & Z' & \longleftarrow & S, \end{array} \tag{8.10}$$

where f is a normalization, and we recall that $S = Z' \backslash W'$. Since W' is as open subset of Z' as shown above, we have $f^{-1}(W') = {W'}^N$. Let $\nu_{Z'} : {Z'}^N \xrightarrow{f} Z' \xrightarrow{\phi'} \widehat{B}_n$ be the composite. We need to show that for some $1 \leq i \leq n-1$, we have

$$[\nu_{Z'}^*(\{y_i - 1\})] \geq (m+1)[\nu_{Z'}^*(\{t = 0\})] \tag{8.11}$$

as Weil divisors on ${Z'}^N$. Since ${Z'}^N$ is an irreducible normal variety and the relation (8.11) holds on ${W'}^N$ by Claim (4), the same relation will hold on ${Z'}^N$ if and only if no component of $\nu_{Z'}^*(\{t = 0\})$ is contained in $f^{-1}(S)$. However, we have shown in Claim (3) that $\dim(S \cap {\phi'}^*(\{t = 0\})) \leq \dim(Z') - 2$. On the other hand, since f is a finite map, if a component D of $\nu_{Z'}^*(\{t=0\})$ is contained in $f^{-1}(S)$, then $f(D) \subset S \cap \phi^*(\{t=0\})$ and we get

$$\dim({Z'}^N) - 1 = \dim(D) = \dim(f(D)) \leq \dim(S \cap {\phi'}^*(\{t=0\}))$$
$$\leq \dim(Z') - 2 = \dim({Z'}^N) - 2,$$

where the second and the last equalities hold by the finiteness and surjectivity of f. This gives a contradiction. This proves the modulus condition M_{ssup} for Z, thus, Claim (5). This completes the proof of the proposition. □

Corollary 8.5. *Let X and Y be smooth projective varieties. Let $V_1 \in \mathrm{Tz}^{q_1}(X, n_1; m)$ and $V_2 \in \mathrm{Tz}^{q_2}(Y, n_2; m)$ be two irreducible admissible cycles. Then $Z = \mu_*(V_1 \times V_2)$ is an admissible additive cycle in $\mathrm{Tz}^q(X \times Y, n; m)$.*

Proof. This follows immediately from Lemma 8.2 and Proposition 8.3. □

8.2. **Shuffle products.** For an integer $r \geq 1$, let $\mathbb{P}\mathrm{erm}_r$ be the group of permutations on the set $\{1, \cdots, r\}$. For integers $s, p_1, p_2, \cdots, p_s \geq 1$, a $(p_1, \cdots, p_s)$-*shuffle* is a permutation $\sigma \in \mathbb{P}\mathrm{erm}_{p_1+\cdots+p_s}$ satisfying the following properties:

$$\begin{cases} \sigma(1) \leq \cdots \leq \sigma(p_1) \\ \qquad \vdots \\ \sigma(p_1 + \cdots + p_{i-1} + 1) \leq \cdots \leq \sigma(p_1 + \cdots + p_{i-1} + p_i), \\ \qquad \vdots \\ \sigma(p + 1 + \cdots + p_{s-1} + 1) \leq \cdots \leq \sigma(p_1 + \cdots + p_{s-1} + p_s) \end{cases}$$

Note that since σ is an automorphism, each of the above inequalities is in fact strict unless some $p_i = 1$ in which case one has $\sigma(p_1+\cdots+p_{i-1}+1) = \sigma(p_1+\cdots+p_{i-1}+p_i)$.

The set of all $(p_1, \cdots, p_s)$-shuffles is denoted by $\mathbb{P}\mathrm{erm}_{(p_1,\cdots,p_s)}$. Here, $\mathbb{P}\mathrm{erm}_r = \mathbb{P}\mathrm{erm}_{(\underbrace{1,\cdots,1}_{r})}$, and $|\mathbb{P}\mathrm{erm}_{(p_1,\cdots,p_s)}| = \frac{(p_1+\cdots+p_s)!}{p_1!\cdots p_s!}$. A permutation $\sigma \in \mathbb{P}\mathrm{erm}_{n-1}$ acts compatibly on the spaces $\square^{n-1}$ and $(\mathbb{P}^1)^{n-1}$ via

$$\sigma \cdot (t_1, \cdots, t_{n-1}) = (t_{\sigma^{-1}(1)}, \cdots, t_{\sigma^{-1}(n-1)}).$$

This generalizes to spaces of the form $Y \times \square^{n-1}$ and $Y \times (\mathbb{P}^1)^{n-1}$ via the trivial action of σ on Y. We obtain induced actions on the groups of algebraic cycles such as $\mathrm{Tz}^q(X, n; m)$, or $z^q(X, n-1)$, etc.

For a permutation σ, the sign $\mathrm{sgn}(\sigma)$ is $+1$ if σ is even, and $\mathrm{sgn}(\sigma)$ is -1 if odd.

8.2.1. *Permutation identities.* The following basic identities on permutations play important roles in proving the associativity of the wedge product, and in proving that the differential operator defined in Section 10, is a graded derivation for the wedge product.

Proposition 8.6. *Let $r, s, t \geq 1$ be integers. Then in the group ring $\mathbb{Z}[\mathbb{P}\mathrm{erm}_{r+s+t}]$, we have two equations*

$$\begin{aligned}\sum_{\nu \in \mathbb{P}\mathrm{erm}_{(r,s,t)}} \mathrm{sgn}(\nu)\nu &= \sum_{\tau \in \mathbb{P}\mathrm{erm}_{(r+s,t)}} \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\tau \cdot (\sigma \times \mathrm{Id}_3),\\ \sum_{\nu \in \mathbb{P}\mathrm{erm}_{(r,s,t)}} \mathrm{sgn}(\nu)\nu &= \sum_{\tau' \in \mathbb{P}\mathrm{erm}_{(r,s+t)}} \sum_{\sigma' \in \mathbb{P}\mathrm{erm}_{(s,t)}} \mathrm{sgn}(\tau')\mathrm{sgn}(\sigma')\tau' \cdot (\mathrm{Id}_1 \times \sigma'),\end{aligned}$$

where Id_1 is the identity function of the set $\{1, \cdots, r\}$, and Id_3 is the identity function of the set $\{r+s+1, \cdots, r+s+t\}$.

We need the following two simple lemmas.

Lemma 8.7. *Let $r, s, t \geq 1$ be integers. For $\tau \in \mathbb{P}\mathrm{erm}_{(r+s,t)}$ and $\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}$, the permutation $\tau \cdot (\sigma \times \mathrm{Id}_3)$ is in $\mathbb{P}\mathrm{erm}_{(r,s,t)}$. Furthermore, the set-theoretic function*

$$\phi : \mathbb{P}\mathrm{erm}_{(r+s,t)} \times \mathbb{P}\mathrm{erm}_{(r,s)} \to \mathbb{P}erm_{(r,s,t)}$$

$$(\tau, \sigma) \mapsto \tau \cdot (\sigma \times \mathrm{Id}_3)$$

is a bijection.

Proof. The first part is obvious. For the second part, consider the following:

Claim. *If $\phi(\tau_1, \sigma_1) = \phi(\tau_2, \sigma_2)$, then $\tau_1 = \tau_1$ and $\sigma_1 = \sigma_2$. That is, ϕ is injective.*

We are given $\tau_1 \cdot (\sigma_1 \times \mathrm{Id}_3) = \tau_2(\sigma_2 \times \mathrm{Id}_3)$. Since $\sigma_1 \times \mathrm{Id}_3, \sigma_2 \times \mathrm{Id}_3$ do not touch the set $S_3 := \{r+s+1, \cdots, r+s+t\}$, we get $\tau_1|_{S_3} = \tau_2|_{S_3}$. However, τ_i are $(r+s,t)$-shuffles so that they are strictly increasing on $\{1, \cdots, r+s\}$. This forces $\tau_1|_{\{1,\cdots,r+s\}} = \tau_2|_{\{1,\cdots,r+s\}}$. Hence, $\tau_1 = \tau_2$. This implies $\sigma_1 = \sigma_2$. Thus the Claim is proved.

Notice that for the function ϕ, the domain and the target have equal cardinalities:

$$|\mathbb{P}\mathrm{erm}_{(r+s,t)}| \times |\mathbb{P}\mathrm{erm}_{(r,s)}| = \frac{(r+s+t)!}{(r+s)!t!} \cdot \frac{(r+s)!}{r!s!} = \frac{(r+s+t)!}{r!s!t!} = |\mathbb{P}\mathrm{erm}_{(r,s,t)}|.$$

Since ϕ is an injective function, this shows that it is automatically bijective. $\square$

Lemma 8.8. *Let $r, s, t \geq 1$ be integers. For $\tau' \in \mathbb{P}\mathrm{erm}_{(r,s+t)}$, $\sigma' \in \mathbb{P}\mathrm{erm}_{(s,t)}$, the permutation $\tau' \cdot (\mathrm{Id}_1 \times \sigma')$ is in $\mathbb{P}\mathrm{erm}_{(r,s,t)}$. Furthermore, the set-theoretic function*

$$\psi : \mathbb{P}\mathrm{erm}_{(r,s+t)} \times \mathbb{P}\mathrm{erm}_{(s,t)} \to \mathbb{P}\mathrm{erm}_{(r,s,t)}$$

$$(\tau', \sigma') \mapsto \tau' \cdot (\mathrm{Id}_1 \times \sigma')$$

is a bijection.

Proof. Its proof is essentially identical to that of Lemma 8.7. $\square$

Proof of Proposition 8.6. This obviously follows from Lemmas 8.7 and 8.8 by observing that $\mathrm{sgn}(\tau \cdot (\sigma \times \mathrm{Id}_3)) = \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)$, and $\mathrm{sgn}(\tau' \cdot (\mathrm{Id}_1 \times \sigma)) = \mathrm{sgn}(\tau')\mathrm{sgn}(\sigma')$. This proves the proposition. $\square$

For the Leibniz rule later, we need the following as well as the above results:

Definition 8.9. For permutations $\sigma \in \mathbb{P}\mathrm{erm}_n$ and $\tau \in \mathbb{P}\mathrm{erm}_{(1,n)}$ with $\tau(1) = i \in \{1, \cdots, n\}$, define the permutation $\sigma_\tau = \sigma[i] \in \mathbb{P}\mathrm{erm}_{n+1}$ by sending

$$j \in \{1, \cdots, n+1\} \mapsto \begin{cases} \sigma(j) & \text{if } j < i, \\ j & \text{if } j = i, \\ \sigma(j-1) & \text{if } j > i. \end{cases}$$

Lemma 8.10. *Let $\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}$ and $\tau \in \mathbb{P}\mathrm{erm}_{(1,r+s)}$. Then the product $\sigma_\tau \cdot \tau$ in $\mathbb{P}\mathrm{erm}_{r+s+1}$ is a $(1, r, s)$-shuffle, i.e., $\sigma_\tau \cdot \tau \in \mathbb{P}\mathrm{erm}_{(1,r,s)}$. Furthermore, the set-theoretic map*

$$\phi_1 : \mathbb{P}\mathrm{erm}_{(r,s)} \times \mathbb{P}\mathrm{erm}_{(1,r+s)} \to \mathbb{P}\mathrm{erm}_{(1,r,s)}$$

$$(\sigma, \tau) \mapsto \sigma_\tau \cdot \tau$$

is a bijection.

Proof. The first statement is obvious. For the second statement, the surjectivity part is obvious by keeping track of where 1 is sent. But since both sides have the cardinality $\frac{(r+s)!}{r!s!}\frac{(r+s+1)!}{(r+s)!} = \frac{(r+s+1)!}{r!s!}$, the map ϕ_1 must be bijective. □

Lemma 8.11. *In the group ring $\mathbb{Z}[\mathbb{P}\mathrm{erm}_{r+s+1}]$, we have*

$$\sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma)) \left(\sum_{\tau \in \mathbb{P}\mathrm{erm}_{(1,r+s)}} (\mathrm{sgn}(\tau)) \sigma_\tau \cdot \tau \right) = \sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}} (\mathrm{sgn}(\nu))\nu.$$

Proof. Note that $\mathrm{sgn}(\sigma_\tau \cdot \tau) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)$. Thus, together with the Lemma 8.10, we get the desired result. □

8.3. **Pre-wedge product via shuffles.** Let X and Y be smooth projective varieties. Consider the groups $\mathrm{Tz}^{q_1}(X, n_1; m)$ and $\mathrm{Tz}^{q_2}(Y, n_2; m)$. Let $n = n_1 + n_2 - 1$ and $q = q_1 + q_2 - 1$. Consider the group of cubical higher Chow cycles, $z^{q+1}(X \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1)$, i.e. cycles of codimension $q+1$ in $X \times Y \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n-1}$ that intersect all faces of $\square^{n-1}$ properly, modulo the degenerate cycles.

Definition 8.12. For two irreducible admissible cycles $V_1 \in \mathrm{Tz}^{q_1}(X, n_1; m)$, and $V_2 \in \mathrm{Tz}^{q_2}(Y, n_2; m)$, the *shuffle product* $V_1 \times_{sh} V_2$ *is defined as a cycle in* $z^{q+1}(X \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1)$, *given by the equation*

$$V_1 \times_{sh} V_2 := \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(n_1-1, n_2-1)}} \mathrm{sgn}(\sigma)\ \sigma \cdot (V_1 \times V_2). \tag{8.12}$$

We can extend this definition $\mathbb{Z}$-bilinearly to get a homomorphism

$$\times_{sh} : \mathrm{Tz}^{q_1}(X, n_1; m) \otimes_{\mathbb{Z}} \mathrm{Tz}^{q_2}(Y, n_2; m) \to z^{q+1}(X \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1).$$

The image of this map is the group of $(n_1 - 1, n_2 - 1)$-shuffles.

Lemma 8.13. *For the cycles V_1, V_2 above and for $\sigma \in \mathbb{P}\mathrm{erm}_{(n_1-1,n_2-1)}$, one has that $\mu_* (\sigma(V_1 \times V_2)) \in \mathrm{Tz}^q(X \times Y, n; m)$.*

Proof. We first observe that σ induces an automorphism of $(\mathbb{P}^1)^{n-1}$ which preserves $\square^{n-1}$ and acts trivially on $X \times Y \times \mathbb{G}_m \times \mathbb{G}_m$. In particular, the actions of μ and σ commute. Hence we have $\mu\left(\sigma(V_1 \times V_2)\right) = \sigma\left(\mu(V_1 \times V_2)\right)$, which in turn implies that $\mu_*\left(\sigma(V_1 \times V_2)\right) = \sigma\left(\mu_*(V_1 \times V_2)\right)$. The lemma now follows from Corollary 8.5. $\square$

This lemma allows one to define the *pre-wedge product* $V_1 \overline{\wedge} V_2$ of V_1 and V_2 in $\mathrm{Tz}^q(X \times Y, n; m)$ by the equation

$$\begin{aligned} V_1 \overline{\wedge} V_2 &:= \mu_*(V_1 \times_{sh} V_2) = \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(n_1-1, n_2-1)}} \mathrm{sgn}(\sigma) \mu_*(\sigma \cdot (V_1 \times V_2)) \\ &= \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(n_1-1, n_2-1)}} \mathrm{sgn}(\sigma) \sigma \cdot (\mu_*(V_1 \times V_2)) \end{aligned}$$

As before, extend it $\mathbb{Z}$-bilinearly to get a homomorphism

$$\overline{\wedge} : \mathrm{Tz}^{q_1}(X, n_1; m) \otimes_{\mathbb{Z}} \mathrm{Tz}^{q_2}(Y, n_2; m) \to \mathrm{Tz}^q(X \times Y, n; m). \tag{8.13}$$

The image of this map is the group of (n_1-1, n_2-1)-pre-wedges of codimension q and modulus m. The group of pre-wedges is simply the image under μ_* of the group of shuffles.

Corollary 8.14. *For $V_i \in \mathrm{Tz}^{q_i}(X_i, n_i; m)$, we have $(V_1 \times_{sh} V_2) \times_{sh} V_3 = V_1 \times_{sh} (V_2 \times_{sh} V_3)$ in $\mathrm{Tz}^q(X_1 \times X_2 \times X_3, n; m)$ for appropriate q and n. The same is true for $\overline{\wedge}$.*

Proof. This follows from the associativity of μ and Proposition 8.6. $\square$

Lemma 8.15. *For two cycles $\xi \in \mathrm{Tz}^{q_1}(X, n_1; m)$ and $\eta \in \mathrm{Tz}^{q_2}(X, n_2; m)$, we have equations*

$$\partial(\xi \bar{\wedge} \eta) = (\partial \xi) \bar{\wedge} \eta + (-1)^{n_1-1} \xi \bar{\wedge} (\partial \eta), \tag{8.14}$$

$$\xi \bar{\wedge} \eta = (-1)^{(n_1-1)(n_2-1)} \eta \bar{\wedge} \xi, \tag{8.15}$$

where ∂ is the boundary map in the definition of additive higher Chow groups.

Proof. For both of the equations, it is enough to prove it for $\times_{sh}$, where we use the fact that ∂ and μ_* commute for the first. But, actually both are just purely combinatorial statements. $\square$

Proof of Proposition 8.1. The external product structure in (8.1) follows directly from the pre-wedge product of cycles in (8.13) and from the first identity of Lemma 8.15. If $X = Y$, the anti-commutativity follows directly from the second identity of Lemma 8.15. $\square$

We now prove the following main result of this section and its consequences.

Theorem 8.16. *Let X be a smooth projective variety over a field k. Then there exists an internal wedge product on the additive higher Chow groups of X.*

$$\wedge_X : \mathrm{TCH}^{q_1}(X, n_1; m) \otimes_{\mathbb{Z}} \mathrm{TCH}^{q_2}(X, n_2; m) \to \mathrm{TCH}^q(X, n; m), \tag{8.16}$$

where $q = q_1 + q_2 - 1$, $n = n_1 + n_2 - 1$, and $q_i, n_i, m \geq 1$ for $i = 1, 2$, which is associative and satisfies the equation

$$\xi \wedge_X \eta = (-1)^{(n_1-1)(n_2-1)} \eta \wedge_X \xi. \tag{8.17}$$

for all classes $\xi \in \mathrm{TCH}^{q_1}(X, n_1; m)$ and $\eta \in \mathrm{TCH}^{q_2}(X, n_2; m)$.

This wedge product is natural with respect to the pull-back maps of additive higher Chow groups and satisfies the projection formula

$$f_*\left(a \wedge_X f^*(b)\right) = f_*(a) \wedge_Y b \tag{8.18}$$

for a morphism $f : X \to Y$ of smooth projective varieties.

Proof. Consider the diagonal map $\Delta_X : X \to X \times X$. Since X is smooth projective, so is $X \times X$. Hence, by applying Theorem 7.1 to Δ_X, we get the pull-back map

$$\Delta_X^* : \mathrm{TCH}^q(X \times X, n; m) \to \mathrm{TCH}^q(X, n; m),$$

for all integer $q \geq 0$. Composing with the pre-wedge product $\overline{\wedge}$, we have

$$\begin{tikzcd} \mathrm{TCH}^{q_1}(X, n_1; m) \otimes \mathrm{TCH}^{q_2}(X, n_2; m) \arrow[r, "\overline{\wedge}"] \arrow[dr, "\wedge_X"'] & \mathrm{TCH}^q(X \times X, n; m) \arrow[d, "\Delta_X^*"] \\ & \mathrm{TCH}^q(X, n; m), \end{tikzcd}$$

where the induced map $\overline{\wedge}$ is well-defined by Proposition 8.1. Now we define $\wedge_X := \Delta_X^* \circ \overline{\wedge}$. This gives the desired wedge product by the second equation of Lemma 8.15. The associativity follows from Corollary 8.14.

We now show the naturality of the wedge product and the projection formula. We first observe that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of smooth projective varieties, then it follows from the contravariance property of the additive higher Chow groups (*cf.* Theorem 7.1) that the naturality of the wedge product with f^* and g^* implies the same with $(g \circ f)^*$. Similarly, the projection formula for f and g implies that

$$\begin{aligned} (g \circ f)_*[a \wedge_X (g \circ f)^*(b)] &= (g_* \circ f_*)[a \wedge_X (f^* \circ g^*(b))] \\ &= g_*[f_*(a) \wedge_Y g^*(b)] \\ &= (g_* \circ f_*)(a) \wedge_Z b \\ &= (g \circ f)_*(a) \wedge_Z b \end{aligned}$$

Hence by factoring a map $f : X \to Y$ as a composite of the closed embedding $X \hookrightarrow X \times Y$ and the projection $X \times Y \to Y$, it suffices to prove the naturality and projection formula when f is one of these two types of morphisms.

To prove the naturality, we can use the contravariance property of the additive higher Chow groups and the construction of the wedge product above to reduce to proving the naturality for the pre-wedge product in Proposition 8.1. In this case, we only need to show that for the flat map $f : X \to Y$, the diagram

$$\begin{tikzcd} z^q(Y \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1) \arrow[r, "\mu_*"] \arrow[d, "f^*"'] & z^q(Y \times Y \times \mathbb{G}_m, n-1) \arrow[d, "f^*"] \\ z^q(X \times X \times \mathbb{G}_m \times \mathbb{G}_m, n-1) \arrow[r, "\mu_*"'] & z^q(X \times X \times \mathbb{G}_m, n-1) \end{tikzcd} \tag{8.19}$$

commutes, which is immediate from the definition of f^* and μ_*.

If f is a closed embedding, we can use Theorem 4.1 to replace $\mathrm{Tz}^q(Y, \bullet; m)$ by $\mathrm{Tz}^q_{\{X\}}(Y, \bullet; m)$. Then, the pull-back map is induced by the Gysin map f^* of Corollary 7.2. In this case, we have for the irreducible admissible $V_i \in \mathrm{Tz}^{q_i}_{\{X\}}(Y, n_i; m)$ that $V_1 \times V_2 \in \mathrm{Tz}^{q+1}_{\{X \times X\}}(Y \times Y, n; m)$ and $f^*(V_1 \times V_2) = f^*(V_1) \times f^*(V_2)$. Thus we only need to show that the Diagram (8.19) commutes where $z^q(Y \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1)$ (resp. $z^q(Y \times Y \times \mathbb{G}_m, n-1)$) is replaced by $z^q_{\{X \times X \times \mathbb{G}_m \times \mathbb{G}_m\}}(Y \times Y \times \mathbb{G}_m \times \mathbb{G}_m, n-1)$ (resp. $z^q_{\{X \times X \times \mathbb{G}_m\}}(Y \times Y \times \mathbb{G}_m, n-1)$). But this follows easily once we know that the diagram

$$\begin{tikzcd} X \times X \times \mathbb{G}_m \times \mathbb{G}_m \arrow[r] \arrow[d, "\mu"'] & Y \times Y \times \mathbb{G}_m \times \mathbb{G}_m \arrow[d, "\mu"] \\ X \times X \times \mathbb{G}_m \arrow[r] & Y \times Y \times \mathbb{G}_m \end{tikzcd} \tag{8.20}$$

is in fact a Cartesian square. This proves the naturality with pull-backs.

To prove the projection formula for the closed embedding $X \xrightarrow{f} Y$, we can again assume that the cycles under consideration intersect X or $X \times X$ properly. Then we have for $V_1 \in z^{q_1}(X \times \mathbb{G}_m, n_1 - 1)$ and $V_2 \in z^{q_2}_{\{X \times \mathbb{G}_m\}}(Y \times \mathbb{G}_m, n_2 - 1)$,

$$\begin{aligned} f_*[\Delta_X^*\{\mu_*^{X \times X}(V_1 \times [X] \cdot V_2)\}] &= \Delta_Y^*[(f \times f)_*\{\mu_*^{X \times X}(V_1 \times [X] \cdot V_2)\}] \\ &= \Delta_Y^*[\mu_*^{Y \times Y}\{(f \times f)_*(V_1 \times [X] \cdot V_2)\}] \\ &= \Delta_Y^*[\mu_*^{Y \times Y}\{f_*(V_1) \times ([X] \cdot V_2)\}] \\ &= \Delta_Y^*[\mu_*^{Y \times Y}\{f_*(V_1) \times V_2)\}], \end{aligned}$$

where the first equality follows from the fact that the the left square in the diagrams

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times X \arrow[d, "f \times f"] & X \times Y \arrow[r, "\mathrm{Id}_X \times \Delta_Y"] \arrow[d, "f"'] & W \arrow[d, "f'"] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y & Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd} \tag{8.21}$$

is Cartesian and the last equality holds since Δ^* commutes with μ_*, as follows from Theorem 4.1. This proves the projection formula for the closed embedding.

Finally, we prove the projection formula for the projection map $f : Z = X \times Y \to Y$. Let $W = X \times Y \times Y$ and let $\mu^W : W \times \mathbb{G}_m \times \mathbb{G}_m \to W \times \mathbb{G}_m$ be the product map. Let $p : Z \times Z \to W$ be the projection map. Then for any irreducible admissible cycles $V_1 \in z^{q_1}(Z \times \mathbb{G}_m, n_1 - 1)$ and $V_2 \in z^{q_2}(Y \times \mathbb{G}_m, n_2 - 1)$, we have

$$\begin{aligned} f_*[\Delta_Z^*\{\mu_*^{Z \times Z}(V_1 \times f^*(V_2))\}] &= f_*[\Delta_Z^*\{\mu_*^{Z \times Z}(V_1 \times (X \times V_2))\}] \\ &= f_*[\Delta_Z^*\{p^*\left(\mu_*^W(V_1 \times V_2)\right)\}] \\ &= f_*[(id_X \times \Delta_Y)^*\left(\mu_*^W(V_1 \times V_2)\right)] \\ &= \Delta_*^Y[f'_*\left(\mu_*^W(V_1 \times V_2)\right)] \qquad (*) \\ &= \Delta_*^Y[\mu_*^{Y \times Y}\{f'_*(V_1 \times V_2)\}] \\ &= \Delta_*^Y[\mu_*^{Y \times Y}\{f_*(V_2) \times V_1)\}], \end{aligned}$$

where the equality $(*)$ follows from the right Cartesian square in (8.21). This proves the projection formula for the projection map. This completes the proof of the theorem. $\square$

For a smooth projective variety X over k, let $\mathrm{TCH}_n(X) = \bigoplus_q \mathrm{TCH}^q(X, n+1; m)$ and let $\mathrm{TCH}(X) = \bigoplus_{n \ge 0} \mathrm{TCH}_n(X)$. Let $A(X) = \bigoplus_q CH^q(X)$ be the ordinary Chow ring of X. As an immediate consequence of Theorem 8.16, we have :

Corollary 8.17. *For X as above, there is a wedge product structure on* $\mathrm{TCH}(X)$

$$\mathrm{TCH}(X) \otimes_{A(X)} \mathrm{TCH}(X) \xrightarrow{\wedge} \mathrm{TCH}(X),$$

that makes $\mathrm{TCH}(X)$ *a graded-commutative algebra.*

Proof. This follows immediately from Theorem 8.16 once we know that the pre-wedge product in (8.13) is bilinear over the ring $A(X)$, where the $A(X)$-module structure on $\mathrm{TCH}(X)$ is given by Theorem 3.1. But this can be easily checked from the construction of the shuffle product in (8.13). $\square$

As another consequence of Theorem 8.16, we get the following result which was widely expected in view of the belief that the additive higher Chow groups compute the relative K-theory of the infinitesimal thickenings of smooth varieties.

Corollary 8.18. *Let X be a smooth projective variety over a field k such that* $\mathrm{char}(k) \neq 2$. *Then for any $q, n, m \ge 1$, the group* $\mathrm{TCH}^q(X, n; m)$ *is a $\mathbb{W}_m(k)$-module, where $\mathbb{W}_m(k)$ is the ring of generalized Witt-vectors of length m over k. In particular,* $\mathrm{TCH}^q(X, n; m)$ *is naturally a k-vector space if* $\mathrm{char}(k) = 0$.

Proof. The follows immediately from Theorem 8.16 by considering the composite map

$$\begin{tikzcd}
\mathrm{TCH}^1(k,1;m) \otimes_{\mathbb{Z}} \mathrm{TCH}^q(X,n;m) \arrow[r, "p^* \otimes \mathrm{Id}"] \arrow[dr] & \mathrm{TCH}^1(X,1;m) \otimes_{\mathbb{Z}} \mathrm{TCH}^q(X,n;m) \arrow[d, "\wedge"] \\
& \mathrm{TCH}^q(X,n;m),
\end{tikzcd}$$

where $p : X \to \mathrm{Spec}(k)$ is the structure map, and using the isomorphism $\mathbb{W}_m(k) \xrightarrow{\cong} \mathrm{TCH}^1(k, 1; m)$ (*cf.* [21]). That this gives a module structure, also follows from Theorem 8.16. In characteristic zero, $\mathbb{W}_m(k)$ is itself a k-module. $\square$

9. Differential operator on additive higher Chow groups

We have shown in Section 8 that the additive higher Chow groups of a smooth projective variety have a structure of naturally defined commutative graded algebra. Our main goal in the remaining part of this paper is to show that these are also equipped with differential operators, one of which turns this algebra into a *differential graded algebra*. We construct one of these differential operators in this section.

Let X be a smooth projective variety. Let $\mathbb{G}_m^\times$ denote the variety $\mathbb{G}_m \backslash \{1\}$. We have natural inclusions of open sets $\mathbb{G}_m^\times \hookrightarrow \square \hookrightarrow \mathbb{P}^1$. For $n, m \ge 1$, define the map

$$\phi_n : X \times \mathbb{G}_m^\times \times \square^{n-1} \to X \times \mathbb{G}_m \times \square^n \tag{9.1}$$

$$(x, t, y_1, \cdots, y_{n-1}) \mapsto (x, t, t^{-1}, y_1, \cdots, y_{n-1}).$$

Note that ϕ_n is not a closed immersion. Rather, it is the composite of the closed immersion $X \times \mathbb{G}_m^\times \times \square^{n-1} \hookrightarrow X \times \mathbb{G}_m^\times \times \mathbb{G}_m^\times \times \square^{n-1}$ followed by the open immersion $X \times \mathbb{G}_m^\times \times \mathbb{G}_m^\times \times \square^{n-1} \hookrightarrow X \times \mathbb{G}_m \times \square^n$. For once and all, we fix the coordinates $(t, y_1, \cdots, y_n)$ of $\mathbb{G}_m \times \square^n \subset \mathbb{P}^1 \times (\mathbb{P}^1)^n$. For any irreducible cycle $Z \subset X \times \mathbb{G}_m \times \square^{n-1}$, let $Z^\times$ denote its restriction to the open set $X \times \mathbb{G}_m^\times \times \square^{n-1}$. Our first observation is the following.

Lemma 9.1. *For any irreducible admissible cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, $\phi_n(Z^\times)$ is closed in $X \times \mathbb{G}_m \times \square^n$.*

Proof. We look at the Zariski closure $W := \overline{\phi_n(Z^\times)}$ of $\phi_n(Z^\times)$ in the bigger space $X \times \mathbb{G}_m \times (\mathbb{P}^1)^n$, and see what happens. The image of $Z^\times$ is clearly closed in $X \times \mathbb{G}_m^\times \times \mathbb{G}_m^\times \times \square^{n-1}$.

Hence $W \backslash \phi_n(Z^\times)$ must be contained in $\{t = 1\} \cup \{y_1 = 1\}$. By the definition of ϕ_n, if a point in $W \backslash \phi_n(Z^\times)$ intersects $\{t = 1\}$, then it must also intersect $\{y_1 = 1\}$ in $X \times \mathbb{G}_m \times (\mathbb{P}^1)^n$. Hence $W \backslash \phi_n(Z^\times)$ is in fact contained in $\{t = 1\} \cap \{y_1 = 1\}$ in $X \times \mathbb{G}_m \times (\mathbb{P}^1)^n$. In particular, $W \backslash \phi_n(Z^\times)$ cannot intersect with $X \times \mathbb{G}_m \times \square^n$. Hence if W' is the Zariski closure of $\phi_n(Z^\times)$ in $X \times \mathbb{G}_m \times \square^n$, then $W' \backslash \phi_n(Z^\times)$, which is a subset of $W \backslash \phi_n(Z^\times)$, does not intersect $X \times \mathbb{G}_m \times \square^n$, either. This shows that $W' = \phi_n(Z^\times)$. Hence, $\phi_n(Z^\times)$ is closed. $\square$

We shall often write the morphisms such as ϕ_n in the sequel simply as rational maps on the ambient space and also write $\phi_n(Z^\times)$ as $\phi_n(Z)$.

Lemma 9.2. *For Z as in Lemma 9.1, the closed subvariety $V := \phi_n(Z^\times)$ satisfies the modulus condition.*

Proof. Consider the following commutative diagram.

(9.2)
$$\begin{tikzcd}
\overline{Z}^N \arrow[rr, "\widetilde{\phi_n}"] \arrow[d] \arrow[dd, bend right, "f"'] & & \overline{V}^N \arrow[d] \arrow[dd, bend left, "g"] \\
\overline{Z} \arrow[rr] \arrow[d] & & \overline{V} \arrow[d] \\
X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \arrow[rr, "\overline{\phi_n}"'] & & X \times \mathbb{P}^1 \times (\mathbb{P}^1)^n
\end{tikzcd}$$

Here $\overline{\phi_n}(x, t, y_1, \cdots, y_{n-1}) = (x, t, t^{-1}, y_1, \cdots, y_{n-1})$ is the natural extension of ϕ_n. Note also that $\widetilde{\phi_n}$ is induced by the dominant map $Z^\times \to V$, which in turn gives the map $\overline{Z} \to \overline{V}$ as $\overline{\phi_n}$ is closed. In particular, $\widetilde{\phi_n}$ is projective and surjective.

Next, it is easy to check from the description of $\overline{\phi_n}$ that $\overline{\phi_n}^*(F_{n+1,0}) = F_{n,0}$ and $\overline{\phi_n}^*(F^1_{n+1,i}) = F^1_{n,i-1}$ for $i \geq 2$. In particular, the modulus condition M_{ssup} for Z implies that

$$\widetilde{\phi_n}^* \circ g^*[F^1_{n+1,i+1} - (m+1)F_{n+1,0}] = f^* \circ \overline{\phi_n}^*[F^1_{n+1,i+1} - (m+1)F_{n+1,0}]$$

$$= f^*[F^1_{n,i} - (m+1)F_{n,0}] \geq 0$$

for some $1 \leq i \leq n-1$. We conclude from the surjectivity of $\widetilde{\phi_n}$ and from an easy variant of Proposition 5.2 that $g^*[F^1_{n+1,i+1} - (m+1)F_{n+1,0}] \geq 0$ for some $1 \leq i \leq n-1$, which is the M_{ssup} condition for V.

If Z satisfies the modulus condition M_{sum}, then following the same argument as above, we get

$$\begin{aligned} & \widetilde{\phi_n}^* \circ g^*[F^1_{n+1} - (m+1)F_{n+1,0}] \\ \geq \quad & \left(\sum_{i=1}^n \widetilde{\phi_n}^* \circ g^*[F^1_{n+1,i}] \right) - \widetilde{\phi_n}^* \circ g^*[(m+1)F_{n+1,0}] \\ = \quad & \left(\sum_{i=2}^n f^* \circ \overline{\phi_n}^*[F^1_{n+1,i}] \right) - f^* \circ \overline{\phi_n}^*[(m+1)F_{n+1,0}] \\ = \quad & f^*[F^1_n - (m+1)F_{n,0}] \geq 0. \end{aligned}$$

We again conclude from the surjectivity of $\widetilde{\phi_n}$ and from an easy variant of Proposition 5.2 that $g^*[F^1_{n+1} - (m+1)F_{n+1,0}] \geq 0$. This shows the modulus condition M_{sum} for V. $\square$

Proposition 9.3. *For any irreducible admissible cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, $\phi_n(Z^\times)$ defines an admissible irreducible cycle in $\underline{\mathrm{Tz}}^{q+1}(X, n+1; m)$, that we denote by $\delta(Z)$. Furthermore, δ and ∂ satisfy the relation $\partial\delta + \delta\partial = 0$.*

Proof. We first prove the following.

Claim : (1) $\partial^\epsilon_{n+1,1} \circ \phi_n = 0$ *for* $\epsilon = 0, \infty$.

(2) $\partial^\epsilon_{n+1,i} \circ \phi_n = \phi_{n-1} \circ \partial^\epsilon_{n,i-1}$ *for* $i \geq 2$ *and* $\epsilon = 0, \infty$.

Here $\partial^\epsilon_{n+1,i}$ is the i-th face ∂^ϵ_i on B_{n+1}. It is easy to see from the definition of ϕ_n that $\phi_n(Z)$ intersects $\{y_1 = 1\}$ if and only if it intersects $\{t = 1\}$. This clearly implies (1).

For (2), we can again observe from the definition of ϕ_n that it just shifts the coordinates of $\square^{n-1}$ by one. In particular, for $i \geq 2$ and $\epsilon = 0, \infty$, the diagram

$$\begin{array}{ccc} \mathbb{G}_m^\times \times \square^{n-2} & \xrightarrow{\iota^\epsilon_{i-1}} & \mathbb{G}_m^\times \times \square^{n-1} \\ {\scriptstyle \phi_{n-1}} \downarrow & & \downarrow {\scriptstyle \phi_n} \\ \mathbb{G}_m \times \square^{n-1} & \xrightarrow[\iota^\epsilon_i]{} & \mathbb{G}_m \times \square^n \end{array} \tag{9.3}$$

is Cartesian. Hence we have for $i \geq 2$,

$$\begin{aligned} \partial^\epsilon_{n+1,i} \circ \phi_n(Z^\times) &= \partial^\epsilon_{n+1,i}\left(\phi_n^{-1}(F^\epsilon_{n+1,i}) \cdot Z^\times\right) = \phi_{n-1}\left(F^\epsilon_{n,i-1} \cdot Z^\times\right) \\ &= \phi_{n-1}\left((F^\epsilon_{n,i-1} \cdot Z)^\times\right) = \phi_{n-1}\left(\left(\partial^\epsilon_{n,i-1}(Z)\right)^\times\right). \end{aligned}$$

This proves the Claim.

Using the Claim and the proper intersection property of Z, we see immediately that $\delta(Z)$ has proper intersections with faces. We also conclude from Lemma 9.2 that $\delta(Z)$ satisfies the modulus condition. Since δ does not change the dimension, we have shown that $\delta(Z)$ is in $\mathrm{Tz}^{q+1}(X, n+1; m)$.

Finally, if we denote the operator δ at the level n of $\mathrm{Tz}^q(X, \bullet; m)$ by δ_n, then we have

$$\begin{aligned}
\partial \circ \delta_n(Z) &= \sum_{i=1}^{n} (-1)^i [\partial^\infty_{n+1,i} \circ \delta_n(Z) - \partial^0_{n+1,i} \circ \delta_n(Z)] \\
&= \sum_{i=2}^{n} (-1)^i [\delta_{n-1} \circ \partial^\infty_{n,i-1}(Z) - \delta_{n-1} \circ \partial^0_{n,i-1}(Z)] \\
&= -\left(\sum_{i=1}^{n-1} (-1)^i [\delta_{n-1} \circ \partial^\infty_{n,i}(Z) - \delta_{n-1} \circ \partial^0_{n,i}(Z)] \right) \\
&= -\delta_{n-1} \left(\sum_{i=1}^{n-1} (-1)^i [\partial^\infty_{n,i}(Z) - \partial^0_{n,i}(Z)] \right) \\
&= -\delta_{n-1} \circ \partial(Z),
\end{aligned}$$

where the second equality follows from the above Claim. This proves the proposition. □

Corollary 9.4. *For every $q \geq 1$, δ defines a chain map*

$$\delta : \mathrm{Tz}^q(X, \bullet; m) \to \mathrm{Tz}^{q+1}(X, \bullet; m)[1].$$

Proof. For an irreducible admissible cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, we define $\delta(Z)$ as in Proposition 9.3 and then extend linearly to $\underline{\mathrm{Tz}}^q(X, n; m)$. It is clear from (9.1) that δ preserves the degenerate cycles. Now the corollary follows from Proposition 9.3. □

9.1. **Computation of δ^2.** Our next goal is show that δ^2 is zero to make it into a differential operator on the additive higher Chow groups. We achieve this by explicitly constructing certain admissible cycles which bound $\delta^2(Z)$ for any irreducible admissible cycle Z. We first define certain 2-cycles in $z^2(\mathbb{G}_m, 3)$ which are all two dimensional analogues of variants of B. Totaro's 1-cycles in [22]. For any general point $t \in \mathbb{G}_m$, the parameter u will always denote t^{-1} in this part of the section.

For $1 \leq j \leq 4$, let $\Gamma^1_j \subset \mathbb{G}_m \times \square^3$ be the the 2-cycles defined by the rational maps $\psi^1_j : \mathbb{G}_m \times \square \to \mathbb{G}_m \times \square^3$ given as follows:

$$\begin{aligned}
\psi^1_1(t,x) &= \left(t, u, x, \tfrac{(1-u)x-(1-u)/(1-t)}{x-(1-u)/(1-t)}\right) \\
\psi^1_2(t,x) &= \left(t, u, x, \tfrac{(1-u)x-1}{x-1}\right) \\
\psi^1_3(t,x) &= \left(t, x, \tfrac{ux-1}{x-1}, 1-u\right) \\
\psi^1_4(t,x) &= \left(t, x, 1-x, \tfrac{u-x}{1-x}\right)
\end{aligned} \tag{9.4}$$

for $t \in \mathbb{G}_m \backslash \{1\}$ and $x \in \square \backslash \{0\}$. We similarly define the 2-cycles $\Gamma_j^2 \subset \mathbb{G}_m \times \square^3$ given by the rational maps

$$
\begin{aligned}
\psi_1^2(t,x) &= \left(t, u^2, x, \tfrac{(1-u^2)x-(1-u^2)/(1-t^2)}{x-(1-u^2)/(1-t^2)}\right)\\
\psi_2^2(t,x) &= \left(t, u^2, x, \tfrac{(1-u^2)x-1}{x-1}\right)\\
\psi_3^2(t,x) &= \left(t, x, \tfrac{u^2x-1}{x-1}, 1-u^2\right)\\
\psi_4^2(t,x) &= \left(t, x, 1-x, \tfrac{u^2-x}{1-x}\right)\\
\psi_5^2(t,x) &= \left(t, x, \tfrac{ux-u^2}{x-u^2}, -u^2\right)\\
\psi_6^2(t,x) &= \left(t, u, x, \tfrac{ux+u^2}{x+u^2}\right)
\end{aligned} \tag{9.5}
$$

for $t \in \mathbb{G}_m \backslash \{1, -1\}$ and $x \in \square \backslash \{0\}$.

For an irreducible 1-cycle $\alpha \subset \mathbb{G}_m \times \square^2$ which is admissible and defined by a rational map

$$
\begin{gathered}
\phi : \mathbb{G}_m \to \mathbb{G}_m \times \square^2\\
\phi(t) = (\phi(t)(0), \phi(t)(1), \phi(t)(2)),
\end{gathered} \tag{9.6}
$$

we shall often write α by the parametrization $(\phi(t)(0), \phi(t)(1), \phi(t)(2))$ to simplify the notations.

It is now easy to check from the definitions that all Γ_j^l are closed in $\mathbb{G}_m \times \square^3$ and they in fact define admissible cycles in $z^2(\mathbb{G}_m, 3)$ (*cf.* Lemma 9.1). Moreover, one can also check in a straightforward way (or using the computations in [22, Section 2]) that these cycles have the following boundaries:

$$
\begin{aligned}
\partial\Gamma_1^l &= \left(t, u^l, \tfrac{1-u^l}{1-t^l}\right) - \left(t, u^l, 1-u^l\right) - \left(t, u, \tfrac{1}{1-t^l}\right)\\
\partial\Gamma_2^l &= \left(t, u^l, 1-t^l\right) - \left(t, u^l, \tfrac{1}{1-t^l}\right)\\
\partial\Gamma_3^l &= \left(t, u^l, 1-t^l\right) - \left(t, t^l, 1-t^l\right)\\
\partial\Gamma_4^l &= \left(t, u^l, 1-u^l\right)\\
\partial\Gamma_5^2 &= 2\,(t, u, -u^2) - (t, u^2, -u^2)\\
\partial\Gamma_6^2 &= (t, u, -u^2) - (t, u, u) - (t, u, -u)
\end{aligned} \tag{9.7}
$$

Since $u = t^{-1}$, note that

$$\frac{1-u^l}{1-t^l} = \frac{1-t^{-l}}{1-t^l} = \frac{t^l-1}{1-t^l} \cdot t^{-l} = -t^{-l} = -u^l. \tag{9.8}$$

Hence, we have

$$\left(t, u^l, -u^l\right) = \left(t, u^l, (1-u^l)/(1-t^l)\right).$$

Using (9.7) together with this, we see at once that for $l = 1, 2$,

$$\left(t, u^l, -u^l\right) \;=\; \partial\Gamma_1^l - \partial\Gamma_2^l - \partial\Gamma_3^l + 2\Gamma_4^l. \tag{9.9}$$

We also obtain from (9.7) that

$$(t, u^2, -u^2) \;=\; 2(t, u, u) - \partial\Gamma_5^2 + 2\Gamma_6^2 + 2(t, u, -u). \tag{9.10}$$

Combining (9.9) and (9.10) together, we obtain that as an element of $z^2(\mathbb{G}_m, 2)$,

$$\begin{aligned} 2(t,u,u) &= \partial\Gamma_1^2 - \partial\Gamma_2^2 - \partial\Gamma_3^2 + 2\partial\Gamma_4^2 + \partial\Gamma_5^2 - 2\partial\Gamma_6^2 \\ & \quad -2(\partial\Gamma_1^1 - \partial\Gamma_2^1 - \partial\Gamma_3^1 + 2\partial\Gamma_4^1) \\ &=: \partial\Gamma. \end{aligned} \tag{9.11}$$

Let X be a smooth projective variety. For any admissible additive cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, we can naturally consider it as a higher Chow cycle in $z^q(X \times \mathbb{G}_m, n-1)$. For any $\Gamma_j^l \in z^2(\mathbb{G}_m, 3)$, we get the exterior product $Z \times \Gamma_j^l \in z^{q+2}(X \times \mathbb{G}_m \times \mathbb{G}_m, n+2)$. Moreover, it also follows from the definition of these cycles that $Z \times \Gamma_j^l$ intersects $X \times \mathbb{G}_m$ properly under the diagonal embedding $\Delta_{\mathbb{G}_m} : X \times \mathbb{G}_m \to X \times \mathbb{G}_m \times \mathbb{G}_m$ given by $(x,t) \mapsto (x,t,t)$. Since X is smooth, we get the pull-back cycle $Z \star \Gamma_j^l = \Delta_{\mathbb{G}_m}^*(Z \times \Gamma_j^l) \in z^{q+2}(X \times \mathbb{G}_m, n+2)$.

Lemma 9.5. *The cycle* $Z \star \Gamma_j^l$ *lies in* $\underline{\mathrm{Tz}}^{q+2}(X, n+3; m)$ *under the natural inclusion* $\underline{\mathrm{Tz}}^{q+2}(X, n+3; m) \hookrightarrow z^{q+2}(X \times \mathbb{G}_m, n+2)$.

Proof. We only need to show that $Z \star \Gamma_j^l$ satisfies the modulus condition. For this, we observe that $Z \star \Gamma_j^l$ is the closure of image of $Z' = \square \times Z$ under the rational map

$$\Psi_j^l : X \times \mathbb{G}_m \times \square^n \to X \times \mathbb{G}_m \times \square^{n+2} \tag{9.12}$$

$$\Psi_j^l(x,t,y,y_1,\cdots,y_{n-1}) = (x, \psi_j^l(0), \psi_j^l(1), \psi_j^l(2), \psi_j^l(3), y_1, \cdots, y_{n-1})$$

in the notation of (9.6). We now follow the proof of Lemma 9.2 to prove the modulus condition for Z'. Let $V' = Z \star \Gamma_j^l = \overline{\Psi_j^l(Z')}$. We consider the following commutative diagram.

$$\begin{tikzcd} \overline{Z}^N \times \mathbb{P}^1 \arrow[r, "\widetilde{\Psi_j^l}"] \arrow[d, "f'"'] & \overline{V}'^N \arrow[d, "g'"] \\ X \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1} \arrow[r, "\overline{\Psi_j^l}"'] & X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n+2} \end{tikzcd} \tag{9.13}$$

Here $f' = f \times \mathrm{Id}$, where $f : \overline{Z}^N \to X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$ is the normalization map for $\overline{Z}$ as in the Diagram (9.2). Note that the map $\overline{\Psi_j^l}$ is defined since all the rational maps ψ_j^l naturally extend to morphisms $\psi_j^l : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times (\mathbb{P}^1)^3$.

If Z satisfies the modulus condition M_{ssup}, then there is some $1 \le i \le n-1$ such that $[f^*(y_i = 1) - (m+1)f^*(t=0)] \ge 0$ on $\overline{Z}^N$. Since f' is identity on $\mathbb{P}^1$, this implies that $[f'^*(y_i = 1) - (m+1)f'^*(t=0)] \ge 0$ on $\overline{Z}^N$ for some $2 \le i \le n$. Since $\overline{\Psi_j^l}$ is identity on the last $(n-1)$ copies of $\mathbb{P}^1$, we conclude that $f'^* \circ \overline{\Psi_j^l}^*[F_{n+3,i}^1 - (m+1)F_{n+3,0}] \ge 0$ for some $4 \le i \le n+2$, which in turn gives $\widetilde{\Psi_j^l}^* \circ g'^*[F_{n+3,i}^1 - (m+1)F_{n+3,0}] \ge 0$ on $\overline{Z}^N \times \mathbb{P}^1$. Since $\widetilde{\Psi_j^l}$ is projective and surjective, we conclude from Proposition 5.2 that $g'^*[F^1_{n+3,i} - (m+1)F_{n+3,0}] \geq 0$ for some $4 \leq i \leq n+2$. This prove the modulus condition M_{ssup} for $Z \star \Gamma^l_j$.

If Z satisfies the modulus condition M_{sum}, then we use the same argument as above plus the proof of the M_{sum} part of Lemma 9.2 to complete the proof of the lemma. □

Our main interest about δ^2 is the following.

Proposition 9.6. *Assume that* $\mathrm{char}(k) \neq 2$ *and let* $\alpha \in \underline{\mathrm{Tz}}^q(X, n; m)$ *be a cycle such that* $\partial(\alpha) = 0$. *Then* $\delta^2(\alpha) = 0$ *as a homology class in* $\mathrm{TCH}^{q+2}(X, n+2; m)$. *In particular,* δ *descends to a natural map of additive higher Chow groups* $\delta : \mathrm{TCH}^q(X, \bullet; m) \to \mathrm{TCH}^{q+1}(X, \bullet; m)[1]$ *such that* $\delta^2 = 0$.

Proof. The last part of the proposition follows from Corollary 9.4 once we prove the first part. Since δ^2 is equal to a boundary in $\underline{\mathrm{Tz}}^q(X, \bullet; m)$ if and only if it is a boundary of an admissible additive cycle in $z^q(X \times \mathbb{G}_m, \bullet - 1)$, we can work with the latter complexes. We begin with the following.

Claim : *For any* $\alpha \in \underline{\mathrm{Tz}}^q(X, n; m)$ *and* Γ *as in* (9.11), *one has*

$$\partial(\alpha \times \Gamma) = \alpha \times \partial\Gamma - \partial\alpha \times \Gamma$$

in $z^{q+2}(X \times \mathbb{G}_m \times \mathbb{G}_m, n+2)$.

This is an elementary computation. Since Γ is a $\mathbb{Z}$-linear combination of Γ^l_j's, it suffices to prove the claim for each Γ^l_j. We can further assume that α is represented by an irreducible cycle Z. Then we note that for $\epsilon \in \{0, \infty\}$, $l \in \{1, 2\}$,

$$\partial^\epsilon_i(Z \times \Gamma^l_j) = \begin{cases} Z \times \partial^\epsilon_i \Gamma^l_j & \text{if } 1 \leq i \leq 3 \\ \partial^\epsilon_{i-3}(Z) \times \Gamma^l_j & \text{if } 4 \leq i \leq n+2. \end{cases}$$

This in turn gives

$$\begin{aligned}
\partial(Z \times \Gamma^l_j) &= \sum_{i=1}^{n+2} (-1)^i [\partial^\infty_i(Z \times \Gamma^l_j) - \partial^0_i(Z \times \Gamma^l_j)] \\
&= Z \times \left\{ \sum_{i=1}^{3} (-1)^i [\partial^\infty_i(\Gamma^l_j) - \partial^0_i(\Gamma^l_j)] \right\} \\
&\quad + \left\{ \sum_{i=4}^{n+2} (-1)^i [\partial^\infty_{i-3}(Z) - \partial^0_{i-3}(Z)] \right\} \times \Gamma^l_j \\
&= Z \times \partial(\Gamma^l_j) + \left\{ \sum_{i=4}^{n+2} (-1)^i [\partial^\infty_{i-3}(Z) - \partial^0_{i-3}(Z)] \right\} \times \Gamma^l_j \\
&= Z \times \partial(\Gamma^l_j) + \left\{ \sum_{i=1}^{n-1} (-1)^{i+3} [\partial^\infty_i(Z) - \partial^0_i(Z)] \right\} \times \Gamma^l_j \\
&= Z \times \partial(\Gamma^l_j) - \partial Z \times \Gamma^l_j.
\end{aligned}$$

This proves the claim.

Next, we see from the definition of ϕ_n in (9.1) that for any irreducible admissible cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, $\delta^2(Z)$ is just the image of Z under the rational map

$$X \times \mathbb{G}_m \times \square^{n-1} \to X \times \mathbb{G}_m \times \square^{n+1} \tag{9.14}$$

$$\begin{aligned}(x, t, y_1, \cdots, y_{n-1}) &\mapsto (x, t, t^{-1}, t^{-1}, y_1, \cdots, y_{n-1}) \\ &= (x, t, u, u, y_1, \cdots, y_{n-1}).\end{aligned}$$

In particular, we see from (9.12) that $\delta^2(Z)$ is the cycle $Z \star (t, u, u) = \Delta^*_{\mathbb{G}_m}(Z \times (t, u, u))$. Hence for any $\alpha \in \underline{\mathrm{Tz}}^q(X, n; m)$, we have $\delta^2(\alpha) = \alpha \star (t, u, u)$ as an element of $z^{q+2}(X \times \mathbb{G}_m, n+1)$. Since the diagram

$$\begin{tikzcd}
\mathbb{G}_m \times \square^{n+1} \arrow[r, "\iota_i^\epsilon"] \arrow[d, "\Delta_{\mathbb{G}_m}"'] & \mathbb{G}_m \times \square^{n+2} \arrow[d, "\Delta_{\mathbb{G}_m}"] \\
\mathbb{G}_m \times \mathbb{G}_m \times \square^{n+1} \arrow[r, "\iota_i^\epsilon"'] & \mathbb{G}_m \times \mathbb{G}_m \times \square^{n+2}
\end{tikzcd}$$

is Cartesian, we see in $z^{q+2}(X \times \mathbb{G}_m, n+1)$ that for any $\alpha \in \underline{\mathrm{Tz}}^q(X, n; m)$ with $\partial(\alpha) = 0$, we have

$$\begin{aligned}
2\delta^2(\alpha) &= 2\alpha \star (t, u, u) \\
&= \alpha \star 2(t, u, u) \\
&= \alpha \star \partial\Gamma & \text{(by (9.11))} \\
&= \Delta^*_{\mathbb{G}_m}(\alpha \times \partial\Gamma) \\
&= \Delta^*_{\mathbb{G}_m}(\alpha \times \partial\Gamma - \partial\alpha \times \Gamma) \\
&= \Delta^*_{\mathbb{G}_m}(\partial(\alpha \times \Gamma)) & \text{(by the Claim)} \\
&= \partial\left(\Delta^*_{\mathbb{G}_m}(\alpha \times \Gamma)\right) \\
&= \partial(\alpha \star \Gamma).
\end{aligned}$$

Since $\alpha \star \Gamma \in \underline{\mathrm{Tz}}^{q+2}(X, n+3; m)$ by Lemma 9.5, we conclude that $2\delta^2(\alpha) = 0$ as a class in $\mathrm{TCH}^{q+2}(X, n+2, m)$. Since $\mathrm{char}(k) \neq 2$, we conclude from Corollary 8.18 that the homology class of $\delta^2(\alpha)$ is zero in $\mathrm{TCH}^{q+2}(X, n+2, m)$. $\square$

The following is the main result of this section.

Theorem 9.7. *Let X be a smooth projective variety over a field k such that char$(k) \neq 2$. Then the additive higher Chow groups $(\mathrm{TCH}(X), \wedge)$ is a graded-commutative algebra which is equipped with a differential operator δ of degree one satisfying $\delta^2 = 0$. Moreover, this differential operator commutes with the pull-back and push-forward maps of additive higher Chow groups.*

Proof. It follows directly from Corollary 8.17 and Proposition 9.6. The commutativity of δ with the pull-back and push-forward maps can be directly checked from its definition. $\square$

Remark 9.8. It seems that the assumption $\mathrm{char}(k) \neq 2$ in Corollary 8.18 and Theorem 9.7 is not serious and can be removed using the infinite pro-l extension of the field for $l \neq 2$. We do not go into this here.

10. Differential operator and Leibniz rule

In this section we introduce another differential operator on the additive cycle complexes. This differential is an analogue of the Connes' boundary operator in the theory of Hochschild and cyclic homology (*cf.* [17, Chapter 2]). We shall show that this satisfies the Leibniz rule for the wedge product on the admissible cycle classes. We shall comment about the relation between the two differential operators towards the end of this section.

For $1 \leq i \leq n$, let σ_i be the permutation

$$\sigma_i(j) = \begin{cases} i & \text{if } j = 1 \\ j-1 & \text{if } 2 \leq j \leq i \\ j & \text{if } j > i. \end{cases}$$

Let $\delta_i : X \times B_n \to X \times B_{n+1}$ be the rational map $\phi_n^i = \sigma_i \circ \phi_n$, where ϕ_n is defined in (9.1). In particular, we have $\phi_n^1 = \phi_n$. Since σ_i defines an automorphism of $X \times \overline{B}_n$ which preserves the modulus condition and proper intersection, it follows from Proposition 9.3 that for any admissible cycle $Z \in \underline{\mathrm{Tz}}^q(X, n; m)$, $V = \phi_n^i(Z^\times)$ defines an admissible cycle $\delta_i(Z) = V \in \underline{\mathrm{Tz}}^{q+1}(X, n+1; m)$. Thus $\delta_i = \sigma_i^* \circ \delta$. We put

$$\delta_{alt} = \sum_{i=1}^{n} (-1)^i \delta_i : \ \mathrm{Tz}^q(X, n; m) \to \mathrm{Tz}^{q+1}(X, n+1; m). \tag{10.1}$$

These δ_i's satisfy the following identities.

Lemma 10.1. *For $i, j \in \{1, \cdots, n\}$, we have*

$$\begin{cases} \delta_i \delta_j = \delta_{j+1} \delta_i, & \text{if } i \leq j, \\ \delta_i \delta_j = \delta_j \delta_{i-1}, & \text{if } i > j. \end{cases} \tag{10.2}$$

Proof. This is obvious from the definition of δ_i's. $\square$

Lemma 10.2. *We have $\delta_{alt}^2 = 0$.*

Proof. Indeed δ_{alt}^2 is,

$$\left(\sum_{i=1}^{n+1} (-1)^i \delta_i \right) \left(\sum_{j=1}^{n} (-1)^j \delta_j \right) = \sum_{i \leq j} (-1)^{i+j} \delta_i \delta_j + \sum_{i \geq j+1} (-1)^{i+1} \delta_i \delta_j$$

$$= \sum_{i \leq j} (-1)^{i+1} \delta_i \delta_j + \sum_{i \geq j+1} (-1)^{i+j} \delta_j \delta_{i-1}.$$

For the right hand side, use the substitution $i - 1 = j'$ and $j = i'$ so that we have

$$\delta_{alt}^2 = \sum_{i \leq j} (-1)^{i+j} \delta_i \delta_j + \sum_{i' \leq j'} (-1)^{i'+j'+1} \delta_{i'} \delta_{j'} = 0.$$

$\square$

One major drawback of δ_{alt} is that unlike $\delta = \delta_1$, it does not have good commutativity (or anti-commutativity) relations with the boundary operator ∂ of the additive cycle complex. We shall show in the next section that δ_{alt} still defines an operator on the homology groups. At this stage, we note that δ_{alt} and ∂ satisfy the following properties.

Lemma 10.3. *The following identities hold, where $\epsilon \in \{0, \infty\}$:*

$$\begin{cases} \partial_i^\epsilon \delta_k = \delta_{k-1} \partial_i^\epsilon, & \text{if } i < k, \\ \partial_i^\epsilon \delta_k = 0, & \text{if } i = k, \\ \partial_i^\epsilon \delta_k = \delta_k \partial_{i-1}^\epsilon, & \text{if } i > k. \end{cases} \tag{10.3}$$

Equivalently,

$$\begin{cases} \delta_k \partial_i^\epsilon = \partial_{i+1}^\epsilon \delta_k, & \text{if } k \le i, \\ \delta_k \partial_i^\epsilon = \partial_i^\epsilon \delta_{k+1}, & \text{if } k \ge i. \end{cases} \tag{10.4}$$

In particular, $\partial_{i+1}^\epsilon \delta_i = \partial_i^\epsilon \delta_{i+1}$.

Proof. This is straightforward, while it takes some patience to keep track of the indices correctly. ∎

We will come back to this issue about the interaction of δ_{alt} with ∂ in the next section. See Lemma 11.7 and Question 11.8.

10.0.1. *Leibniz rule.* We now show that the differential δ_{alt} is in fact a derivation for the wedge product on the additive cycle complex. We first define a new operator on a pair of additive cycles which is the cycle theoretic analog of the *cyclic shuffle product* in the Hochschild complex in [17, Section 4.3.2]. Recall that this cyclic shuffle product is used to show that the Conne's boundary operator is a derivation for the wedge product on the Hochschild homology. We prove here the analogous statement for the additive higher Chow groups.

Consider the rational map

$$\mu' : X \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n_1+n_2-2} \times \square \to X \times X \times \mathbb{G}_m \times \square^{n_1+n_2} \tag{10.5}$$

$$\mu'(x, t_1, t_2, y_1, \cdots, y_{n_1+n_2-2}, y) = \left(x, t_1 t_2, y, \frac{t_1 y - 1}{t_1 t_2 y - 1}, y_1, \cdots, y_{n_1+n_2-2} \right).$$

For two irreducible admissible cycles $Z_i \in \mathrm{Tz}^{q_i}(X, n_i, m)$ for $i = 1, 2$, let $Z_1 \times' Z_2$ be the closure of $\mu'((Z_1 \times Z_2) \times \square)$ in $X \times X \times \mathbb{G}_m \times \square^{n_1+n_2}$, where we omit a suitable transposition from our notations. As before, we put $n = n_1 + n_2 - 1$ and $q = q_1 + q_2 - 1$.

Proposition 10.4. *$Z_1 \times' Z_2$ is an admissible cycle in $\mathrm{Tz}^q(X \times X, n + 2; m)$.*

Proof. We first prove the modulus condition for $Z = Z_1 \times' Z_2$. We consider the commutative diagram

$$\begin{array}{ccc} X \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n_1+n_2-2} \times \square & \xrightarrow{\mu'} & X \times X \times \mathbb{G}_m \times \square^{n_1+n_2} \\ \downarrow & & \downarrow \\ X \times X \times \mathbb{G}_m \times \mathbb{G}_m \times \square^{n_1+n_2-2} & \xrightarrow[\mu]{} & X \times X \times \mathbb{G}_m \times \square^{n_1+n_2-2}, \end{array} \tag{10.6}$$

where the vertical arrows are the natural projections. In particular, we get the map $Z \to \mu_*(Z_1 \times Z_2)$ under the projection map. Let $\overline{Z}$ and $\overline{\mu_*(Z_1 \times Z_2)}$ denote the closures of Z and $\mu_*(Z_1 \times Z_2)$ in $X \times \widehat{B}_{n+2}$ and $X \times \widehat{B}_n$ respectively. Thus we get a commutative diagram

$$\begin{tikzcd} \overline{Z} \arrow[r] \arrow[d] & X \times X \times \widehat{B}_{n+2} \arrow[d, "p"] \\ \overline{\mu_*(Z_1 \times Z_2)} \arrow[r] & X \times X \times \widehat{B}_n. \end{tikzcd} \tag{10.7}$$

We have shown in Corollary 8.5 that $\mu_*(Z_1 \times Z_2)$ satisfies the modulus condition. Since $p^*(F^1_{n,i}) = F^1_{n+2,i}$ and $p^*(F_{n,0}) = F_{n+2,0}$, we see that the modulus condition for $\mu_*(Z_1 \times Z_2)$ implies the same for $p^*(\mu_*(Z_1 \times Z_2))$. The modulus condition for Z now follows from Proposition 2.4.

Now we compute the various boundaries of Z. It is easy to see from (10.5) that

$$\partial_1^0(Z) = 0,\ \partial_1^\infty(Z) = \sigma_{n_1} \cdot (\mu_*(Z_1 \times \delta(Z_2))),$$

$$\partial_2^0(Z) = \mu_*(\delta(Z_1) \times Z_2),\ \partial_2^\infty(Z) = \delta(\mu_*(Z_1 \times Z_2)).$$

For $3 \le i \le n+1$, we have

$$\partial_i^\epsilon(Z) = \begin{cases} \partial_{i-2}^\epsilon(Z_1)\times' Z_2 & \text{if } 3 \le i \le n_1+1 \\ Z_1 \times' \partial^\epsilon_{i-n_1-1}(Z_1) & \text{if } n_1+2 \le i \le n+1. \end{cases}$$

Since Z_i's are admissible cycles, the above automatically imply the proper intersection property of Z. $\square$

Using Proposition 10.4, we can define the our *cyclic shuffle product* as

$$Z_1 \bar{\wedge}' Z_2 := \sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,n_1-1,n_2-1)}} \mathrm{sgn}(\nu)\nu \cdot (Z_1 \times' Z_2) \in \mathrm{Tz}^{q+1}(X \times X, n+2; m), \tag{10.8}$$

where the permutations $\nu \in \mathbb{P}\mathrm{erm}_{(1,n_1-1,n_2-1)}$ act on the given set of n objects $\{(1,2), 3, \cdots, n+1\}$ in the obvious way, treating the element $(1,2)$ as a single object. This induces the action $\nu \cdot (\xi \times' \eta)$. We extend this bilinearly to get the cyclic shuffle product

$$\mathrm{Tz}^{q_1}(X, n_1; m) \otimes \mathrm{Tz}^{q_2}(X, n_2; m) \xrightarrow{\bar{\wedge}'} \mathrm{Tz}^{q+1}(X \times X, n+2; m). \tag{10.9}$$

Proposition 10.5 (Leibniz rule)**.** *Let $\xi \in \mathrm{Tz}^{q_1}(X, n_1; m), \eta \in \mathrm{Tz}^{q_2}(X, n_2; m)$. Then, in the group $\mathrm{Tz}^{q+1}(X \times X, n+1; m)$, we have*

$$\begin{aligned} &\delta_{alt}(\xi \bar{\wedge} \eta) - (\delta_{alt}\xi)\bar{\wedge}\eta - (-1)^{n_1-1}\xi\bar{\wedge}(\delta_{alt}\eta) \\ &\qquad = \partial(\xi\bar{\wedge}'\eta) - (\partial\xi)\bar{\wedge}'\eta - (-1)^{n_1-1}\xi\bar{\wedge}'(\partial\eta), \end{aligned} \tag{10.10}$$

where $n = n_1 + n_2 - 1$, $q = q_1 + q_2 - 1$.

Proof. It follows from Proposition 10.4 that

$$
\begin{aligned}
\partial\left(\xi \times' \eta\right) &= \sum_{i=1}^{n+1}(-1)^i(\partial_i^{\infty}-\partial_i^0)\left(\xi \times' \eta\right) \\
&= \delta\left(\mu_*(\xi\times\eta)\right)-\left[\sigma_{n_1}\cdot\left(\mu_*\left(\xi\times\delta\eta\right)\right)+\mu_*\left(\delta\xi\times\eta\right)\right] \\
&\quad+\sum_{i=3}^{n_1+1}(-1)^i\{(\partial_{i-2}^{\infty}-\partial_{i-2}^0)(\xi)\}\times'\eta \\
&\quad+\sum_{i=n_1+2}^{n+1}(-1)^i\xi\times'\{(\partial_{i-n_1-1}^{\infty}-\partial_{i-n_1-1}^0)(\eta)\} \\
&= \delta\left(\mu_*(\xi\times\eta)\right)-\left[\sigma_{n_1}\cdot\left(\mu_*\left(\xi\times\delta\eta\right)\right)+\mu_*\left(\delta\xi\times\eta\right)\right] \\
&\quad+\partial\xi\times'\eta+(-1)^{n_1-1}\xi\times'\partial\eta.
\end{aligned}
\tag{10.11}
$$

In particular, we have

$$
\begin{aligned}
&\delta\left(\mu_*(\xi\times\eta)\right)-\mu_*\left(\delta\xi\times\eta\right)-\sigma_{n_1}\cdot\left(\mu_*\left(\xi\times\delta\eta\right)\right) \\
&\qquad=\partial\left(\xi\times'\eta\right)-\partial\xi\times'\eta-(-1)^{n_1-1}\xi\times'\partial\eta.
\end{aligned}
\tag{10.12}
$$

Since the desired identity (10.10) of the proposition and (10.12) differ only by the action of various permutations, it is now enough to show that the identity holds on the coordinates of $X \times B_{n+2}$.

One ingredient in the proof is the application of Proposition 8.6 and Lemma 8.11 to the triple shuffles $\mathbb{P}\mathrm{erm}_{(1,r,s)}$. Observe that the map δ_{alt} can be written as a sum over the set $\mathbb{P}\mathrm{erm}_{(1,n)}$ of double shuffles. Indeed, for the coordinate $(t, y_1, \cdots, y_n)$, we have

$$
\begin{aligned}
\delta_{alt}(t,y_1,\cdots,y_n) &= \sum_{i=1}^{n+1}(-1)^i(t,y_1,\cdots,y_{i-1},\underbrace{\frac{1}{t}}_{i^{\text{th}}},y_i,\cdots,y_n) \\
&= -\sum_{\tau\in\mathbb{P}\mathrm{erm}_{(1,n)}}(\mathrm{sgn}(\tau))\tau\cdot\left(t,\frac{1}{t},y_1,\cdots,y_n\right).
\end{aligned}
$$

We first compute the term on the left hand side of the identity (10.10) of the proposition on the level the coordinates of B_{n_1}, B_{n_2}. Since the variety X doesn't play a role in the calculation, we shrink points of X from our notations. Let $\xi = (t_1, y_1, \cdots, y_{n_1-1}), \eta = (t_2, y_{n_1}, \cdots, y_{n-1})$. We then have

$$
\begin{cases}
\xi\times\eta=(t_1,t_2,y_1,\cdots,y_{n-1}) \\
\xi\times'\eta=C_{t_1,t_2}\times(y_1,\cdots,y_{n-1}).
\end{cases}
$$

Here $C_{t_1,t_2} := \{t_1\}\times'\{t_2\} \subset \mathbb{G}_m \times \square^2$ is the parameterized curve

$$
C_{t_1,t_2}=\left\{\left(t_1t_2,y,\frac{t_2y-1}{t_1t_2y-1}\right)|y\in k\right\}
$$

for any given points $t_1, t_2 \in \mathbb{G}_m$. As shown before, this is an admissible 1-cycle and its boundary is given by (*cf.* [19, Lemma 2.5])

$$\partial C_{t_1,t_2} = \left(t_1t_2, \frac{1}{t_1}\right) + \left(t_1t_2, \frac{1}{t_2}\right) - \left(t_1t_2, \frac{1}{t_1t_2}\right). \tag{10.13}$$

This property will play an important role in the calculation.

To simplify the notations, we introduce new indices r, s, u by letting $r := n_1 - 1, s := n_2 - 1$, and $u := n - 1 = r + s$.

Then, by a direct calculation we have for the first term of (10.10),

$$\begin{aligned}
&\delta_{alt}(\xi \wedge \eta) \\
= \ & \delta_{alt}\left(\sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma))\sigma \cdot (\mu_* (\xi \times \eta))\right) \\
= \ & \delta_{alt}\left(\sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma))\sigma \cdot (t_1t_2, y_1, \cdots, y_u)\right) \\
= \ & \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma))\delta_{alt}(t_1t_2, y_{\sigma^{-1}(1)}, \cdots, y_{\sigma^{-1}(u))}) \\
= \ & - \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma)) \sum_{\tau \in \mathbb{P}\mathrm{erm}_{(1,u)}} (\mathrm{sgn}(\tau))\tau \cdot (xy, \frac{1}{t_1t_2}, y_{\sigma^{-1}(1)}, \cdots, y_{\sigma^{-1}(u)}) \\
= \ & - \sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s)}} (\mathrm{sgn}(\sigma)) \sum_{\tau \in \mathbb{P}\mathrm{erm}_{(1,u)}} (\mathrm{sgn}(\tau))(\sigma_\tau \cdot \tau) \cdot (t_1t_2, \frac{1}{t_1t_2}, y_1, \cdots, y_u) \\
= \ & - \left(\sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}} (\mathrm{sgn}(\nu))\nu\right) \cdot (t_1t_2, \frac{1}{t_1t_2}, y_1, \cdots, y_u),
\end{aligned}$$

where σ_τ and the last equality are from Lemma 8.11.

The second term of (10.10) is,

$$
\begin{aligned}
&(\delta_{alt}\xi)\wedge\eta \\
= \; & \mu\left(\sum_{\sigma\in\mathbb{P}\mathrm{erm}_{(r+1,s)}}(\mathrm{sgn}(\sigma))\sigma\cdot((\delta_{alt}\xi)\times\eta)\right) \\
= \; & -\mu_*\sum_{\sigma\in\mathbb{P}\mathrm{erm}_{(r+1,s)}}(\mathrm{sgn}(\sigma))\sigma\cdot \\
& \left(\sum_{\tau\in(_{(1,r)}}(\mathrm{sgn}(\tau))\tau\cdot(t_1,\frac{1}{t_1},y_1,\cdots,y_r)\right)\times(t_2,y_{r+1},\cdots,y_u) \\
= \; & -\left(\sum_{\sigma\in\mathbb{P}\mathrm{erm}_{(r+1,s)}}(\mathrm{sgn}(\sigma))\sigma\right)\cdot \\
& \left(\sum_{\tau\in\mathbb{P}\mathrm{erm}_{(1,r)}}(\mathrm{sgn}(\tau))(\tau\times\mathrm{Id}_s)\right)\cdot(t_1t_2,\frac{1}{t_1},y_1,\cdots,y_u) \\
= \; & -\left(\sum_{\nu\in\mathbb{P}\mathrm{erm}_{(1,r,s)}}(\mathrm{sgn}(\nu))\nu\right)\cdot(t_1t_2,\frac{1}{t_1},y_1,\cdots,y_u),
\end{aligned}
$$

where the last equality follows from Proposition 8.6.

Before we compute the third term of (10.10), first note that

$$
\begin{aligned}
\delta_{alt}(t_2,y_{r+1},\cdots,y_n) &= \sum_{i=r+1}^{u+1}(-1)^{i-r}(t_2,y_{r+1},\cdots,y_{i+r-1},\underbrace{\frac{1}{t_2}}_{i+r^{\mathrm{th}}},y_{i+r},\cdots,y_n) \\
&= -(-1)^r\sum_{\tau\in\mathbb{P}\mathrm{erm}_{(1,s)}}(\mathrm{sgn}(\tau))\tau\cdot(t_2,\frac{1}{t_2},y_{r+1},\cdots,y_u).
\end{aligned}
$$

Hence, for the third term of (10.10), we have

$$\begin{aligned}
& (-1)^r \xi \wedge (\delta_{alt}\eta) \\
= \ & - \left(\sum_{\sigma \in \mathbb{P}\mathrm{erm}_{(r,s+1)}} (\mathrm{sgn}(\sigma))\sigma \right) \cdot \\
& \left(\sum_{\tau \in \mathbb{P}\mathrm{erm}_{(1,s)}} (\mathrm{sgn}(\tau))(\mathrm{Id}_r \times \tau) \right) \cdot (t_1 t_2, \frac{1}{t_2}, y_1, \cdots, y_u) \\
= \ & - \left(\sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}} (\mathrm{sgn}(\nu))\nu \right) \cdot (t_1 t_2, \frac{1}{t_2}, y_1, \cdots, y_u),
\end{aligned}$$

where the last equality follows from Proposition 8.6.

Thus, the left hand side of the equation (10.10) is compactified into

$$\begin{aligned}
& \delta_{alt}(\xi \wedge \eta) - (\delta_{alt}\xi) \wedge \eta - (-1)^r \xi \wedge (\delta_{alt}\eta) \qquad (10.14) \\
= - & \sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}} (\mathrm{sgn}(\nu))\nu \cdot \left((t_1 t_2, \frac{1}{t_1}) + (t_1 t_2, \frac{1}{t_2}) - (t_1 t_2, \frac{1}{t_1 t_2}) \right) \times (y_1, \cdots, y_u).
\end{aligned}$$

On the other hand, for the coordinate points, we have for the first term of the right hand side of (10.10),

$$\partial(\xi \bar{\wedge}' \eta) = \partial \left(\sum_{\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}} (\mathrm{sgn}(\nu))\nu \right) \cdot C_{t_1,t_2} \times (y_1, \cdots, y_u). \qquad (10.15)$$

Since we have by (10.13) the equation

$$\left(t_1 t_2, \frac{1}{t_1} \right) + \left(t_1 t_2, \frac{1}{t_2} \right) - \left(t_1 t_2, \frac{1}{t_1 t_2} \right) = \partial C_{t_1,t_2},$$

for each $\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}$, the four faces of $\nu \cdot (\xi \times' \eta)$ in the sum (10.15) that interact with C_{t_1,t_2}, *i.e.*, ∂_i^ϵ with $i \in \nu(1,2)$, cancel out the corresponding terms of (10.14). This process cancels all terms in (10.14), thus all the terms of the left hand side of (10.10). Hence, we need to see what happens for the remaining faces of (10.15). But we have already seen in (10.11) that for any general admissible cycles ξ and η,

$$\begin{aligned}
\sum_{i=3}^{n+1} (-1)^i (\partial_i^\infty - \partial_i^0)(\xi \times' \eta) &= \left(\sum_{i=3}^{n_1+1} (-1)^i (\partial_{i-2}^\infty - \partial_{i-2}^0)\xi \right) \times' \eta \\
&+ \xi \times' \left(\sum_{i=n_1+2}^{n+1} (-1)^i (\partial_{i-n_1-1}^\infty - \partial_{i-n_1-1}^0)\eta \right) \\
&= (\partial\xi) \times' \eta + (-1)^{n_1-1} \xi \times' (\partial\eta).
\end{aligned}$$

We apply this argument for the faces ∂_i^ϵ with $i \notin \nu(1,2)$ to each $\nu \cdot (\xi \times' \eta)$, where $\nu \in \mathbb{P}\mathrm{erm}_{(1,r,s)}$, and take the signed sum. This gives the remaining terms $(\partial\xi)\bar{\wedge}'\eta + (-1)^{n_1-1}\xi\bar{\wedge}'(\partial\eta)$ of the right hand side of (10.10). This completes the proof of Proposition 10.5. □

11. Normalized additive cycle complex

We have seen in the previous section that the differential operator δ_{alt} on the additive cycle complex has all the nice properties except that it does not commute (or anti-commute) with the boundary map ∂. In this section, we rectify this anomaly by introducing the normalized version of the additive cycle complex. This is analogous to the similar construction of S. Bloch in [3, Theorem 4.4.2]. It turns out that δ_{alt} indeed has good behaviors with respect to the boundary operator of the normalized complex. Our final goal is then achieved by showing that the homology of the normalized additive cycle complex does not change our additive higher Chow groups. We begin with the following construction of M. Levine which appeared in [15] to study Bloch's higher Chow groups. This is essentially equivalent to the method of S. Bloch in [3]. We suitably adapt this Levine's construction to the additive world in what follows next.

11.1. **Homotopy variety.** In the following construction, we shall make an identification between $\square$ and $\mathbb{A}^1$ via the map

$$\square \to \mathbb{A}^1; \quad y \mapsto 1/(1-y). \tag{11.1}$$

This gives the isomorphism $(\mathbb{P}^1, \{0,1,\infty\}) \cong (\mathbb{P}^1, \{1,\infty,0\})$. The boundary map of the corresponding cycle complex under this identification is given by $\sum_i (-1)^i(\partial_i^0 - \partial_i^1)$. Let X be a smooth projective variety and let $i_n : W_n^X \to X \times \mathbb{G}_m \times \square^{n+1} \times \mathbb{P}^1$ be the closed subvariety defined by the equation

$$t_0(1-y_n)(1-y_{n+1}) = t_0 - t_1, \tag{11.2}$$

where $(y_1, \cdots, y_n)$ are the coordinates of $\square^n$ and $(t_0 : t_1)$ are the homogeneous coordinates of $\mathbb{P}^1$. Let $\pi_n : W_n^X \to X \times \mathbb{G}_m \times \square^n$ be the map defined by

$$\pi_n(x, t, y_1, \cdots, y_{n+1}, (t_0 : t_1)) = (x, t, y_1, \cdots, y_{n-1}, y_n + y_{n+1} - y_n y_{n+1}). \tag{11.3}$$

Let $\left((u_0^1 : u_1^1), \cdots, (u_0^{n+1} : u_1^{n+1})\right)$ denote the homogeneous coordinate of $(\mathbb{P}^1)^{n+1}$. We identify $\square^{n+1}$ to the open subset of $(\mathbb{P}^1)^{n+1}$ given by $\prod_{i=1}^{n+1} \{u_0^i \neq 0\}$, and we set $y_i = u_1^i/u_0^i, y = t_1/t_0$. In terms of these homogeneous coordinates, the projectivization $\overline{W_n^X}$ of W_n^X in $X \times \square \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1$ is given by the equation

$$t_0(u_0^n - u_1^n)(u_0^{n+1} - u_1^{n+1}) = u_0^n u_0^{n+1}(t_0 - t_1). \tag{11.4}$$

Let $\overline{\theta}_n : X \times \square \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \to X \times \square \times (\mathbb{P}^1)^{n-1} \times \mathbb{P}^1$ be the natural projection map given by

$$\overline{\theta}_n \left(x, t, (u_0^1; u_1^1), \cdots, (u_0^{n+1} : u_1^{n+1}), (t_0 : t_1)\right) = \left(x, t, (u_0^1; u_1^1), \cdots, (u_0^{n-1} : u_1^{n-1}), (t_0 : t_1)\right) \tag{11.5}$$

and let θ_n be its restriction to the open set $X \times \mathbb{G}_m \times \square^{n+1} \times \square$. Here we identify $\mathbb{G}_m$ as $\square \backslash \{1\}$. Let $\overline{\pi}_n : \overline{W_n^X} \to X \times \square \times (\mathbb{P}^1)^{n-1} \times \mathbb{P}^1$ be the restriction of $\overline{\theta}_n$ to $\overline{W_n^X}$.

Let $p_n : X \times \mathbb{G}_m \times \square^{n+1} \times \mathbb{P}^1 \to X \times \mathbb{G}_m \times \square^{n+1}$ be the natural projection.

Lemma 11.1. *$W_n^X \cap \{t_0 = 0\} = \emptyset$ and hence W_n^X is in fact contained in the open subset $X \times \mathbb{G}_m \times \square^{n+1} \times \square$. The variety $\overline{W_n^X}$ (and hence W_n^X) is smooth. Moreover, π_n and $\overline{\pi}_n$ are flat and surjective morphisms of relative dimension one.*

Proof. The first assertion is immediate from the defining equation of W_n^X. Using this assertion, we can write the restriction of p_n on W_n^X as

$$W_n^X \hookrightarrow X \times \mathbb{G}_m \times \square^{n+1} \times \square \to X \times \mathbb{G}_m \times \square^{n+1},$$

where the first inclusion is given by the equation $y = y_n + y_{n+1} - y_n y_{n+1}$. Since X is smooth, it is now easy to see using the Jacobian criterion that W_n^X is smooth and the above composite map is an isomorphism. Furthermore, under this isomorphism, the map π_n is just the projection $(x, t, y_1, \cdots, y_n, y) \mapsto (x, t, y_1, \cdots, y_{n-1}, y)$, as can be checked from the equation of W_n^X. This also shows that π_n is in fact smooth and surjective map of relative dimension one. To prove the smoothness of $\overline{W_n^X}$, we can check it locally on an open set of points with coordinates $\big(x, t, (u_0^1 : u_1^1), \cdots, (u_0^{n+1} : u_1^{n+1}), (t_0 : t_1)\big)$ where either of u_i^n, u_i^{n+1}, t_i is non-zero for $i = 0, 1$. In any such open set, $\overline{W_n^X}$ has the equation of the form that defines W_n^X and hence is smooth. It is also easy to check using these local coordinates that $\overline{\pi}_n$ is of relative dimension one. Moreover, as $\overline{\theta}_n$ is projective and π_n is surjective, we see that $\overline{\pi}_n$ is projective and surjective. In particular, it is flat (*cf.* [11, Exercise III-10.9]). This proves the lemma. $\square$

Lemma 11.2. *The diagram*

$$\begin{tikzcd} W_n^X \arrow[r, "i_n"] \arrow[dr, "\pi_n"'] & X \times \mathbb{G}_m \times \square^{n+1} \times \square \arrow[r, "j_n"] \arrow[d, "\theta_n"] & X \times \square \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \arrow[d, "\overline{\theta}_n"] \\ & X \times \mathbb{G}_m \times \square^n \arrow[r, "j'_n"'] & X \times \square \times (\mathbb{P}^1)^n \end{tikzcd} \tag{11.6}$$

commutes.

Proof. By Lemma 11.1, W_n^X is contained in the open subset $X \times \mathbb{G}_m \times \square^{n+1} \times \square$, where it is given by the equation $y = y_n + y_{n+1} - y_n y_{n+1}$. It is clear from the definition of θ_n in (11.5) that the triangle on the left commutes. The right square commutes by the definitions of θ_n and $\overline{\theta}_n$. Hence the outer trapezium also commutes. $\square$

Lemma 11.3. *Let $Z \subset X \times \mathbb{G}_m \times \square^n$ be a closed subvariety which satisfies the modulus condition M_{sum}. Let $Z' = (i_n)_* (\pi_n{}^*(Z))$. Then Z' also satisfies the modulus condition M_{sum}.*

Proof. Let $\overline{Z}$ and $\overline{Z}'$ denote the closures of Z and Z' in $X \times \square \times (\mathbb{P}^1)^n$ and $X \times \square \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1$ respectively. Let $\overline{Z}^N$ and $\overline{Z}'^N$ denote the normalizations of $\overline{Z}$ and $\overline{Z}'$ respectively. Using Lemmas 11.1, 11.2 and the projectivity of the map $\overline{\theta}_n$, we see that $\overline{\theta}_n(\overline{Z}') = \overline{Z}$ in the Diagram (11.6). Since $\overline{W^X_n}$ smooth, we get the following commutative diagram:

$$\begin{tikzcd}
\overline{Z}'^N \arrow[r, "g"'] \arrow[rr, bend left, "\nu'_Z"] \arrow[d, "f"] & \overline{W^X_n} \arrow[r, "\bar{i}_n"'] \arrow[dr, "\overline{\pi}_n"] & X \times \square \times (\mathbb{P}^1)^{n+1} \times \mathbb{P}^1 \arrow[d, "\overline{\theta}_n"] \\
\overline{Z}^N \arrow[rr, "\nu_Z"'] & & X \times \square \times (\mathbb{P}^1)^n,
\end{tikzcd} \tag{11.7}$$

where f is the map of normal k-schemes induced by the surjective map $\overline{Z}' \to \overline{Z}$. As before, let $F^{\infty}_{n+1,i}$ denote the Cartier divisor on $(\mathbb{P}^1)^n$ defined by $\{y_i = \infty\}$ for $1 \le i \le n$ and we have similar Cartier divisors $F^{\infty}_{n+2,i}$ on $(\mathbb{P}^1)^{n+1}$ for $1 \le i \le n+1$. It is then easy to see from the defining equation of $\overline{W^X_n}$ in (11.4) that

$$\begin{aligned}
\overline{\pi}^*_n(F^{\infty}_{n+1,n}) &= \bar{i}^*_n(y = \infty) \\
&= \bar{i}^*_n(t_0 = 0) \\
&\le \bar{i}^*_n[(u^n_0 = 0) + (u^{n+1}_0 = 0)] \\
&= \bar{i}^*_n[(y_n = \infty) + (y_{n+1} = \infty)] \\
&= \bar{i}^*_n[F^{\infty}_{n+2,n} + F^{\infty}_{n+2,n+1}].
\end{aligned} \tag{11.8}$$

Since $\overline{Z}'^N \to \overline{W^X_n}$ is a map of normal k-schemes, and since $\overline{\theta}^*_n(F^{\infty}_{n+1,i}) = F^{\infty}_{n+2,i}$ for $1 \le i \le n-1$, we have

$$\begin{aligned}
\nu^*_{Z'} \circ \overline{\theta}^*_n(F^{\infty}_{n+1}) &= \sum_{i=1}^{n} \nu^*_{Z'} \circ \overline{\theta}^*_n(F^{\infty}_{n+1,i}) \\
&= \nu^*_{Z'}[\sum_{i=1}^{n-1} F^{\infty}_{n+2,i}] + g^* \circ \overline{\pi}^*_n(F^{\infty}_{n+1,n}) \\
&\le \nu^*_{Z'}[\sum_{i=1}^{n-1} F^{\infty}_{n+2,i}] + g^* \circ \bar{i}^*_n[F^{\infty}_{n+2,n} + F^{\infty}_{n+2,n+1}] \\
&= \nu^*_{Z'}[\sum_{i=1}^{n+1} F^{\infty}_{n+2,i}] \\
&= \nu^*_{Z'}(F^{\infty}_{n+2}).
\end{aligned} \tag{11.9}$$

Now the modulus condition for Z implies that $\nu^*_Z[(m+1)F_{n+1,0}] \le \nu^*_Z[F^{\infty}_{n+1}]$ which implies that $f^* \circ \nu^*_Z[(m+1)F_{n+1,0}] \le f^* \circ \nu^*_Z[F^{\infty}_{n+1}]$. This in turn implies that $\nu^*_{Z'} \circ \overline{\theta}^*_n[(m+1)F_{n+1,0}] \le \nu^*_{Z'} \circ \overline{\theta}^*_n[F^{\infty}_{n+1}]$. Since $\overline{\theta}^*_n(F_{n+1,0}) = F_{n+2,0}$, we conclude from (11.9) that $\nu^*_{Z'}[(m+1)F_{n+2,0}] \le \nu^*_{Z'}[F^{\infty}_{n+2}]$ which proves the modulus condition for Z'. ∎

For any closed subvariety $Z \subset X \times \mathbb{G}_m \times \square^n$, let

$$W_n^X(Z) := (p_n)_* \circ (i_n)_* \circ \pi_n^*(Z). \tag{11.10}$$

Note that $W_n^X(Z)$ is a closed subvariety of $X \times \mathbb{G}_m \times \square^{n+1}$ since p_n is projective.

Lemma 11.4 (*cf.* [15])**.** *For Z as above, one has*
(1) $W_n^X(Z) \cdot \{y_n = 0\} = Z = W_n^X(Z) \cdot \{y_{n+1} = 0\}$.
(2) $Z \in z^q(X \times \mathbb{G}_m, n) \Rightarrow W_n^X(Z) \in z^q(X \times \mathbb{G}_m, n+1)$.

Proof. Let $Z' = (i_n)_* \circ \pi_n^*(Z)$. Let $\square_i$ denote the ith factor of $\square^{n+1}$ in $X \times \mathbb{G}_m \times \square^{n+1} \times \square$ in Diagram (11.6). Then we see from Diagram (11.6) that

$$Z' = (Z \times \square_n \times \square_{n+1}) \cdot W_n^X = (Z \times \square_n \times \square_{n+1}) \cdot \{y = y_n + y_{n+1} - y_n y_{n+1}\}. \tag{11.11}$$

Combining this with the equation of W_n^X in (11.2), we see that

$$Z' \cdot \{y_n = \epsilon\} = (Z \times \square_n \times \square_{n+1}) \cdot \{y = y_{n+1}\} \ \text{ for } \ \epsilon = 0, 1.$$

Thus we get

$$W_n^X(Z) \cdot \{y_n = 0\} = p_n(Z') \cdot \{y_n = 0\} = (p_n)_*[(Z \times \{0\} \times \square_{n+1}) \cdot \{y = y_{n+1}\}] = Z.$$

Using the same steps, we also see that

$$W_n^X(Z) \cdot \{y_n = 1\} = (Z \cdot \{y = 1\}) \times \square.$$

Since $Z' \cdot \{y_{n+1} = \epsilon\} = (Z \times \square_n \times \square_{n+1}) \cdot \{y = y_n\}$, the same calculation as before shows that $W_n^X(Z) \cdot \{y_{n+1} = 0\} = Z$ and $W_n^X(Z) \cdot \{y_{n+1} = 1\} = (Z \cdot \{y = 1\}) \times \square$. This proves (1). This in particular shows that the intersection $W_n^X(Z) \cdot \{y_i = \epsilon\}$ is proper for $i = n, n+1$ and $\epsilon = 0, 1$.

Now we calculate the other boundaries of $W_n^X(Z)$. It follows directly from (11.11) that for $1 \le i \le n-1$, one has $W_n^X(Z) \cdot \{y_i = \epsilon\} = W_{n-1}^X(Z \cdot \{y_i = \epsilon\})$. Since π_{n-1} is flat of relative dimension one as shown in Lemma 11.1, we see that this intersection is proper. This proves (2), thus the lemma. $\square$

Proposition 11.5. *For the modulus condition M_{sum}, let $Z \in \underline{\mathrm{Tz}}^q(X, n+1; m)$ be an irreducible admissible cycle. Then $W_n^X(Z) \in \underline{\mathrm{Tz}}^q(X, n+2; m)$.*

Proof. This follows by combining Lemmas 11.3 and 11.4. $\square$

Using Proposition 11.5, we can define for every $n \ge 0$, a group homomorphism

$$W_n^X : \underline{\mathrm{Tz}}^q(X, n+1; m)_{sum} \to \underline{\mathrm{Tz}}^q(X, n+2; m)_{sum} \tag{11.12}$$

by extending W_n^X linearly. This homomorphism has the property that $\partial_i^0 \circ W_n^X = \mathrm{Id}$ for $i = n, n+1$ as shown in Lemma 11.4.

11.2. **Normalized additive cycle complex.** We now define the normalized version of our additive cycle complexes and study their properties. These complexes are the additive analogues of the similar constructions of S. Bloch and M. Levine [3, 15] for higher Chow groups. It turns out that the normalized additive cycle complex has better properties. One expects that the resulting additive Chow groups also form a kind of Witt complex so that all the expected maps from the relative K-groups of the infinitesimal deformations of smooth schemes to the additive higher Chow groups actually factors through these normalized additive higher Chow groups. Another outcome of using the normalized additive cycle complex is the invention of the motivic version of the cycilc homology of A. Connes [17]. We shall deal with this aspect in the end of this section.

Definition 11.6. Let X be a smooth projective variety and let M be any of the modulus conditions $M_{sum}, M_{sup}, M_{ssup}$. For $n, m \geq 1$, let $\mathrm{Tz}^q_N(X, n; m)$ be the subgroup of $\mathrm{Tz}^q(X, n; m)$ of cycles α such that $\partial_i^0(\alpha) = 0$ for $1 \leq i \leq n-1$ and $\partial_i^\infty(\alpha) = 0$ for $2 \leq i \leq n-1$. Using the simplicial structure of the additive cycle complex, it easy to see that for $\alpha \in \mathrm{Tz}^q_N(X, n; m)$, one has $\partial_1^\infty \circ \partial_1^\infty(\alpha) = 0$. Writing ∂_1^∞ as ∂^N, we thus get a subcomplex $\left(\mathrm{Tz}^q_N(X, \bullet; m), \partial^N\right)$ of $(\mathrm{Tz}^q(X, \bullet; m), \partial)$. We shall call $\mathrm{Tz}^q_N(X, \bullet; m)$ the "*normalized additive cycle complex*" for any given modulus condition.

We define the *normalized* additive higher Chow groups of X by $\mathrm{TCH}^q_N(X, n; m) = H_n\left(\mathrm{Tz}^q_N(X, \bullet; m)\right)$.

The point of using this normalized complex is the following lemma regarding its interaction with δ_{alt} in the previous section.

Lemma 11.7. (1) *For all $i \in \{1, \cdots, n\}$, we have $\delta_i(\mathrm{Tz}^q_N(X, n; m)) \subset \mathrm{Tz}^{q+1}_N(X, n+1; m)$.*
(2) *For $\partial^N = \partial_1^\infty$, we have $\delta_{alt}\partial^N + \partial^N\delta_{alt} = 0$ on $\mathrm{Tz}^*_N(X, \bullet; m)$.*

Proof. (1) follows immediately from (10.3). To prove (2), we have for any in $\mathrm{Tz}^q_N(X, n; m)$,

$$\begin{aligned}
\delta_{alt}\partial^N &= \delta_{alt}\partial_1^\infty \\
&= \sum_{i=1}^{n+1} (-1)^i \delta_i \partial_1^\infty \\
&= \sum_{i=1}^{n+1} (-1)^i \partial_1^\infty \delta_{i+1} \\
&= \partial_1^\infty \left(\sum_{i=1}^{n+1} (-1)^i \delta_{i+1} \right) \\
&= -\partial_1^\infty \delta_1 - \partial_1^\infty \left(\sum_{i=2}^{n+2} (-1)^i \delta_i \right) \\
&= -\partial_1^\infty \left(\sum_{i=1}^{n+2} (-1)^i \delta_i \right) \\
&= -\partial^N \delta_{alt},
\end{aligned}$$

where the third equality holds from (10.4) and the fifth one follows from (10.3). This finishes the proof of (2). □

It was shown in [3, Theorem 4.4.2] that the normalized cycle complex for the usual higher Chow groups is quasi-isomorphic to the original cycle complex. This prompts one to ask the following.

Question 11.8. *Is the natural inclusion* $\left(\mathrm{Tz}^q_N(X, \bullet; m), \partial^N\right) \hookrightarrow (\mathrm{Tz}^q(X, \bullet; m), \partial)$ *of complexes a quasi-isomorphism?*

Our next goal is to show that the answer to this Question 11.8 is indeed affirmative for the modulus condition M_{sum}. We also derive some crucial consequences of this for the additive higher Chow groups. Although we can not prove this for the other two modulus conditions at this moment for certain technical reasons, we definitely expect this to be the case for all modulus conditions.

For $n \geq 1$, let $C(n-1) = \oplus_q \underline{\mathrm{Tz}}^q(X, n; m)$. Let $D(n-1) \subset C(n-1)$ be the subgroup of degenerate cycles. Let $C(n-1)^0 = \bigcap_{i=1}^{n-1} \ker(\partial_i^0) \subset C(n-1)$. Note that $\bigoplus_{n \geq 1} C(n-1)^0$ is a subcomplex of $\bigoplus_{n\geq 1} C(n-1)$ with respect to the boundary map $\partial = \sum\limits_{i=1}^{n-1} (-1)^i \partial_i^\infty$. We shall write this subcomplex as $(C(*), \partial)$.

Proposition 11.9. $C(n-1) = D(n-1) \oplus C(n-1)^0$. *Thus, we can identify*

$$\bigoplus_q \mathrm{Tz}^q(X, n; m) = \bigoplus_q \underline{\mathrm{Tz}}^q(X, n; m) / \underline{\mathrm{Tz}}^q(X, n; m)_{\mathrm{degn}}$$

with its subgroup $C(n-1)^0 = \bigcap_{i=1}^{n-1} \ker(\partial_i^0)$.

Proof. We first prove that $C(n-1) = D(n-1) + C(n-1)^0$. Let $z \in C(n-1)$, and suppose that $\partial^0_{r+1}(z) = \cdots = \partial^0_{n-1}(z) = 0$, to use a backward induction argument on the subscripts. Let $z' := z - \pi_r \circ \partial^0_r(z)$, where π_r is the pull-back via the projection $(x, t, y_1, \cdots, y_{n-1}) \mapsto (x, t, y_1, \cdots, \widehat{y_i}, \cdots, y_{n-1})$. One easily checks that $\partial^j_\nu \circ \pi_\nu = 1$. Hence, $\partial^0_r(z') = \partial^0_r(z) - \partial^0_r(\pi \circ \partial^0_r(z)) = \partial^0_r(z) - \partial^0_r(z) = 0$. For $s > r$, one first easily sees that $\partial^k_{\mu-1} \circ \partial^j_\nu = \partial^j_\nu \circ \partial^k_\mu$ and $\pi_\nu \circ \partial^j_{\mu-1} = \partial^j_\mu \circ \pi_\nu$ for $\nu < \mu$ from the standard cubical identities. Hence, using these two and the induction hypothesis that $\partial^0_s(z) = 0$, we obtain $\partial^0_s(z') = \partial^0_s(z) - \partial^0_s(\pi_r \circ \partial^0_r(z)) = 0 - \pi_r \circ \partial^0_{s-1} \circ \partial^0_r(z) = -\pi_r \partial^0_r \partial^0_s(z) = 0$. Hence, by induction we may write z as a sum of elements in $D(n-1)$ and $C(n-1)^0$.

To prove that the sum is direct, let r be the minimum such that there is a non-zero $z \in C(n-1)^0$ with $z = \sum_{i=1}^r \pi_i w_i$ for some w_i. Since $\partial^0_r = 0$ and $\partial^0_r \circ \pi_r = 1$, by applying ∂^0_r to z, we obtain

$$w_r = -\sum_{i=1}^{r-1} \partial^0_r \circ \pi_i w_i = -\sum_{i=1}^{r-1} \pi_i \left(\partial^0_{r-1} w_i\right),$$

where for the second equation we used the cubical identity $\pi_\nu \circ \partial^j_{\mu-1} = \partial^j_\mu \circ \pi_\nu$ for $\nu < \mu$. Hence, by plugging this back into the expression of z, we have

$$z = \sum_{i=1}^{r-1} \pi_i w_i + \pi_r w_r = \sum_{i=1}^{r-1} \pi_i \left(w_i - \pi_{r-1} \circ \partial^0_{r-1} w_i\right),$$

where for the second equation we used the cubical identity $\pi_\nu \circ \pi_{\mu-1} = \pi_\mu \circ \pi_\nu$ for $\nu < \mu$. This contradicts the minimality of r. Hence the sum is direct. This proves the proposition. $\square$

Theorem 11.10. *Let X be a smooth projective variety. Then for the modulus condition M_{sum}, the natural inclusion $\mathrm{Tz}^q_N(X, \bullet; m) \hookrightarrow \mathrm{Tz}^q(X, \bullet; m)$ of complexes is a quasi-isomorphism.*

Proof. In this proof, we again temporarily identify $\square$ with $\mathbb{A}^1$ via the isomorphism of (11.1). Recall that this gives the isomorphism $(\mathbb{P}^1, \{0, 1, \infty\}) \cong (\mathbb{P}^1, \{1, \infty, 0\})$. Thus we need to show that the inclusion $\left(\mathrm{Tz}^q_N(X, \bullet; m), \partial^N\right) \hookrightarrow (\mathrm{Tz}^q(X, \bullet; m), \partial)$ is a quasi-isomorphism, where $\partial = \sum_{i=1}^{n-1} (-1)^i (\partial_i^0 - \partial_i^1)$. We make the appropriate identification for $C(*)^0$ as well.

Using Proposition 11.9, we only need to show that the inclusion $\mathrm{Tz}^q_N(X, \bullet; m) \hookrightarrow C(*)^0$ is a quasi-isomorphism. In particular, we can assume that for all cycles $\alpha \in \mathrm{Tz}^q(X, n+1; m)$, one has $\partial_i^1(\alpha) = 0$ for $1 \le i \le n$. For $i \ge 0$, let

$$C(*)^0_i = \{\alpha \in C(*)^0 | \partial_j^0(\alpha) = 0 \text{ for } j \ge i+2\}.$$

Let $\tau_j \in \mathbb{P}\mathrm{erm}_n$ be the permutation such that

$$\tau_j(i) = \begin{cases} i & \text{if } i < j \\ i-1 & \text{if } i > j \\ n & \text{if } i = j. \end{cases}$$

For any $0 \le i \le n$ and admissible cycle $\alpha \in \mathrm{Tz}^q(X, n+1; m)$, let

$$H_i^n(\alpha) = (-1)^{n+1-i} \tau_{n+1-i} \cdot W_n^X(\alpha).$$

By Proposition 11.5, $W_n^X(\alpha) \in \mathrm{Tz}^q(X, n+2; m)$. Since τ clearly preserves the admissibility, we get a homomorphism $H_i^n : \mathrm{Tz}^q(X, n+1; m) \to \mathrm{Tz}^q(X, n+2; m)$. Now we use the computations of the boundaries of $W_n^X(\alpha)$ in Lemma 11.4 to see that H_i^n restricts to a map

$$H_i^n : C(*)^0_n \to C(*)^0_{n+1}. \tag{11.13}$$

Define $\psi_0 : C(*)^0_n \to C(*)^0_n$ by $\psi_0 = \mathrm{Id} - (\partial \circ H_0 + H_0 \circ \partial)$ and we inductively define $\psi_{i+1} = (\mathrm{Id} - (\partial \circ H_n + H_n \circ \partial)) \circ \psi_i$. In particular, we have

$$\psi := \psi_n = (\mathrm{Id} - (\partial \circ H_n + H_n \circ \partial)) \circ \cdots \circ (\mathrm{Id} - (\partial \circ H_0 + H_0 \circ \partial)),$$

where $\partial = \sum (-1)^i \partial_i^0$. Thus ψ defines an endomorphism of $C(*)^0$ which is homotopic to identity. Furthermore, Lemma 11.4 implies that the restriction of ψ on $C(*)^0_0 = \mathrm{Tz}^q_N(X, n+1; m)$ is identity. The proof of the theorem will now be complete from the following.

Claim : $\psi_i \left(C(*)^0_{n-i-1}\right) \subset C(*)^0_{n-i-2}$ *for* $0 \le i \le n-2$.

We prove it for ψ_0 and other cases are exactly similar and can be proved inductively. We have

$$\begin{aligned} H_0 \circ \partial &= H_0^{n-1} \circ \partial \\ &= (-1)^n W_{n-1}^X \circ \partial \\ &= (-1)^n \sum_{i=1}^{n-1} (-1)^i [W_{n-1}^X \circ \partial_i^0] + (-1)^{2n} [W_{n-1}^X \circ \partial_n^0]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \partial \circ H_0 &= \partial \circ H_0^n \\ &= (-1)^{n+1} \partial \circ W_n^X \\ &= (-1)^{n+1} \sum_{i=1}^{n+1} (-1)^i \partial_i^0 \circ W_n^X \\ &= (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i [\partial_i^0 \circ W_n^X] \\ &\quad + (-1)^{2n+1} \partial_i^0 \circ W_n^X + (-1)^{2n+2} \partial_i^0 \circ W_n^X \\ &= (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i [W_{n-1}^X \circ \partial_i^0] \\ &\quad + (-1)^{2n+1} id + (-1)^{2n+2} \mathrm{Id} \\ &= (-1)^{n+1} \sum_{i=1}^{n-1} (-1)^i [W_{n-1}^X \circ \partial_i^0], \end{aligned}$$

where the fifth equality follows from Lemma 11.4 and (11.12). Thus we get $H_0 \circ \partial + \partial \circ H_0 = W_{n-1}^X \circ \partial_n^0$. However, we see from Lemma 11.4 again that $\partial_n^0 \circ W_{n-1}^X \circ \partial_n^0 = \partial_n^0$. This shows that $\psi_0 \left(C(*)_{n-1}^0 \right) \subset C(*)_{n-2}^0$. This proves the claim and the theorem. □

Although we are unable to prove Theorem 11.10 for the modulus conditions M_{sup} and M_{ssup} in this paper, the following result gives a partial answer to Question 11.8 in these cases.

Theorem 11.11. *Let X be a smooth projective variety over k. Then for any modulus condition and for any $n, m \geq 1$, the natural map $\mathrm{TCH}_N^q(X, n; m) \to \mathrm{TCH}^q(X, n; m)$ is injective. In particular, the map $\mathrm{TCH}_N^n(k, n; m) \to \mathrm{TCH}^n(k, n; m)$ is an isomorphism.*

Proof. Let $X^0 = X \times \mathbb{G}_m$. We have seen before that forgetting the modulus condition, one has a natural inclusion of cubical objects

$$(\underline{n} \mapsto (\underline{\mathrm{Tz}}^q(X, n; m), \partial_i^\epsilon)) \to \left(\underline{n} \mapsto (\underline{z}^q(X^0, n), \partial_i^\epsilon) \right),$$

where the right side is the cubical object for the higher Chow group. This induces a natural inclusion of chain complexes $i_X : \underline{\mathrm{Tz}}^q(X, \bullet; m) \hookrightarrow \underline{z}^q(X^0, \bullet)$ such that a cycle $z \in \underline{\mathrm{Tz}}^q(X, \bullet; m)$ is degenerate if and only if $i_X(z) \in \underline{z}^q(X^0, \bullet)$ is so. Considering the normalized version of these complexes, we get a commutative diagram

$$\begin{array}{ccc} \mathrm{Tz}^q_N(X,n;m) & \longrightarrow & z^q_N(X^0,n-1) \\ \downarrow & & \downarrow \\ \mathrm{Tz}^q(X,n;m) & \longrightarrow & z^q(X^0,n-1) \end{array} \tag{11.14}$$

which is clearly Cartesian and all arrows are injective. Let

$$\overline{\mathrm{Tz}}^q(X,n;m) = \mathrm{Image}\left(z^q_N(X^0,n-1) \oplus \mathrm{Tz}^q(X,n;m) \to z^q(X^0,n-1)\right).$$

Then we get the exact sequence of complexes

$$0 \to \mathrm{Tz}^q_N(X,\bullet;m) \longrightarrow z^q_N(X^0,\bullet) \oplus \mathrm{Tz}^q(X,\bullet;m) \longrightarrow \overline{\mathrm{Tz}}^q(X,\bullet;m) \to 0.$$

The theorem would follow if we show that the map

$$H_n\left(z^q_N(X^0,\bullet)\right) \to H_n\left(\overline{\mathrm{Tz}}^q(X,\bullet;m)\right)$$

is injective for all $n \geq 1$. For this, we consider the inclusions

$$z^q_N(X^0,\bullet) \hookrightarrow \overline{\mathrm{Tz}}^q(X,\bullet;m) \hookrightarrow z^q(X^0,\bullet).$$

By [3, Theorem 4.4.2], the composite map is a quasi-isomorphism. Hence the map $H_n\left(z^q_N(X^0,\bullet)\right) \to H_n\left(\overline{\mathrm{Tz}}^q(X,\bullet;m)\right)$ is in fact split injective. This finishes the proof of the theorem.

To prove the isomorphism of the additive higher Chow groups of zero cycles, we simply note that the inclusion map $\mathrm{Tz}^n_N(k,n;m) \hookrightarrow \mathrm{Tz}^n(k,n;m)$ is in fact an isomorphism and hence the map $\mathrm{TCH}^n_N(k,n;m) \to \mathrm{TCH}^n(k,n;m)$ is surjective too. □

For our purposes, the following is the main application of Theorems 11.10 and 11.11.

Corollary 11.12. *For the modulus condition M_{sum}, the map δ_{alt} defines a homomorphism*

$$\delta_{alt} : \mathrm{TCH}^q(X,n;m) \to \mathrm{TCH}^{q+1}(X,n+1;m)$$

satisfying $\delta^2_{alt} = 0$.
For the modulus condition M_{ssup}, δ_{alt} defines a homomorphism

$$\delta_{alt} : \mathrm{TCH}^n(k,n;m) \to \mathrm{TCH}^{n+1}(k,n+1;m)$$

satisfying $\delta^2_{alt} = 0$.

Proof. This follows immediately by combining Lemmas 10.2, 10.3, Lemma 11.7, and Theorem 11.10. The second statement about the additive zero cycles follows in the same way, where we now use Theorem 11.11 in place of Theorem 11.10. □

We complete our study of CDGA structures on the additive higher Chow groups with the following main result of this section.

Theorem 11.13. *Let X be a smooth projective variety over a field k. Then for the modulus condition M_{sum}, the additive higher Chow groups $(\mathrm{TCH}(X), \wedge, \delta_{alt})$ form a graded-commutative differential graded algebra, where δ_{alt} is the graded derivation for the wedge product $\wedge$. This derivation commutes with the pull-back and push-forward maps of additive higher Chow groups.*

Proof. This follows directly from Corollary 8.17, Proposition 10.5 and Corollary 11.12. The commutativity property of δ_{alt} with the pull-back and push-forward maps follows from the similar property of δ in Theorem 9.7. □

Remark 11.14. It follows from the above results that for the modulus condition M_{sum}, $\mathrm{TCH}(X)$ has two differentials, namely, δ of Theorem 9.7 and δ_{alt} of Theorem 11.13. But one can in fact show that the latter is just a finitely many copies of the former. To see this, one needs to know that each $\sigma \in \mathbb{P}\mathrm{erm}_n$ acts on $\mathrm{TCH}^q(X, n+1, m)$ as $\mathrm{sgn}(\sigma) \cdot \mathrm{Id}$. Using this, one can also see that while δ is analogous to the exterior derivation of Kähler differentials, δ_{alt} corresponds to the exterior derivation of the Hochschild homology. These differential operators are related by a kind of anti-symmetrizer maps defined via permutations (*cf.* [17]).

Remark 11.15. We see from Corollary 8.17 and Proposition 10.5 that $(\mathrm{TCH}(X), \wedge, \delta_{alt})$ is a differential graded algebra also for the modulus condition M_{ssup} if the answer to Question 11.8 is affirmative. As we have already remarked before, this is very much expected and a proof of this should be available in a near future. For now, it does follow from Corollary 11.12 that the groups $\left(\{\mathrm{TCH}^n(k, n; m)\}_{n \geq 1}, \wedge, \delta_{alt}\right)$ form a CDGA.

11.3. **Motivic cyclic homology.** We end this section by showing how one can use the normalized additive Chow groups and Theorem 11.10 to define a motivic version of the cyclic homology of A. Connes. We see from Lemmas 10.2 and 11.7 that there is a bicomplex

(11.15)
$$\begin{tikzcd}
\vdots \arrow[d, "\partial^N"'] & \vdots \arrow[d, "\partial^N"'] & \vdots \arrow[d, "\partial^N"'] & \\
\mathrm{Tz}^{q+2}_N(n+2) \arrow[d, "\partial^N"'] & \mathrm{Tz}^{q+1}_N(n+1) \arrow[l, "\delta_{alt}"'] \arrow[d, "\partial^N"'] & \mathrm{Tz}^q_N(n) \arrow[l, "-\delta_{alt}"'] \arrow[d, "\partial^N"'] & \cdots \arrow[l, "\delta_{alt}"'] \\
\mathrm{Tz}^{q+2}_N(n+1) \arrow[d, "\partial^N"'] & \mathrm{Tz}^{q+1}_N(n) \arrow[l, "-\delta_{alt}"'] \arrow[d, "\partial^N"'] & \mathrm{Tz}^q_N(n-1) \arrow[l, "\delta_{alt}"'] \arrow[d, "\partial^N"'] & \cdots \arrow[l, "-\delta_{alt}"'] \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

where $\mathrm{Tz}^q_N(n) := \mathrm{Tz}^q_N(X, n; m)$. This allows us to propose a cyclic analogue of additive cycle complex, regarding the additive cycle complex as the motivic analogue of the Hochschild complex. Let $\mathrm{Tz}(n) := \bigoplus^q \mathrm{Tz}^q(n)$. Note that ∂^N decreases only n by 1, while δ_{alt} increases both q and n by $+1$. The above bicomplex then reads,

(11.16)
$$\begin{tikzcd}
\vdots \arrow[d, "\partial^N"'] & \vdots \arrow[d, "\partial^N"'] & \vdots \arrow[d, "\partial^N"'] & \vdots \arrow[d, "\partial^N"'] \\
\mathrm{Tz}(4) \arrow[d, "\partial^N"'] & \mathrm{Tz}(3) \arrow[l, "\delta_{alt}"'] \arrow[d, "\partial^N"'] & \mathrm{Tz}(2) \arrow[l, "-\delta_{alt}"'] \arrow[d, "\partial^N"'] & \mathrm{Tz}(1) \arrow[l, "\delta_{alt}"'] \\
\mathrm{Tz}(3) \arrow[d, "\partial^N"'] & \mathrm{Tz}(2) \arrow[l, "-\delta_{alt}"'] \arrow[d, "\partial^N"'] & \mathrm{Tz}(1) \arrow[l, "\delta_{alt}"'] & \\
\mathrm{Tz}(2) \arrow[d, "\partial^N"'] & \mathrm{Tz}(1) \arrow[l, "\delta_{alt}"'] & & \\
\mathrm{Tz}(1). & & &
\end{tikzcd}$$

Let $\mathcal{BZ}$ be this bicomplex. This is a mixed complex in the sense of A. Connes (*cf.* [17, p. 79]). We apply the usual formalism of mixed complexes to $\mathcal{BZ}$. By definition, its homology $H_n(\mathcal{BZ})$ is the homology of the first column and this is just the additive higher Chow groups $\mathrm{TCH}_N^*(X, n-1; m)$. Its *cyclic homology* $HC_n(\mathcal{BZ})$ is the homology $H_n(Tot(\mathcal{BZ}))$ of the total complex. Notice that the bicomplex (11.15) itself is not a mixed complex, but since $\mathcal{BZ}$ is the direct sum of these, the groups $H_n(\mathcal{BZ})$ and $HC_n(\mathcal{BZ})$ have natural decompositions.

We define the *motivic cyclic homology* $\mathrm{CCH}^*(X, n; m)$ as the cyclic homology $HC_n(\mathcal{BZ})$ of the bicomplex $\mathcal{BZ}$. In this analogy, we could as well call our additive higher Chow groups as *motivic Hochschild homology*. The group $\mathrm{CCH}^q(X, n; m)$ is the direct summand of $\mathrm{CCH}^*(X, n; m)$ that comes from the diagonal of (11.15) that contains $\mathrm{Tz}_N^q(n)$ in the first column. Note that, despite the double index (q, n), the group $\mathrm{CCH}^q(X, n; m)$ contains cycles not just from $\mathrm{Tz}_N^q(n)$, but also from
$$\bigoplus_{i\geq 0}^{\min\left\{q, \left\lfloor \frac{n-1}{2} \right\rfloor\right\}} \mathrm{Tz}_N^{q-i}(n-2i).$$
Following the formalism of mixed complexes (*cf.* [17, 2.5.3]), we have the long exact sequence of complexes
$$0 \to (\mathrm{Tz}(*), \partial^N) \xrightarrow{I} Tot(\mathcal{BZ}) \xrightarrow{S} Tot\left(\mathcal{BZ}[1,1]\right) \to 0.$$
Notice that $Tot\left(\mathcal{BZ}[1,1]\right) = \left(Tot(\mathcal{BZ})\right)[2]$. Thus, we obtain its long exact sequence, which is the Connes' periodicity exact sequence, that decomposes as follows:

Corollary 11.16. *Suppose the modulus condition is M_{sum}. Then there is a Connes' periodicity exact sequence involving* TCH *and* CCH*:*
$$\cdots \xrightarrow{B} \mathrm{TCH}^q(X, n; m) \xrightarrow{I} \mathrm{CCH}^q(X, n; m) \xrightarrow{S} \mathrm{CCH}^{q-1}(X, n-2; m) \xrightarrow{B}$$
$$\mathrm{TCH}^q(X, n-1; m) \xrightarrow{I} \cdots,$$
where the maps I, S, B have bidegrees $(0,0), (-1,-2), (+1,+1)$ in (q, n) respectively.

As a consequence, we have the following motivic interpretation of the top Hodge piece $HC_{n-1}^{n-1}(k)$ of the cyclic homology $HC_{n-1}(k)$ of the ground field.

Corollary 11.17. *Assume that char$(k) \neq 2$. Then, for any modulus condition and for $n, m \geq 1$, there is an isomorphism*

$$\mathrm{CCH}^n(k, n; m) \xrightarrow{\cong} \mathbb{W}_m\Omega^{n-1}_{k/\mathbb{Z}}/d\mathbb{W}_m\Omega^{n-2}_{k/\mathbb{Z}}.$$

Proof. By definition,

$$\mathrm{CCH}^n(k, n; m) = \frac{\mathrm{Tz}^n_N(k, n)}{\partial^N \mathrm{Tz}^n_N(k, n+1) + \delta_{alt}\mathrm{Tz}^{n-1}_N(k, n-1)}.$$

By Theorems 3.4 and 11.11, we have $\mathrm{Tz}^n_N(k, n)/\partial^N \mathrm{Tz}^n_N(k, n+1) \simeq \mathbb{W}_m\Omega^{n-1}_{k/\mathbb{Z}}$. On the other hand, combining these two theorems with Theorem 11.11 and Remark 11.15, we see that the elements of the group $\delta_{alt}\mathrm{Tz}^{n-1}_N(k, n-1)$ are exact de Rham-Witt forms. This finishes the proof. □

Further study of the above motivic cyclic homology using algebraic cycles and its applications to additive higher Chow groups will be taken up in a sequel.

12. Remarks and computations

12.1. **Moving modulus conditions.** We saw that M_{sum} and M_{ssup} apparently have much better structural behavior than the modulus condition M_{sup} studied in [14, 18], and this makes the former better suited for being a motivic cohomology. On the other hand, in the main theorem of [18], the regulators on 1-cycles were defined with the modulus condition M_{sup}. Although we have seen that this regulator map does exist and has good properties with the modulus condition M_{ssup}, its construction doesn't automatically generalize to the groups with M_{sum}. So, one may ask the following :

Question 12.1. *Given an M_{sum}-admissible cycle ξ with $\partial\xi = 0$, can one find M_{sup}-admissible cycles η and Γ such that $\xi = \eta + \partial\Gamma$?*

A positive answer to this question will immediately solve one part of Conjecture 2.9. This is a kind of *deeper moving lemma* than we have proved in this paper. This moving lemma allows one to move the modulus as well as the proper intersection property when we move a cycle. On the other hand, the moving lemma of this paper does not allow changing the modulus conditions. We expect the answer to the above question to be much harder.

12.2. **Examples.**

Example 12.2. We give an example where the homotopy used in [1, 16] doesn't preserve the modulus conditions for additive Chow groups of quasi-projective varieties, even for the simplest possible cases.

Take $X = \mathbb{A}^1_k$ and $n = 1$, so, we are interested in admissible cycles in $X \times \widetilde{B}_1 = X \times \mathbb{A}^1_k$. Admissible closed subvarieties $Z \subset X \times \mathbb{A}^1_k$ are given by the condition $Z \cap (X \times \{0\}) = \emptyset$. Let $\mathbb{G}_{a,k} = \mathbb{A}^1_k$ act on X by translation, and take its function field $K = k(s)$, s transcendental over k. Take the line $\phi : \square^1_K \to \mathbb{G}_{a,K}$ defined by $y \mapsto sy/(y-1)$ that sends 0 to 0 and ∞ to the k-generic point s of $\mathbb{G}_{a,k}$, which is K-rational in $\mathbb{G}_{a,K}$.

Take Z given by the ideal $(xt+1) \subset k[x,t]$, which is in $\mathrm{Tz}^1(\mathbb{A}^1, 1; m)$. Then, Z_K is given by $(xt+1) \subset K[x,t]$ and $\mathrm{pr}'^* Z_K$ is given by $(xt+1) \subset K[x,t,y/(y-1)]$. Pulling back through μ_ϕ, we get $(x + sy/(y-1))\, t + 1 = 0$. This is the equation for our homotopy variety Z'. Rewriting it as $1 - y = t((y-1)x + sy)$, we see that it doesn't satisfy any of the given modulus conditions $M_{sum}, M_{sup}, M_{ssup}$. For instance, for a given $m \geq 1$, we need $1-y$ to be divisible by at least t^{1+m} where $m \geq 1$, which is obviously false in this case. Hence $Z' \notin \mathrm{Tz}^1(\mathbb{A}^1_K, 2; m)$.

Example 12.3. Recall from Remark 5.3 that if X is projective, then admissible cycles in $X \times \widetilde{B}_1 = X \times \mathbb{A}^1$ have a very simple description : an admissible irreducible closed subvariety Z should be of the form $Y \times \{*\} \subset X \times \mathbb{A}^1$ for some closed subvariety $Y \subset X$, and a closed point $\{*\} \neq \{0\}$ of $\mathbb{A}^1$. This variety obviously satisfies all of the modulus conditions.

Note that the admissible variety Z in Example 12.2 is not of the form $Y \times \{*\}$: this happens because $X = \mathbb{A}^1_k$ is not complete.

These two examples show that the additive higher Chow groups of quasi-projective varieties may have a bit more complicated structures than those of projective varieties. The authors don't know yet what extra-structures one can expect in general for this quasi-projective case.

12.3. **A computation.** We finish the paper with a calculation of some additive higher Chow groups, which the authors had worked out while working on this paper. The following extends [5, Theorem 6.4, p. 153] to affine spaces.

Theorem 12.4. *Let M be the modulus condition M_{sum}, M_{sup}, or M_{ssup}. Let $X = \mathbb{A}^r_k$, and let $m = 1$. Then, the additive higher Chow groups of zero-dimensional cycles of X are the absolute Kähler differentials of k:*

$$\mathrm{TCH}^{r+n+1}(X, n; 1) \simeq \Omega^{n-1}_{k/\mathbb{Z}}.$$

Remark 12.5. Note that, although it looks similar, this theorem does not imply that additive higher Chow groups have $\mathbb{A}^1$-homotopy invariance. For the structure morphism $\mathbb{A}^r_k \to \mathrm{Spec}(k)$, the pull-backs of 0-cycles on $\mathrm{Spec}(k) \times \widetilde{B}_n$ to $X \times \widetilde{B}_n$ are r-cycles, not 0-cycles.

Proof. The proof is very similar to that of [5, Theorem 6.4, p. 153]. For a closed point $p \in X \times \widetilde{B}_n$ that does not intersect the faces and the divisor $\{t = 0\}$, we define a homomorphism by setting

$$\psi(p) := \mathrm{Tr}_{k(p)/k} \left(\frac{1}{t} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}} \right) (p) \in \Omega^{n-1}_{k/\mathbb{Z}}.$$

In other words, we ignore the coordinate of X. This defines a homomorphism $\psi : \mathrm{Tz}^{r+n+1}(X, n; 1) \to \Omega^{n-1}_{k/\mathbb{Z}}$.

Claim (1): *The composition*

$$\psi \circ \partial : \mathrm{Tz}^{r+n+1}(X, n+1; 1) \xrightarrow{\partial} \mathrm{Tz}^{r+n+1}(X, n; 1) \xrightarrow{\psi} \Omega^{n-1}_{k/\mathbb{Z}}$$

is zero.

It just follows from [5, Proposition 6.2, p. 150].

Claim (2): *Any two closed admissible points $p, p' \in X \times \widetilde{B}_n$ for which only the coordinates of X differ are equivalent as additive higher Chow cycles.*
Abusing notations, write $p = (a, b, s_1, \cdots, s_{n-1}), p' = (a', b, s_1, \cdots, s_{n-1})$, where a, a' are closed points of X, where $b \neq 0$, $s_i \neq 0, \infty$. Consider a parametrized line

$$C = \left\{ \left(a\frac{y}{y-1} + a'\left(1 - \frac{y}{y-1}\right), b, y, s_1, \cdots, s_{n-1} \right) \in X \times \widetilde{B}_{n+1} |\ \ y \in \square^1 \right\}.$$

This 1-cycle satisfies all the modulus conditions $M_{sum}, M_{sup}, M_{ssup}$ having $b \neq 0$, and it intersects all faces properly having constant y_i-coordinate values s_i.

By direct calculations, $\partial_1^0(C) = p', \partial_1^\infty(C) = p$, and $\partial_i^\epsilon(C) = 0$ for $i \geq 2$ and $\epsilon \in \{0, \infty\}$. Hence, $\partial(C) = p' - p$ proving Claim (2).

Given Claim (2), by [5, Proposition 6.3] and the rest of the arguments of [5, Theorem 6.4], this theorem follows. $\square$

We remark that the same arguments work for any variety X as long as we can prove Claim (2). In particular, for any connected union of affine spaces, irreducible or not, we can conclude the same results.

Acknowledgments. The authors would like to thank Spencer Bloch and Marc Levine for some valuable comments. JP would like to thank TIFR and KAIST for their hospitality and reduced teaching loads, and Juya for supports throughout this work.

For this research, JP was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2009-0063180).

References

[1] Bloch, S., *Algebraic cycles and higher K-theory*, Adv. Math., **61**, (1986), 267-304.
[2] Bloch, S., *The moving lemma for higher Chow groups*, J. Algebraic Geom., **3**, (1994), no. 3, 537-568.
[3] Bloch, S., *Lecture notes on higher Chow groups*, available at Bloch's Homepage, University of Chicago.
[4] Bloch, S., Esnault, H., *An additive version of higher Chow groups*, Ann. Sci. Éc. Norm. Supér., (4) **36**, (3), (2003), 463-477.
[5] Bloch, S., Esnault, H., *The additive dilogarithm*, Doc. Math., J., DMV **Extra Vol.**, (2003), 131-155.
[6] Chow, W.-L., *On equivalence classes of cycles in an algebraic variety*, Ann. of Math., (2) **64**, (1956), 450-479.
[7] Fantechi, B., Göttsche, L., Illusie, L., Kleiman, S. L., Nitsure, N., Vistoli, A., *Fundamental algebraic geometry. Grothendieck's FGA explained*, Mathematical Surveys and Monographs, **123**, American Mathematical Society, Providence, RI, 2005.
[8] Fulton, W., *Intersection theory*, Second Edition, Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge. A Series of Modern Surveys in Mathematics, **2**, Springer-Verlag, Berlin, 1998.
[9] Grothendieck, A., *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. IV.*, Inst. Hautes Études Sci. Publ. Math., **32**, (1967).
[10] Guillén, F., Navarro Aznar, V., *Un critère d'extension des foncteurs définis sur les schémas lisses*, Publ. Math. Inst. Hautes Études Sci., **95**, (2002), 1-91.

[11] Hartshorne, R., *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**, Springer-Verlag, New York-Heidelberg, 1977.
[12] Hesselholt L., *K-theory of truncated polynomial algebras*, Handbook of K-Theory, **1**, Springer-Verlag, Berlin, 2005, 71-110.
[13] Hesselholt, L., Madsen, I., *The K-theory of nilpotent endomorphisms*, Homotopy methods in algebraic topology (Boulder, CO, 1999), 127-140, Contemp. Math., **271**, Amer. Math. Soc., Providence, RI, 2001.
[14] Krishna, A., Levine, M., *Additive higher Chow groups of schemes*, J. Reine Angew. Math., **619,** (2008) 75-140.
[15] Levine, M., *Bloch's higher Chow groups revisited*, K-theory (Strasbourg, 1992). Astérisque, **226**, (1994), 10, 235-320.
[16] Levine, M., *Mixed Motives*, Mathematical Surveys and Monographs, **57**, American Mathematical Society, Providence, RI, 1998.
[17] Loday, J.-L. *Cyclic homology*, Second edition. Grundlehren der Mathematischen Wissenschaften, **301**, Springer-Verlag, Berlin, 1998.
[18] Park, J., *Regulators on additive higher Chow groups*, Amer. J. Math., **131,** (2009) no. 1, 257-276.
[19] Park, J., *Algebraic cycles and additive dilogarithm*, Int. Math. Res. Not., **18**, (2007), Article ID rnm067.
[20] Roberts, J., *Chow's moving lemma*, Appendix 2 to: "Motives", Algebraic geometry, Oslo, 1970, (Proc. Fifth Nordic Summer School in Math.), pp. 53-82, Wolters-Noordhoff, Groningen, 1972.
[21] Rülling, K., *The generalized de Rham-Witt complex over a field is a complex of zero-cycles*, J. Algebraic Geom., **16**, (2007), no. 1, 109-169.
[22] Totaro, B., *Milnor K-theory is the simplest part of algebraic K-theory*, K-theory, **6**, (2), (1992), 177-189.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai, India
E-mail address: amal@math.tifr.res.in

Department of Mathematical Sciences, KAIST, Daejeon, 305-701, Republic of Korea (South)
E-mail address: jinhyun@mathsci.kaist.ac.kr; jinhyun@kaist.edu